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Approximation Results for Equilibrium Problems Involving Strongly Pseudomonotone Bifunction in Real Hilbert Spaces

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Abstract: A plethora of applications in non-linear analysis, including minimax problems, mathematical programming, the fixed-point problems, saddle-point problems, penalization and complementary problems, may be framed as a problem of equilibrium. Most of the methods used to solve equilibrium problems involve iterative methods, which is why the aim of this article is to establish a new iterative method by incorporating an inertial term with a subgradient extragradient method to solve the problem of equilibrium, which includes a bifunction that is strongly pseudomonotone and meets the Lipschitz-type condition in a real Hilbert space. Under certain mild conditions, a strong convergence theorem is proved, and a required sequence is generated without the information of the Lipschitz-type cost bifunction constants. Thus, the method operates with the help of a slow-converging step size sequence. In numerical analysis, we consider various equilibrium test problems to validate our proposed results.

Keywords: equilibrium problem; variational inequalities; strongly pseudomonotone bifunction; Lipschitz-type conditions

1. Background

Assume that a bifunction $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ satisfying the conditions $f(v,v) = 0$ for each $v \in \mathcal{K}$. A *equilibrium problem* [1,2] for f on $\mathcal{K}$ is said to be:

$$\text{Find } v^* \in \mathcal{K} \text{ such that } f(v^*, v) \geq 0, \ \forall\, v \in \mathcal{K}. \tag{1}$$

where $\mathcal{K}$ is a non-empty closed and convex subset of a Hilbert space $\mathcal{H}$. Next, we present the definitions of the important classification of the problems of equilibrium [1,3]. A function $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ on $\mathcal{K}$ for $\gamma > 0$ is said to be

(i) strongly monotone if

$$f(v_1, v_2) + f(v_2, v_1) \leq -\gamma \|v_1 - v_2\|^2, \ \forall\, v_1, v_2 \in \mathcal{K};$$

(ii) monotone if

$$f(v_1, v_2) + f(v_2, v_1) \leq 0, \ \forall\, v_1, v_2 \in \mathcal{K};$$

(iii) γ-strongly pseudo-monotone if

$$f(v_1, v_2) \geq 0 \Longrightarrow f(v_2, v_1) \leq -\gamma \|v_1 - v_2\|^2, \; \forall \, v_1, v_2 \in \mathcal{K};$$

(iv) pseudo-monotone if

$$f(v_1, v_2) \geq 0 \Longrightarrow f(v_2, v_1) \leq 0, \forall \, v_1, v_2 \in \mathcal{K};$$

and

(v) satisfy the Lipschitz-type conditions on $\mathcal{K}$ for $L_1, L_2 > 0$, such that

$$f(v_1, v_3) - L_1 \|v_1 - v_2\|^2 - L_2 \|v_2 - v_3\|^2 \leq f(v_1, v_2) + f(v_2, v_3), \forall \, v_1, v_2, v_3 \in \mathcal{K}.$$

The above well-defined simple mathematical problem (1) includes many mathematical and applied sciences problems as a special case, consisting of the fixed point problems, vector and scalar minimization problems, problems of variational inequalities (VIP), the complementarity problems, the Nash equilibrium problems in non-cooperative games, and inverse optimization problems [1,4,5]. This problem is also seen as a problem of Ky Fan inequality based on his initial contribution [2]. Several researchers have developed and generalized numerous findings on the nature of a solution to an equilibrium problem. (e.g., see [2,4,6,7]). Due to the basic formulation of a problem (1) and its application in both the theoretical and applied sciences, it has been extensively studied in recent times by several authors [8,9] (see also [10–16]).

Many methods have been previously established and considered their convergence investigation to deal with the problem (1). There is an impressive number of numerical methods have been designed along with their well-defined convergence analysis and theoretical properties to solve the problem (1) in different dimensional spaces [17–22]. Regularization is one of the most significant methods to figure out various ill-posed problems in the many fields of pure and applied mathematics. The prominent aspect of the regularization method is to employ it on monotone equilibrium problems and the initial problem converts into strongly monotone equilibrium sub-problem. Therefore, each computationally efficient sub-problem is strongly monotone and a unique solution exists.

A proximal method is another approach to deal with equilibrium problems that rely on numerical minimization problems [23]. This method has also been identified as the extragradient method [24] based on the initial contribution of the Korpelevich [25] method to solve the saddle point problems. Hieu [26] established an algorithmic sequence $\{u_n\}$ as follows:

$$\begin{cases} u_0 \in \mathcal{K} \\ v_n = \arg\min_{v \in \mathcal{K}} \{\zeta_n f(u_n, v) + \frac{1}{2}\|u_n - v\|^2\}, \\ u_{n+1} = \arg\min_{v \in \mathcal{K}} \{\zeta_n f(v_n, v) + \frac{1}{2}\|u_n - v\|^2\}, \end{cases} \tag{2}$$

while $\{\zeta_n\}$ meet the following conditions:

$$\mathcal{C}_1 : \lim_{n \to +\infty} \zeta_n = 0 \;\text{ and }\; \mathcal{C}_2 : \sum_{n=1}^{+\infty} \zeta_n = +\infty. \tag{3}$$

Inertial-like methods are two-step iterative methods, where the next iteration is carried out by employing the previous two iterations [27,28]. The inertial interpolation term is required to boost the sequence and help to improve the convergence rate of the iterative sequence. Such inertial methods are essentially used to speed up the iterative sequence to the appropriate solution and to improve the convergence rate. Numerical descriptions demonstrate that inertial effects also enhance the numerical performance. Such impressive attributes increase the curiosity of researchers in creating inertial methods. Recently, various inertial methods have also been established for specific types of equilibrium problems [29–32].

In this paper, we use the projection method that is simple to carry out due to its low cost and efficient numerical computations. Inspired by the works of Fan et al. [33], Thong and Hieu [34], and Censor et al. [35], we set up an accelerated extragradient-like algorithm to solve the problem (1) and other special class of equilibrium problem, such as variational inequalities. We prove a strong convergence theorem corresponding to the sequence generated to solve the problem of equilibrium under certain mild conditions. At the end, the computational tests show that the algorithm is more efficient than the current ones [26,29,36–38].

The rest of the article has been organized as follows. Section 2 consists of some basic results which are used throughout the article. Section 3 includes our proposed method and its convergence analysis. Section 4 includes numerical experiments that demonstrate practical effectiveness.

2. Preliminaries

Assume that a convex function $g : \mathcal{K} \to \mathbb{R}$ and subdifferential of g on $v_1 \in \mathcal{K}$ is defined as follows:

$$\partial g(v_1) = \{v_3 \in \mathcal{H} : g(v_2) - g(v_1) \geq \langle v_3, v_2 - v_1\rangle, \forall\, v_2 \in \mathcal{K}\}.$$

A *normal cone for* $\mathcal{K}$ on $v_1 \in \mathcal{K}$ is defined as follows:

$$N_{\mathcal{K}}(v_1) = \{v_3 \in \mathcal{H} : \langle v_3, v_2 - v_1\rangle \leq 0, \forall\, v_2 \in \mathcal{K}\}.$$

Lemma 1 ([39]). *Assume the three sequences α_n, β_n and γ_n are in $[0, +\infty)$ such that*

$$\alpha_{n+1} \leq \alpha_n + \beta_n(\alpha_n - \alpha_{n-1}) + \gamma_n, \text{for all} n \geq 1, \;\; \textit{having} \;\; \sum_{n=1}^{+\infty} \gamma_n < +\infty,$$

where $0 < \beta$ with $0 \leq \beta_n \leq \beta < 1$ for each $n \in \mathbb{N}$. Thus, we have

(i) $\sum_{n=1}^{+\infty} [\alpha_n - \alpha_{n-1}]_+ < +\infty$, *with* $[q]_+ := \max\{q, 0\}$;
(ii) $\lim_{n\to+\infty} \alpha_n = \alpha^* \in [0, +\infty)$.

Lemma 2 ([40]). *For each $v_1, v_2 \in \mathcal{H}$ and $r \in \mathbb{R}$, the following equality holds*

$$\|rv_1 + (1-r)v_2\|^2 = r\|v_1\|^2 + (1-r)\|v_2\|^2 - r(1-r)\|v_1 - v_2\|^2.$$

Lemma 3 ([41]). *Let $\{p_n\}$ and $\{q_n\} \subset [0, +\infty)$ be two sequences such that*

$$\sum_{n=1}^{+\infty} p_n = +\infty \quad \textit{and} \quad \sum_{n=1}^{+\infty} p_n q_n < +\infty.$$

Then, $\liminf_{n\to+\infty} q_n = 0$.

Lemma 4 ([42]). *Assume that a function $h : \mathcal{K} \to \mathbb{R}$ is subdifferentiable, convex, and lower semi-continuous on $\mathcal{K}$. Then, $v_1 \in \mathcal{K}$ is a function h minimizer if and only if $0 \in \partial h(v_1) + N_{\mathcal{K}}(v_1)$ while $\partial h(v_1)$ and $N_{\mathcal{K}}(v_1)$ stand for the subdifferential of h on $v_1 \in \mathcal{K}$ and a normal cone of $\mathcal{K}$ at v_1, respectively.*

Suppose that $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ satisfies the following conditions:

(C1) $f(v_1, v_1) = 0$, for all $v_1 \in \mathcal{K}$ and f is strongly pseudomonotone on $\mathcal{K}$;
(C2) f meet the Lipschitz-type condition with two constants L_1 and L_2; and
(C3) $f(v_1, .)$ is convex and sub-differentiable on $\mathcal{H}$ for fixed each $v_1 \in \mathcal{H}$.

