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Article

On Fractional q-Extensions of Some q-Orthogonal Polynomials

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Abstract: In this paper, we introduce a fractional q-extension of the q-differential operator $D_{q^{-1}}$ and prove some of its main properties. Next, fractional q-extensions of some classical q-orthogonal polynomials are introduced and some of the main properties of the newly-defined functions are given. Finally, a fractional q-difference equation of Gaussian type is introduced and solved by means of the power series method.

Keywords: Q-fractional calculus; Q-hypergeometric functions; Q-orthogonal polynomials; fractional q-difference equations

1. Introduction

Q-fractional calculus is the field of mathematical analysis which deals with the investigation and applications of derivatives and integrals of arbitrary (real or complex) order (see [1–8] and the references therein). It is an interesting topic having interconnections with various problems in function theory, integral and differential equations, and other branches of analysis. It has been continually developed, stimulated by ideas and results in various fields of mathematical analysis. This is demonstrated by the many publications—hundreds of papers in the past years—and by the many conferences devoted to the problems of fractional calculus.

A family $\{P_n(x)\}$, $(n \in \mathbb{N} := \{0, 1, 2 \ldots\}, k_n \neq 0)$ of polynomials of degree exactly n is a family of classical q-orthogonal polynomials of the q-Hahn class if it is the solution of a q-differential equation of the type

$$\sigma(x) D_q D_{1/q} P_n(x) + \tau(x) D_q P_n(x) + \lambda_n P_n(x) = 0, \tag{1}$$

where $\sigma(x) = ax^2 + bx + c$ is a polynomial of at most second order and $\tau(x) = dx + e$ is a polynomial of first order. Here, The q-difference operator D_q is defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0, \, q \neq 1,$$

and $D_q f(0) := f'(0)$ by continuity, provided $f'(0)$ exists.

The polynomial systems that are a solution of (1) form the q-Hahn tableau (see [9,10] and the references therein). These systems are contained in the so-called Askey–Wilson scheme [11]. The following systems are members of the q-Hahn tableau: the Big q-Jacobi polynomials, the q-Hahn polynomials, the Big q-Laguerre polynomials, the Little q-Jacobi polynomials, the q-Meixner polynomials, the Quantum q-Krawtchouk polynomials, the q-Krawtchouk polynomials, the Affine q-Krawtchouk polynomials, the Little q-Laguerre polynomials, the q-Laguerre polynomials, the Alternative q-Charlier (also called q-Bessel) polynomials, the q-Charlier polynomials, the Al-Salam–Carlitz I polynomials, the Al-Salam–Carlitz II polynomials, the Stieltjes–Wigert polynomials, the Discrete q-Hermite I polynomials, and the Discrete q-Hermite II polynomials.

In [12], the authors have defined some fractional extensions of the Jacobi polynomials from their Rodrigues representation and provided several properties of these new functions. They introduced a fractional version of the Gauss hypergeometric differential equation and used the modified power series method to provide some of its solutions. Note that, very recently, these Jacobi functions were used in [3] to provide new connection formulas for Jacobi polynomials. Note also that the previous Jacobi functions contain ultraspherical, Chebyshev of first, second, third and fourth kinds and Legendre functions as special cases.

In [13], the authors defined the C-Laguerre functions from the Rodrigues representation of the Laguerre polynomials by replacing the ordinary derivative by a fractional type derivative, then they gave several properties of the new defined functions.

In this work, we introduce a new q-differential operator of fractional order and use it to introduce the Little q-Jacobi, the Little q-Laguerre and the q-Laguerre functions. The hypergeometric representations of the new defined functions are given and the limit transitions are provided. Note that we obtained the results for the Big q-Jacobi, the Big q-Laguerre, the Big q-Legendre, the Al-Salam Carlitz-I and II and the Stieltjes–Wigert polynomials but did not include them because they are not of nice form.

The paper is organised as follows:

1. In Section 2, we present the preliminary results and definitions that are useful for a better reading of this manuscript.
2. In Section 3, we introduce the fractional q-calculus
3. In Section 4, we introduce a new fractional q-differential operator $D^{\alpha}_{q^{-1}}$ and apply it to some functions,
4. In Section 5, fractional q-extensions of some q-orthogonal polynomials are given and their basic hypergeometric representation provided. We prove for some of these new defined functions some limit transitions.
5. In Section 6, we introduce a fractional q-extension of the q-hypergeometric q-difference equation and provide some of its solution.

2. Preliminary Definitions and Results

This section contains some preliminary definitions and results that are useful for a better reading of the manuscript. The q-hypergeometric series, a fractional q-derivative and fractional q-integral are defined. The reader will consult the References [4,6,11,14] for more informations about these concepts and some applications.

Definition 1 (See [11])**.** *The basic hypergeometric or q-hypergeometric series ${}_r\phi_s$ is defined by the series*

$$ {}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix}\middle| q;z\right) := \sum_{n=0}^{\infty} \frac{(a_1,\cdots,a_r;q)_n}{(b_1,\cdots,b_s;q)_n}\left((-1)^n q^{\binom{k}{2}}\right)^{1+s-r}\frac{z^n}{(q;q)_n}, $$

where

$$ (a_1,\cdots,a_r)_n := (a_1;q)_n\cdots(a_r;q)_n, $$

with

$$ (a_i;q)_n = \begin{cases} \prod\limits_{j=0}^{n-1}(1-a_iq^j) & \text{if } n=1,2,3,\cdots \\ 1 & \text{if } n=0 \end{cases}. $$

For $n=\infty$ we set

$$ (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n),\ |q|<1. $$

The notation $(a;q)_n$ is the so-called q-Pochhammer symbol.

From the definition of $(a;q)_\infty$, it follows that for $0 < |q| < 1$, and for a nonnegative integer n, we have

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

Definition 2 (See [11])**.** *For any complex number λ,*

$$(a;q)_\lambda = \frac{(a;q)_\infty}{(aq^\lambda;q)_\infty}, 0 < |q| < 1,$$

where the principal value of q^λ is taken.

We will also use the following common notations

$$[a]_q = \frac{1-q^a}{1-q}, a \in \mathbb{C}, q \neq 1,$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}}, 0 \leq m \leq n,$$

and

$$(x \ominus y)_q^n = (x-y)(x-qy)\cdots(x-q^{n-1}y).$$

called the q-bracket and the q-binomial coefficients and the q-power respectively.

Proposition 1 ([11], Page 16)**.** *The basic hypergeometric series fulfil the following identities*

$$ {}_1\phi_0\left(\left.\begin{matrix} a \\ - \end{matrix}\right| q;z\right) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad 0 < |q| < 1, \ \ |z| < 1, \tag{2}$$

$$ {}_1\phi_1\left(\left.\begin{matrix} a \\ c \end{matrix}\right| \begin{matrix} c \\ \hline a \end{matrix}\right) = \frac{(c/a;q)_\infty}{(c;q)_\infty}, \quad 0 < |q| < 1. \tag{3}$$

Relation (2) is the so-called q-binomial theorem.

