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State Estimation of Positive Switched Interval Systems with Metzler–Takagi–Sugeno Fuzzy Models

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Abstract: This paper addresses the problem of estimating the state of a class of interval and positive nonlinear switched systems. The considered system class is represented by Metzler–Takagi–Sugeno fuzzy switched models with positive Lipschitz nonlinear functions and bounded disturbance. The fuzzy switching interval observers need real-time measurable values of premise variables. The introduced design method in this paper allows us to compute the lower and upper bounds of the system state under assumption that unknown disturbances are norm-bounded, computing the observer gain to achieve such robustness. Formulations and proofs of the design condition for switching fuzzy positive interval observers document that the diagonal stabilisation principle is implementable by a common set of LMIs in the construction of strictly positive interval observer gains, guaranteeing Metzler and Hurwitz observer system matrices and positiveness of the lower and upper bounds of the estimated system states. Design conditions for the interval-switching observer structures are formulated via linear matrix inequalities to also ensure H_∞-norm disturbance attenuation and corresponding Lipschitz parameter upper bounds. The proposed algorithm structures are informal and easily creatable as is illustrated by a numerical example.

Keywords: Takagi–Sugeno models; switched systems; Metzler systems; parametric constraints; interval observer design

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1. Introduction

Specific state observers for a class of complex systems with nonlinear dynamics are needed for realizing state feedback control to infer online estimation of unmeasurable system states. By using the Takagi–Sugeno (T-S) fuzzy models [1], the state-space representation schemes are preferred when projecting the design task into the fuzzy representation. Consequently, related techniques [2–4], are widely recognised as the effective methods in approximating a great class of nonlinear systems to provide a mathematical formalization within the linear matrix inequalities (LMI) formulation [5].

When maintaining analogous features and platforms of the application for systems with nonnegative states [6,7], the dynamics of the mathematical models of the system should be represented by the theory of Metzler matrices [8,9]. To describe the nature and the nonnegative responses of this class of systems, some additional parametric constraints are necessary to reflect the system positiveness [10]. Benefits of a potential unification are presented in [11], when reflecting the diagonal stabilization and associated Metzler system matrix parametric representations in the construction of LMI-based design techniques.

The system matrix parametric constraints give rise to substantially complex design methods when applied to positive systems with interval-defined model parameters [12]. To demarcate the object of study in this field, Metzler matrix properties are reflected for interval observers analysis in [13]. This result is mainly reflected in [14,15], when formulating and solving the design observer task for systems with T-S fuzzy interval models.

The switched positive systems induce analogous settings, following from the interval bounds on the system state [16]. To use them interchangeably by combining positivity criteria with the premise variable bounds, the independent design lines are formulated in [17–19] when synthesizing stabilizing controllers. Because the positiveness must be maintained, the same framework is used in [20] for switched-interval observer design, wherein the observer structure design uses sufficient stability criteria in the form of LMIs to guarantee observer exponential stability. Due to the continuum nature of the premise, variables in all these solutions are supposed based on the available measurement of the considered premise variables.

The nonoverlapping LMI compatibility in the design of T-S and the Metzler T-S (M-T-S) fuzzy interval switched observers is solved in the paper to form algorithmic platforms for the problem's solution with relation to the incorporated M-T-S system positivity. The presented results substantially extend and strengthen those given in [21] to accomplish the relationships between observer state upper and lower vector states, H_∞ disturbance attenuation and the Metzler constraints. Despite the specific limitations of the concept of diagonal stabilization, the design is formulated as a feasibility problem with standard algorithmic aspects. The proposed approach constitutes a unified algorithmic structure with specific parametric representations to design an interval switching observer for systems with T-S and M-T-S fuzzy models, preferring the asymptotic convergence of the interval estimation errors.