3. Main Results

The following is the main method (Algorithm 1) in more detail.

Algorithm 1. Modified subgradient extragradient method for equilibrium problems.

Step 0: Choose $u_{-1}, u_0 \in \mathcal{H}$ arbitrarily. Let ζ_n satisfy the conditions (3). $\{\theta_n\}$ and $\{\vartheta_n\}$ are control parameter sequences.

Step 1: Compute

$$v_n = \arg\min_{v \in \mathcal{K}} \{\zeta_n f(w_n, v) + \frac{1}{2}\|w_n - v\|^2\},$$

where $w_n = u_n + \theta_n(u_n - u_{n-1})$. If $v_n = w_n$, then STOP and $w_n \in EP(f, \mathcal{K})$.

Step 2: Compute a set

$$\mathcal{H}_n = \{z \in \mathcal{H} : \langle w_n - \zeta_n t_n - v_n, z - v_n\rangle \leq 0\},$$

where $t_n \in \partial_2 f(w_n, v_n)$.

Step 3: Compute

$$\eta_n = \arg\min_{v \in \mathcal{H}_n} \{\zeta_n f(v_n, v) + \frac{1}{2}\|w_n - v\|^2\}.$$

Step 4: Compute

$$u_{n+1} = (1 - \vartheta_n)w_n + \vartheta_n \eta_n,$$

where $\{\vartheta_n\}$ and $\{\theta_n\}$ are real sequences meet the conditions:

(i) $\{\theta_n\}$ sequence is non-decreasing and $0 \leq \theta_n \leq \theta < 1$ for each $n \geq 1$;

(ii) there exists $\vartheta, \delta, \sigma > 0$ such that

$$\delta > \frac{4\theta\big[\theta(1+\theta) + \sigma\big]}{1 - \theta^2}, \tag{4}$$

and

$$0 < \vartheta \leq \vartheta_n \leq \frac{\delta - 4\theta\big[\theta(1+\theta) + \sigma + \frac{1}{4}\theta\delta\big]}{4\delta\big[\theta(1+\theta) + \sigma + \frac{1}{4}\theta\delta\big]}. \tag{5}$$

Set $n := n + 1$ and switch to **Step 1**.

Lemma 5. *Suppose that $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ satisfies the conditions* (C1)-(C3). *For $v^* \in EP(f, \mathcal{K}) \neq \emptyset$, we have*

$$\|\eta_n - v^*\|^2 \leq \|w_n - v^*\|^2 - (1 - 2L_1\zeta_n)\|w_n - v_n\|^2 - (1 - 2L_2\zeta_n)\|\eta_n - v_n\|^2 - 2\gamma\zeta_n\|v_n - v^*\|^2.$$

Proof. By value of η_n and Lemma 4, we have

$$0 \in \partial_2\Big\{\zeta_n f(v_n, v) + \frac{1}{2}\|w_n - v\|^2\Big\}(\eta_n) + N_{\mathcal{H}_n}(\eta_n).$$

Thus, there exists $\omega \in \partial f(v_n, \eta_n)$ and $\overline{\omega} \in N_{\mathcal{H}_n}(\eta_n)$ such that

$$\zeta_n\omega + \eta_n - w_n + \overline{\omega} = 0.$$

Thus, the above implies that

$$\langle w_n - \eta_n, v - \eta_n\rangle = \zeta_n\langle \omega, v - \eta_n\rangle + \langle \overline{\omega}, v - \eta_n\rangle, \ \forall\, v \in \mathcal{H}_n.$$

Since $\overline{\omega} \in N_{\mathcal{H}_n}(\eta_n)$, it implies that $\langle \overline{\omega}, v - \eta_n \rangle \leq 0$, for all $v \in \mathcal{H}_n$. This gives that

$$\zeta_n \langle \omega, v - \eta_n \rangle \geq \langle w_n - \eta_n, v - \eta_n \rangle, \ \forall \, v \in \mathcal{H}_n. \tag{6}$$

By $\omega \in \partial f(v_n, \eta_n)$, we have

$$f(v_n, v) - f(v_n, \eta_n) \geq \langle \omega, v - \eta_n \rangle, \ \forall \, v \in \mathcal{H}. \tag{7}$$

From (6) and (7), we obtain

$$\zeta_n f(v_n, v) - \zeta_n f(v_n, \eta_n) \geq \langle w_n - \eta_n, v - \eta_n \rangle, \ \forall \, v \in \mathcal{H}_n. \tag{8}$$

By the use of $v = v^*$, we get

$$\zeta_n f(v_n, v^*) - \zeta_n f(v_n, \eta_n) \geq \langle w_n - \eta_n, v^* - \eta_n \rangle. \tag{9}$$

By given $v^* \in EP(f, \mathcal{K})$, $f(v^*, v_n) \geq 0$, which implies that $f(v_n, v^*) \leq -\gamma \|v_n - v^*\|^2$. From the expression (9), we obtain

$$\langle w_n - \eta_n, \eta_n - v^* \rangle \geq \zeta_n f(v_n, \eta_n) + \gamma \zeta_n \|v_n - v^*\|^2. \tag{10}$$

Due to the Lipschitz-type continuity of a bifunction f,

$$f(w_n, \eta_n) \leq f(w_n, v_n) + f(v_n, \eta_n) + L_1 \|w_n - v_n\|^2 + L_2 \|v_n - \eta_n\|^2. \tag{11}$$

Expressions (10) and (11) gives that

$$\begin{aligned} \langle w_n - \eta_n, \eta_n - v^* \rangle \geq \zeta_n \{ f(w_n, \eta_n) - f(w_n, v_n) \} \\ - L_1 \zeta_n \|w_n - v_n\|^2 - L_2 \zeta_n \|v_n - \eta_n\|^2 + \gamma \zeta_n \|v_n - v^*\|^2. \end{aligned} \tag{12}$$

By value $\eta_n \in \mathcal{H}_n$,

$$\langle w_n - \zeta_n t_n - v_n, \eta_n - v_n \rangle \leq 0.$$

The above implies that

$$\langle w_n - v_n, \eta_n - v_n \rangle \leq \zeta_n \langle t_n, \eta_n - v_n \rangle. \tag{13}$$

$t_n \in \partial_2 f(w_n, v_n)$ gives that

$$f(w_n, v) - f(w_n, v_n) \geq \langle t_n, v - v_n \rangle, \forall v \in \mathcal{H}.$$

Substituting $v = \eta_n$ into the above expression,

$$f(w_n, \eta_n) - f(w_n, v_n) \geq \langle t_n, \eta_n - v_n \rangle. \tag{14}$$

Expressions (13) and (14) imply that

$$\zeta_n \{ f(w_n, \eta_n) - f(w_n, v_n) \} \geq \langle w_n - v_n, \eta_n - v_n \rangle. \tag{15}$$

Combining expressions (12) and (15) implies that

$$\begin{aligned} \langle w_n - \eta_n, \eta_n - v^* \rangle \geq \langle w_n - v_n, \eta_n - v_n \rangle \\ - L_1 \zeta_n \|w_n - v_n\|^2 - L_2 \zeta_n \|v_n - \eta_n\|^2 + \gamma \zeta_n \|v_n - v^*\|^2. \end{aligned} \tag{16}$$

We have the following facts:

$$2\langle w_n - \eta_n, \eta_n - v^* \rangle = \|w_n - v^*\|^2 - \|\eta_n - w_n\|^2 - \|\eta_n - v^*\|^2.$$

$$2\langle v_n - w_n, v_n - \eta_n \rangle = \|w_n - v_n\|^2 + \|\eta_n - v_n\|^2 - \|w_n - \eta_n\|^2.$$

Thus, we finally obtain

$$\begin{aligned}\|\eta_n - v^*\|^2 \le \|w_n - v^*\|^2 &- (1 - 2L_1\zeta_n)\|w_n - v_n\|^2 - (1 - 2L_2\zeta_n)\|\eta_n - v_n\|^2 \\ &- 2\gamma\zeta_n\|v_n - v^*\|^2.\end{aligned}$$

□

Theorem 1. *The sequences $\{w_n\}$, $\{v_n\}$, $\{\eta_n\}$ and $\{u_n\}$ generated by Algorithm 1 strongly converge to v^*.*

Proof. By the value of u_{n+1}, we have

$$\begin{aligned}\|u_{n+1} - v^*\|^2 &= \|(1 - \vartheta_n)w_n + \vartheta_n\eta_n - v^*\|^2 \\ &= \|(1 - \vartheta_n)(w_n - v^*) + \vartheta_n(\eta_n - v^*)\|^2 \\ &= (1 - \vartheta_n)\|w_n - v^*\|^2 + \vartheta_n\|\eta_n - v^*\|^2 - \vartheta_n(1 - \vartheta_n)\|w_n - \eta_n\|^2 \\ &\le (1 - \vartheta_n)\|w_n - v^*\|^2 + \vartheta_n\|\eta_n - v^*\|^2. \end{aligned} \tag{17}$$

