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Strong Convergence of Extragradient-Type Method to Solve Pseudomonotone Variational Inequalities Problems

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Abstract: A number of applications from mathematical programmings, such as minimax problems, penalization methods and fixed-point problems can be formulated as a variational inequality model. Most of the techniques used to solve such problems involve iterative algorithms, and that is why, in this paper, we introduce a new extragradient-like method to solve the problems of variational inequalities in real Hilbert space involving pseudomonotone operators. The method has a clear advantage because of a variable stepsize formula that is revised on each iteration based on the previous iterations. The key advantage of the method is that it works without the prior knowledge of the Lipschitz constant. Strong convergence of the method is proved under mild conditions. Several numerical experiments are reported to show the numerical behaviour of the method.

Keywords: pseudomonotone mapping; subgradient extragradient method; strong convergence; Hilbert spaces; variational inequality problems

1. Introduction

In this article, we consider the classic variational inequalities problems (VIPs) [1,2] for an operator $\mathcal{F} : \mathcal{E} \to \mathcal{E}$ is formulated in the following way:

$$\text{Find } u^* \in \mathcal{K} \text{ such that } \langle \mathcal{F}(u^*), y - u^* \rangle \geq 0, \ \forall y \in \mathcal{K}, \tag{1}$$

where $\mathcal{K}$ is a nonempty, convex and closed subset of a real Hilbert space $\mathcal{E}$. The inner product and induced norm on $\mathcal{E}$ are denoted by $\langle ., . \rangle$ and $\|.\|$, respectively. Moreover, the set of real and natural numbers are denoted by $\mathcal{R}$ and $\mathcal{N}$, respectively. It is important to note that solving the problem (1) is equivalent to solving the following problem:

$$\text{Find an element } u^* \in \mathcal{K} \text{ such that } u^* = P_{\mathcal{K}}[u^* - \zeta \mathcal{F}(u^*)].$$

We assume that the following requirements have been fulfilled:

(B1) The solution set of the problem (1), represented by SVIP is nonempty.

(B2) A mapping $\mathcal{F} : \mathcal{E} \to \mathcal{E}$ is called to be pseudomonotone, i.e.,

$$\langle \mathcal{F}(y_1), y_2 - y_1 \rangle \geq 0 \Longrightarrow \langle \mathcal{F}(y_2), y_1 - y_2 \rangle \leq 0, \ \forall y_1, y_2 \in \mathcal{K}.$$

(B3) A mapping $\mathcal{F} : \mathcal{E} \to \mathcal{E}$ is said to be Lipschitz continuous, i.e., there exists $L > 0$ such that

$$\|\mathcal{F}(y_1) - \mathcal{F}(y_2)\| \leq L\|y_1 - y_2\|, \ \forall y_1, y_2 \in \mathcal{K}.$$

(B4) A mapping $\mathcal{F} : \mathcal{E} \to \mathcal{E}$ is called to be sequentially weakly continuous, i.e., $\{\mathcal{F}(u_n)\}$ converges weakly to $\mathcal{F}(u)$, where $\{u_n\}$ weakly converges to u.

The concept of variational inequalities has been used as a powerful tool to study different subjects, i.e., physics, engineering, economics and optimization theory. The problem (1) was firstly introduced by Stampacchia [1] in 1964 and also provided that this problem (1) is a crucial problem in nonlinear analysis. This is an efficient mathematical technique that integrates several key elements of applied mathematics, i.e., the problems of network equilibrium, the necessary optimality conditions, the complementarity problems and the systems of non-linear equations (for more details [3–9]). On the other hand, the projection methods are important to find the numerical solution of variational inequalities. Many authors have proposed and studied different projection methods to solve the problem of variational inequalities (see for more details [10–20]) and others in [21–32]. In particular, Karpelevich [10] and Antipin [33] introduced the following extragradient method:

$$\begin{cases} u_n \in \mathcal{K}, \\ v_n = P_{\mathcal{K}}[u_n - \zeta \mathcal{F}(u_n)], \\ u_{n+1} = P_{\mathcal{K}}[u_n - \zeta \mathcal{F}(v_n)]. \end{cases} \tag{2}$$

Recently, the subgradient extragradient algorithm was established by Censor et al. [12] for solving problem (1) in real Hilbert space. Their method has the form

$$\begin{cases} u_n \in \mathcal{K}, \\ v_n = P_{\mathcal{K}}[u_n - \zeta \mathcal{F}(u_n)], \\ u_{n+1} = P_{\mathcal{E}_n}[u_n - \zeta \mathcal{F}(v_n)]. \end{cases} \tag{3}$$

where $\mathcal{E}_n = \{z \in \mathcal{E} : \langle u_n - \zeta \mathcal{F}(u_n) - v_n, z - v_n \rangle \leq 0\}$. Migorski et al. [34] proposed a viscosity-type subgradient extragradient method to solve monotone variational inequalities problems. The main contribution is the presence of a viscosity scheme in the algorithm that was used to improve the convergence rate of the iterative sequence and provide strong convergence theorem. The iterative sequence $\{u_n\}$ was generated in the following way: (i) Let $u_0 \in \mathcal{K}$, $\mu \in (0,1)$, $\zeta_0 > 0$ and a sequence $\gamma_n \subset (0,1)$ with $\gamma_n \to 0$ and $\sum_n^\infty \gamma_n = +\infty$. (ii) Compute

$$\begin{cases} v_n = P_{\mathcal{K}}[u_n - \zeta_n \mathcal{F}(u_n)], \\ w_n = P_{\mathcal{E}_n}[u_n - \zeta_n \mathcal{F}(v_n)], \\ u_{n+1} = \gamma_n f(u_n) + (1 - \gamma_n) w_n, \end{cases} \tag{4}$$

where

$$\mathcal{E}_n = \{z \in \mathcal{E} : \langle u_n - \zeta_n \mathcal{F}(u_n) - v_n, z - v_n \rangle \leq 0\}.$$

(iii) Revised the stepsize in the following way:

$$\zeta_{n+1} = \begin{cases} \min\left\{\zeta_n, \frac{\mu\|u_n - v_n\|}{\|\mathcal{F}(u_n) - \mathcal{F}(v_n)\|}\right\} & \text{if} \quad \mathcal{F}(u_n) \neq \mathcal{F}(v_n), \\ \zeta_n & \quad \text{otherwise.} \end{cases}$$

In this paper, inspired by the iterative methods in [12,16,35,36], a modified subgradient extragradient algorithm is proposed for solving variational inequalities problems involving pseudomonotone mapping in real Hilbert space. It is important to note that our proposed scheme is effective. In particular, by comparing the results of Migorski et al. [34], our algorithm can solve pseudomonotone variational inequalities. Similar to the results of Migorski et al. [34] the proof of strong convergence of the proposed algorithm is proved without knowing the Lipschitz constant of the operator $\mathcal{F}$. The proposed algorithm could be seen as a modification of the methods that are appeared in [10,12,34–36]. Under mild conditions, a strong convergence theorem is proved. Numerical experiments have been shown that the new approach tends to be more successful than the existing one [34].

