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Fixed Points of g-Interpolative Ćirić–Reich–Rus-Type Contractions in b-Metric Spaces

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Abstract: We use interpolation to obtain a common fixed point result for a new type of Ćirić–Reich–Rus-type contraction mappings in metric space. We also introduce a new concept of g-interpolative Ćirić–Reich–Rus-type contractions in b-metric spaces, and we prove some fixed point results for such mappings. Our results extend and improve some results on the fixed point theory in the literature. We also give some examples to illustrate the given results.

Keywords: fixed point; Ćirić–Reich–Rus-type contractions; interpolation; b-metric space

MSC: 46T99; 47H10; 54H25

1. Introduction and Preliminaries

Banach's contraction principle [1] has been applied in several branches of mathematics. As a result, researching and generalizing this outcome has proven to be a research area in nonlinear analysis (see [2–6]). It is a well-known fact that a map that satisfies the Banach contraction principle is necessarily continuous. Therefore, it was natural to wonder if in a complete metric space, a discontinuous map satisfying somewhat similar contractual conditions may have a fixed point. Kannan [7] answered yes to this question by introducing a new type of contraction. The concept of the interpolation Kannan-type contraction appeared with Karapinar [8] in 2018; this concept appealed to many researchers [8–14], making them invest in various types of contractions: interpolative Ćirić–Reich–Rus-type contraction [9–11,13], interpolative Hardy–Rogers [15]; and they used it on various spaces: metric space, b-metric space, and the Branciari distance.

In this paper, we will generalize some of the related findings to the interpolation Ćirić–Reich–Rus-type contraction in Theorems 1 and 2. In addition, we use a new concept of interpolative weakly contractive mapping to generalize some findings about the interpolation Kannan-type contraction in Theorem 3.

Now, we recall the concept of b-metric spaces as follows:

Definition 1 ([16,17]). *Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric if for all $x, y, z \in X$, the following conditions are satisfied:*

(b_1) $d(x, y) = 0$ *if and only if* $x = y$;
(b_2) $d(x, y) = d(y, x)$;
(b_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.
The pair (X, d) is called a b-metric space.

Note that the class of b-metric spaces is larger than that of metric spaces.

The notions of b-convergent and b-Cauchy sequences, as well as of b-complete b-metric spaces are defined exactly the same way as in the case of usual metric spaces (see, e.g., [18]).

Definition 2 ([19,20]). *Let $\{x_n\}$ be a sequence in a b-metric space (X, d). $g, h{:}X \to X$, are self-mappings, and $x \in X$. x is said to be the coincidence point of pair $\{g, h\}$ if $gx = hx$.*

Definition 3 ([10,11]). *Let Ψ be denoted as the set of all non-decreasing functions ψ: $[0, \infty) \to [0, \infty)$, such that $\sum_{k=0}^{\infty} \psi^k(t) < \infty$ for each $t > 0$. Then:*

(i) $\psi(0) = 0$,
(ii) $\psi(t) < t$ *for each* $t > 0$.

Remark 1 ([18]). *In a b-metric space (X, d), the following assertions hold:*

1. *A b-convergent sequence has a unique limit.*
2. *Each b-convergent sequence is a b-Cauchy sequence.*
3. *In general, a b-metric is not continuous.*

The fact in the last remark requires the following lemma concerning the b-convergent sequences to prove our results:

Lemma 1 ([19]). *Let (X, d) be a b-metric space with $s \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x, y, respectively, then we have:*

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n \to \infty} d(x_n, y_n) \leq \limsup_{n \to \infty} d(x_n, y_n) \leq s^2 d(x, y).$$

In particular, if $x = y$, then we have $\lim_{n \to \infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have:

$$\frac{1}{s} d(x, z) \leq \liminf_{n \to \infty} d(x_n, z) \leq \limsup_{n \to \infty} d(x_n, z) \leq s d(x, z).$$

2. Results

We denote by Φ the set of functions $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t) < t$ for every $t > 0$. Our main result is the following theorem:

Theorem 1. *Let (X, d) be a complete metric space, and T is a self-mapping on X such that:*

$$d(Tx, Ty) \leq \phi([d(x, y)]^{\alpha} [d(x, Tx)]^{\beta} [d(y, Ty)]^{\gamma}) \tag{1}$$

is satisfied for all $x, y \in X \setminus Fix(T)$; where $Fix(T) = \{a \in X | Ta = a\}$, $\alpha, \beta, \gamma \in (0, 1)$ such that $\alpha + \beta + \gamma > 1$, and $\phi \in \Phi$.

If there exists $x \in X$ such that $d(x, Tx) < 1$, then T has a fixed point in X.

Proof. We define a sequence $\{x_n\}$ by $x_0 = x$ and $x_{n+1} = Tx_n$ for all integers n, and we assume that $x_n \neq Tx_n$, for all n.