From Lemma 5, we obtain

$$\begin{aligned}\|\eta_n - v^*\|^2 \le \|w_n - v^*\|^2 &- (1 - 2L_1\zeta_n)\|w_n - v_n\|^2 - (1 - 2L_2\zeta_n)\|\eta_n - v_n\|^2 \\ &- 2\gamma\zeta_n\|v_n - v^*\|^2. \end{aligned} \tag{18}$$

By combining expressions (17) and (18), we get

$$\begin{aligned}\|u_{n+1} - v^*\|^2 \le{} & (1 - \vartheta_n)\|w_n - v^*\|^2 + \vartheta_n\|w_n - v^*\|^2 - 2\gamma\vartheta_n\zeta_n\|v_n - v^*\|^2 \\ & - \vartheta_n(1 - 2L_1\zeta_n)\|w_n - v_n\|^2 - \vartheta_n(1 - 2L_2\zeta_n)\|\eta_n - v_n\|^2 && (19) \\ ={} & \|w_n - v^*\|^2 - \vartheta_n(1 - b\zeta_n)\left[\|w_n - v_n\|^2 + \|\eta_n - v_n\|^2\right] && (20) \\ ={} & \|w_n - v^*\|^2 - \frac{\vartheta_n(1 - b\zeta_n)}{2}\left[2\|w_n - v_n\|^2 + 2\|\eta_n - v_n\|^2\right] \\ \le{} & \|w_n - v^*\|^2 - \frac{\vartheta_n(1 - b\zeta_n)}{2}\left[\|w_n - v_n\| + \|\eta_n - v_n\|\right]^2 \\ \le{} & \|w_n - v^*\|^2 - \frac{\vartheta_n(1 - b\zeta_n)}{2}\|\eta_n - w_n\|^2, && (21)\end{aligned}$$

where $b = \max\{2L_1, 2L_2\}$. It continues from u_{n+1} such that

$$\|u_{n+1} - w_n\| = \|(1 - \vartheta_n)w_n + \vartheta_n\eta_n - w_n\| = \|\vartheta_n(\eta_n - w_n)\|. \tag{22}$$

Combining (21) and (22), we have

$$\|u_{n+1} - v^*\|^2 \le \|w_n - v^*\|^2 - \frac{(1 - b\zeta_n)}{2\vartheta_n}\|u_{n+1} - w_n\|^2. \tag{23}$$

Since $\zeta_n \to 0$, thus there is $n_0 > 0$ in order that $\zeta_n \le \frac{1}{2b}$ for each $n \ge n_0$. This implies $\frac{1 - b\zeta_n}{2} \ge \frac{1}{4}$ for every $n \ge n_0$. The expression (23) for $n \ge n_0$, turn as

$$\|u_{n+1} - v^*\|^2 \le \|w_n - v^*\|^2 - \frac{1}{4\vartheta_n}\|u_{n+1} - w_n\|^2. \tag{24}$$

By description of w_n, we have

$$\begin{aligned} \|w_n - v^*\|^2 &= \|u_n + \theta_n(u_n - u_{n-1}) - v^*\|^2 \\ &= \|(1+\theta_n)(u_n - v^*) - \theta_n(u_{n-1} - v^*)\|^2 \\ &= (1+\theta_n)\|u_n - v^*\|^2 - \theta_n\|u_{n-1} - v^*\|^2 + \theta_n(1+\theta_n)\|u_n - u_{n-1}\|^2. \end{aligned} \tag{25}$$

By value of w_n, we have

$$\begin{aligned} \|u_{n+1} - w_n\|^2 &= \|u_{n+1} - u_n - \theta_n(u_n - u_{n-1})\|^2 \\ &= \|u_{n+1} - u_n\|^2 + \theta_n^2\|u_n - u_{n-1}\|^2 + 2\theta_n\langle u_n - u_{n+1}, u_n - u_{n-1}\rangle \qquad (26) \\ &\geq \|u_{n+1} - u_n\|^2 + \theta_n^2\|u_n - u_{n-1}\|^2 - \rho_n\theta_n\|u_{n+1} - u_n\|^2 - \frac{\theta_n}{\rho_n}\|u_n - u_{n-1}\|^2 \\ &\geq (1 - \rho_n\theta_n)\|u_{n+1} - u_n\|^2 + \left(\theta_n^2 - \frac{\theta_n}{\rho_n}\right)\|u_n - u_{n-1}\|^2, \qquad (27) \end{aligned}$$

where $\rho_n = \frac{1}{\delta\vartheta_n + \theta_n}$. Combining (24), (25), and (27) gives that

$$\begin{aligned} \|u_{n+1} - v^*\|^2 \leq\ & (1+\theta_n)\|u_n - v^*\|^2 - \theta_n\|u_{n-1} - v^*\|^2 + \theta_n(1+\theta_n)\|u_n - u_{n-1}\|^2 \\ & - \frac{1}{4\vartheta_n}\left[(1-\rho_n\theta_n)\|u_{n+1} - u_n\|^2 + \left(\theta_n^2 - \frac{\theta_n}{\rho_n}\right)\|u_n - u_{n-1}\|^2\right] \qquad (28) \\ =\ & (1+\theta_n)\|u_n - v^*\|^2 - \theta_n\|u_{n-1} - v^*\|^2 - \frac{1}{4\vartheta_n}(1-\rho_n\theta_n)\|u_{n+1} - u_n\|^2 \\ & + \left[\theta_n(1+\theta_n) - \frac{1}{4\vartheta_n}\left(\theta_n^2 - \frac{\theta_n}{\rho_n}\right)\right]\|u_n - u_{n-1}\|^2 \\ =\ & (1+\theta_n)\|u_n - v^*\|^2 - \theta_n\|u_{n-1} - v^*\|^2 - \frac{1}{4\vartheta_n}(1-\rho_n\theta_n)\|u_{n+1} - u_n\|^2 \\ & + \gamma_n\|u_n - u_{n-1}\|^2, \qquad (29) \end{aligned}$$

where

$$\gamma_n = \theta_n(1+\theta_n) - \frac{1}{4\vartheta_n}\left(\theta_n^2 - \frac{\theta_n}{\rho_n}\right) = \theta_n(1+\theta_n) + \frac{1}{4\vartheta_n}\left(\frac{\theta_n}{\rho_n} - \theta_n^2\right) > 0. \tag{30}$$

By the above expression and the choice of $\{\rho_n\}$, we have

$$\gamma_n = \theta_n(1+\theta_n) + \frac{1}{4\vartheta_n}\left(\frac{\theta_n}{\rho_n} - \theta_n^2\right) \leq \theta(1+\theta) + \frac{1}{4}\theta\delta. \tag{31}$$

We substitute

$$\Psi_n = \|u_n - p\|^2 - \theta_n\|u_{n-1} - p\|^2 + \gamma_n\|u_n - u_{n-1}\|^2.$$

It follows (29) such that

$$\begin{aligned} \Psi_{n+1} - \Psi_n &= \|u_{n+1} - p\|^2 - \theta_{n+1}\|u_n - p\|^2 + \gamma_{n+1}\|u_{n+1} - u_n\|^2 \\ &\quad - \|u_n - p\|^2 + \theta_n\|u_{n-1} - p\|^2 - \gamma_n\|u_n - u_{n-1}\|^2 \\ &\leq \|u_{n+1} - p\|^2 - (1+\theta_n)\|u_n - p\|^2 + \theta_n\|u_{n-1} - p\|^2 \\ &\quad + \gamma_{n+1}\|u_{n+1} - u_n\|^2 - \gamma_n\|u_n - u_{n-1}\|^2 \\ &= -\left(\frac{1}{4\vartheta_n}(1-\rho_n\theta_n) - \gamma_{n+1}\right)\|u_{n+1} - u_n\|^2. \end{aligned} \tag{32}$$

We claim that

$$\frac{1}{4\vartheta_n}(1-\rho_n\theta_n) - \gamma_{n+1} \geq \sigma.$$

The above inequality implies that

$$\begin{aligned}&\frac{1}{4\vartheta_n}(1-\rho_n\theta_n)-\gamma_{n+1}\geq\sigma\\ \text{iff}\quad&(1-\rho_n\theta_n)-4\vartheta_n\gamma_{n+1}\geq 4\vartheta_n\sigma\\ \text{iff}\quad&(1-\rho_n\theta_n)-4\vartheta_n(\gamma_{n+1}+\sigma)\geq 0\\ \text{iff}\quad&\frac{\delta\vartheta_n}{\delta\vartheta_n+\theta_n}-4\vartheta_n(\gamma_{n+1}+\sigma)\geq 0\\ \text{iff}\quad&-4(\gamma_{n+1}+\sigma)(\delta\vartheta_n+\theta_n)\geq-\delta\end{aligned}\tag{33}$$

(31) and (5) give that

$$-4(\gamma_{n+1}+\sigma)(\delta\vartheta_n+\theta_n)\geq-4\Big[\theta(1+\theta)+\frac{1}{4}\theta\delta+\sigma\Big](\delta\vartheta_n+\theta_n)\geq-\delta.\tag{34}$$

Expression (32) implies that

$$\Psi_{n+1}-\Psi_n\leq-\sigma\|u_{n+1}-u_n\|^2\leq 0,\quad\text{for all} n\geq n_0.\tag{35}$$