The presented procedure primarily changes the theoretical base of the synthesis of interval observers for positive and cooperative systems with the T-S models using the diagonal stabilization principle, and it in no way affects the basic requirements for the construction of fuzzy premise variables and the synthesis of switched fuzzy controllers. A common gain matrix of the interval observer delivers anticipated performances in both mathematical analysis and practical use and can be considered as a compulsion of conservatism. The application limitation lies in the measurability of the premise variables, but this measurability is also necessary for the synthesis of T-S fuzzy switched regulators. Because the methodology is based only on LMIs, its computational complexity is standard (see, for example [22,23]), which addresses the problem of switched-interval observer asymptotic stability. There are still insufficient results in the literature for considering the effects of unmeasured premise variables to design the controller and observers for the switched T-S fuzzy system [24,25] and, despite the presented results, the synthesis of the switched positive fuzzy observers still remains for authors an open question that requires solutions that are as robust as possible against this effect in the M-T-S fuzzy system.

The paper is organized as follows. In Section 2 the constructive switching observer structure and fundamental design conditions for T-S fuzzy switching systems is adduced. To adapt the principle of fuzzy switched positive interval observers, the major theoretical starting points and redeployments are introduced in Section 3 to construct the basis of the observer positiveness in LMI representation. In Section 4, the conditions that allow us to design M-T-S interval switched observers by using the transformed set of LMIs are formulated. The illustrative solution is included into Section 5. The results section outlines the illustrative solution of given problems in Section 5, underlying Metzler–Hurwitz switched observer mode dynamics. The substantive concepts provided by the methodology are summarized in Section 6, outlining as well some topics in the future works.

Throughout the paper, the following notations are used: $\boldsymbol{x}^{\mathrm{T}}$ and $\boldsymbol{X}^{\mathrm{T}}$ denote the transpose of the vector $\boldsymbol{x}$, and the matrix $\boldsymbol{X}$, respectively. Here, diag $[\,\cdot\,]$ marks a (block) diagonal matrix, and a square symmetric matrix $\boldsymbol{X} \prec 0$ means that $\boldsymbol{X}$ is a negative definite matrix. The symbol $*$ is used as an ellipsis in a block symmetric matrix, $\boldsymbol{I}_n$ indicates the nth order unit matrix, $\mathbb{R}$ ($\mathbb{R}_+$) qualifies the set of (nonnegative) real numbers, $\mathbb{R}^{n\times n}$ ($\mathbb{R}_+^{n\times n}$) refers to the set of (nonnegative) real matrices, and $\mathbb{R}_{-+}^{n\times n}$ covers the set of Metzler matrices.

2. Takagi–Sugeno Fuzzy Switching Observer

The considered system class covers a variety of switching multiinput and multioutput (MIMO) nonlinear dynamic systems, which is representable for a more complex system behavior by the continuous-time T-S fuzzy models

$$\dot{\boldsymbol{q}}(t) = \sum_{i=1}^{s} h_i^{\sigma}(\boldsymbol{\vartheta}(t))(\boldsymbol{A}_i^{\sigma}\boldsymbol{q}(t) + \boldsymbol{B}_i^{\sigma}\boldsymbol{u}(t)) + \boldsymbol{f}^{\sigma}(\boldsymbol{q}(t)) + \boldsymbol{D}\boldsymbol{d}(t)\,, \tag{1}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}^{\sigma}\boldsymbol{q}(t)\,, \tag{2}$$

where $\boldsymbol{q}(t) \in \mathbb{R}^n$, $\boldsymbol{y}(t) \in \mathbb{R}^m$ are vectors of the state and output variables, $\boldsymbol{d}(t) \in \mathbb{R}^d$ is a norm-bounded disturbance, and the local mode state-dependent nonlinearities $\boldsymbol{f}^{\sigma}(\boldsymbol{q}(t)) \in \mathbb{R}^n$ are Lipschitz. The system matrix parameters involved in the state-space model are $\boldsymbol{A}_i^{\sigma} \in \mathbb{R}^{n\times n}$, $\boldsymbol{B}_i^{\sigma} \in \mathbb{R}^{n\times r}$, $\boldsymbol{C}^{\sigma} \in \mathbb{R}^{m\times n}$, $\boldsymbol{D} \in \mathbb{R}^{n\times d}$, $\sigma \in \Sigma$ denotes a concrete active mode from the set of switched modes $\Sigma = \{1, \dots n_w\}$ with the total number of modes n_w.