We have:

$$d(x_n, x_{n+1}) \leq \phi([d(x_{n-1}, x_n)]^{\alpha} [d(x_{n-1}, x_n)]^{\beta} [d(x_n, x_{n+1})]^{\gamma}). \tag{2}$$

Using the fact $\phi(t) < t$ for each $t > 0$, from (2), we obtain:

$$d(x_n, x_{n+1}) < [d(x_{n-1}, x_n)]^{\alpha} [d(x_{n-1}, x_n)]^{\beta} [d(x_n, x_{n+1})]^{\gamma}.$$

which implies:

$$[d(x_n, x_{n+1})]^{1-\gamma} < [d(x_{n-1}, x_n)]^{\alpha+\beta}. \tag{3}$$

We have $d(x_0, x_1) < 1$, so that there exists a real $\lambda \in (0,1)$ such that $d(x_0, x_1) \leq \lambda$ and $\lambda = \frac{d(x_0,x_1)+1}{2}$.

By (3), we obtain:

$$d(x_1, x_2) < [d(x_0, x_1)]^{\frac{\alpha+\beta}{1-\gamma}} \leq \lambda^{\frac{\alpha+\beta}{1-\gamma}}.$$

By (3), we find:

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1})^{1+\epsilon}$$

for all n, with $\epsilon = \frac{\alpha+\beta}{1-\gamma} - 1 > 0$.

Now, we prove by induction that for all n,

$$d(x_{n+1}, x_n) \leq \lambda^{(1+\epsilon)^n}$$

where $0 < \lambda < 1$. For $n = 1$, this is the inequality at the bottom of page 3. The induction step is:

$$d(x_{n+2}, x_{n+1}) \leq d(x_{n+1}, x_n)^{1+\epsilon} \leq \left(\lambda^{(1+\epsilon)^n}\right)^{1+\epsilon} = \lambda^{(1+\epsilon)^{n+1}}$$

Since $(1+\epsilon)^n \geq 1 + n\epsilon$ by Bernoulli's inequality and since $\lambda < 1$, this implies:

$$d(x_{n+1}, x_n) \leq \lambda^{1+n\epsilon} = \lambda\rho^n$$

for all n, where $\rho = \lambda^\epsilon < 1$. This implies:

$$d(x_{n+k}, x_n) \leq \lambda(\rho^{n+k-1} + \rho^{n+k-2} + \cdots + \rho^n) = \lambda\rho^n \left(\frac{1-\rho^k}{1-\rho}\right) = C\rho^n,$$

where $C = \lambda\left(\frac{1-\rho^k}{1-\rho}\right)$ for some integer k, from which it follows that $\{x_n\}$ forms a Cauchy sequence in (X, d), and then, it converges to some $z \in X$. Assume that $z \neq Tz$.

By letting $x = x_n$ and $y = z$ in (1), we obtain:

$$\begin{aligned} d(x_{n+1}, Tz) &\leq \phi([d(x_n, z)]^\alpha [d(x_n, x_{n+1})]^\beta [d(z, Tz)]^\gamma) \\ &< [d(x_n, z)]^\alpha [d(x_n, x_{n+1})]^\beta [d(z, Tz)]^\gamma \end{aligned}$$

for all n, which leads to $d(z, Tz) = 0$, which is a contradiction. Then, $Tz = z$. □

Example 1. *Let $X = [0,2]$ be endowed with metric $d : X \times X \to [0,\infty)$, defined by:*

$$d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ \frac{2}{3}, & \text{if } x, y \in [0,1] \text{ and } x \neq y; \\ 2, & \text{otherwise.} \end{cases}$$

Consider that the self-mapping $T : X \to X$ is defined by:

$$Tx = \begin{cases} \frac{1}{2}, & \text{if } x \in [0,1]; \\ \frac{x}{2}, & \text{if } x \in (1,2]; \end{cases}$$

and the function $\phi(t) = 0,4t^2$ *for all* $t \in [0, \infty)$.

For $\alpha = 0,8$, $\beta = 0,2$, *and* $\gamma = 0,25$.

We discus the following cases:

***Case* 1.** *If* $x, y \in [0,1]$ *or* $x = y$ *for all* $x, y \in [0,2]$; *it is obvious.*

***Case* 2.** *If* $x, y \in (1,2]$ *and* $x \neq y$.

We have:

$$d(Tx, Ty) = \frac{2}{3}$$

and:

$$\phi([d(x,y)]^{\alpha}[d(x,Tx)]^{\beta}[d(y,Ty)]^{\gamma}) = \phi(2^{\alpha+\beta+\gamma}) = \frac{2^{3,5}}{5} \geq \frac{2}{3}.$$

Then:

$$d(Tx, Ty) \leq \phi([d(x,y)]^{\alpha}[d(x,Tx)]^{\beta}[d(y,Ty)]^{\gamma})$$

for all $x, y \in (1,2]$.