Thus, we obtain a non-increasing sequence $\{\Psi_n\}$ for $n\geq n_0$. By the value of Ψ_{n+1}, we have

$$\begin{aligned}\Psi_{n+1}&=\|u_{n+1}-p\|^2-\theta_{n+1}\|u_n-p\|^2+\gamma_{n+1}\|u_{n+1}-u_n\|^2\\&\geq-\theta_{n+1}\|u_n-p\|^2.\end{aligned}\tag{36}$$

By the value of Ψ_n, we have

$$\begin{aligned}\Psi_n&=\|u_n-p\|^2-\theta_n\|u_{n-1}-p\|^2+\gamma_n\|u_n-u_{n-1}\|^2\\&\geq\|u_n-p\|^2-\theta_n\|u_{n-1}-p\|^2.\end{aligned}\tag{37}$$

Thus, expression (37) for $n\geq n_0$ is such that

$$\begin{aligned}\|u_n-p\|^2&\leq\Psi_n+\theta_n\|u_{n-1}-p\|^2\\&\leq\Psi_{n_0}+\theta\|u_{n-1}-p\|^2\\&\leq\cdots\leq\Psi_{n_0}(\theta^{n-n_0}+\cdots+1)+\theta^{n-n_0}\|u_{n_0}-p\|^2\\&\leq\frac{\Psi_{n_0}}{1-\theta}+\theta^{n-n_0}\|u_{n_0}-p\|^2.\end{aligned}\tag{38}$$

By (36) and (38) for all $n\geq n_0$, we get

$$\begin{aligned}-\Psi_{n+1}&\leq\theta_{n+1}\|u_n-p\|^2\\&\leq\theta\|u_n-p\|^2\\&\leq\theta\frac{\Psi_{n_0}}{1-\theta}+\theta^{n-n_0+1}\|u_{n_0}-p\|^2.\end{aligned}\tag{39}$$

It follows from (35) and (39) that

$$\begin{aligned}\sigma\sum_{n=n_0}^{k}\|u_{n+1}-u_n\|^2&\leq\Psi_{n_0}-\Psi_{k+1}\\&\leq\Psi_{n_0}+\theta\frac{\Psi_{n_0}}{1-\theta}+\theta^{n-n_0+1}\|u_{n_0}-p\|^2\\&\leq\frac{\Psi_{n_0}}{1-\theta}+\|u_{n_0}-p\|^2.\end{aligned}\tag{40}$$

Sending $k \to +\infty$ implies that

$$\sum_{n=1}^{+\infty} \|u_{n+1} - u_n\|^2 < +\infty. \tag{41}$$

It continues from that

$$\lim_{n \to +\infty} \|u_{n+1} - u_n\| = 0. \tag{42}$$

Equations (26) and (42) provide that

$$\lim_{n \to +\infty} \|u_{n+1} - w_n\| = 0. \tag{43}$$

By the value of u_{n+1}, we have

$$\|u_{n+1} - w_n\| = \|(1 - \vartheta_n)w_n + \vartheta_n \eta_n - w_n\| = \vartheta_n \|\eta_n - w_n\|. \tag{44}$$

By Equations (43) and (44), we obtain

$$\lim_{n \to +\infty} \|\eta_n - w_n\| = 0. \tag{45}$$

By the use of triangular inequality and (42) with (43), we obtain

$$\lim_{n \to +\infty} \|u_n - w_n\| \leq \lim_{n \to +\infty} \|u_n - u_{n+1}\| + \lim_{n \to +\infty} \|u_{n+1} - w_n\| = 0 \tag{46}$$

and

$$\lim_{n \to +\infty} \|u_n - \eta_n\| \leq \lim_{n \to +\infty} \|u_n - w_n\| + \lim_{n \to +\infty} \|w_n - \eta_n\| = 0. \tag{47}$$

Expressions (28) and (41) with Lemma 1 imply that

$$\lim_{n \to +\infty} \|u_n - v^*\|^2 = b \quad \text{for some} \quad b \geq 0. \tag{48}$$

Expressions (46) and (47) imply that

$$\lim_{n \to +\infty} \|w_n - v^*\|^2 = \lim_{n \to +\infty} \|\eta_n - v^*\|^2 = b. \tag{49}$$

Thus, Lemma 5 implies that

$$(1 - 2L_2\zeta)\|w_n - v_n\|^2 \leq \|w_n - v^*\|^2 - \|\eta_n - v^*\|^2. \tag{50}$$

The above expression with (48) and (49) gives that

$$\lim_{n \to +\infty} \|w_n - v_n\| = 0 \quad \text{and} \quad \lim_{n \to +\infty} \|v_n - v^*\|^2 = b. \tag{51}$$

The argument referred to above concludes that the sequences $\{w_n\}$, $\{v_n\}$, $\{\eta_n\}$, and $\{\eta_n\}$ are bounded for each $v^* \in EP(f, \mathcal{K})$ the $\lim_{n \to +\infty} \|u_n - v^*\|^2$ exists. It follows from (19) and (25) that we have

$$\begin{aligned} 2\gamma\vartheta_n\zeta_n\|v_n - v^*\|^2 \leq& -\|u_{n+1} - v^*\|^2 + (1 + \theta_n)\|u_n - v^*\|^2 - \theta_n\|u_{n-1} - v^*\|^2 \\ &+ \theta_n(1 + \theta_n)\|u_n - u_{n-1}\|^2 \\ \leq& (\|u_n - v^*\|^2 - \|u_{n+1} - v^*\|^2) + 2\theta\|u_n - u_{n-1}\|^2 \\ &+ (\theta_n\|u_n - v^*\|^2 - \theta_{n-1}\|u_{n-1} - v^*\|^2). \end{aligned} \tag{52}$$

The above expression for $k \geq n_0$ gives that

$$\sum_{n=n_0}^{k} 2\gamma\vartheta_n\zeta_n\|v_n - v^*\|^2 \leq (\|u_{n_0} - v^*\|^2 - \|u_{k+1} - v^*\|^2) + 2\theta \sum_{n=n_0}^{k} \|u_n - u_{n-1}\|^2$$
$$+ (\theta_k\|u_k - v^*\|^2 - \theta_0\|u_{n_0} - v^*\|^2)$$
$$\leq \|u_{n_0} - v^*\|^2 + \theta\|u_k - v^*\|^2 + 2\theta \sum_{n=n_0}^{k} \|u_n - u_{n-1}\|^2, \quad (53)$$

letting $k \to +\infty$ in (53), we obtain

$$\sum_{n=n_0}^{k} 2\gamma\vartheta_n\zeta_n\|v_n - v^*\|^2 < +\infty. \quad (54)$$

From Lemma 3 and (54),

$$\liminf \|v_n - p\| = 0. \quad (55)$$

By expressions (46), (47), (49), (51) and (55),

$$\lim_{n\to+\infty} \|v_n - p\| = \lim_{n\to+\infty} \|w_n - p\| = \lim_{n\to+\infty} \|\eta_n - p\| = \lim_{n\to+\infty} \|u_n - p\| = 0. \quad (56)$$

This completes the proof. □

Next, we consider the application of our results to solve variational inequality problems. A function $G : \mathcal{H} \to \mathcal{H}$ is said to be

(G1) *strongly pseudo-monotone* over $\mathcal{K}$ for $\gamma > 0$ if

$$\langle G(v_1), v_2 - v_1\rangle \geq 0 \quad \text{implies that} \quad \langle G(v_2), v_1 - v_2\rangle \leq -\gamma\|v_1 - v_2\|^2, \forall v_1, v_2 \in \mathcal{K};$$

and

(G2) *L-Lipschitz continuity* on C if

$$\|G(v_1) - G(v_2)\| \leq L\|v_1 - v_2\|, \forall v_1, v_2 \in \mathcal{K}.$$

Let a bifunction $f(v_1, v_2) := \langle G(v_1), v_2 - v_1\rangle$ for all $v_1, v_2 \in \mathcal{K}$ then equilibrium problem turns into problem of variational inequality with $L = 2L_1 = 2L_2$. By the value of v_n,

$$\begin{aligned} v_n &= \arg\min_{v\in\mathcal{K}}\Big\{\zeta_n f(w_n, v) + \frac{1}{2}\|w_n - v\|^2\Big\} \\ &= \arg\min_{v\in\mathcal{K}}\Big\{\zeta_n\langle G(w_n), v - w_n\rangle + \frac{1}{2}\|w_n - v\|^2\Big\} \\ &= \arg\min_{v\in\mathcal{K}}\Big\{\zeta_n\langle G(w_n), v - w_n\rangle + \frac{1}{2}\|w_n - v\|^2 + \frac{\zeta_n^2}{2}\|G(w_n)\|^2 - \frac{\zeta_n^2}{2}\|G(w_n)\|^2\Big\} \\ &= \arg\min_{v\in\mathcal{K}}\Big\{\frac{1}{2}\|v - (w_n - \zeta_n G(w_n)\|^2\Big\} - \frac{\zeta_n^2}{2}\|G(w_n)\|^2 \\ &= P_{\mathcal{K}}(w_n - \zeta_n G(w_n)). \end{aligned} \quad (57)$$