The next proposition gives some important Heine transformation formulas for basic hypergeometric series.

Proposition 2 ([15], p. 10)**.** *The following transformation formulas hold*

$$ {}_2\phi_1\left(\left.\begin{matrix} a,b \\ c \end{matrix}\right| z\right) = \frac{(az,b;q)_\infty}{(c,z;q)_\infty} {}_2\phi_1\left(\left.\begin{matrix} c/b,z \\ az \end{matrix}\right| q;b\right), \quad |z| < 1, \tag{4}$$

$$ {}_2\phi_1\left(\left.\begin{matrix} a,b \\ c \end{matrix}\right| q;z\right) = \frac{(abz/c;q)_\infty}{(z;q)_\infty} {}_2\phi_1\left(\left.\begin{matrix} c/a,b/a \\ c \end{matrix}\right| q;abz/c\right), \tag{5}$$

$$ {}_2\phi_1\left(\left.\begin{matrix} a,b \\ c \end{matrix}\right| z\right) = \frac{(az;q)_\infty}{(z;q)_\infty} {}_2\phi_2\left(\left.\begin{matrix} a,c/b \\ c,az \end{matrix}\right| bz\right), \quad |z| < 1,$$

$$ {}_2\phi_1\left(\left.\begin{matrix} a,b \\ 0 \end{matrix}\right| z\right) = \frac{(az;q)_\infty}{(z;q)_\infty} {}_1\phi_1\left(\left.\begin{matrix} a \\ az \end{matrix}\right| q;bz\right), \quad |z| < 1. \tag{6}$$

Proposition 3 ([16])**.** *The q-hypergeometric q-difference equation*

$$z(q^c - q^{a+b+1}z)(D_q^2 u)(z) \ \ + \left([c]_q - (q^b[a]_q + q^a[b+1]_q)z\right)(D_q u)(z) - [a]_q[b]_q u(z) = 0 \tag{7}$$

has particular solutions

$$u_1(z) = {}_2\phi_1\left(\begin{matrix} q^a, q^b \\ q^c \end{matrix}\middle|\, q; z\right), \quad \textit{and} \quad u_2(z) = z^{1-c}{}_2\phi_1\left(\begin{matrix} q^{1+a-c}, q^{1+b-c} \\ q^{2-c} \end{matrix}\middle|\, q; z\right).$$

Definition 3 (See [11]). *The q-Gamma function is defined by*

$$\Gamma_q(x) := \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \;\; 0 < q < 1. \tag{8}$$

Remark 1. *From Definition 2, the q-Gamma function is also represented by*

$$\Gamma_q(x) = (1-q)^{1-x}(q;q)_{x-1}.$$

Note also that the q-Gamma function satisfies the functional equation

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x), \quad \textit{with} \quad \Gamma_q(1) = 1.$$

Note that for arbitrary complex α,

$$\begin{bmatrix} \alpha \\ k \end{bmatrix}_q = \frac{(q^{-\alpha};q)_k}{(q;q)_k}(-1)^k q^{\alpha k - \binom{k}{2}} = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(k+1)\Gamma_q(\alpha-1)}. \tag{9}$$

The exponential function has two different natural q-extensions, denoted by $e_q(z)$ and $E_q(z)$, which can be defined by

$$e_q(z) := {}_1\phi_0\left(\begin{matrix} 0 \\ - \end{matrix}\middle|\, q; z\right) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty}, \quad 0 < |q| < 1, \;\; |z| < 1, \tag{10}$$

and

$$E_q(z) := {}_0\phi_0\left(\begin{matrix} - \\ - \end{matrix}\middle|\, q, -z\right) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n} z^n = (-z;q)_\infty, \quad 0 < |q| < 1. \tag{11}$$

These q-analogues of the exponential function are clearly related by

$$e_q(z)E_q(-z) = 1.$$

Corollary 1. *The following expansions apply.*

$$\begin{aligned} \frac{e_q(\alpha z)}{e_q(\beta z)} &= \sum_{n=0}^{\infty} \frac{(\beta\alpha^{-1};q)_n}{(q;q)_n}(\alpha z)^n = {}_1\phi_0\left(\begin{matrix} \beta\alpha^{-1} \\ - \end{matrix}\middle|\, q; \alpha z\right); \\ \frac{e_q(\alpha z)}{E_q(\beta z)} &= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \left(-\frac{\alpha}{\beta}\right)^k\right]\frac{(-\beta z)^n}{(q;q)_n}. \end{aligned}$$

Next, we recall some basic knowledge about fractional q-calculus. The usual starting point for a definition of fractional operators in q-calculus taken in [1,2,7,8,17], is the q-analogue of the Riemann–Liouville fractional integral

$$I_q^\alpha f(z) = \frac{z^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^z (tq/z;q)_{\alpha-1} f(t)d_qt = \frac{1}{\Gamma_q(\alpha)}\int_0^z (z \ominus qt)_q^{\alpha-1} f(t)d_qt. \tag{12}$$

This q-integral was motivated from the q-analogue of the Cauchy formula for a repeated q-integral

$$\begin{aligned} I^n_{q,a} f(z) &= \int_a^z d_q t \int_a^t d_q t_{n-1} \int_a^{t_{n-1}} d_q t_{n-2} \cdots \int_a^{t_2} f(t_1) d_q t_1 \\ &= \frac{z^{n-1}}{[n-1]_q!]} \int_0^z (tq/z;q)_{n-1} f(t) d_q t. \end{aligned} \tag{13}$$

The reduction of the multiple q-integral to a single one was considered by Al-Salam in [18]. In [8], the authors allow the lower parameter in (12) to be any real number $a \in (0,z)$. There are several definitions of the fractional q-integral and fractional q-derivatives. We adopt in this work the definition of the fractional q-integral given in [7].

Definition 4 (See [7])**.** *The fractional q-integral is*

$$(I^\alpha_{q,c} f)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_c^x (qt/x;q)_{\alpha-1} f(t) d_q t = \frac{1}{\Gamma_q(\alpha)} \int_c^x (x \ominus qt)_q^{\alpha-1} f(t) d_q t, \quad (\alpha \in \mathbb{R}^+).$$

In [7], it is proved that for $\alpha \in \mathbb{R}^+$, $\lambda, \lambda+\alpha \in \mathbb{R} \setminus \{-1,-2,\cdots\}$, equation is valid:

$$I^\alpha_{q,c} (x \ominus c)_q^\lambda = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)} (x \ominus c)_q^{\alpha+\lambda}, \quad (0 < c < x). \tag{14}$$

Definition 5 (See [7])**.** *The fractional q-derivative of Riemann–Liouville type of order $\alpha \in \mathbb{R}^+$ is*

$$(D^\alpha_{q,c} f)(x) = (D_q^{\lceil \alpha \rceil} I_{q,c}^{\lceil \alpha \rceil - \alpha} f)(x), \tag{15}$$

where $\lceil \alpha \rceil$ denotes the smallest integer greater or equal to α.