***Case* 3.** *If* $x \in [0,1]$ *and* $y \in (1,2]$ *with* $x \neq \frac{1}{2}$.

We have:

$$d(Tx, Ty) = \frac{2}{3}$$

and:

$$\phi([d(x,y)]^{\alpha}[d(x,Tx)]^{\beta}[d(y,Ty)]^{\gamma}) = \phi\left(2^{\alpha+\gamma}\left(\frac{2}{3}\right)^{\beta}\right) = \frac{2^{3,5}}{5.3^{0,2}} \geq \frac{2}{3}.$$

Then:

$$d(Tx, Ty) \leq \phi([d(x,y)]^{\alpha}[d(x,Tx)]^{\beta}[d(y,Ty)]^{\gamma})$$

for all $x \in [0,1] \backslash \{\frac{1}{2}\}$ *and* $y \in (1,2]$.

***Case* 4.** *If* $x \in (1,2]$ *and* $y \in [0,1]$ *with* $y \neq \frac{1}{2}$.

We have:

$$d(Tx, Ty) = \frac{2}{3}$$

and:

$$\phi([d(x,y)]^{\alpha}[d(x,Tx)]^{\beta}[d(y,Ty)]^{\gamma}) = \phi\left(2^{\alpha+\beta}\left(\frac{2}{3}\right)^{\gamma}\right) = \frac{2^{3,5}}{5.3^{0,25}} \geq \frac{2}{3}.$$

Then:

$$d(Tx, Ty) \leq \phi([d(x,y)]^{\alpha}[d(x,Tx)]^{\beta}[d(y,Ty)]^{\gamma})$$

for all $x \in (1,2]$ *and* $y \in [0,1] \backslash \{\frac{1}{2}\}$.

Therefore, all the conditions of Theorem 1 are satisfied, and T *has a fixed point,* $x = \frac{1}{2}$.

Example 2. *Let* $X = \{a, q, r, s\}$ *be endowed with the metric defined by the following table of values:*

$d(x,y)$	a	q	r	s
a	0	$\frac{1}{3}$	$\frac{10}{3}$	$\frac{5}{3}$
q	$\frac{1}{3}$	0	3	2
r	$\frac{10}{3}$	3	0	5
s	$\frac{5}{3}$	2	5	0

Consider the self-mapping T on X as:

$$T\colon \begin{pmatrix} a & q & r & s \\ a & a & q & s \end{pmatrix}.$$

For $\psi(t) = \frac{2^t-1}{2^t+1}$ *for all* $t \in [0,\infty)$; $\alpha = 0,6$; $\beta = 0,9$; *and* $\gamma = 0,7$.

We have:

$$d(Tu,Tv) \leq \psi([d(u,v)]^{\alpha}[d(u,Tu)]^{\beta}[d(v,Tv)]^{\gamma})$$

for all $u,v \in X \setminus \{a,s\}$.

Then, T has two fixed points, which are a and s.

If we take $\psi(t) = kt$ in Theorem (1) with $k \in (0,1)$, then we have the following corollary:

Corollary 1. *Let* (X,d) *be a complete metric space, and T is a self-mapping on X such that:*

$$d(Tx,Ty) \leq k[d(x,y)]^{\alpha}[d(x,Tx)]^{\beta}[d(y,Ty)]^{\gamma}$$

is satisfied for all $x,y \in X \setminus Fix(T)$; *where* $Fix(T) = \{a \in X | Ta = a\}$, *and* $\alpha,\beta,\gamma,k \in (0,1)$ *such that* $\alpha+\beta+\gamma > 1$.

If there exists $x \in X$ *such that* $d(x,Tx) < 1$, *then T has a fixed point in X.*

Example 3. *It is enough to take in Example 1:* $\phi(t) = \frac{57}{58}t$ *for all* $t \in [0,+\infty)$.

Example 4. *Let* $X = \{a,q,r,s\}$ *be endowed with the metric defined by the following table of values:*

$d(x,y)$	a	q	r	s
a	0	0,1	3,1	4
q	0,1	0	3	3,9
r	3,1	3	0	0,9
s	4	3,9	0,9	0

Consider the self-mapping T on X as:

$$T: \begin{pmatrix} a & q & r & s \\ a & a & q & s \end{pmatrix}.$$

For $k = \frac{3}{10}$; $\alpha = 0,7$; $\beta = 0,1$; *and* $\gamma = 0,8$.

We have:

$$d(Tu,Tv) \leq k[d(u,v)]^{\alpha}[d(u,Tu)]^{\beta}[d(v,Tv)]^{\gamma}$$

for all $u,v \in X \setminus \{a,s\}$.