According to the degree of membership, the T-S fuzzy rule-based switching system description establishes for an ith fuzzy rule the normalized membership function $h_i^{\sigma}(\boldsymbol{\vartheta}(t))$ for all $\sigma \in \{1, \dots, n_w\}$, where

$$0 \le h_i^{\sigma}(\boldsymbol{\vartheta}(t)) \le 1, \quad \sum_{i=1}^{s} h_i^{\sigma}(\boldsymbol{\vartheta}(t)) = 1\,, \tag{3}$$

for $i \in \{1, \dots, s\}$, reflecting the total number of possible fuzzy rules s and the set of total number o of premise variables $\boldsymbol{\vartheta}(t) = \begin{bmatrix}\theta_1(t) & \theta_2(t) & \cdots & \theta_o(t)\end{bmatrix}$.

The premise variables govern the blending of the linear submodels, associated with anticipated nonlinear changes in the system operating conditions. The considered class of T-S fuzzy switched systems deals only with measurable premise variables, which are a subset of the system state variables.

To embed into the design task the nonlinearity of corresponding Lipschitz properties, the following definition is suitable.

Definition 1 ([26,27])**.** *Let $\boldsymbol{f}(\boldsymbol{s}) : \mathbb{X} \to \mathbb{R}^n$ be a vector function defined on $\mathbb{X} \subset \mathbb{R}^n$ and $\bar{\boldsymbol{s}} \in \mathbb{X}$. The function $\boldsymbol{f}(\boldsymbol{s})$ is said to satisfy locally Lipschitz condition at $\bar{\boldsymbol{s}}$ if there exists a neighborhood $\mathcal{S}_+$ of $\bar{\boldsymbol{s}}$ and real $\lambda > 0$, such that*

$$\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\| \le \lambda\|\boldsymbol{x} - \boldsymbol{y}\| \text{ for all } \boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}_+\,. \tag{4}$$

Definition 1 is the basic assumption in the formulation of the H_∞-norm based T-S observer design criterion (see, for example [28]).

Specifying the observer performances to be compatible with the common notation of the switching T-S fuzzy system when switching together with a subsystem that is in action, the algorithm of the switching T-S fuzzy observer is prescribed as

$$\dot{\boldsymbol{q}}_e(t) = \sum_{i=1}^{s} h_i^{\sigma}(\boldsymbol{\vartheta}(t))(\boldsymbol{A}_i^{\sigma}\boldsymbol{q}_e(t) + \boldsymbol{B}_i^{\sigma}\boldsymbol{u}(t)) + \boldsymbol{f}^{\sigma}(\boldsymbol{q}_e(t)) + \sum_{i=1}^{s} h_i^{\sigma}(\boldsymbol{\vartheta}(t))\boldsymbol{J}_i^{\sigma}\boldsymbol{C}^{\sigma}(\boldsymbol{q}(t) - \boldsymbol{q}_e(t))\,, \tag{5}$$

$$\boldsymbol{y}_e(t) = \boldsymbol{C}^{\sigma}\boldsymbol{q}_e(t)\,, \tag{6}$$

where $\boldsymbol{q}_e(t) \in \mathbb{R}^n$ is the observer state vector and the design subjects are the matrices $\boldsymbol{J}_i^{\sigma} \in \mathbb{R}^{n\times m}$, $i \in \{1, \dots, s\}$, $\sigma \in \{1, \dots, n_w\}$.