The rest of this article has been arranged as follows: Section 2 contains some definitions and basic results that have been used throughout the paper. Section 3 contains our main algorithm and a strong convergence theorem. Section 4 presents the numerical results showing the algorithmic efficacy of the proposed method.

2. Preliminaries

This section contains useful lemmas and basic identities that have been used throughout the article. The metric projection $P_{\mathcal{K}}(u_1)$ for $u_1 \in \mathcal{E}$ onto a closed and convex subset $\mathcal{K}$ of $\mathcal{E}$ is defined by

$$P_{\mathcal{K}}(u_1) = \arg\min\{\|u_2 - u_1\| : u_2 \in \mathcal{K}\}.$$

Lemma 1. [37,38] *Assume $\mathcal{K}$ is a nonempty, convex and closed subset of a real Hilbert space $\mathcal{E}$ and $P_{\mathcal{K}} : \mathcal{E} \to \mathcal{K}$ is a metric projection from $\mathcal{E}$ onto $\mathcal{K}$.*

(i) *Let $u_1 \in \mathcal{K}$ and $u_2 \in \mathcal{E}$, we have*

$$\|u_1 - P_{\mathcal{K}}(u_2)\|^2 + \|P_{\mathcal{K}}(u_2) - u_2\|^2 \leq \|u_1 - u_2\|^2.$$

(ii) *$u_3 = P_{\mathcal{K}}(u_1)$ if and only if*

$$\langle u_1 - u_3, u_2 - u_3\rangle \leq 0, \ \forall u_2 \in \mathcal{K}.$$

(iii) *For $u_2 \in \mathcal{K}$ and $u_1 \in \mathcal{E}$*

$$\|u_1 - P_{\mathcal{K}}(u_1)\| \leq \|u_1 - u_2\|.$$

Lemma 2. [37] *Let $u, v \in \mathcal{E}$ and $\varpi \in \mathcal{R}$.*

(i) $\|\varpi u + (1-\varpi)v\|^2 = \varpi\|u\|^2 + (1-\varpi)\|v\|^2 - \varpi(1-\varpi)\|u - v\|^2$.

(ii) $\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v\rangle$.

Lemma 3. [39] *Assume that $\{\chi_n\}$ be a sequence of non-negative real numbers satisfying*

$$\chi_{n+1} \leq (1 - \tau_n)\chi_n + \tau_n\delta_n, \ \forall n \in \mathcal{N},$$

where $\{\tau_n\} \subset (0, 1)$ and $\{\delta_n\} \subset \mathcal{R}$ satisfy the following conditions:

$$\lim_{n\to\infty} \tau_n = 0, \ \sum_{n=1}^{\infty} \tau_n = \infty, \ \textit{and} \ \limsup_{n\to\infty} \delta_n \leq 0.$$

Then, $\lim_{n\to\infty} \chi_n = 0$.

Lemma 4. [40] *Assume that* $\{\chi_n\}$ *is a sequence of real numbers such that there exists a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that* $\chi_{n_i} < \chi_{n_{i+1}}$ *for all* $i \in \mathcal{N}$*. Then, there exists a non decreasing sequence* $m_k \subset \mathcal{N}$ *such that* $m_k \to \infty$ *as* $k \to \infty$*, and the following conditions are fulfilled by all (sufficiently large) numbers* $k \in \mathcal{N}$*:*

$$\chi_{m_k} \leq \chi_{m_{k+1}} \text{ and } \chi_k \leq \chi_{m_{k+1}}.$$

In fact, $m_k = \max\{j \leq k : \chi_j \leq \chi_{j+1}\}$*.*

Lemma 5. [41] *Assume that* $\mathcal{F} : \mathcal{K} \to \mathcal{E}$ *is a pseudomonotone and continuous mapping. Then,* u^* *is a solution of the problem* (1) *if and only if* u^* *is a solution of the following problem.*

$$\text{Find } x \in \mathcal{K} \text{ such that } \langle \mathcal{F}(y), y - x \rangle \geq 0, \ \forall y \in \mathcal{K}.$$

3. Main Results

We provide a method consisting of two convex minimization problems through a viscosity scheme and an explicit stepsize formula which is being used to improve the convergence rate of the iterative sequence and to make the method independent of the Lipschitz constants. The detailed method is provided in Algorithm 1.

Algorithm 1 (Explicit method for pseudomonotone variational inequalities problems).

Step 0: Let $u_0 \in \mathcal{K}$, $\mu \in (0,1)$, $\zeta_0 > 0$ and a sequence $\gamma_n \subset (0,1)$ satisfying

$$\lim_{n\to\infty} \gamma_n = 0 \quad \text{and} \quad \sum_{n}^{\infty} \gamma_n = +\infty.$$

Step 1: Evaluate

$$v_n = P_{\mathcal{K}}[u_n - \zeta_n \mathcal{F}(u_n)].$$

If $u_n = v_n$; STOP. Otherwise, go to **Step 2**.

Step 2: Evaluate

$$w_n = P_{\mathcal{E}_n}[u_n - \zeta_n \mathcal{F}(v_n)],$$

where $\mathcal{E}_n = \{z \in \mathcal{E} : \langle u_n - \zeta_n \mathcal{F}(u_n) - v_n, z - v_n \rangle \leq 0\}$.

Step 3: Compute

$$u_{n+1} = \gamma_n f(u_n) + (1 - \gamma_n) w_n.$$

Step 4: Evaluate

$$\zeta_{n+1} = \begin{cases} \min\left\{\zeta_n, \frac{\mu\|u_n - v_n\|^2 + \mu\|w_n - v_n\|^2}{2\langle \mathcal{F}(u_n) - \mathcal{F}(v_n), w_n - v_n \rangle}\right\} & \text{if} \quad \langle \mathcal{F}(u_n) - \mathcal{F}(v_n), w_n - v_n \rangle > 0, \\ \zeta_n & \text{else.} \end{cases}$$

Lemma 6. *The stepsize sequence* $\{\zeta_n\}$ *is monotonically decreasing with a lower bound* $\min\{\frac{\mu}{L}, \zeta_0\}$ *and converges to a fixed* $\zeta > 0$*.*