Mahmoud Annaby and Zeinab Mansour ([17], p. 148) prove that the Riemann–Liouville fractional operator $D^\alpha_{q,0}$ coincides with a q-analogue of the Grünwald Letnikov fractional operator defined by

$$\begin{aligned} (\mathcal{D}^\alpha_q f)(x) &= \frac{1}{x^\alpha (1-q)^\alpha} \sum_{n=0}^{\infty} (-1)^n \begin{bmatrix} \alpha \\ n \end{bmatrix}_q \frac{f(q^n x)}{q^{\frac{n(n-1)}{2} + n(\alpha-n)}} \\ &= \frac{1}{x^\alpha (1-q)^\alpha} \sum_{n=0}^{\infty} \frac{(q^{-\alpha};q)_n}{(q;q)_n} q^n f(q^n x). \end{aligned} \tag{16}$$

3. More q-Fractional Operator

Since many Rodrigues-type formulas for some of the orthogonal polynomials of the q-Hahn class are expressed in terms of the q-operator $D_{q^{-1}}$ instead of D_q, and since our new functions are defined using the Rodrigues-type formula of each family, there is a need to develop a fractional calculus for $D_{q^{-1}}$. The more natural way to do it is to start by the power derivative of $D_{q^{-1}}$. The following proposition (see [19]) gives the result.

Proposition 4 (See [19])**.** *Let $n \in \mathbb{N}_0$ and f a given function defined on $\{q^k,\ k \in \mathbb{Z}\}$. Then the following power derivative rule for $D_{q^{-1}}$ applies*

$$D^n_{q^{-1}} f(x) = \frac{q^{\binom{n+1}{2}}}{(1-q)^n x^n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2} - (n-1)k} f(q^{k-n} x). \tag{17}$$

Proof. This proof is also from [19]. The result is clear for $n=0$. Assume the assertion is true for $n \geq 0$, then:

$$\begin{aligned}
D_{q^{-1}}^{n+1} f(x) &= \frac{q^{\binom{n+1}{2}}}{(1-q)^n} D_{q^{-1}} \left(x^{-n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}-(n-1)k} f(q^{k-n}x) \right) \\
&= \frac{q^{\binom{n+1}{2}}}{(1-q)^n} \frac{q}{(1-q)x} \left(x^{-n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}-(n-1)k} f(q^{k-n}x) \right. \\
&\quad \left. -q^n x^{-n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}-(n-1)k} f(q^{k-n-1}x) \right) \\
&= \frac{-q^{\binom{n+1}{2}} q^{n+1}}{(1-q)^{n+1} x^{n+1}} \left(\sum_{k=1}^{n+1} (-1)^{k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q q^{\binom{k-1}{2}-(n-1)(k-1)-n} f(q^{k-n-1}x) \right. \\
&\quad \left. - \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}-(n-1)k} f(q^{k-n-1}x) \right) \\
&= \frac{q^{\binom{n+2}{2}}}{(1-q)^{n+1} x^{n+1}} \left(\sum_{k=1}^{n+1} (-1)^{k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q q^{\binom{k-1}{2}-(n-1)(k-1)-n} f(q^{k-n-1}x) \right. \\
&\quad \left. - \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}-(n-1)k} f(q^{k-n-1}x) \right) \\
&= \frac{q^{\binom{n+2}{2}}}{(1-q)^{n+1} x^{n+1}} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n+1 \\ k \end{bmatrix}_q q^{\binom{k}{2}-nk} f(q^{k-n-1}x).
\end{aligned}$$

□

Note that, using the obvious relation $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$, and reversing the order of summation, (17) reads

$$\begin{aligned}
D_{q^{-1}}^{n} f(x) &= \frac{q^{\binom{n+1}{2}}}{(1-q)^n x^n} \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ n-k \end{bmatrix}_q q^{\binom{n-k}{2}-(n-1)(n-k)} f(xq^{-k}) \\
&= \frac{q^{\binom{n+1}{2}}}{(q-1)^n x^n} \sum_{k=0}^{n} (-1)^{k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}-(n-1)(n-k)} f(xq^{-k}).
\end{aligned}$$

Next, using the fact that

$$\binom{n-k}{2} - (n-1)(n-k) = \frac{(n-k)(n-k-1) - 2(n-1)(n-k)}{2} = -\binom{n}{2} + \binom{k}{2},$$

we get

$$\begin{aligned}
D_{q^{-1}}^{n} f(x) &= \frac{q^{\binom{n+1}{2}}}{(q-1)^n x^n} \sum_{k=0}^{n} (-1)^{k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{-\binom{n}{2}+\binom{k}{2}} f(xq^{-k}) \\
&= \frac{q^n}{(q-1)^n x^n} \sum_{k=0}^{n} (-1)^{k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} f(xq^{-k}). \qquad (18)
\end{aligned}$$

Remark 2 (See [19,20]). *It is known that*

$$D_q^n f(x) = \frac{1}{(1-q)^n x^n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}-(n-1)k} f(q^k x). \qquad (19)$$

Note that replacing q by q^{-1} in (19), it follows that

$$\begin{aligned} D^n_{q^{-1}} f(x) &= \frac{1}{(1-q^{-1})^n x^n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} q^{-\binom{k}{2}-(n-1)k} f(q^{-k}x) \\ &= \frac{q^n}{(q-1)^n x^n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} q^{-\binom{k}{2}-(n-1)k} f(q^{-k}x). \end{aligned}$$

Taking care that

$$[n]_{q^{-1}} = \frac{1-q^{-n}}{1-q^{-1}} = \frac{1}{q^{n-1}} [n]_q,$$

it follows that

$$[n]_{q^{-1}}! = q^{-\binom{n}{2}} [n]_q!,$$

and so we get

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = \frac{q^{-\binom{n}{2}} [n]_q!}{q^{-\binom{k}{2}} [k]_q! q^{-\binom{n-k}{2}} [n-k]_q!} = q^{\binom{k}{2}+\binom{n-k}{2}-\binom{n}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

Now, we can write

$$\begin{aligned} D^n_{q^{-1}} f(x) &= \frac{1}{(1-q^{-1})^n x^n} \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}+\binom{n-k}{2}-\binom{n}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{-\binom{k}{2}-(n-1)k} f(q^{-k}x) \\ &= \frac{q^n}{(q-1)^n x^n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}-\binom{n}{2}-(n-1)k} f(q^{-k}x) \\ &= \frac{q^n}{(q-1)^n x^n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}-2\binom{n}{2}+(n-1)(n-k)-(n-1)k} f(q^{-k}x) \\ &= \frac{q^n}{(q-1)^n x^n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} f(xq^{-k}). \end{aligned}$$

This is exactely another way to write the result (17) obtained in [19] thanks to (18).