Similar to above, the value of η_n turns into

$$\eta_n = P_{\mathcal{H}_n}(w_n - \zeta_n G(v_n)).$$

Corollary 1. *Assume that an operator* $G : \mathcal{K} \to \mathcal{H}$ *satisfies Conditions* (G1)–(G2). *Let* $\{w_n\}$, $\{v_n\}$, $\{\eta_n\}$, *and* $\{u_n\}$ *be the sequences generated as follows:*

(S1) *Let* $u_{-1}, u_0 \in \mathcal{H}$ *arbitrarily.*
(S2) *Choose* ζ_n *satisfying condition* (3) *and* $\{\theta_n\}$, $\{\vartheta_n\}$ *are control parameters.*
(S3) *Compute*

$$v_n = P_{\mathcal{K}}(w_n - \zeta_n G(w_n)),$$

where $w_n = u_n + \theta_n(u_n - u_{n-1})$. *If* $v_n = w_n$, *then STOP.*
(S4) *Determine a half space first* $\mathcal{H}_n = \{z \in \mathcal{H} : \langle w_n - \zeta_n G(w_n) - v_n, z - v_n\rangle \le 0\}$ *and evaluate*

$$\eta_n = P_{\mathcal{H}_n}(w_n - \zeta_n G(v_n)).$$

(S5) *Compute*

$$u_{n+1} = (1 - \vartheta_n)w_n + \vartheta_n \eta_n,$$

where $\{\theta_n\}$ *and* $\{\vartheta_n\}$ *satisfies the following conditions:*

(i) *non-decreasing sequence* $\{\theta_n\}$ *through* $0 \le \theta_n \le \theta < 1$, *for each* $n \ge 1$; *and*
(ii) *there exists* $\vartheta, \delta, \sigma > 0$, *thus that*

$$\delta > \frac{4\theta\big[\theta(1+\theta) + \sigma\big]}{1 - \theta^2} \tag{58}$$

and

$$0 < \vartheta \le \vartheta_n \le \frac{\delta - 4\theta\big[\theta(1+\theta) + \sigma + \frac{1}{4}\theta\delta\big]}{4\delta\big[\theta(1+\theta) + \sigma + \frac{1}{4}\theta\delta\big]}. \tag{59}$$

Then, $\{w_n\}$, $\{v_n\}$, $\{\eta_n\}$, *and* $\{u_n\}$ *strongly converge to* $v^* \in VI(G, \mathcal{K})$.

4. Numerical Illustration

Numerical findings are summarized in this section to demonstrate the effectiveness of the proposed methods. The following control parameters are used in this section.

(1) For Hieu et al. [26] (Hieu-EgA), we use $D_n = \|u_n - v_n\|^2$.

(2) For Hieu et al. [29] (Hieu-mEgA), we use $\theta = 0.5$ and $D_n = \max\{\|u_{n+1} - v_n\|^2, \|u_{n+1} - w_n\|^2\}$.

(3) For Algorithm 1 (iEgA), we use $\alpha_n = 0.50$, $\beta_n = 0.80$, and $D_n = \|w_n - v_n\|^2$.

Example 1. *Let bifunction* f *have the following form*

$$f(u, v) = \langle Au + Bv + c, v - u\rangle$$

where $c \in \mathbb{R}^5$ *and* A *and* B *are*

$$A = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

and

$$c = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{pmatrix}$$

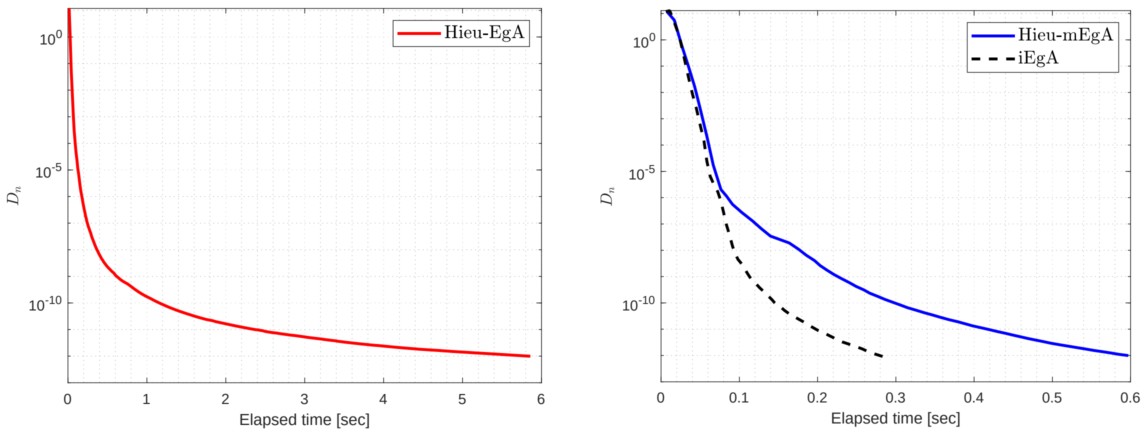
where Lipschitz parameters $L_1 = L_2 = \frac{1}{2}\|A - B\|$ *[26]. The feasible set* $\mathcal{K} \subset \mathbb{R}^5$ *is*

$$\mathcal{K} := \{u \in \mathbb{R}^5 : -5 \leq u_i \leq 5\}.$$

Table 1 and Figures 1–3 show the numerical results by $u_{-1} = u_0 = v_0 = (1, \cdots, 1)$, *and* $TOL = 10^{-12}$.

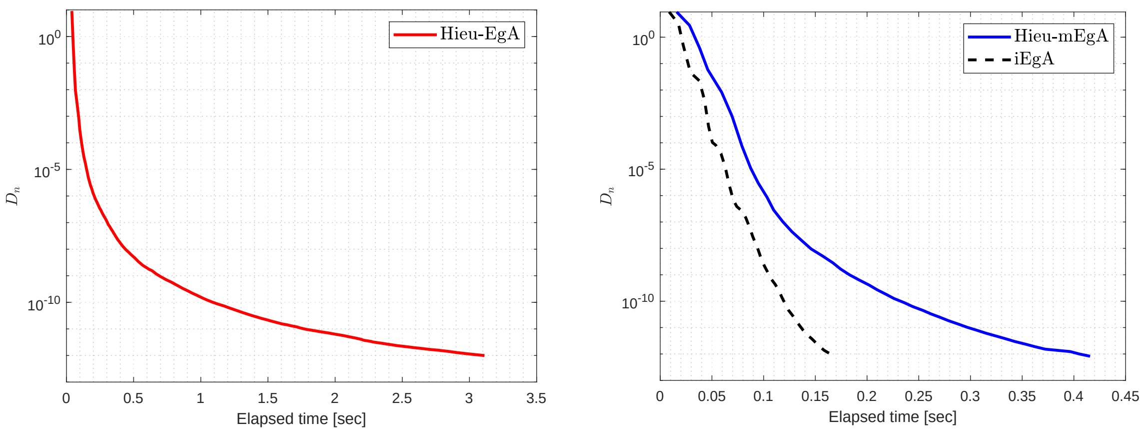
Table 1. Example 1: Numerical values for Figures 1–3.

			Hieu-EgA [26]		Hieu-mEgA [29]		iEgA	Algorithm 1
n	**TOL**	ζ_n	**Iter.**	**Time**	**Iter.**	**Time**	**Iter.**	**Time**
5	10^{-12}	$\frac{1}{\log(n+3)(n+1)}$	320	5.8584	59	0.5979	64	0.2830
5	10^{-12}	$\frac{1}{n+1}$	222	3.1116	43	0.4158	39	0.1696
5	10^{-12}	$\frac{\log(n+3)}{n+1}$	122	1.5466	40	0.3732	33	0.1581



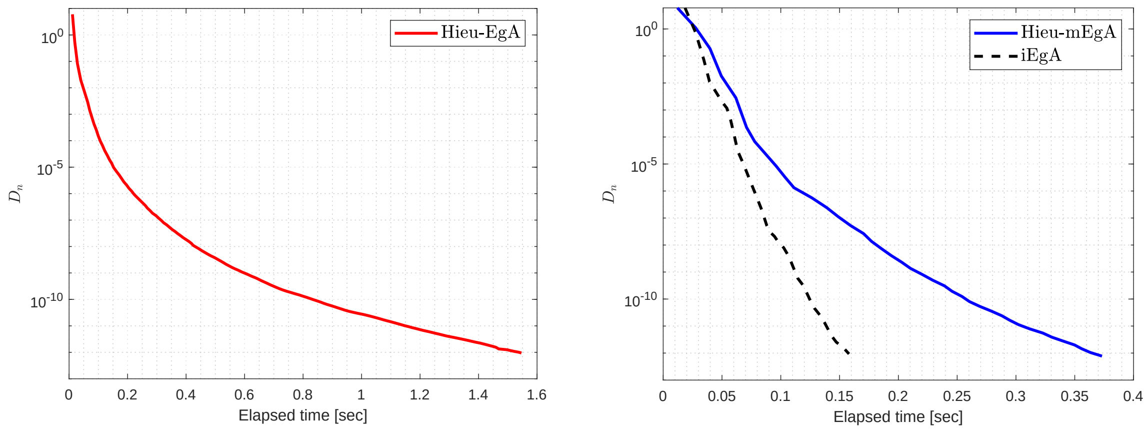
(**a**) CPU time in seconds (**b**) CPU time in seconds

Figure 1. Example 1: Numerical comparison for Algorithm 1 while $\zeta_n = \frac{1}{(n+1)\log(n+3)}$.