Proof. Let $\langle \mathcal{F}(u_n) - \mathcal{F}(v_n), w_n - v_n \rangle > 0$, such that

$$\begin{aligned} \frac{\mu(\|u_n - v_n\|^2 + \|w_n - v_n\|^2)}{2\langle \mathcal{F}(u_n) - \mathcal{F}(v_n), w_n - v_n \rangle} &\geq \frac{2\mu\|u_n - v_n\|\|w_n - v_n\|}{2\|\mathcal{F}(u_n) - \mathcal{F}(v_n)\|\|w_n - v_n\|} \\ &\geq \frac{2\mu\|u_n - v_n\|\|w_n - v_n\|}{2\|u_n - v_n\|\|w_n - v_n\|} \\ &\geq \frac{\mu}{L}. \end{aligned} \tag{5}$$

Clearly, from above we can conclude that $\{\zeta_n\}$ has a lower bound $\min\{\frac{\mu}{L}, \zeta_0\}$. Moreover, there exists a real number $\zeta > 0$, such that $\lim_{n\to\infty} \zeta_n = \zeta$. □

Lemma 7. *Assume that* $\mathcal{F} : \mathcal{E} \to \mathcal{E}$ *satisfies the conditions* (B1)–(B4). *For a given* $u^* \in SVIP \neq \emptyset$, *we have*

$$\|w_n - u^*\|^2 \leq \|u_n - u^*\|^2 - \Big(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\Big)\|u_n - v_n\|^2 - \Big(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\Big)\|w_n - v_n\|^2.$$

Proof. Consider that

$$\begin{aligned} \| w_n - u^* \|^2 &= \| P_{\mathcal{E}_n}[u_n - \zeta_n\mathcal{F}(v_n)] - u^* \|^2 \\ &= \| P_{\mathcal{E}_n}[u_n - \zeta_n\mathcal{F}(v_n)] + [u_n - \zeta_n\mathcal{F}(v_n)] - [u_n - \zeta_n\mathcal{F}(v_n)] - u^* \|^2 \\ &= \| [u_n - \zeta_n\mathcal{F}(v_n)] - u^* \|^2 + \| P_{\mathcal{E}_n}[u_n - \zeta_n\mathcal{F}(v_n)] - [u_n - \zeta_n\mathcal{F}(v_n)] \|^2 \\ &\quad + 2\langle P_{\mathcal{E}_n}[u_n - \zeta_n\mathcal{F}(v_n)] - [u_n - \zeta_n\mathcal{F}(v_n)], [u_n - \zeta_n\mathcal{F}(v_n)] - u^*\rangle . \end{aligned} \tag{6}$$

Given that $u^* \in SVIP \subset \mathcal{K} \subset \mathcal{E}_n$, we get

$$\begin{aligned} &\| P_{\mathcal{E}_n}[u_n - \zeta_n\mathcal{F}(v_n)] - [u_n - \zeta_n\mathcal{F}(v_n)] \|^2 \\ &\quad + \langle P_{\mathcal{E}_n}[u_n - \zeta_n\mathcal{F}(v_n)] - [u_n - \zeta_n\mathcal{F}(v_n)], [u_n - \zeta_n\mathcal{F}(v_n)] - u^*\rangle \\ &= \langle [u_n - \zeta_n\mathcal{F}(v_n)] - P_{\mathcal{E}_n}[u_n - \zeta_n\mathcal{F}(v_n)], u^* - P_{\mathcal{E}_n}[u_n - \zeta_n\mathcal{F}(v_n)]\rangle \leq 0, \end{aligned} \tag{7}$$

which implies that

$$\begin{aligned} &\langle P_{\mathcal{E}_n}[u_n - \zeta_n\mathcal{F}(v_n)] - [u_n - \zeta_n\mathcal{F}(v_n)], [u_n - \zeta_n\mathcal{F}(v_n)] - u^*\rangle \\ &\leq -\| P_{\mathcal{E}_n}[u_n - \zeta_n\mathcal{F}(v_n)] - [u_n - \zeta_n\mathcal{F}(v_n)] \|^2. \end{aligned} \tag{8}$$

Using expressions (6) and (8), we obtain

$$\begin{aligned} \| w_n - u^* \|^2 &\leq \| u_n - \zeta_n\mathcal{F}(v_n) - u^* \|^2 - \| P_{\mathcal{E}_n}[u_n - \zeta_n\mathcal{F}(v_n)] - [u_n - \zeta_n\mathcal{F}(v_n)] \|^2 \\ &\leq \|u_n - u^* \|^2 - \| u_n - w_n \|^2 + 2\zeta_n\langle \mathcal{F}(v_n), u^* - w_n\rangle . \end{aligned} \tag{9}$$

Since u^* is the solution of problem (1), we have

$$\langle \mathcal{F}(u^*), y - u^*\rangle \geq 0, \text{ for all } y \in \mathcal{K}.$$

Due to the pseudomonotonicity of $\mathcal{F}$ on $\mathcal{K}$, we get

$$\langle \mathcal{F}(y), y - u^*\rangle \geq 0, \text{ for all } y \in \mathcal{K}.$$

By substituting $y = v_n \in \mathcal{K}$, we get

$$\langle \mathcal{F}(v_n), v_n - u^*\rangle \geq 0.$$

Thus, we have

$$\big\langle \mathcal{F}(v_n), u^* - w_n\big\rangle = \big\langle \mathcal{F}(v_n), u^* - v_n\big\rangle + \big\langle \mathcal{F}(v_n), v_n - w_n\big\rangle \leq \big\langle \mathcal{F}(v_n), v_n - w_n\big\rangle. \tag{10}$$

Combining expressions (9) and (10), we obtain

$$\begin{aligned} \| w_n - u^* \|^2 &\leq \|u_n - u^* \|^2 - \| u_n - w_n \|^2 + 2\zeta_n\langle \mathcal{F}(v_n), v_n - w_n\rangle \\ &\leq \|u_n - u^* \|^2 - \| u_n - v_n + v_n - w_n \|^2 + 2\zeta_n\langle \mathcal{F}(v_n), v_n - w_n\rangle \\ &\leq \|u_n - u^* \|^2 - \|u_n - v_n \|^2 - \| v_n - w_n \|^2 + 2\langle u_n - \zeta_n\mathcal{F}(v_n) - v_n, w_n - v_n\rangle . \end{aligned} \tag{11}$$