We are about to define a fractional extension of $D_{q^{-1}}$. Since $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ for $k > n$, we can write (18) as

$$D^n_{q^{-1}} f(x) = \frac{q^n}{(q-1)^n x^n} \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} f(xq^{-k}). \tag{20}$$

We extend Equation (20) to any arbitrary complex number α, $D^\alpha_{q^{-1}}$ in the following way:

$$D^\alpha_{q^{-1}} f(x) = \frac{q^\alpha}{(q-1)^\alpha x^\alpha} \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix} \alpha \\ k \end{bmatrix}_q q^{\binom{k}{2}} f(xq^{-k}),$$

provided that the infinite series of the right hand side converges. Now, using Equation (9), we obtain

$$\begin{aligned} D^\alpha_{q^{-1}} f(x) &= \frac{q^\alpha}{(q-1)^\alpha x^\alpha} \sum_{k=0}^{\infty} (-1)^k \frac{(q^{-\alpha};q)_k}{(q;q)_k} (-1)^k q^{\alpha k-\binom{k}{2}} q^{\binom{k}{2}} f(xq^{-k}) \\ &= \frac{q^\alpha}{(q-1)^\alpha x^\alpha} \sum_{k=0}^{\infty} \frac{(q^{-\alpha};q)_k}{(q;q)_k} q^{\alpha k} f(xq^{-k}). \end{aligned}$$

We then set the following definition.

Definition 6. *For any complex number α, we define the fractional operator $D^{\alpha}_{q^{-1}}$ by*

$$D^{\alpha}_{q^{-1}} f(x) = \frac{q^{\alpha}}{(q-1)^{\alpha} x^{\alpha}} \sum_{k=0}^{\infty} \frac{(q^{-\alpha};q)_k}{(q;q)_k} q^{\alpha k} f(xq^{-k}), \tag{21}$$

provided that the right hand side of (21) converges.

Note also that we could use directly (17) to write

$$\begin{aligned} {}_{\star}D^{\alpha}_{q^{-1}} f(x) &= \frac{q^{\frac{\alpha(\alpha-1)}{2}}}{(1-q)^{\alpha} x^{\alpha}} \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix} \alpha \\ k \end{bmatrix}_q q^{\binom{k}{2}-(\alpha-1)k} f(q^{k-\alpha}x) \\ &= \frac{q^{\frac{\alpha(\alpha-1)}{2}}}{(1-q)^{\alpha} x^{\alpha}} \sum_{k=0}^{\infty} (-1)^k \frac{(q^{-\alpha};q)_k}{(q;q)_k} (-1)^k q^{\alpha k - \binom{k}{2}} q^{\binom{k}{2}-(\alpha-1)k} f(q^{k-\alpha}x) \\ &= \frac{q^{\frac{\alpha(\alpha-1)}{2}}}{(1-q)^{\alpha} x^{\alpha}} \sum_{k=0}^{\infty} \frac{(q^{-\alpha};q)_k}{(q;q)_k} q^k f(q^{k-\alpha}x), \end{aligned}$$

which may be looked as another fractional extension of $D_{q^{-1}}$.

Proposition 5. *For $\alpha \in \mathbb{R} \setminus \{1,2,3,\cdots\}$, the following derivative rule applies*

$$D^{\lambda}_{q^{-1}} x^{\alpha} = \begin{cases} \dfrac{q^{\lambda}}{(q-1)^{\lambda}} \dfrac{(q^{-\alpha};q)_{\infty}}{(q^{\lambda-\alpha};q)_{\infty}} x^{\alpha-\lambda}, & \text{if} \quad \alpha-\lambda \in \mathbb{R}\setminus\mathbb{N} \\ 0, & \text{if} \quad \alpha-\lambda \in \mathbb{N} \end{cases}. \tag{22}$$

Proof. From (21), we get

$$\begin{aligned} D^{\lambda}_{q^{-1}} x^{\alpha} &= \frac{q^{\lambda}}{(q-1)^{\lambda} x^{\lambda}} \sum_{k=0}^{\infty} \frac{(q^{-\lambda};q)_k}{(q;q)_k} q^{\lambda k} (xq^{-k})^{\alpha} \\ &= \frac{q^{\lambda} x^{\alpha}}{(q-1)^{\lambda} x^{\lambda}} \sum_{k=0}^{\infty} \frac{(q^{-\lambda};q)_k}{(q;q)_k} q^{(\lambda-\alpha)k} \\ &= \frac{q^{\lambda}}{(q-1)^{\lambda}} x^{\alpha-\lambda} {}_1\phi_0 \left(\begin{matrix} q^{-\lambda} \\ - \end{matrix} \middle| q; q^{\lambda-\alpha} \right) \\ &= \frac{q^{\lambda}}{(q-1)^{\lambda}} \frac{(q^{-\alpha};q)_{\infty}}{(q^{\lambda-\alpha};q)_{\infty}} x^{\alpha-\lambda}. \end{aligned}$$

□

Before we state the semi-group property of the operator $D^{\alpha}_{q^{-1}}$, we state the following summation results that will be useful for the proof.

Lemma 1. (See [14], Lemma 10) *The following relation applies:*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k,n-k). \tag{23}$$

Lemma 2. (See [15], Exercise 1.3) *The following multiplication formula holds true*

$$(ab;q)_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q b^k (a;q)_k (b;q)_{n-k} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q a^{n-k} (a;q)_k (b;q)_{n-k}.$$

Using Lemmas 1 and 2 and the definition of $D^{\lambda}_{q^{-1}}$, we prove the following proposition.

Proposition 6 (Semi-group property)**.** *The following equation applies*

$$D^{\alpha}_{q^{-1}} D^{\beta}_{q^{-1}} f(x) = D^{\alpha+\beta}_{q^{-1}} f(x). \tag{24}$$