(**a**) CPU time in seconds (**b**) CPU time in seconds

Figure 2. Example 1: Numerical comparison for Algorithm 1 while $\zeta_n = \frac{1}{n+1}$.



(**a**) CPU time in seconds (**b**) CPU time in seconds

Figure 3. Example 1: Numerical comparison for Algorithm 1 while $\zeta_n = \frac{\log(n+3)}{n+1}$.

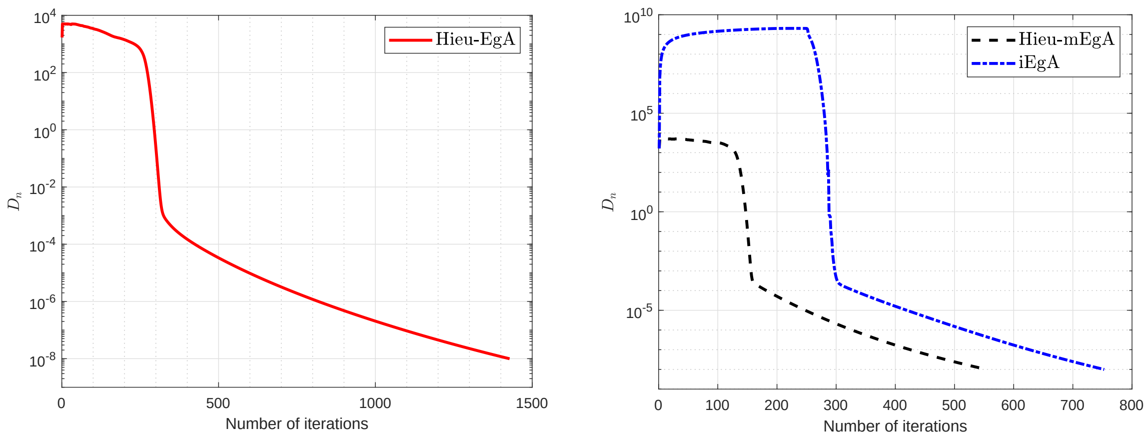
Example 2. *Let a bifunction f be defined on the convex set $\mathcal{K}$ as*

$$f(u, v) = \langle (BB^T + S + D)u, v - u \rangle,$$

where B is a 50×50 matrix, S is a 50×50 skew-symmetric matrix, and D is a 50×50 diagonal matrix. The set $\mathcal{K} \subset \mathbb{R}^{50}$ is defined by

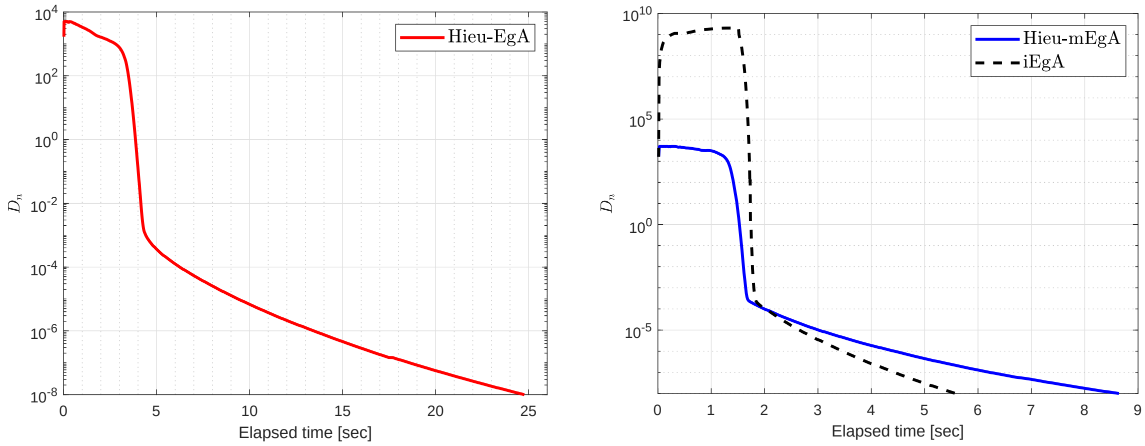
$$\mathcal{K} := \{u \in \mathbb{R}^{50} : Au \leq b\}$$

with matrix A as 100×50 and vector b as a non-negative vector. Observe that f is monotone and Lipschitz-type constants are $c_1 = c_2 = \frac{\|BB^T+S+D\|}{2}$. We generate random matrices in our case $[B = rand(n), C = rand(n), S = 0.5C - 0.5C^T, D = diag(rand(n, 1))]$ and the numerical findings regarding Example 2 are shown in Figures 4–7 with $u_{-1} = u_0 = v_0 = (1, \cdots, 1)$ and $TOL = 10^{-12}$.



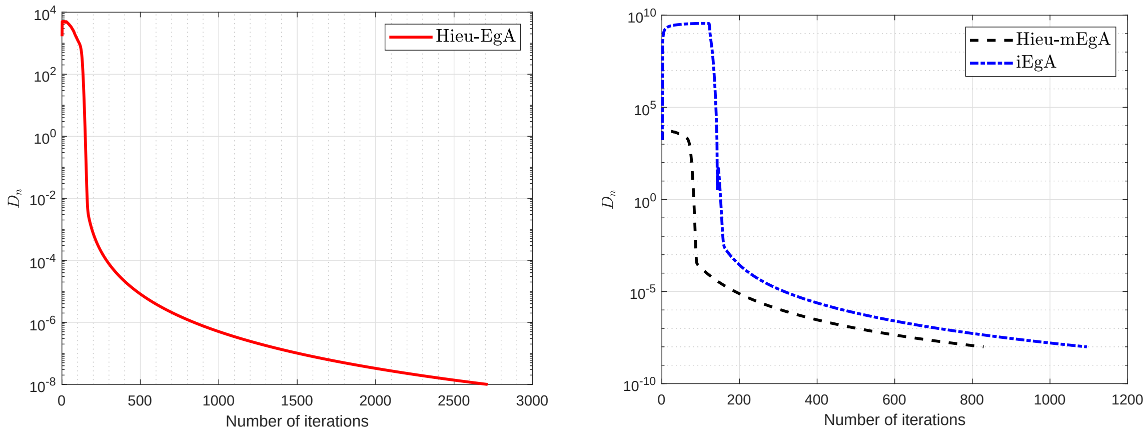
(**a**) Number of iterations (**b**) Number of iterations

Figure 4. Example 2: Numerical comparison for Algorithm 1 while $\zeta_n = \frac{1}{n+1}$.



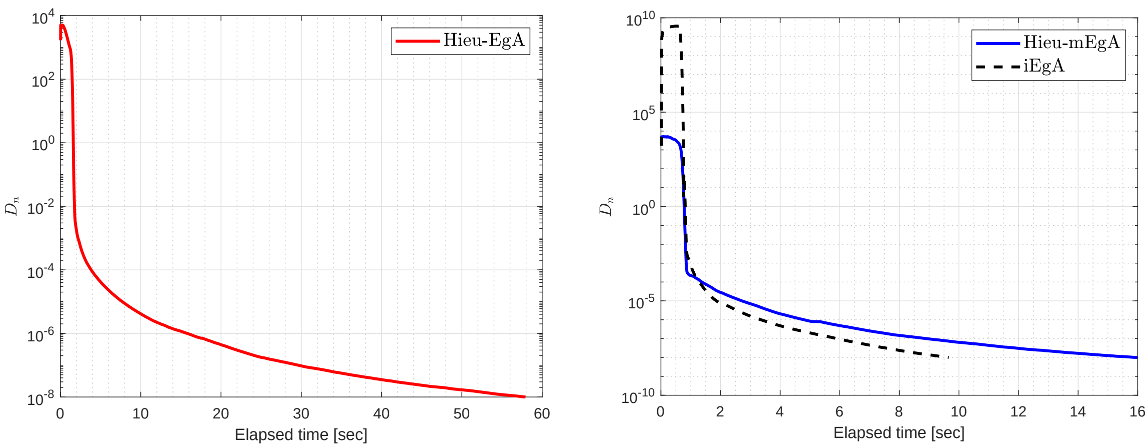
(**a**) CPU time in seconds (**b**) CPU time in seconds

Figure 5. Example 2: Numerical comparison for Algorithm 1 while $\zeta_n = \frac{1}{n+1}$.



(**a**) Number of iterations (**b**) Number of iterations

Figure 6. Example 2: Numerical comparison for Algorithm 1 while $\zeta_n = \frac{\log(n+3)}{n+1}$.



(**a**) CPU time in seconds (**b**) CPU time in seconds

Figure 7. Example 2: Numerical comparison for Algorithm 1 while $\zeta_n = \frac{\log(n+3)}{n+1}$.