Then, T has two fixed points, which are a and s.

Definition 4. *Let (X, d, s) be a b-metric space and $T, g : X \to X$ be self-mappings on X. We say that T is a g-interpolative Ćirić–Reich–Rus-type contraction, if there exists a continuous $\psi \in \Psi$ and $\alpha, \beta \in (0,1)$ such that:*

$$d(Tx, Ty) \leq \psi([d(gx, gy)]^{\alpha}[d(gx, Tx)]^{\beta}[d(gy, Ty)]^{1-\alpha-\beta}) \tag{4}$$

is satisfied for all $x, y \in X$ such that $Tx \neq gx, Ty \neq gy$, and $gx \neq gy$.

Theorem 2. *Let (X, d, s) be a b-complete b-metric space, and T is a g-interpolative Ćirić–Reich–Rus-type contraction. Suppose that $TX \subseteq gX$ such that gX is closed. Then, T and g have a coincidence point in X.*

Proof. Let $x \in X$; since $TX \subseteq gX$, we can define inductively a sequence $\{x_n\}$ such that:

$$x_0 = x, \text{ and } gx_{n+1} = Tx_n, \text{ for all integer } n.$$

If there exists $n \in \{0, 1, 2, \ldots\}$ such that $gx_n = Tx_n$, then x_n is a coincidence point of g and T. Assume that $gx_n \neq Tx_n$, for all n. By (4), we obtain:

$$\begin{aligned} d(Tx_{n+1}, Tx_n) &\leq \psi([d(gx_{n+1}, gx_n]^{\alpha}[d(gx_{n+1}, Tx_{n+1}]^{\beta}[d(gx_n, Tx_n]^{1-\alpha-\beta}) \\ &= \psi([d(Tx_n, Tx_{n-1}]^{\alpha}[d(Tx_n, Tx_{n+1}]^{\beta}[d(Tx_{n-1}, Tx_n]^{1-\alpha-\beta}) \\ &= \psi([d(Tx_n, Tx_{n-1}]^{1-\beta}[d(Tx_n, Tx_{n+1}]^{\beta}). \end{aligned}$$

Using the fact $\psi(t) < t$ for each $t > 0$,

$$\begin{aligned} d(Tx_{n+1}, Tx_n) &\leq \psi([d(Tx_n, Tx_{n-1})]^{1-\beta}[d(Tx_n, Tx_{n+1})]^{\beta}) \\ &< [d(Tx_n, Tx_{n-1})]^{1-\beta}[d(Tx_n, Tx_{n+1})]^{\beta}. \end{aligned} \tag{5}$$

which implies:

$$[d(Tx_{n+1}, Tx_n)]^{1-\beta} < [d(Tx_n, Tx_{n-1})]^{1-\beta}.$$

Thus,

$$d(Tx_{n+1}, Tx_n) < d(Tx_n, Tx_{n-1}) \text{ for all } n \geq 1. \tag{6}$$

That is, the positive sequence $\{d(Tx_{n+1}, Tx_n)\}$ is monotone decreasing, and consequently, there exists $c \geq 0$ such that $\lim_{n\to\infty} d(Tx_{n+1}, Tx_n) = c$. From (6), we obtain:

$$\begin{aligned} [d(Tx_n, Tx_{n-1})]^{1-\beta}[d(Tx_n, Tx_{n+1})]^{\beta} &\leq [d(Tx_n, Tx_{n-1})]^{1-\beta}[d(Tx_n, Tx_{n-1})]^{\beta} \\ &= d(Tx_n, Tx_{n-1}). \end{aligned}$$

Therefore, with (5) together with the nondecreasing character of ψ, we get:

$$\begin{aligned} d(Tx_{n+1}, Tx_n) &\leq \psi([d(Tx_n, Tx_{n-1})]^{1-\beta}[d(Tx_n, Tx_{n+1})]^{\beta}) \\ &\leq \psi(d(Tx_n, Tx_{n-1})). \end{aligned}$$

By repeating this argument, we get:

$$d(Tx_{n+1}, Tx_n) \leq \psi(d(Tx_n, Tx_{n-1})) \leq \psi^2(d(Tx_{n-1}, Tx_{n-2})) \leq \cdots \leq \psi^n(d(Tx_1, Tx_0)). \tag{7}$$

Taking $n \to \infty$ in (7) and using the fact $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t > 0$, we deduce that $c = 0$, that is,

$$\lim_{n\to\infty} d(Tx_{n+1}, Tx_n) = 0. \tag{8}$$

Then, $\{Tx_n\}$ is a b-Cauchy sequence. Suppose on the contrary that there exists an $\epsilon > 0$ and subsequences $\{Tx_{m_k}\}$ and $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that n_k is the smallest integer for which:

$$n_k > m_k > k, \;\; d(Tx_{n_k}, Tx_{m_k}) \geq \epsilon, \;\; \text{and} \;\; d(Tx_{n_k-1}, Tx_{m_k}) < \epsilon.$$