Note that $w_n = P_{\mathcal{E}_n}[u_n - \zeta_n \mathcal{F}(v_n)]$ and by the definition of ζ_{n+1}, we have

$$
\begin{aligned}
&2\left\langle u_n - \zeta_n \mathcal{F}\left(v_n\right) - v_n, w_n - v_n\right\rangle \\
&= 2\left\langle u_n - \zeta_n \mathcal{F}\left(u_n\right) - v_n, w_n - v_n\right\rangle + 2\zeta_n \left\langle \mathcal{F}\left(u_n\right) - \mathcal{F}\left(v_n\right), w_n - v_n\right\rangle \\
&\leq \frac{2\zeta_n}{\zeta_{n+1}} \zeta_{n+1} \left\langle \mathcal{F}\left(u_n\right) - \mathcal{F}\left(v_n\right), w_n - v_n\right\rangle \leq \frac{\zeta_n}{\zeta_{n+1}}\left[\mu \left\|u_n - v_n\right\|^2 + \mu \left\| w_n - v_n\right\|^2\right].
\end{aligned}
\tag{12}
$$

Combining expressions (11) and (12), we obtain

$$
\begin{aligned}
&\| w_n - u^* \|^2 \\
&\leq \|u_n - u^* \|^2 - \|u_n - v_n \|^2 - \| v_n - w_n \|^2 + \frac{\zeta_n}{\zeta_{n+1}}\left[\mu \|u_n - v_n \|^2 + \mu \| w_n - v_n \|^2\right] \\
&\leq \|u_n - u^* \|^2 - \left(1 - \frac{\mu \zeta_n}{\zeta_{n+1}}\right) \| u_n - v_n \|^2 - \left(1 - \frac{\mu \zeta_n}{\zeta_{n+1}}\right) \| w_n - v_n \|^2.
\end{aligned}
\tag{13}
$$

□

Lemma 8. *Suppose that conditions* (B1)–(B4) *hold. Let* $\{u_n\}$ *be a sequence generated by Algorithm 1. If there is a subsequence* $\{u_{n_k}\}$ *which is weakly convergent to* $\hat{u} \in \mathcal{E}$ *and* $\lim_{n\to\infty} \|u_n - v_n\| = 0$, *then* $\hat{u} \in SVIP$.

Proof. We have

$$v_{n_k} = P_{\mathcal{K}}[u_{n_k} - \zeta_{n_k} \mathcal{F}(u_{n_k})], \tag{14}$$

which is equivalent to

$$\langle u_{n_k} - \zeta_{n_k} \mathcal{F}(u_{n_k}) - v_{n_k}, y - v_{n_k}\rangle \leq 0, \ \forall y \in \mathcal{K}. \tag{15}$$

From expression (15), we can write

$$\langle u_{n_k} - v_{n_k}, y - v_{n_k}\rangle \leq \zeta_{n_k} \langle \mathcal{F}(u_{n_k}), y - v_{n_k}\rangle, \ \forall y \in \mathcal{K}. \tag{16}$$

Therefore, we get

$$\frac{1}{\zeta_{n_k}} \langle u_{n_k} - v_{n_k}, y - v_{n_k}\rangle + \langle \mathcal{F}(u_{n_k}), v_{n_k} - u_{n_k}\rangle \leq \langle \mathcal{F}(u_{n_k}), y - u_{n_k}\rangle, \ \forall y \in \mathcal{K}. \tag{17}$$

Due to the boundedness of the sequence $\{u_{n_k}\}$ so does $\{\mathcal{F}(u_{n_k})\}$. By using the facts $\lim_{n\to\infty} \|u_{n_k} - v_{n_k}\| = 0$, and $\lim_{k\to\infty} \zeta_{n_k} = \zeta > 0$, limit as $k \to \infty$ in (17), we get

$$\liminf_{k\to\infty} \langle \mathcal{F}(u_{n_k}), y - u_{n_k}\rangle \geq 0, \ \forall y \in \mathcal{K}. \tag{18}$$

Moreover, we have

$$\langle \mathcal{F}(v_{n_k}), y - v_{n_k}\rangle = \langle \mathcal{F}(v_{n_k}) - \mathcal{F}(u_{n_k}), y - u_{n_k}\rangle + \langle \mathcal{F}(u_{n_k}), y - u_{n_k}\rangle + \langle \mathcal{F}(v_{n_k}), u_{n_k} - v_{n_k}\rangle. \tag{19}$$

Since $\lim_{n\to\infty} \|u_{n_k} - v_{n_k}\| = 0$, and $\mathcal{F}$ is L-Lipschitz continuous on $\mathcal{E}$, we get

$$\lim_{n\to\infty} \|\mathcal{F}(u_{n_k}) - \mathcal{F}(v_{n_k})\| = 0. \tag{20}$$

From (19) and (20), we obtain

$$\liminf_{k\to\infty} \langle \mathcal{F}(v_{n_k}), y - v_{n_k}\rangle \geq 0, \ \forall y \in \mathcal{K}. \tag{21}$$

Next, we show that $u^* \in SVIP$. We choose a sequence $\{\epsilon_k\}$ of positive numbers decreasing and tending to 0. For each k, we denote by m_k the smallest positive integer such that

$$\liminf_{k\to\infty} \langle \mathcal{F}(u_{n_i}), y - u_{n_i} \rangle + \epsilon_k \geq 0, \ \forall i \geq m_k. \tag{22}$$

Due to $\{\epsilon_k\}$ being decreasing, the sequence $\{m_k\}$ is increasing.

Case 1: If there is a subsequence $u_{n_{m_{k_j}}}$ of $u_{n_{m_k}}$ such that $\mathcal{F}(u_{n_{m_{k_j}}}) = 0$ $(\forall j)$. Letting $j \to \infty$, we obtain

$$\langle \mathcal{F}(u^*), y - u^* \rangle = \lim_{j\to\infty} \langle \mathcal{F}(u_{n_{m_{k_j}}}), y - u^* \rangle = 0. \tag{23}$$

Hence $u^* \in \mathcal{K}$, therefore we have $u^* \in SVIP$.

Case 2: If there exists N_0 such that for all $n_{m_k} \geq N_0$, $\mathcal{F}(u_{n_{m_k}}) \neq 0$. Suppose that

$$\Theta_{n_{m_k}} = \frac{\mathcal{F}(u_{n_{m_k}})}{\|\mathcal{F}(u_{n_{m_k}})\|^2}, \ \forall n_{m_k} \geq N_0. \tag{24}$$

Due to the above definition, we obtain

$$\langle \mathcal{F}(u_{n_{m_k}}), \mathcal{F}(\Theta_{n_{m_k}}) \rangle = 1, \ \forall n_{m_k} \geq N_0. \tag{25}$$

From (18) and (25), for all $n_{m_k} \geq N_0$, we have

$$\langle \mathcal{F}(u_{n_{m_k}}), \ y + \epsilon_k \Theta_{n_{m_k}} - u_{n_{m_k}} \rangle \geq 0. \tag{26}$$

Due to pseudomonotonicity of $\mathcal{F}$ for $n_{m_k} \geq N_0$, we obtain

$$\langle \mathcal{F}(y + \epsilon_k \Theta_{n_{m_k}}), \ y + \epsilon_k \Theta_{n_{m_k}} - u_{n_{m_k}} \rangle \geq 0. \tag{27}$$