Example 3. *Let* $G : \mathbb{R}^5 \to \mathbb{R}^5$ *be defined by*

$$G(u) = Au + B(u) + c,$$

where $n \times n$ symmetric semi-definite matrix A and $B(u)$ is the function depends on the proximal operator [43] through $h(u) = \frac{1}{4}\|u\|^4$ such that

$$B(u) = \arg\min_{v \in \mathbb{R}^n} \left\{ \frac{\|u\|^4}{4} + \frac{1}{2}\|v - u\|^2 \right\}.$$

The feasible set $\mathcal{K}$ is considered as

$$\mathcal{K} := \{u \in \mathbb{R}^5 : -2 \leq u_i \leq 5\}.$$

The entries of A and c are taken as follows:

$$A = \begin{pmatrix} 3 & 1 & 0 & 1 & 2 \\ 1 & 5 & -1 & 0 & 1 \\ 0 & 1 & -4 & 2 & -2 \\ 1 & 0 & 2 & 6 & -1 \\ 2 & 1 & -2 & -1 & 4 \end{pmatrix} \quad c = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{pmatrix}$$

Figures 8–11 and Table 2 show the numerical results by using $u_{-1} = u_0 = v_0 = (1, \cdots, 1)$ and $TOL = 10^{-12}$.

Table 2. Example 3: Numerical results for Figures 8–11.

			Hieu-EgA [26]		Hieu-mEgA [29]		iEgA	Algorithm 1
n	**TOL**	ζ_n	**Iter.**	**Time**	**Iter.**	**Time**	**Iter.**	**Time**
5	10^{-10}	$\frac{1}{(n+1)\log(n+3)}$	440	29.7625	190	16.2712	247	10.8531
5	10^{-10}	$\frac{1}{n+1}$	198	13.8482	104	11.8096	145	5.8483
5	10^{-10}	$\frac{\log(n+3)}{n+1}$	178	12.2979	98	7.8478	120	5.2870
5	10^{-10}	$\frac{1}{\sqrt{n+1}}$	251	16.7337	110	9.6097	148	6.0004

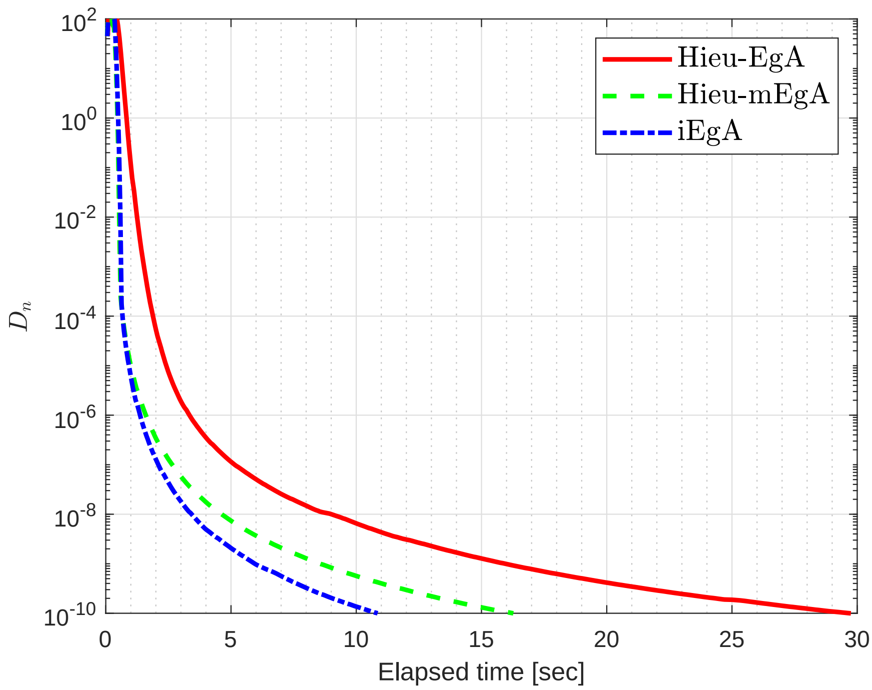


Figure 8. Example 3: Numerical comparison for Algorithm 1 while $\zeta_n = \frac{1}{(n+1)\log(n+3)}$.

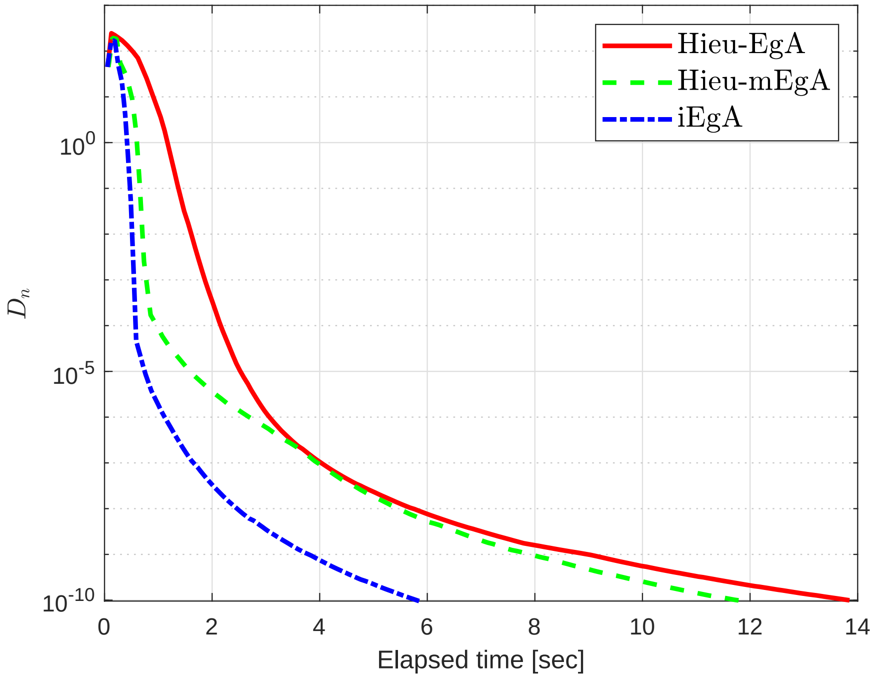


Figure 9. Example 3: Numerical comparison for Algorithm 1 while $\zeta_n = \frac{1}{n+1}$.

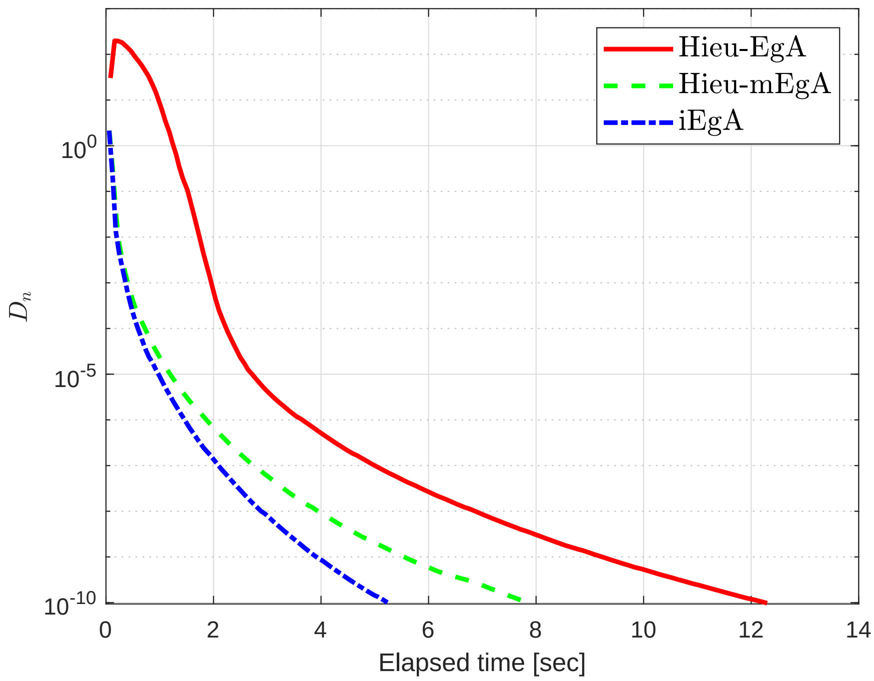


Figure 10. Example 3: Numerical comparison for Algorithm 1 while $\zeta_n = \frac{\log(n+3)}{n+1}$.

Example 4. *Suppose that $\mathcal{K} \subset G : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by*

$$G \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 + v_2 + sin(v_1) \\ -v_1 + v_2 + sin(v_2) \end{pmatrix}, \text{ for all } (v_1, v_2) \in \mathbb{R}^2.$$

where $\mathcal{K} = [-5, 5] \times [-5, 5]$. It is easy that G is Lipschitz continuous and strongly pseudomonotone operator. Figures 12–15 show the numerical results with $u_{-1} = u_0 = v_0$ and $TOL = 10^{-10}$.