Then, we have:

$$\begin{aligned} d(gx_{n_k}, gx_{m_k}) = d(Tx_{n_k-1}, Tx_{m_k-1}) &\leq sd(Tx_{n_k-1}, Tx_{m_k}) + sd(Tx_{m_k}, Tx_{m_k-1}) \\ &\leq s\epsilon + sd(Tx_{m_k}, Tx_{m_k-1}). \end{aligned}$$

Using (8) in the inequality above, we obtain:

$$\limsup_{k\to\infty} d(Tx_{n_k-1}, Tx_{m_k-1}) = \limsup_{k\to\infty} d(gx_{n_k}, gx_{m_k}) \leq s\epsilon. \tag{9}$$

Putting $x = x_{n_k}$ and $y = x_{m_k}$ in (4), we have:

$$\begin{aligned} \epsilon \leq d(Tx_{n_k}, Tx_{m_k}) &\leq \psi([d(gx_{n_k}, gx_{m_k})]^{\alpha}[d(gx_{n_k}, Tx_{n_k})]^{\beta}[d(gx_{m_k}, Tx_{m_k})]^{1-\alpha-\beta}) \\ &= \psi([d(Tx_{n_k-1}, Tx_{m_k-1})]^{\alpha}[d(Tx_{n_k-1}, Tx_{n_k})]^{\beta}[d(Tx_{m_k-1}, Tx_{m_k})]^{1-\alpha-\beta}). \end{aligned} \tag{10}$$

Taking the upper limit as $k \to \infty$ in (10) and using (8) and (9) and the property of ψ, we get:

$$\epsilon \leq \limsup_{k\to\infty} d(Tx_{n_k}, Tx_{m_k}) \leq \psi(0) = 0,$$

which implies that $\epsilon = 0$, a contradiction with $\epsilon > 0$. We deduce that $\{Tx_n\}$ is a b-Cauchy sequence, and consequently, $\{gx_n\}$ is also a b-Cauchy sequence. Let $z \in X$ such that,

$$\lim_{n\to\infty} d(Tx_n, z) = \lim_{n\to\infty} d(gx_{n+1}, z) = 0.$$

Since $z \in gX$, there exists $u \in X$ such that $z = gu$. We claim that u is a coincidence point of g and T. For this, if we assume that $gu \neq Tu$, we obtain:

$$\begin{aligned} d(Tx_n, Tu) &\leq \psi([d(gx_n, gu)]^{\alpha}[d(gx_n, Tx_n)]^{\beta}[d(gu, Tu)]^{1-\alpha-\beta}) \\ &< [d(gx_n, gu)]^{\alpha}[d(gx_n, Tx_n)]^{\beta}[d(gu, Tu)]^{1-\alpha-\beta}. \end{aligned}$$

At the limit as $n \to \infty$ and using Lemma 1, we get:

$$\begin{aligned} \frac{1}{s} d(z, Tu) \leq \liminf_{n\to\infty} d(Tx_n, Tu) &\leq \limsup_{n\to\infty} [d(gx_n, gu)]^{\alpha}[d(gx_n, Tx_n)]^{\beta}[d(gu, Tu)]^{1-\alpha-\beta} \\ &\leq [sd(z, gu)]^{\alpha}[s^2 d(z, z)]^{\beta}[d(gu, Tu)]^{1-\alpha-\beta} = 0, \end{aligned}$$

which is a contradiction, which implies that:

$$Tu = z = gu.$$

Then, u is a coincidence point in X of T and g. □

Example 5. *Let $X = [0, +\infty)$ and $d : X \times X \to [0, \infty)$ be defined by:*

$$d(x, y) = \begin{cases} (x + y)^2, & \text{if } x \neq y; \\ 0, & \text{if } x = y. \end{cases}$$

Then, (X, d) is a complete b-metric space.

Define two self-mappings T and g on X by $g(x) = x^2$; for all $x \in X$ and:

$$Tx = \begin{cases} 1, & \text{if } x \in [0,2]; \\ \frac{1}{x}, & \text{if } x \in (2,+\infty). \end{cases}$$

T is a g-interpolative Ćirić–Reich–Rus-type contraction for $\alpha = 0,7$, $\beta = 0,4$, and:

$$\psi(t) = \begin{cases} \frac{3}{20}t^2, & \text{if } t \in [0, \frac{89}{20}]; \\ \frac{3^{t+1}-1}{3^t+1}, & \text{if } t \in (\frac{89}{20}, +\infty). \end{cases}$$

For this, we discuss the following cases:

***Case* 1.** *If $x, y \in [0,2]$ or $x = y$ for all $x \in [0,+\infty)$. It is obvious.*

***Case* 2.** *If $x, y \in (2,+\infty)$ and $x \neq y$.*

We have:

$$d(Tx, Ty) = (\frac{1}{x} + \frac{1}{y})^2 \leq 1.$$

Using the property of ψ, we get:

$$\begin{aligned} \psi([d(gx,gy)]^\alpha [d(gx,Tx)]^\beta [d(gy,Ty)]^{1-\alpha-\beta}) &= \psi((x^2+y^2)^{2\alpha}(x^2+\frac{1}{x})^{2\beta}(y^2+\frac{1}{y})^{2(1-\alpha-\beta)}) \\ &\geq \psi(8^{2\alpha}.(\frac{9}{2})^{2(1-\alpha)}) \geq 1. \end{aligned}$$

Therefore,

$$d(Tx, Ty) \leq \psi([d(gx,gy)]^\alpha [d(gx,Tx)]^\beta [d(gy,Ty)]^{1-\alpha-\beta}).$$

***Case* 3.** *If $x \in [0,2] \backslash \{1\}$ and $y \in (2,+\infty)$.*

We have:

$$d(Tx, Ty) = (1 + \frac{1}{y})^2 \leq (\frac{3}{2})^2 = \frac{9}{4},$$

and:

$$\begin{aligned} \psi([d(gx,gy)]^\alpha [d(gx,Tx)]^\beta [d(gy,Ty)]^{1-\alpha-\beta}) &= \psi((x^2+y^2)^{2\alpha}(x^2+1)^{2\beta}(y^2+\frac{1}{y})^{2(1-\alpha-\beta)}) \\ &\geq \psi(4^{2\alpha}.1^{2\beta}.(\frac{9}{2})^{2(1-\alpha-\beta)}) \geq \frac{9}{4}. \end{aligned}$$

Therefore,

$$d(Tx, Ty) \leq \psi([d(gx,gy)]^\alpha [d(gx,Tx)]^\beta [d(gy,Ty)]^{1-\alpha-\beta}).$$

***Case* 4.** *If $x \in (2,+\infty)$ and $y \in [0,2] \backslash \{1\}$.*

We have:

$$d(Tx, Ty) = (1 + \frac{1}{x})^2 \leq \frac{9}{4},$$

and:

$$\begin{aligned} \psi([d(gx,gy)]^\alpha [d(gx,Tx)]^\beta [d(gy,Ty)]^{1-\alpha-\beta}) &= \psi((x^2+y^2)^{2\alpha}(x^2+\frac{1}{x})^{2\beta}(y^2+1)^{2(1-\alpha-\beta)}) \\ &\geq \psi(4^{2\alpha}.(\frac{9}{2})^{2\beta}.1^{2(1-\alpha-\beta)}) \geq \frac{9}{4}. \end{aligned}$$

Therefore,

$$d(Tx,Ty) \leq \psi([d(gx,gy)]^{\alpha}[d(gx,Tx)]^{\beta}[d(gy,Ty)]^{1-\alpha-\beta}).$$

Then, it is clear that g,T satisfies (4) for all $u,v \in X \setminus \{1\}$. Moreover, one is a coincidence point of g and T.

Example 6. *Let the set $X = \{a,b,q,r\}$ and a function $d : X \times X \to [0,\infty)$ be defined as follows:*

$d(x,y)$	a	b	q	r
a	0	1	16	$\frac{49}{4}$
b	1	0	9	$\frac{25}{4}$
q	16	9	0	$\frac{1}{4}$
r	$\frac{49}{4}$	$\frac{25}{4}$	$\frac{1}{4}$	0

By a simple calculation, one can verify that the function d is a b-metric, for $s = 2$. We define the self-mappings g,T on X, as:

$$g : \begin{pmatrix} a & b & q & r \\ a & r & q & q \end{pmatrix}, \qquad T : \begin{pmatrix} a & b & q & r \\ q & r & r & q \end{pmatrix}.$$

For $\alpha = 0,3$; $\beta = 0,8$; and $\psi(t) = \frac{t}{1+t}$ for all $t \in [0,\infty)$.

It is clear that g,T satisfies (4) for all $u,v \in X \setminus \{b,r\}$. Moreover, b and r are two coincidence points of g and T.