For all $n_{m_k} \geq N_0$, we have

$$\langle \mathcal{F}(y), y - u_{n_{m_k}} \rangle \geq \langle \mathcal{F}(y) - \mathcal{F}(y + \epsilon_k \Theta_{n_{m_k}}), \ y + \epsilon_k \Theta_{n_{m_k}} - u_{n_{m_k}} \rangle - \epsilon_k \langle \mathcal{F}(y), \Theta_{n_{m_k}} \rangle. \tag{28}$$

Since $\{u_{n_k}\}$ converges weakly to $u^* \in \mathcal{K}$ and $\mathcal{F}$ is sequentially weakly continuous on $\mathcal{K}$, we have $\{\mathcal{F}(u_{n_k})\}$ converges weakly to $\mathcal{F}(u^*)$. We can suppose that $\mathcal{F}(u^*) \neq 0$. Since the norm mapping is sequentially weakly lower semicontinuous, we have

$$\|\mathcal{F}(u^*)\| \leq \liminf_{k\to\infty} \|\mathcal{F}(u_{n_k})\|. \tag{29}$$

Since $\{u_{n_{m_k}}\} \subset \{u_{n_k}\}$ and $\lim_{k\to\infty} \epsilon_k = 0$, we have

$$0 \leq \lim_{k\to\infty} \|\epsilon_k \Theta_{n_{m_k}}\| = \lim_{k\to\infty} \frac{\epsilon_k}{\|\mathcal{F}(u_{n_{m_k}})\|} \leq \frac{0}{\|\mathcal{F}(u^*)\|} = 0. \tag{30}$$

Now, letting $k \to \infty$ in (28), we obtain

$$\langle \mathcal{F}(y), y - u^* \geq 0, \ \forall y \in \mathcal{K}. \tag{31}$$

Applying the well-known Lemma 5, we can deduce that $u^* \in SVIP$. □

Theorem 1. *Assume that $\mathcal{F} : \mathcal{K} \to \mathcal{E}$ satisfies the conditions* (B1)–(B4). *Moreover, assume that u^* belongs to the solution set SVIP. Then, the sequences $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ generated by Algorithm 1 converge strongly to u^*.*

Proof. By using Lemma 7, we have

$$\|w_n - u^*\|^2 \le \|u_n - u^*\|^2 - \left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right)\|u_n - v_n\|^2 - \left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right)\|w_n - v_n\|^2. \quad (32)$$

Due to $\zeta_n \to \zeta$, there exists a fixed number $\epsilon \in (0, 1-\mu)$ such that

$$\lim_{n\to\infty}\left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right) = 1 - \mu > \epsilon > 0.$$

Then, there exists a finite number $N_1 \in \mathcal{N}$ such that

$$\left(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\right) > \epsilon > 0, \ \forall n \ge N_1. \quad (33)$$

Hence, we obtain

$$\|w_n - u^*\|^2 \le \|u_n - u^*\|^2, \ \forall n \ge N_1. \quad (34)$$

From the definition of the sequence $\{u_{n+1}\}$ and the fact that f is a contraction with constant $\rho \in [0,1)$ and $n \ge N_1$, we obtain

$$\begin{aligned}\|u_{n+1} - u^*\| &= \|\gamma_n f(u_n) + (1-\gamma_n)w_n - u^*\| \\ &= \|\gamma_n[f(u_n) - u^*] + (1-\gamma_n)[w_n - u^*]\| \\ &= \|\gamma_n[f(u_n) + f(u^*) - f(u^*) - u^*] + (1-\gamma_n)[w_n - u^*]\| \\ &\le \gamma_n\|f(u_n) - f(u^*)\| + \gamma_n\|f(u^*) - u^*\| + (1-\gamma_n)\|w_n - u^*\| \\ &\le \gamma_n\rho\|u_n - u^*\| + \gamma_n\|f(u^*) - u^*\| + (1-\gamma_n)\|w_n - u^*\|.\end{aligned} \quad (35)$$

From expressions (34) and (36) and $\gamma_n \subset (0,1)$, we obtain

$$\begin{aligned}\|u_{n+1} - u^*\| &\le \gamma_n\rho\|u_n - u^*\| + \gamma_n\|f(u^*) - u^*\| + (1-\gamma_n)\|u_n - u^*\| \\ &= [1 - \gamma_n + \rho\gamma_n]\|u_n - u^*\| + \gamma_n(1-\rho)\frac{\|f(u^*) - u^*\|}{(1-\rho)} \\ &\le \max\left\{\|u_n - u^*\|, \frac{\|f(u^*) - u^*\|}{(1-\rho)}\right\} \\ &\le \max\left\{\|u_{N_1} - u^*\|, \frac{\|f(u^*) - u^*\|}{(1-\rho)}\right\}.\end{aligned} \quad (36)$$

Hence, we conclude that the sequence $\{u_n\}$ is bounded. Next, the reflexivity of $\mathcal{E}$ and the boundedness of the sequence $\{u_n\}$ guarantee that there exists a subsequence $\{u_{n_k}\}$ such that $\{u_{n_k}\} \rightharpoonup u^* \in \mathcal{E}$ as $k \to \infty$. Now, we prove the strong convergence of the sequence iterative sequence $\{u_n\}$ generated by Algorithm 1. Due to the continuity and pseudomonotonicity of the operator $\mathcal{F}$ imply that the solution set $SVIP$ is a closed and convex set (for more details see [42,43]). Since the mapping f is a contraction, $P_{SVIP} \circ f$ is a contraction. The Banach contraction theorem guarantee the existence of a fixed point of $u^* \in SVIP$ such that

$$u^* = P_{SVIP}(f(u^*)).$$

By using Lemma 1 (ii), we have

$$\langle f(u^*) - u^*, y - u^*\rangle \le 0, \ \forall y \in SVIP. \quad (37)$$

From given $u_{n+1} = \gamma_n f(u_n) + (1-\gamma_n)w_n$, and using Lemma 2 (i) and Lemma 7, we have

$$\begin{aligned}
&\left\|u_{n+1} - u^*\right\|^2 \\
&= \left\|\gamma_n f(u_n) + (1-\gamma_n)w_n - u^*\right\|^2 \\
&= \left\|\gamma_n[f(u_n) - u^*] + (1-\gamma_n)[w_n - u^*]\right\|^2 \\
&= \gamma_n\|f(u_n) - u^*\|^2 + (1-\gamma_n)\|w_n - u^*\|^2 - \gamma_n(1-\gamma_n)\|f(u_n) - w_n\|^2 \\
&\le \gamma_n\|f(u_n) - u^*\|^2 + (1-\gamma_n)\Big[\|u_n - u^*\|^2 - \Big(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\Big)\|u_n - v_n\|^2 \\
&\quad - \Big(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\Big)\|w_n - v_n\|^2\Big] - \gamma_n(1-\gamma_n)\|f(u_n) - w_n\|^2 \\
&\le \gamma_n\|f(u_n) - u^*\|^2 + \|u_n - u^*\|^2 - (1-\gamma_n)\Big(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\Big)\Big[\|w_n - v_n\|^2 + \|u_n - v_n\|^2\Big].
\end{aligned} \tag{38}$$