Because the observation produces the observer errors

$$\boldsymbol{e}(t) = \boldsymbol{q}(t) - \boldsymbol{q}_e(t), \quad \boldsymbol{e}_y(t) = \boldsymbol{C}^{\sigma}\boldsymbol{e}(t)\,, \tag{7}$$

and then, because the premise variables are measurable and $\boldsymbol{q}_e(0) = \boldsymbol{0}$ is freely assignable,

$$\begin{aligned} \dot{\boldsymbol{e}}(t) &= \sum_{i=1}^{s} h_j^{\sigma}(\boldsymbol{\vartheta}(t))(\boldsymbol{A}_i^{\sigma} - \boldsymbol{J}_i^{\sigma}\boldsymbol{C}^{\sigma})\boldsymbol{e}(t) + \Delta\boldsymbol{f}^{\sigma}(t) + \boldsymbol{D}\boldsymbol{d}(t) \\ &= \sum_{i=1}^{s} h_i^{\sigma}(\boldsymbol{\vartheta}(t))\boldsymbol{A}_{ei}^{\sigma}\boldsymbol{e}(t) + \Delta\boldsymbol{f}^{\sigma}(t) + \boldsymbol{D}\boldsymbol{d}(t)\,, \end{aligned} \tag{8}$$

where the following notation is associated

$$\boldsymbol{A}_{ei}^{\sigma} = \boldsymbol{A}_i^{\sigma} - \boldsymbol{J}_i^{\sigma}\boldsymbol{C}^{\sigma}, \quad \Delta\boldsymbol{f}^{\sigma}(t) = \boldsymbol{f}^{\sigma}(\boldsymbol{q}(t)) - \boldsymbol{f}^{\sigma}(\boldsymbol{q}_e(t))\,. \tag{9}$$

Note, (8) promotes also a switching structure.

Remark 1. *The observer error dependence of the nonlinear function can be obtained via representation in the square norm on the differentiability of (4) for $\kappa = \lambda^2$ as*

$$\begin{aligned} \|\boldsymbol{f}(\boldsymbol{q}(t)) - \boldsymbol{f}(\boldsymbol{q}_e(t))\|^2 &= (\boldsymbol{f}(\boldsymbol{q}(t)) - \boldsymbol{f}(\boldsymbol{q}_e(t))^{\mathrm{T}}(\boldsymbol{f}(\boldsymbol{q}(t)) - \boldsymbol{f}(\boldsymbol{q}_e(t)) \\ &\le \kappa(\boldsymbol{q}(t) - \boldsymbol{q}_e(t))^{\mathrm{T}}(\boldsymbol{q}(t) - \boldsymbol{q}_e(t)) \\ &= \kappa\,\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{I}_n\boldsymbol{e}(t) \\ &> 0\,. \end{aligned} \tag{10}$$

If the previous assumptions are reflected, it is possible to prove the following theorem, which is the basic fundamental element of the proposed approach.

Theorem 1. *If there exists a symmetric positive definite matrix $\boldsymbol{P} \in \mathbb{R}^{n\times n}$, matrices $\boldsymbol{V}_i^{\sigma} \in \mathbb{R}^{n\times m}$ and a positive scalars $\kappa, \mu \in \mathbb{R}_+$, such that for all $i \in \{1,\ldots,s\}$, $\sigma \in \{1,\ldots,n_w\}$*

$$\boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}} \succ 0\,, \quad \mu > 0\,, \tag{11}$$

$$\begin{bmatrix} \boldsymbol{P}\boldsymbol{A}_i^{\sigma} + \boldsymbol{A}_i^{\sigma\mathrm{T}}\boldsymbol{P} - \boldsymbol{V}_i^{\sigma}\boldsymbol{C}^{\sigma} - \boldsymbol{C}^{\sigma\mathrm{T}}\boldsymbol{V}_i^{\sigma\mathrm{T}} + \kappa\boldsymbol{I}_n & * & * & * \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P} & -\mu\boldsymbol{I}_d & * & * \\ \boldsymbol{C}^{\sigma} & \boldsymbol{0} & -\mu\boldsymbol{I}_m & * \\ \boldsymbol{P} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I}_n \end{bmatrix} \prec 0\,, \tag{12}$$

then, in a feasible case, the switching T-S fuzzy observer (5), (6) for the system (1), (2) is asymptotically stable and the set of observer gains for $i \in \{1,\ldots,s\}$, $\sigma \in \{1,\ldots,n_w\}$ can be found by the rules

$$\boldsymbol{J}_i^{\sigma} = \boldsymbol{P}^{-1}\boldsymbol{V}_i^{\sigma}\,. \tag{13}$$