Definition 5. *Let (X,d) is a metric space. A self-mapping T: $X \to X$ is said to be an interpolative weakly contractive mapping if there exists a constant $\alpha \in (0,1)$ such that:*

$$\zeta(d(Tx,Ty)) \leq \zeta([d(x,Tx)]^{\alpha}[d(y,Ty)]^{1-\alpha}) - \varphi([d(x,Tx)]^{\alpha}[d(y,Ty)]^{1-\alpha}), \tag{11}$$

for all $x,y \in X \setminus Fix(T)$, where
$Fix(T) = \{a \in X | Ta = a\}$,
ζ: $[0,\infty) \to [0,\infty)$ is a continuous monotone nondecreasing function with $\zeta(t) = 0$ if and only if $t = 0$,
φ: $[0,\infty) \to [0,\infty)$ is a lower semi-continuous function with $\varphi(t) = 0$ if and only if $t = 0$.

Theorem 3. *Let (X,d) be a complete metric space. If $T : X \to X$ is a interpolative weakly contractive mapping, then T has a fixed point.*

Proof. For any $x_0 \in X$, we define a sequence $\{x_n\}$ by $x = x_0$ and $x_{n+1} = Tx_n$, $n = 0,1,2,\ldots$
If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then x_{n_0} is clearly a fixed point in X. Otherwise, $x_{n+1} \neq x_n$ for each $n \geq 0$.

Substituting $x = x_n$ and $y = x_{n-1}$ in (11), we obtain that:

$$\begin{aligned} \zeta(d(x_{n+1},x_n)) &\leq \zeta([d(x_n,x_{n+1})]^{\alpha}[d(x_{n-1},x_n)]^{1-\alpha}) - \varphi([d(x_n,x_{n+1})]^{\alpha}[d(x_{n-1},x_n)]^{1-\alpha}) \\ &\leq \zeta([d(x_n,x_{n+1})]^{\alpha}[d(x_{n-1},x_n)]^{1-\alpha}). \end{aligned} \tag{12}$$

Using property of function ζ, we get:

$$d(x_{n+1},x_n) \leq [d(x_n,x_{n+1})]^{\alpha}[d(x_{n-1},x_n)]^{1-\alpha}.$$

We derive:

$$[d(x_{n+1},x_n)]^{1-\alpha} \leq [d(x_{n-1},x_n)]^{1-\alpha}.$$

Therefore:

$$d(x_{n+1},x_n) \leq d(x_{n-1},x_n), \quad \text{for all } n \geq 1.$$

It follows that the positive sequence $\{d(x_{n+1}, x_n)\}$ is decreasing. Eventually, there exists $c \geq 0$ such that $\lim_n d(x_{n+1}, x_n) = c$.

Taking $n \to \infty$ in the inequality (12), we obtain:

$$\zeta(c) \leq \zeta(c) - \varphi(c).$$

We deduce that $c = 0$. Hence:

$$\lim_n d(x_{n+1}, x_n) = 0. \tag{13}$$

Therefore, $\{x_n\}$ is a Cauchy sequence. Suppose it is not. Then, there exists a real number $\epsilon > 0$, for any $k \in \mathbb{N}, \exists m_k \geq n_k \geq k$ such that:

$$d(x_{m_k}, x_{n_k}) \geq \epsilon. \tag{14}$$

Putting $x = x_{n_k - 1}$ and $y = x_{m_k - 1}$ in (11) and using (14), we get:

$$\zeta(\epsilon) \leq \zeta(d(x_{m_k}, x_{n_k})) \leq \zeta([d(x_{m_k-1}, x_{m_k})]^{\alpha}[d(x_{n_k-1}, x_{n_k})]^{1-\alpha}) - \varphi([d(x_{m_k-1}, x_{m_k})]^{\alpha}[d(x_{n_k-1}, x_{n_k})]^{1-\alpha}).$$

Letting $k \to \infty$ and using (13), we conclude:

$$\zeta(\epsilon) \leq \zeta(0) - \varphi(0) = 0,$$

which is contradiction with $\epsilon > 0$; thus, $\{x_n\}$ is a Cauchy sequence; since (X, d) is complete, we obtain $z \in X$ such that $\lim_n d(x_n, z) = 0$, and assuming that $Tz \neq z$, we have:

$$\zeta(d(x_{n+1}, Tz)) \leq \zeta([d(x_n, x_{n+1})]^{\alpha}[d(z, Tz)]^{1-\alpha}) - \varphi([d(x_n, x_{n+1})]^{\alpha}[d(z, Tz)]^{1-\alpha}) \quad \text{for all } \; n.$$

Letting $n \to \infty$, we get:

$$\zeta(d(z, Tz)) \leq \zeta([d(z,z)]^{\alpha}[d(z, Tz)]^{1-\alpha}) - \varphi([d(z,z)]^{\alpha}[d(z, Tz)]^{1-\alpha}) = \zeta(0) - \varphi(0) = 0,$$

which is a contradiction; thus, $Tz = z$. □

Example 7. *Let the set $X = [0,3]$ and a function $\delta : X \times X \to [0, \infty)$ be defined as follows:*

$$\delta(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 3, & \text{if } x, y \in [0,1) \text{ and } x \neq y; \\ 2, & \text{otherwise.} \end{cases}$$

Then, (X, δ) is a complete metric space.