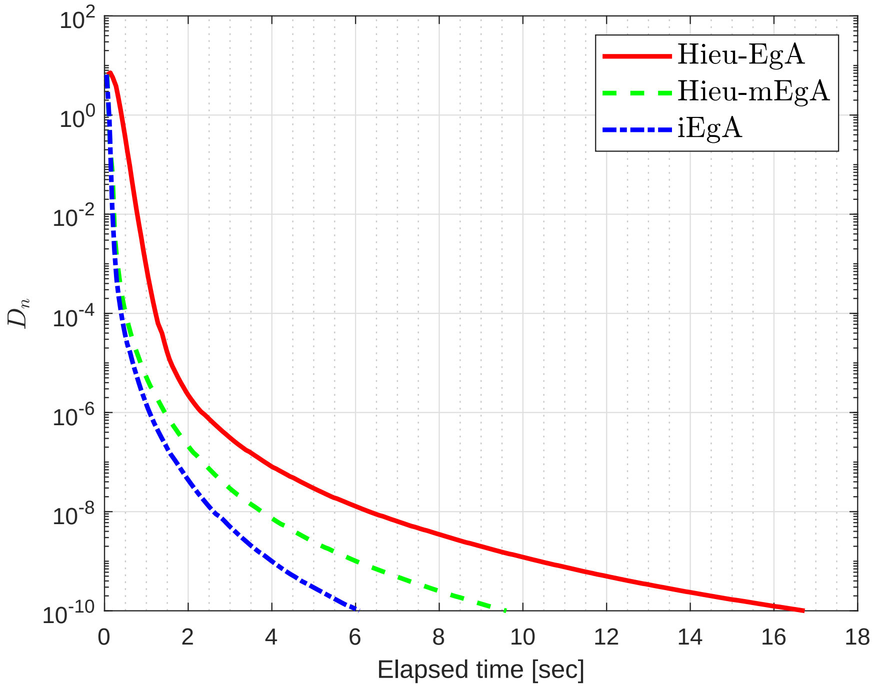
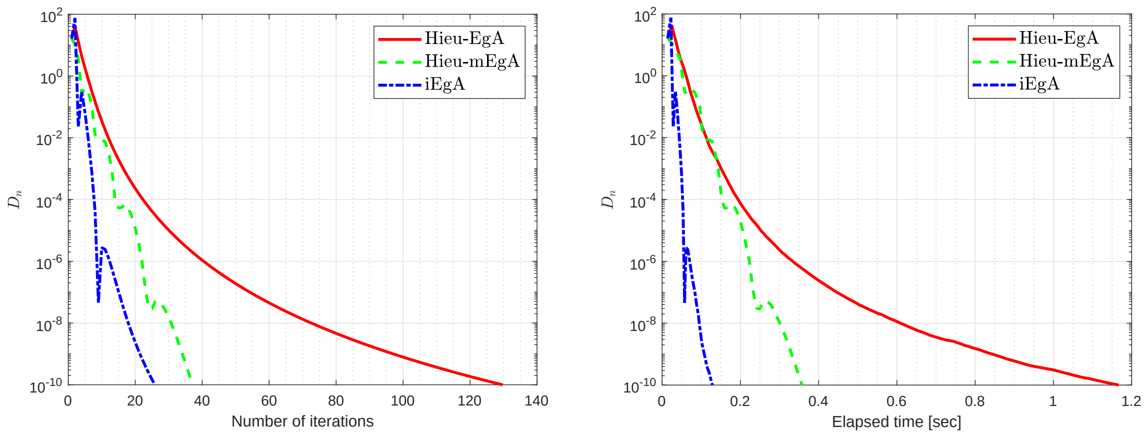
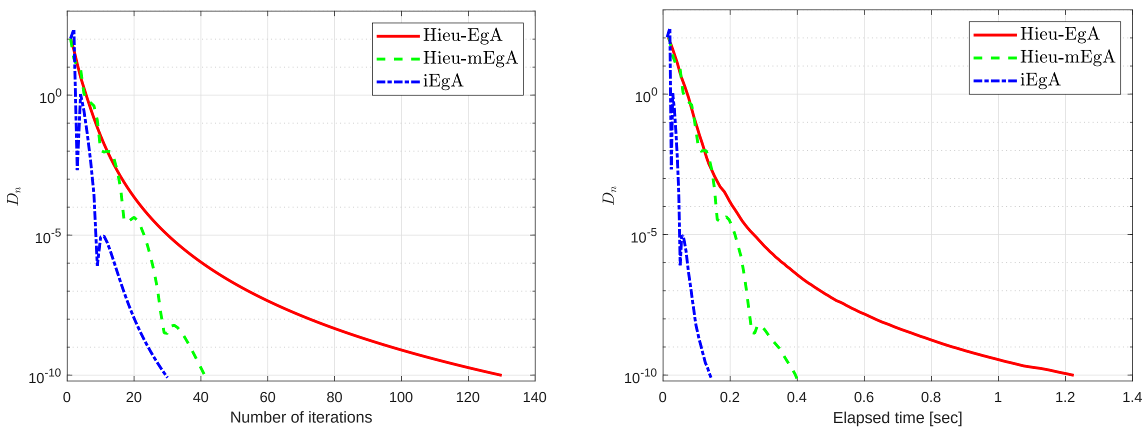


Figure 11. Example 3: Numerical comparison for Algorithm 1 while $\zeta_n = \frac{1}{\sqrt{n+1}}$.



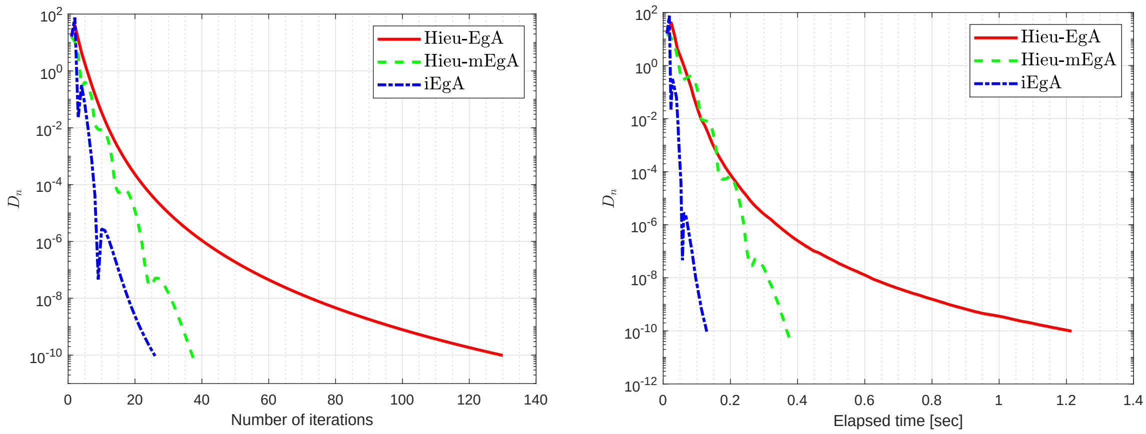
(**a**) Number of iterations (**b**) CPU time in seconds

Figure 12. Example 4: Numerical comparison for Algorithm 1 while $u_0 = (1, 1)$ and $\zeta_n = \frac{1}{n+1}$.



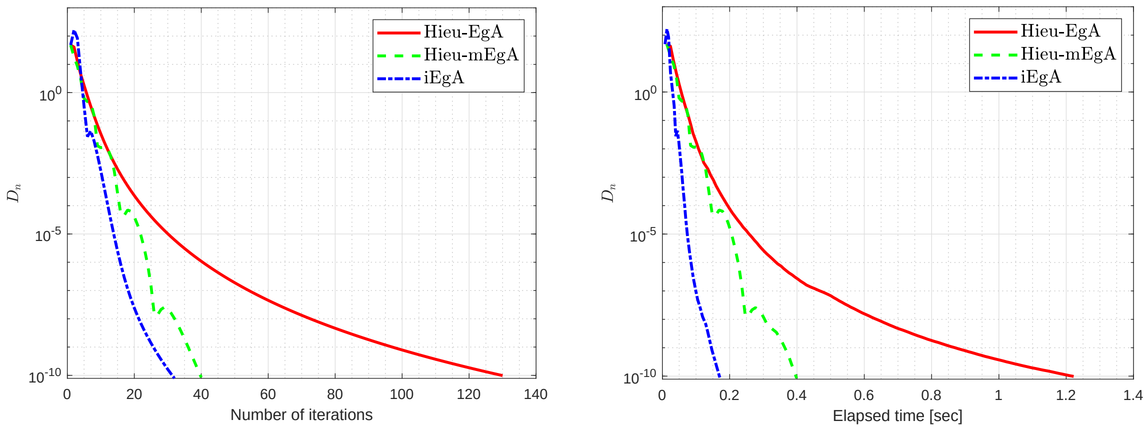
(**a**) Number of iterations (**b**) CPU time in seconds

Figure 13. Example 4: Numerical comparison for Algorithm 1 while $u_0 = (4, 4)$ and $\zeta_n = \frac{1}{n+1}$.



(**a**) Number of iterations (**b**) CPU time in seconds

Figure 14. Example 4: Numerical comparison for Algorithm 1 while $u_0 = (-1, -1)$ and $\zeta_n = \frac{1}{n+1}$.



(**a**) Number of iterations (**b**) CPU time in seconds

Figure 15. Example 4: Numerical comparison for Algorithm 1 while $u_0 = (-2, -2)$ and $\zeta_n = \frac{1}{n+1}$.

5. Conclusions

In this paper, we set up a new method by combining an inertial term with an extragradient method for solving a family of strongly pseudomonotone equilibrium problems. The introduced method involves a sequence of diminishing and non-summable step size rule and the method operates without previous information of the Lipschitz-type constants. Four numerical examples are described to show the computational performance of the proposed method in relation to other existing methods. Numerical experiments clearly point out that the method with an inertial term performs better than those without an inertial term.

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