The rest of the proof shall be divided into the following two parts:

Case 1: Assume that there exists a fixed number $N_2 \in \mathcal{N}$ $(N_2 \ge N_1)$ such that

$$\|u_{n+1} - u^*\| \le \|u_n - u^*\|, \ \forall n \ge N_2. \tag{39}$$

Thus, $\lim_{n\to\infty}\|u_n - u^*\|$ exists and let $\lim_{n\to\infty}\|u_n - u^*\| = l$. From expression (38), we have

$$\begin{aligned}
&(1-\gamma_n)\Big(1 - \frac{\mu\zeta_n}{\zeta_{n+1}}\Big)\Big[\|w_n - v_n\|^2 + \|u_n - v_n\|^2\Big] \\
&\le \gamma_n\|f(u_n) - u^*\|^2 + \|u_n - u^*\|^2 - \|u_{n+1} - u^*\|^2.
\end{aligned} \tag{40}$$

Due to the existence of $\lim_{n\to\infty}\|u_n - u^*\| = l$, and $\gamma_n \to 0$, we deduce that

$$\lim_{n\to\infty}\|u_n - v_n\| = \lim_{n\to\infty}\|w_n - v_n\| = 0. \tag{41}$$

From expression (41), we have

$$\lim_{n\to\infty}\|u_n - w_n\| \le \lim_{n\to\infty}\|u_n - v_n\| + \lim_{n\to\infty}\|v_n - w_n\| = 0. \tag{42}$$

It follows that

$$\begin{aligned}
\| u_{n+1} - u_n \| &= \| \gamma_n f(u_n) + (1-\gamma_n) w_n - u_n \| \\
&= \| \gamma_n [f(u_n) - u_n] + (1-\gamma_n)[w_n - u_n] \| \\
&\le \gamma_n \| f(u_n) - u_n \| + (1-\gamma_n) \| w_n - u_n \| \longrightarrow 0.
\end{aligned} \tag{43}$$

Thus, the sequences $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded. Thus, we can take a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ weakly converges to some $\hat{u} \in \mathcal{E}$. Moreover, due to $\|u_n - v_n\| \to 0$ and using Lemma 8, we have $\hat{u} \in SVIP$. By following expression (37), we consider that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle f(u^*) - u^*, u_n - u^*\rangle \\
&= \limsup_{k\to\infty}\langle f(u^*) - u^*, u_{n_k} - u^*\rangle = \langle f(u^*) - u^*, \hat{u} - u^*\rangle \le 0.
\end{aligned} \tag{44}$$

We have $\lim_{n\to\infty}\left\|u_{n+1} - u_n\right\| = 0$. It follows (44) that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle f(u^*) - u^*, u_{n+1} - u^*\rangle \\
&\le \limsup_{k\to\infty}\langle f(u^*) - u^*, u_{n+1} - u_n\rangle + \limsup_{k\to\infty}\langle f(u^*) - u^*, u_n - u^*\rangle \le 0.
\end{aligned} \tag{45}$$

From Lemma 2 (ii) and Lemma 7 for all $n \geq N_2$, we get

$$\begin{aligned}
&\left\|u_{n+1} - u^*\right\|^2 \\
&= \left\|\gamma_n f(u_n) + (1-\gamma_n)w_n - u^*\right\|^2 \\
&= \left\|\gamma_n[f(u_n) - u^*] + (1-\gamma_n)[w_n - u^*]\right\|^2 \\
&\leq (1-\gamma_n)^2\left\|w_n - u^*\right\|^2 + 2\gamma_n\langle f(u_n) - u^*, (1-\gamma_n)[w_n - u^*] + \gamma_n[f(u_n) - u^*]\rangle \\
&= (1-\gamma_n)^2\left\|w_n - u^*\right\|^2 + 2\gamma_n\langle f(u_n) - f(u^*) + f(u^*) - u^*, u_{n+1} - u^*\rangle \\
&= (1-\gamma_n)^2\left\|w_n - u^*\right\|^2 + 2\gamma_n\langle f(u_n) - f(u^*), u_{n+1} - u^*\rangle + 2\gamma_n\langle f(u^*) - u^*, u_{n+1} - u^*\rangle \\
&\leq (1-\gamma_n)^2\left\|w_n - u^*\right\|^2 + 2\gamma_n\rho\left\|u_n - u^*\right\|\left\|u_{n+1} - u^*\right\| + 2\gamma_n\langle f(u^*) - u^*, u_{n+1} - u^*\rangle \\
&\leq (1+\gamma_n^2 - 2\gamma_n)\left\|u_n - u^*\right\|^2 + 2\gamma_n\rho\left\|u_n - u^*\right\|^2 + 2\gamma_n\langle f(u^*) - u^*, u_{n+1} - u^*\rangle \\
&= (1-2\gamma_n)\left\|u_n - u^*\right\|^2 + \gamma_n^2\left\|u_n - u^*\right\|^2 + 2\gamma_n\rho\left\|u_n - u^*\right\|^2 + 2\gamma_n\langle f(u^*) - u^*, u_{n+1} - u^*\rangle \\
&= \left[1 - 2\gamma_n(1-\rho)\right]\left\|u_n - u^*\right\|^2 + 2\gamma_n(1-\rho)\left[\frac{\gamma_n\left\|u_n - u^*\right\|^2}{2(1-\rho)} + \frac{\langle f(u^*) - u^*, u_{n+1} - u^*\rangle}{1-\rho}\right].
\end{aligned} \tag{46}$$

It follows from expressions (45) and (46), we obtain

$$\limsup_{n\to\infty}\left[\frac{\gamma_n\left\|u_n - u^*\right\|^2}{2(1-\rho)} + \frac{\langle f(u^*) - u^*, u_{n+1} - u^*\rangle}{1-\rho}\right] \leq 0. \tag{47}$$

Choose $n \geq N_3 \in \mathcal{N}$ ($N_3 \geq N_2$) large enough such that $2\gamma_n(1-\rho) < 1$. Now, using expressions (46) and (47) and applying Lemma 3, we conclude that $\left\|u_n - u^*\right\| \to 0$, as $n \to \infty$.