Proof. We have

$$\begin{aligned}
D^{\beta}_{q^{-1}} D^{\alpha}_{q^{-1}} f(x) &= D^{\beta}_{q^{-1}} \left[D^{\alpha}_{q^{-1}} f(x) \right] \\
&= \frac{q^{\alpha}}{(q-1)^{\alpha}} D^{\beta}_{q^{-1}} \left[\frac{1}{x^{\alpha}} \sum_{k=0}^{\infty} \frac{(q^{-\alpha};q)_k}{(q;q)_k} q^{\alpha k} f(xq^{-k}) \right] \\
&= \frac{q^{\alpha}}{(q-1)^{\alpha}} \frac{q^{\beta}}{(q-1)^{\beta} x^{\beta}} \sum_{n=0}^{\infty} \frac{(q^{-\beta};q)_n}{(q;q)_n} q^{\beta n} \left(\frac{1}{(xq^{-n})^{\alpha}} \sum_{k=0}^{\infty} \frac{(q^{-\alpha};q)_k}{(q;q)_k} q^{\alpha k} f(xq^{-n-k}) \right) \\
&= \frac{q^{\alpha+\beta}}{(q-1)^{\alpha+\beta} x^{\alpha+\beta}} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q^{-\beta};q)_n}{(q;q)_n} \frac{(q^{-\alpha};q)_k}{(q;q)_k} q^{\alpha(n+k)} q^{\beta n} f(xq^{-n-k}) \\
&= \frac{q^{\alpha+\beta}}{(q-1)^{\alpha+\beta} x^{\alpha+\beta}} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{(q^{-\beta};q)_{n-k}}{(q;q)_{n-k}} \frac{(q^{-\alpha};q)_k}{(q;q)_k} q^{-\beta k} \right) q^{(\alpha+\beta)n} f(xq^{-n}) \\
&= \frac{q^{\alpha+\beta}}{(q-1)^{\alpha+\beta} x^{\alpha+\beta}} \sum_{n=0}^{\infty} \frac{(q^{-(\alpha+\beta)};q)_n}{(q;q)_n} q^{(\alpha+\beta)n} f(xq^{-n}) = D^{\alpha+\beta}_{q^{-1}} f(x).
\end{aligned}$$

This proves the proposition. □

4. Fractional q-Extensions of Some q-Orthogonal Polynomials

In this section we introduce some fractional q-extensions of some orthogonal polynomials of the q-Hahn class. The families that are of interest here are those which use D_q and $D_{q^{-1}}$ in their Rodrigues representations.

4.1. The Little q-Jacobi Functions

The Little q-Jacobi polynomials have the q-hypergeometric representation ([11], p. 482)

$$p_n(x;a,b|q) = {}_2\phi_1 \left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} \middle| q; qx \right).$$

They can also be represented by the Redrigues-type formula

$$w(x;\alpha,\beta;q) p_n(x;q^{\alpha},q^{\beta}|q) = \frac{q^{n\alpha+\binom{n}{2}}(1-q)^n}{(q^{\alpha+1};q)_n} D^n_{q^{-1}} [w(x;\alpha+n,\beta+n;q)],$$

where

$$w(x;\alpha,\beta;q) = \frac{(qx;q)_{\infty}}{(q^{\beta+1}x;q)_{\infty}} x^{\alpha}.$$

Definition 7. *Let $\lambda \in \mathbb{R}$, we define the fractional Little q-Jacobi functions by*

$$P_{\lambda}(x;q^{\alpha},q^{\beta}|q) = \frac{q^{\lambda\alpha+\frac{\lambda(\lambda-1)}{2}}(1-q)^{\lambda}}{w(x;\alpha,\beta;q)(q^{\alpha+1};q)_{\lambda}} D^{\lambda}_{q^{-1}} [w(x;\alpha+\lambda,\beta+\lambda;q)]. \tag{25}$$

Proposition 7. *The fractional Little q-Jacobi functions defined by relation (25) have the following basic hypergeometric representation*

$$P_\lambda(x;q^\alpha,q^\beta|q)=(-1)^\lambda q^{\lambda\alpha+\frac{\lambda(\lambda+1)}{2}}\frac{(q^{-(\alpha+\lambda)};q)_\infty}{(q^{\alpha+1};q)_\lambda(q^{-\alpha};q)_\infty}{}_2\phi_1\left(\begin{matrix}q^{-\lambda},q^{\alpha+\beta+\lambda+1}\\ q^{\alpha+1}\end{matrix}\middle|\,q;qx\right).$$

Proof. Applying the q-binomial theorem (2), we have

$$\begin{aligned}w(x;\alpha,\beta;q)&=x^\alpha\frac{(qx;q)_\infty}{(q^{\beta+1}x;q)_\infty}\\&=\sum_{n=0}^{\infty}\frac{(q^{-\beta};q)_n}{(q;q)_n}q^{n(\beta+1)}x^{n+\alpha}.\end{aligned}$$

Thus,

$$w(x;\alpha+\lambda,\beta+\lambda;q)=\sum_{n=0}^{\infty}\frac{(q^{-(\beta+\lambda)};q)_n}{(q;q)_n}q^{n(\beta+\lambda+1)}x^{n+\alpha+\lambda}.$$

Hence,

$$\begin{aligned}&D^\lambda_{q^{-1}}[w(x;\alpha+\lambda,\beta+\lambda;q)]\\&\quad=\sum_{n=0}^{\infty}\frac{(q^{-(\beta+\lambda)};q)_n}{(q;q)_n}q^{n(\beta+\lambda+1)}D^\lambda_{q^{-1}}\left[x^{n+\alpha+\lambda}\right]\\&\quad=\frac{q^\lambda}{(q-1)^\lambda}\sum_{n=0}^{\infty}\frac{(q^{-(\beta+\lambda)};q)_n}{(q;q)_n}\frac{(q^{-n}q^{-(\alpha+\lambda)};q)_\infty}{(q^{-n}q^{-\alpha};q)_\infty}q^{n(\beta+\lambda+1)}x^{n+\alpha}.\end{aligned}$$

Using the fact that

$$(aq^{-n};q)_\infty=(-a)^n(a^{-1}q;q)_nq^{-\binom{n+1}{2}}(a;q)_\infty,$$

we have

$$\frac{(q^{-n}q^{-(\alpha+\lambda)};q)_\infty}{(q^{-n}q^{-\alpha};q)_\infty}=\frac{(q^{-(\alpha+\lambda)};q)_\infty}{(q^{-\alpha};q)_\infty}\frac{(q^{\alpha+\lambda+1};q)_n}{(q^{\alpha+1};q)_n}q^{-\lambda n}.$$

Whence

$$\begin{aligned}&D^\lambda_{q^{-1}}[w(x;\alpha+\lambda,\beta+\lambda;q)]\\&\quad=\frac{q^\lambda x^\alpha}{(q-1)^\lambda}\frac{(q^{-(\alpha+\lambda)};q)_\infty}{(q^{-\alpha};q)_\infty}\sum_{n=0}^{\infty}\frac{(q^{-(\beta+\lambda)};q)_n}{(q;q)_n}\frac{(q^{\alpha+\lambda+1};q)_n}{(q^{\alpha+1};q)_n}q^{n(\beta+1)}x^n\\&\quad=\frac{q^\lambda x^\alpha}{(q-1)^\lambda}\frac{(q^{-(\alpha+\lambda)};q)_\infty}{(q^{-\alpha};q)_\infty}{}_2\phi_1\left(\begin{matrix}q^{-(\beta+\lambda)},q^{\alpha+\lambda+1}\\ q^{\alpha+1}\end{matrix}\middle|\,q;xq^{\beta+1}\right).\end{aligned}$$