Let T: $X \to X$ be defined as:

$$Tx = \begin{cases} 0, & \text{if } x \in [0,1); \\ 1, & \text{if } x \in [1,3]. \end{cases}$$

For $\zeta(t) = t^2$, $\varphi(t) = \frac{1}{2}t$ for all $t \in [0, +\infty)$ and $\alpha = 0,6$.

We discuss the following cases.

***Case* 1.** *If $x = y$ or $x, y \in (0,1)$, or $x, y \in (1,3]$ with $x \neq y$. It is obvious.*

***Case* 2.** *If $x \in (0,1)$ and $y \in (1,3]$.*

We have:

$$\zeta(\delta(Tx, Ty)) = \zeta(\delta(0,1)) = \zeta(2) = 4,$$

and:

$$[\delta(x,Tx)]^{\alpha}[\delta(y,Ty)]^{1-\alpha} = [\delta(x,0)]^{\alpha}[\delta(y,1)]^{1-\alpha} = 2.(\frac{3}{2})^{\alpha}.$$

Therefore:

$$\zeta([\delta(x,Tx)]^{\alpha}[\delta(y,Ty)]^{1-\alpha}) - \varphi([\delta(x,Tx)]^{\alpha}[\delta(y,Ty)]^{1-\alpha}) = (\frac{3}{2})^{\alpha}[4.(\frac{3}{2})^{\alpha} - 1] \geq 4 = \zeta(2) = \zeta(\delta(Tx,Ty)).$$

***Case* 3.** *If* $x \in (1,3]$ *and* $y \in (0,1)$.

We have:

$$\zeta(\delta(Tx,Ty)) = \zeta(\delta(1,0)) = \zeta(2) = 4,$$

and:

$$[\delta(x,Tx)]^{\alpha}[\delta(y,Ty)]^{1-\alpha} = [\delta(x,1)]^{\alpha}[\delta(y,0)]^{1-\alpha} = 3.(\frac{2}{3})^{\alpha}.$$

Therefore,

$$\zeta([\delta(x,Tx)]^{\alpha}[\delta(y,Ty)]^{1-\alpha}) - \varphi([\delta(x,Tx)]^{\alpha}[\delta(y,Ty)]^{1-\alpha}) = (\frac{2}{3})^{\alpha}[9.(\frac{2}{3})^{\alpha} - \frac{3}{2}] \geq 4 = \zeta(2) = \zeta(\delta(Tx,Ty)).$$

Thus,

$$\zeta(d(Tu,Tv)) \leq \zeta([d(u,Tu)]^{\alpha}[d(v,Tv)]^{1-\alpha}) - \varphi([d(u,Tu)]^{\alpha}[d(v,Tv)]^{1-\alpha}),$$

for all $u, v \in X \setminus \{0,1\}$.

Then, T has two fixed points, which are zero and one.

Example 8. *Let* $X = \{a,b,r,s\}$ *be endowed with the metric defined by the following table of values:*

$d(x,y)$	a	b	r	s
a	0	1	4	1
b	1	0	5	2
r	4	5	0	3
s	1	2	3	0

Consider the self-mapping T on X as:

$$T : \begin{pmatrix} a & b & r & s \\ a & s & a & s \end{pmatrix}.$$

For $\zeta(t) = e^t - 1$ *and* $\varphi(t) = 2^t - 1$ *for all* $t \in [0,\infty)$; $\alpha = 0,3$.

We have:

$$\zeta(d(Tu,Tv)) \leq \zeta([d(u,Tu)]^{\alpha}[d(v,Tv)]^{1-\alpha}) - \varphi([d(u,Tu)]^{\alpha}[d(v,Tv)]^{1-\alpha}),$$

for all $u, v \in X \setminus \{a,s\}$.

Then, T has two fixed points, which are a and s.

If $\zeta(t) = t$ in Theorem (3), then we have the following corollary:

Corollary 2. *Let (X, d) be a complete metric space and $T : X \to X$ a self-mapping on X. If there exists a constant $\alpha \in (0, 1)$ such that:*

$$d(Tx, Ty) \leq [d(x, Tx)]^{\alpha}[d(y, Ty)]^{1-\alpha} - \varphi([d(x, Tx)]^{\alpha}[d(y, Ty)]^{1-\alpha}),$$

for all $x, y \in X$ and $x \neq Tx, y \neq Ty$.
$\varphi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\varphi(t) = 0$ if and only if $t = 0$.

Then, T has a fixed point.

Remark 2. *In Corollary 2, if we take $\varphi(t) = (1 - \lambda)t$ for a constant $\lambda \in (0, 1)$, then the result of Theorem [8] is obtained.*

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