Case 2: Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|u_{n_i} - u^*\| \leq \|u_{n_{i+1}} - u^*\|, \ \forall i \in \mathcal{N}.$$

Thus, by Lemma 4, there exits a sequence $\{m_k\} \subset \mathcal{N}$ and $\{m_k\} \to \infty$, such that

$$\|u_{m_k} - u^*\| \leq \|u_{m_{k+1}} - u^*\| \quad \text{and} \quad \|u_k - u^*\| \leq \|u_{m_{k+1}} - u^*\|, \ \forall k \in \mathcal{N}. \tag{48}$$

Similar to Case 1, using (38), we have

$$\begin{aligned}
&(1-\gamma_{m_k})\Big(1 - \frac{\mu\zeta_{m_k}}{\zeta_{m_k+1}}\Big)\left[\|w_{m_k} - v_{m_k}\|^2 + \|u_{m_k} - v_{m_k}\|^2\right] \\
&\leq \gamma_{m_k}\|f(u_{m_k}) - u^*\|^2 + \|u_{m_k} - u^*\|^2 - \|u_{m_k+1} - u^*\|^2.
\end{aligned} \tag{49}$$

Due to $\gamma_{m_k} \to 0$ and $\Big(1 - \frac{\mu\zeta_{m_k}}{\zeta_{m_k+1}}\Big) \to 1-\mu$, we can deduce the following:

$$\lim_{n\to\infty}\|u_{m_k} - v_{m_k}\| = \lim_{k\to\infty}\|w_{m_k} - v_{m_k}\| = 0. \tag{50}$$

From expression (50), we have

$$\lim_{k\to\infty}\|u_{m_k} - w_{m_k}\| \leq \lim_{k\to\infty}\|u_{m_k} - v_{m_k}\| + \lim_{k\to\infty}\|v_{m_k} - w_{m_k}\| = 0. \tag{51}$$

Hence, we obtain

$$\begin{aligned}\|u_{m_k+1}-u_{m_k}\| &= \|\gamma_{m_k}f(u_{m_k})+(1-\gamma_{m_k})w_{m_k}-u_{m_k}\| \\ &= \|\gamma_{m_k}[f(u_{m_k})-u_{m_k}]+(1-\gamma_{m_k})[w_{m_k}-u_{m_k}]\| \\ &\leq \gamma_{m_k}\|f(u_{m_k})-u_{m_k}\|+(1-\gamma_{m_k})\|w_{m_k}-u_{m_k}\| \longrightarrow 0. \end{aligned} \tag{52}$$

We have to use the same justification as in the Case 1, such that

$$\limsup_{k\to\infty}\langle f(u^*)-u^*, u_{m_k+1}-u^*\rangle \leq 0. \tag{53}$$

Using (46) and (48), we have

$$\begin{aligned}\|u_{m_k+1}-u^*\|^2 &\leq [1-2\gamma_{m_k}(1-\rho)]\|u_{m_k}-u^*\|^2 \\ &\quad +2\gamma_{m_k}(1-\rho)\left[\frac{\gamma_{m_k}\|u_{m_k}-u^*\|^2}{2(1-\rho)}+\frac{\langle f(u^*)-u^*, u_{m_k+1}-u^*\rangle}{1-\rho}\right] \\ &\leq [1-2\gamma_{m_k}(1-\rho)]\|u_{m_k+1}-u^*\|^2 \\ &\quad +2\gamma_{m_k}(1-\rho)\left[\frac{\gamma_{m_k}\|u_{m_k}-u^*\|^2}{2(1-\rho)}+\frac{\langle f(u^*)-u^*, u_{m_k+1}-u^*\rangle}{1-\rho}\right]. \end{aligned} \tag{54}$$

It follows that

$$\|u_{m_k+1}-u^*\|^2 \leq \frac{\gamma_{m_k}\|u_{m_k}-u^*\|^2}{2(1-\rho)}+\frac{\langle f(u^*)-u^*, u_{m_k+1}-u^*\rangle}{1-\rho}. \tag{55}$$

Since $\gamma_{m_k}\to 0$ and $\|u_{m_k}-u^*\|$ is a bounded sequence. Thus, expressions (53) and (55) implies that

$$\|u_{m_k+1}-u^*\|^2 \to 0, \text{ as } k\to\infty. \tag{56}$$

From the inequality (48), we have

$$\lim_{n\to\infty}\|u_k-u^*\|^2 \leq \lim_{n\to\infty}\|u_{m_k+1}-u^*\|^2 \leq 0. \tag{57}$$

Consequently, $u_n\to u^*$. This completes the proof of the theorem. □

4. Numerical Experiments

Numerical investigations present in this section to demonstrate the efficiency of the introduced Algorithm 1 in four test problems, all of which are pseudomonotone. The MATLAB program has been performed on a PC (with Intel(R) Core(TM)i3-4010U CPU @ 1.70 GHz, RAM 4.00 GB) in MATLAB version 9.5 (R2018b). We use the built-in MATLAB Quadratic programming to solve the minimization problems.

Example 1. *Consider the non-linear complementarity problem of Kojima—Shindo where the feasible set $\mathcal{K}$ which is defined by*

$$\mathcal{K}=\{u\in\mathcal{R}^4 : 1\leq u_i\leq 5,\ i=1,2,3,4\}.$$

The mapping $\mathcal{F} : \mathcal{R}^4 \to \mathcal{R}^4$ is defined by

$$\mathcal{F}(u) = \begin{pmatrix} u_1 + u_2 + u_3 + u_4 - 4u_2u_3u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_3u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_2u_4 \\ u_1 + u_2 + u_3 + u_4 - 4u_1u_2u_3 \end{pmatrix}.$$

It is easy to see that $\mathcal{F}$ is not monotone on the set $\mathcal{K}$. By using the Monte Carlo approach [44], it can be shown that $\mathcal{F}$ is pseudomonotone on $\mathcal{K}$, This problem has a unique solution $u^* = (5,5,5,5)^T$. Generate many pairs of points u and v uniformly in $\mathcal{K}$ satisfying $\mathcal{F}(u)^T(v-u) \geq 0$ and then check if $\mathcal{F}(v)^T(v-u) \geq 0$. In this experiment, we take different initial points and $D_n = \|u_n - v_n\|$. Moreover, control parameters $\zeta_0 = 0.33$, $\mu = 0.25$, $\gamma_n = \frac{1}{100(n+2)}$ and $f(u) = \frac{u}{2}$ for Algorithm 1. Numerical investigation regarding the first example was shown in Table 1.

Table 1. Numerical behaviour of Algorithm 1 using different starting points for Example 1.