Now, applying the Heine–Euler transformation formula (5), we have

$${}_2\phi_1\left(\begin{matrix}q^{-(\beta+\lambda)},q^{\alpha+\lambda+1}\\ q^{\alpha+1}\end{matrix}\middle|\,q;xq^{\beta+1}\right)=\frac{(qx;q)_\infty}{(q^{\beta+1};q)_\infty}{}_2\phi_1\left(\begin{matrix}q^{-\lambda},q^{\alpha+\beta+\lambda+1}\\ q^{\alpha+1}\end{matrix}\middle|\,q;qx\right).$$

So we have

$$\frac{D^\lambda_{q^{-1}}[w(x;\alpha+\lambda,\beta+\lambda;q)]}{w(x;\alpha,\beta;q)}=\frac{q^\lambda}{(q-1)^\lambda}\frac{(q^{-(\alpha+\lambda)};q)_\infty}{(q^{-\alpha};q)_\infty}{}_2\phi_1\left(\begin{matrix}q^{-\lambda},q^{\alpha+\beta+\lambda+1}\\ q^{\alpha+1}\end{matrix}\middle|\,q;qx\right).$$

We finally obtain

$$P_\lambda(x;q^\alpha,q^\beta;q) = (-1)^\lambda q^{\lambda\alpha+\frac{\lambda(\lambda+1)}{2}} \frac{(q^{-(\alpha+\lambda)};q)_\infty}{(q^{\alpha+1};q)_\lambda (q^{-\alpha};q)_\infty} {}_2\phi_1\left(\begin{matrix} q^{-\lambda}, q^{\alpha+\beta+\lambda+1} \\ q^{\alpha+1}\end{matrix}\middle|\, q;qx\right).$$

□

Proposition 8. *The following limit transition holds:*

$$\lim_{\lambda\to n} P_\lambda(x;q^\alpha,q^\beta;q) = p_n(x;q^\alpha,q^\beta;q),$$

where n is a nonnegative integer.

Proof. It is not difficult to see that

$$\begin{aligned}\lim_{\lambda\to n}&\left((-1)^\lambda q^{\lambda\alpha+\frac{\lambda(\lambda+1)}{2}} \frac{(q^{-(\alpha+\lambda)};q)_\infty}{(q^{\alpha+1};q)_\lambda (q^{-\alpha};q)_\infty}\right)\\ &= (-1)^n q^{n\alpha+\binom{n+1}{2}} \frac{(q^{-(\alpha+n)};q)_\infty}{(q^{\alpha+1};q)_n (q^{-\alpha};q)_\infty}\\ &= (-1)^n q^{n\alpha+\binom{n+1}{2}} \frac{(-1)^n q^{-\alpha n} q^{-\binom{n+1}{2}} (q^{\alpha+1};q)_n (q^{-\alpha};q)_\infty}{(q^{\alpha+1};q)_n (q^{-\alpha};q)_\infty} = 1,\end{aligned}$$

so,

$$\lim_{\lambda\to n} P_\lambda(x;q^\alpha,q^\beta;q) = p_n(x;q^\alpha,q^\beta;q).$$

□

4.2. The Little q-Laguerre Functions

The Little q-Laguerre polynomials have the q-hypergeometric representation ([11], p. 518)

$$p_n(x,a|q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, 0 \\ aq\end{matrix}\middle|\, q;qx\right).$$

They can also be represented by the Rodrigues-type formula ([11], p. 520)

$$w(x;\alpha;q)p_n(x;q^\alpha;q) = \frac{q^{n\alpha+\binom{n}{2}}(1-q)^n}{(q^{\alpha+1};q)_n} D^n_{q^{-1}}[w(x;\alpha+n;q)],$$

with

$$w(x;\alpha;q) = x^\alpha (qx;q)_\infty.$$

Definition 8. *Let $\lambda \in \mathbb{R}$, we define the fractional Little q-Laguerre functions by*

$$P_\lambda(x;q^\alpha;q) = \frac{q^{\lambda\alpha+\binom{\lambda}{2}}(1-q)^\lambda}{w(x;\alpha;q)(q^{\alpha+1};q)_\lambda} D^\lambda_{q^{-1}}[w(x;\alpha+\lambda;q)]. \tag{26}$$

Proposition 9. *The fractional Little q-Laguerre functions defined by relation (26) have the following representation*

$$P_\lambda(x;q^\alpha|q) = (-1)^\lambda q^{\lambda\alpha+\frac{\lambda(\lambda+1)}{2}} \frac{(q^{-(\alpha+\lambda)};q)_\infty}{(q^{\alpha+1};q)_\lambda (q^{-\alpha};q)_\infty} {}_2\phi_1\left(\begin{matrix} q^{-\lambda}, 0 \\ q^{\alpha+1}\end{matrix}\middle|\, qx\right).$$

Proof. It is easy to see that

$$w(x;\alpha;q) = x^{\alpha}(qx;q)_{\infty} = \sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}}}{(q;q)_n}(-1)^n x^{n+\alpha}.$$

So,

$$w(x;\alpha+\lambda;q) = \sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}}}{(q;q)_n}(-1)^n x^{n+\alpha+\lambda}.$$

Thus, using the definitions of the fractional Little q-Laguerre functions and the fractional q-derivative (21), combined with the transformations (4) and (6) we have:

$$\begin{aligned}
D_{q^{-1}}^{\lambda}[w(x;\alpha+\lambda;q)] &= \sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}}}{(q;q)_n}(-1)^n D_{q^{-1}}^{\lambda}[x^{n+\alpha+\lambda}] \\
&= \sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}}}{(q;q)_n}(-1)^n \frac{q^{\lambda}}{(q-1)^{\lambda}} \frac{(q^{-n}q^{-(\alpha+\lambda)};q)_{\infty}}{(q^{-n}q^{-\alpha};q)_{\infty}} x^{n+\alpha} \\
&= \frac{x^{\alpha}q^{\lambda}}{(q-1)^{\lambda}} \frac{(q^{-(\alpha+\lambda)};q)_{\infty}}{(q^{-\alpha};q)_{\infty}} \sum_{n=0}^{\infty} \frac{(q^{\alpha+\lambda+1};q)_n}{(q^{\alpha+1};q)_n} \left((-1)^n q^{\binom{n}{2}}\right) \frac{(q^{1-\lambda}x)^n}{(q;q)_n} \\
&= \frac{x^{\alpha}q^{\lambda}}{(q-1)^{\lambda}} \frac{(q^{-(\alpha+\lambda)};q)_{\infty}}{(q^{-\alpha};q)_{\infty}} {}_1\phi_1\left(\begin{matrix} q^{\alpha+\lambda+1} \\ q^{\alpha+1} \end{matrix} \middle| q; xq^{-\lambda+1}\right) \\
&\overset{(6)}{=} \frac{x^{\alpha}q^{\lambda}}{(q-1)^{\lambda}} \frac{(q^{-(\alpha+\lambda)};q)_{\infty}}{(q^{-\alpha};q)_{\infty}} \frac{(q^{-\lambda};q)_{\infty}}{(q^{\alpha+1};q)_{\infty}} {}_2\phi_0\left(\begin{matrix} q^{\alpha+\lambda+1}, qx \\ 0 \end{matrix} \middle| q^{-\lambda}\right) \\
&\overset{(4)}{=} \frac{x^{\alpha}(qx;q)_{\infty}q^{\lambda}}{(q-1)^{\lambda}} \frac{(q^{-(\alpha+\lambda)};q)_{\infty}}{(q^{-\alpha};q)_{\infty}} {}_2\phi_1\left(\begin{matrix} q^{-\lambda}, 0 \\ q^{\alpha+1} \end{matrix} \middle| qx\right) \\
&= (-1)^{\lambda} q^{\lambda\alpha+\frac{\lambda(\lambda+1)}{2}} \frac{(q^{-(\alpha+\lambda)};q)_{\infty}}{(q^{\alpha+1};q)_{\lambda}(q^{-\alpha};q)_{\infty}} {}_2\phi_1\left(\begin{matrix} q^{-\lambda}, 0 \\ q^{\alpha+1} \end{matrix} \middle| qx\right).
\end{aligned}$$

□

Proposition 10. *Let n be a nonnegative integer, then the following limit transition holds true*

$$\lim_{\lambda\to n} P_{\lambda}(x;q^{\alpha};q) = p_n(x;q^{\alpha};q).$$

Proof. Since

$$\begin{aligned}
&\lim_{\lambda\to n}\left((-1)^{\lambda} q^{\lambda\alpha+\frac{\lambda(\lambda+1)}{2}} \frac{(q^{-(\alpha+\lambda)};q)_{\infty}}{(q^{\alpha+1};q)_{\lambda}(q^{-\alpha};q)_{\infty}}\right) \\
&\qquad = (-1)^n q^{n\alpha+\binom{n+1}{2}} \frac{(q^{-(\alpha+n)};q)_{\infty}}{(q^{\alpha+1};q)_n(q^{-\alpha};q)_{\infty}} \\
&\qquad = (-1)^n q^{n\alpha+\binom{n+1}{2}} \frac{(-1)^n q^{-\alpha n} q^{-\binom{n+1}{2}} (q^{\alpha+1};q)_n (q^{-\alpha};q)_{\infty}}{(q^{\alpha+1};q)_n(q^{-\alpha};q)_{\infty}} = 1,
\end{aligned}$$

we have, $\lim_{\lambda\to n} P_{\lambda}(x;q^{\alpha};q) = p_n(x;q^{\alpha};q)$. □

4.3. The q-Laguerre Functions

The q-Laguerre polynomials have the q-hypergeometric representation ([11], p. 522)

$$L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} {}_1\phi_1\left(\begin{matrix} q^{-n} \\ q^{\alpha+1} \end{matrix} \middle| q; -q^{n+\alpha+1}x\right).$$

They can also be represented by the Rodrigues-type formula ([11], p. 524)

$$w(x;\alpha;q)L_n^{(\alpha)}(x;q) = \frac{(1-q)^n}{(q;q)_n} D_q^n[w(x;\alpha+n;q)],$$

with

$$w(x;\alpha;q) = \frac{x^\alpha}{(-x;q)_\infty}.$$

Definition 9. *Let $\lambda \in \mathbb{R}$, we define the fractional q-Laguerre functions by*

$$L_\lambda^\alpha(x;q) = \frac{(1-q)^\lambda\,(-x,;q)_\infty}{(q;q)_\lambda x^\alpha} D_q^\lambda\left[\frac{x^{\alpha+\lambda}}{(-x;q)_\infty}\right]. \tag{27}$$

Proposition 11. *The fractional q-Laguerre functions defined by relation (27) have the following basic hypergeometric representation*

$$\begin{aligned} L_\lambda^\alpha(x;q) &= \frac{1}{(q;q)_\lambda} {}_2\phi_1\left(\begin{matrix} q^{-\lambda}, -x \\ 0 \end{matrix} \middle| q; q^{\alpha+\lambda+1}\right) \\ &= \frac{(q^{\alpha+1};q)_\lambda}{(q;q)_\lambda} {}_1\phi_1\left(\begin{matrix} q^{-\lambda} \\ q^{\alpha+1} \end{matrix} \middle| q; -xq^{\alpha+\lambda+1}\right). \end{aligned}$$

Proof. From the definition of the q-exponential (10), it follows that

$$\frac{x^{\alpha+\lambda}}{(-x;q)_\infty} = x^{\alpha+\lambda} e_q(-x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{\alpha+\lambda+n}}{(q;q)_n}.$$

Then,

$$\begin{aligned} D_{q,0}^\lambda\left[\frac{x^{\alpha+\lambda}}{(-x;q)_\infty}\right] &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(q;q)_n} \frac{\Gamma_q(\alpha+\lambda+n+1)}{\Gamma_q(\alpha+n+1)} x^{\alpha+n} \\ &= x^\alpha \frac{\Gamma_q(\alpha+\lambda+1)}{\Gamma_q(\alpha+1)} \sum_{n=0}^{\infty} \frac{(q^{\alpha+\lambda+1};q)_n}{(q;q)_n (q^{\alpha+1};q)_n} (-x)^n \\ &= x^\alpha \frac{\Gamma_q(\alpha+\lambda+1)}{\Gamma_q(\alpha+1)} {}_2\phi_1\left(\begin{matrix} 0, q^{\alpha+\lambda+1} \\ q^{\alpha+1} \end{matrix} \middle| q; -x\right) \\ &= \frac{x^\alpha}{(1-q)^\lambda} \frac{(q^{\alpha+1};q)_\infty}{(q^{\alpha+\lambda+1};q)_\infty} {}_2\phi_1\left(\begin{matrix} 0, q^{\alpha+\lambda+1} \\ q^{\alpha+1} \end{matrix} \middle| q; -x\right). \end{aligned}$$

Using the Heine transformation (4), we have

$${}_2\phi_1\left(\begin{matrix} 0, q^{\alpha+\lambda+1} \\ q^{\alpha+1} \end{matrix} \middle| q; -x\right) = \frac{(q^{\alpha+\lambda+1};q)_\infty}{(q^{\alpha+1};q)_\infty (-x;q)_\infty} {}_2\phi_1\left(\begin{matrix} q^{-\lambda}; -x \\ 0 \end{matrix} \middle| q; q^{\alpha+\lambda+1}\right).$$