TOL u_0	10^{-2} **Iter.**	10^{-3} **Iter.**	10^{-4} **Iter.**	10^{-5} **Iter.**	10^{-2} **Time**	10^{-3} **Time**	10^{-4} **Time**	10^{-5} **Time**
$[-2,2,8,10]^T$	13	51	501	5001	0.079821	0.247776	3.251465	43.637834
$[-1,1,5,6]^T$	12	51	501	5001	0.083870	0.236924	2.684370	39.651178
$[-5,2,-1,2]^T$	9	51	501	5001	0.065422	0.235173	3.034747	43.630625
$[1,2,3,4]^T$	6	1004	1004	5001	0.040866	8.051234	6.686632	42.431705

Example 2. *Consider the quadratic fractional programming problem in the following form [44]:*

$$\begin{cases} \min f(u) = \dfrac{u^TQu + a^Tu + a_0}{b^Tu + b_0}, \\ \textit{subject to } u \in \mathcal{K} = \{u \in \mathcal{R}^4 : b^Tu + b_0 > 0\}, \end{cases}$$

where

$$Q = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}, \quad a = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \ a_0 = -2, \quad \textit{and} \quad b_0 = 4.$$

It is easy to verify that Q is symmetric and positive definite on $\mathcal{R}^4$ and consequently f is pseudo-convex on $\mathcal{K}$. Therefore, ∇f is pseudomonotone. Using the quotient rule, we obtain

$$\nabla f(u) = \frac{(b^Tu + b_0)(2Qu + a) - b(u^TQ + a^Tu + a_0)}{(b^Tu + b_0)^2}. \tag{58}$$

In this point of view, we can set $\mathcal{F} = \nabla f$ in Theorem 1. We minimize f over $\mathcal{K} = \{u \in \mathcal{R}^4 : 1 \leq u_i \leq 10,\ i = 1,2,3,4\}$. This problem has a unique solution $u^* = (1,1,1,1)^T \in \mathcal{K}$. In this experiment, we take different initial points and $D_n = \|u_n - v_n\|$. Moreover, control parameters $\zeta_0 = 0.33$, $\mu = 0.25$, $\gamma_n = \frac{1}{100(n+2)}$ and $f(u) = \frac{u}{2}$ for Algorithm 1. Numerical investigation regarding the second example is shown in Table 2.

Table 2. Numerical behaviour of Algorithm 1 using different starting points for Example 2.

TOL u_0	10^{-2} **Iter.**	10^{-3} **Iter.**	10^{-4} **Iter.**	10^{-5} **Iter.**	10^{-2} **Time**	10^{-3} **Time**	10^{-4} **Time**	10^{-5} **Time**
$[10,10,10,10]^T$	43	46	99	989	0.289149	0.249285	0.475520	8.480530
$[10,20,30,40]^T$	41	46	99	989	0.211707	0.187559	0.445240	6.898924
$[20,-20,20,-20]^T$	29	32	99	989	0.138575	0.169190	0.394654	7.168460

Example 3. *The third example was taken from [45] where $\mathcal{F}:\mathcal{R}^2 \to \mathcal{R}^2$ is defined by*

$$\mathcal{F}(u) = \begin{pmatrix} 0.5u_1u_2 - 2u_2 - 10^7 \\ -4u_1 - 0.1u_2^2 - 10^7 \end{pmatrix},$$

on $\mathcal{K} = \{u \in \mathcal{R}^2 : (u_1-2)^2 + (u_2-2)^2 \leq 1\}$. It can easily see that $\mathcal{F}$ is Lipschitz continuous with $L = 5$ and $\mathcal{F}$ is not monotone on $\mathcal{K}$ but pseudomonotone. The above problem has a unique solution $u^* = (2.707, 2.707)^T$. In this experiment, we take different initial points and $D_n = \|u_n - v_n\|$. Moreover, control parameters $\zeta_0 = 0.33$, $\mu = 0.25$, $\gamma_n = \frac{1}{100(n+2)}$ and $f(u) = \frac{u}{3}$ for Algorithm 1. Numerical investigations regarding the third example is shown in Table 3.

Table 3. Numerical behaviour of Algorithm 1 using different starting points for Example 3.

TOL u_0	10^{-2} **Iter.**	10^{-3} **Iter.**	10^{-4} **Iter.**	10^{-5} **Iter.**	10^{-2} **Time**	10^{-3} **Time**	10^{-4} **time**	10^{-5} **Time**
$[0,0]^T$	8	27	265	2566	0.606917	1.907212	14.120655	107.506926
$[10,10]^T$	7	27	265	2591	0.286659	1.057623	10.764532	116.258335
$[-5,-5]^T$	8	26	258	2596	0.388227	1.190191	11.424257	107.584978

Example 4. *The fourth example was taken from [45] where $\mathcal{F}:\mathcal{R}^2 \to \mathcal{R}^2$ is defined by*

$$\mathcal{F}(u) = \begin{pmatrix} (u_1^2 + (u_2-1)^2)(1+u_2) \\ -u_1^3 - u_1(u_2-1)^2 \end{pmatrix},$$

where $\mathcal{K} = \{u \in \mathcal{R}^2 : -10 \leq u_i \leq 10,\ i = 1,2\}$. It can easily see that $\mathcal{F}$ is Lipschitz continuous with $L = 5$ and $\mathcal{F}$ is not monotone on $\mathcal{K}$ but pseudomonotone. In this experiment, we take different initial points and $D_n = \|u_n - v_n\|$. Moreover, control parameters $\zeta_0 = 0.33$, $\mu = 0.25$, $\gamma_n = \frac{1}{100(n+2)}$ and $f(u) = \frac{u}{4}$ for Algorithm 1. Numerical investigations regarding the fourth example is shown in Table 4.

Table 4. Numerical behaviour of Algorithm 1 using different starting points for Example 4.

TOL u_0	10^{-2} **Iter.**	10^{-3} **Iter.**	10^{-4} **Iter.**	10^{-5} **Iter.**	10^{-2} **Time**	10^{-3} **Time**	10^{-4} **Time**	10^{-5} **Time**
$[0,0]^T$	16	220	2231	29253	0.21543	2.35401	29.86562	224.95083
$[10,10]^T$	27	190	2072	25762	0.25322	2.64742	26.84528	198.26446
$[-5,-5]^T$	43	411	3801	47891	0.78262	4.77116	42.41738	427.904781

5. Conclusions

We have developed an extragradient-like method to solve pseudomonotone variational inequalities in real Hilbert space. The method had an explicit formula for an appropriate and effective stepsize evaluation on each step. For each iteration, the stepsize formula is modified based on the previous iterations. The numerical investigation was presented to explain the numerical effectiveness of our algorithm relative to other methods. These numerical studies suggest that viscosity schemes in this sense generally improve the effectiveness of the iterative sequence.

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