Hence, we have

$$D_{q,0}^{\lambda}\left[\frac{x^{\alpha+\lambda}}{(-x;q)_\infty}\right]=\frac{x^{\alpha}}{(1-q)^{\lambda}(-x;q)_\infty}{}_2\phi_1\left(\begin{matrix}q^{-\lambda};-x\\0\end{matrix}\middle|q;q^{\alpha+\lambda+1}\right).$$

Finally it follows that

$$L_{\lambda}^{(\alpha)}(x;q)=\frac{1}{(q;q)_\lambda}{}_2\phi_1\left(\begin{matrix}q^{-\lambda};-x\\0\end{matrix}\middle|q;q^{\alpha+\lambda+1}\right).$$

Next, using the transformation formula (6) we get

$$\begin{aligned}\frac{1}{(q;q)_\lambda}{}_2\phi_1\left(\begin{matrix}q^{-\lambda};-x\\0\end{matrix}\middle|q;q^{\alpha+\lambda+1}\right) &= \frac{1}{(q;q)_\lambda}\frac{(q^{\alpha+1};q)_\infty}{(q^{\alpha+\lambda+1};q)_\infty}{}_1\phi_1\left(\begin{matrix}q^{-\lambda}\\q^{\alpha+1}\end{matrix}\middle|q;-xq^{\alpha+\lambda+1}\right)\\ &= \frac{(q^{\alpha+1};q)_\lambda}{(q;q)_\lambda}{}_1\phi_1\left(\begin{matrix}q^{-\lambda}\\q^{\alpha+1}\end{matrix}\middle|q;-xq^{\alpha+\lambda+1}\right).\end{aligned}$$

This ends the proof of the proposition. □

5. Fractional q-Gauss Differential Equation

In this section, we give a fractional version of the q-hypergeometric q-difference equation given by Koorwinder in [16]. Next, we solve this fractional q-difference equation by means of modified power series.

Definition 10. *The fractional q-hypergeometric q-difference equation is defined for* $0<\lambda\leq 1$ *by*

$$\begin{aligned}&z^{\lambda}(q^{c}-q^{a+b+1}z^{\lambda})(D_q^{2\lambda}u)(z)\\&\quad+\left([c]_q-(q^{b}[a]_q+q^{a}[b+1]_q)z^{\lambda}\right)(D_q^{\lambda}u)(z)-[a]_q[b]_q u(z)=0.\end{aligned}\tag{28}$$

Definition 11. *The fractional q-Gauss function is defined as the series*

$${}_2\phi_1^{\mu}\left(\begin{matrix}q^{a},q^{b}\\q^{c}\end{matrix}\middle|q;z\right)=u_0 z^{\rho}\sum_{n=0}^{\infty}\prod_{k=0}^{n}\frac{g_{q,k}(\rho)}{f_{q,k+1}(\rho)}z^{n\lambda},\quad 0<\lambda\leq 1,\tag{29}$$

where

$$f_{q,k}(\rho)=q^{c}\frac{\Gamma_q(1+\rho+k\lambda)}{\Gamma_q(1+\rho+(k-2)\lambda)}+[c]_q\frac{\Gamma_q(1+\rho+k\lambda)}{\Gamma_q(1+\rho+(k-1)\lambda)},\tag{30}$$

$$\begin{aligned}g_{q,k}(\rho) &= q^{a+b+1}\frac{\Gamma_q(1+\rho+k\lambda)}{\Gamma_q(1+\rho+(k-2)\lambda)}\\ &\quad+(q^{b}[a]_q+q^{a}[b+1]_q)\frac{\Gamma_q(1+\rho+k\lambda)}{\Gamma_q(1+\rho+(k-1)\lambda)}+[a]_q[b]_q,\end{aligned}\tag{31}$$

and $\rho>-1$ *satisfies the equation*

$$f_{q,0}(\rho)=\frac{\Gamma_q(1+\rho)}{\Gamma_q(1+\rho-2\lambda)}+[c]_q\frac{\Gamma_q(1+\rho)}{\Gamma_q(1+\rho-\lambda)}=0.\tag{32}$$

The following assertion is valid.

Theorem 1. *The fractional q-Gauss function* (29) *is a solution of the fractonal q-Gauss hypergeometric Equation* (28) *where* $f_{q,k}(\rho)$ *and* $g_{q,k}(\rho)$ *are given by* (30) *and* (31)*, repectively and* ρ *satisfies the condition* (32).

Proof. We look for the solution under the following modified formal power series form

$$u(z) = \sum_{n=0}^{\infty} d_n z^{n\lambda+\rho}.$$

Then,

$$D_q^{\lambda} u(z) = \sum_{n=0}^{\infty} d_n \frac{\Gamma_q(n\lambda+\rho+1)}{\Gamma_q((n-1)\lambda+\rho+1)} z^{(n-1)\lambda+\rho},$$

and

$$D_q^{2\lambda} u(z) = \sum_{n=0}^{\infty} d_n \frac{\Gamma_q(n\lambda+\rho+1)}{\Gamma_q((n-2)\lambda+\rho+1)} z^{(n-2)\lambda+\rho}.$$

Inserting these fractional q-derivatives in (28), we obtain the following recurrence relation for the coefficients a_n,

$$f_{q,n}(\rho) d_{n+1} - g_{q,n+1}(\rho) d_n = 0,$$

with $f_{q,0}(\rho) = 0$, where $f_{q,n}(\rho)$ and $g_{q,n}(\rho)$ are given by (30) and (31) respectively. The theorem follows easily. □

6. Conclusions and Further Perspectives

In this work we have introduced a new fractional q-differential operator $D^{\alpha}_{q^{-1}}$ and have proved some of its main important properties. Then, we have used it to extend some families of classical q-orthogonal polynomials. We have also defined a fractional q-Gauss differential equation, extending the one introduced by Koorwinder in [16], and solve them by means of the power series method. It should be noted that we obtained the results the Big q-Jacobi, the Big q-Laguerre, the Big q-Legendre, the Al-Salam Carlitz-I and II and the Stieltjes–Wigert polynomials but did not include them because they are not of nice form.

As future works, we plan to provide similar extensions to classical orthogonal polynomials on quadratic and q-quadratic lattices. To do it, it will be necessary to introduce new differential operators of fractional order, namely, the Wilson operator [11,21]

$$\mathbf{D} f(x) = \frac{f(x+i/2) - f(x-i/2)}{2ix}$$

or the Askey–Wilson divided difference operator [11]

$$\mathcal{D}_q f(x) = \frac{\hat{f}(q^{1/2}e^{i\theta}) - \hat{f}(q^{-1/2}e^{i\theta})}{i(q^{1/2} - q^{-1/2}) \sin\theta}$$

with

$$\hat{f}(e^{i\theta}) = f(x), \quad x = \cos\theta.$$

For these operators, the paper of Cooper [22] will help to define their fractional extensions. This work is ongoing.

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