Proof. A following positive $v(\boldsymbol{e}(t))$ can be served as the Lyapunov function for (8) and any $\sigma \in \{1,\ldots,n_w\}$ when using a symmetric positive definite matrix $\boldsymbol{P} \in \mathbb{R}^{n\times n}$ and a positive scalar $\mu \in \mathbb{R}_+$ in such a way that

$$v(\boldsymbol{e}(t)) = \boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}(t) + \mu^{-1}\int_0^t (\boldsymbol{e}_y^{\mathrm{T}}(\tau)\boldsymbol{e}_y(\tau) - \mu^2\boldsymbol{d}^{\mathrm{T}}(\tau)\boldsymbol{d}(\tau))\mathrm{d}\tau > 0\,, \tag{14}$$

whose time-derivative for all observer error trajectories satisfies

$$\dot{v}(\boldsymbol{e}(t)) = \dot{\boldsymbol{e}}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}(t) + \boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\dot{\boldsymbol{e}}(t) + \mu^{-1}\boldsymbol{e}_y^{\mathrm{T}}(t)\boldsymbol{e}_y(t) - \mu\boldsymbol{d}^{\mathrm{T}}(t)\boldsymbol{d}(t) < 0\,. \tag{15}$$

Note, the variable μ is an upper bound of H_∞ norm of the disturbance transfer function matrix and therefore the integrand in the definite integral in (14) is always positive [29]. From an optimization point of view, it can be defined as a tuning parameter in design.

Applying by substitution into inequality (15) the observer error dynamics (8) get the following result

$$\begin{aligned}\dot{v}(\boldsymbol{e}(t)) = &\sum_{i=1}^{s} h_i^{\sigma}(\boldsymbol{\vartheta}(t))\boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{A}_{ei}^{\sigma\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_{ei}^{\sigma})\boldsymbol{e}(t)+ \\ &+ \sum_{i=1}^{s} h_i^{\sigma}(\boldsymbol{\vartheta}(t))(\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{D}\boldsymbol{d}(t) + \boldsymbol{d}^{\mathrm{T}}(t)\boldsymbol{D}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{e}(t))+ \\ &+ \sum_{i=1}^{s} h_i^{\sigma}(\boldsymbol{\vartheta}(t))(\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\Delta\boldsymbol{f}^{\sigma}(t) + \Delta\boldsymbol{f}^{\sigma\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}(t))+ \\ &+ \mu^{-1}\sum_{i=1}^{s} h_i^{\sigma}(\boldsymbol{\vartheta}(t))\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{C}^{\sigma\mathrm{T}}\boldsymbol{C}^{\sigma}\boldsymbol{e}(t) - \mu\boldsymbol{d}^{\mathrm{T}}(t)\boldsymbol{d}(t) \\ &< 0\,.\end{aligned} \tag{16}$$

To be altered to suit the needs of an LMI structure, the following elements can be written by using (4) and (10) as

$$\begin{aligned}\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\Delta\boldsymbol{f}^{\sigma}(t) + \Delta\boldsymbol{f}^{\sigma\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}(t) &\le \boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{P}\boldsymbol{e}(t) + \Delta\boldsymbol{f}^{\sigma\mathrm{T}}(t)\Delta\boldsymbol{f}^{\sigma}(t) \\ &\le \boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{P}\boldsymbol{e}(t) + \kappa\,\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{I}_n\boldsymbol{e}(t)\end{aligned} \tag{17}$$

and constructing a common notation which is readily representable for all variables

$$\boldsymbol{e}_d^{\mathrm{T}}(t) = \begin{bmatrix}\boldsymbol{e}^{\mathrm{T}}(t) & \boldsymbol{d}^{\mathrm{T}}(t)\end{bmatrix}, \tag{18}$$

and then there are reasonable grounds to conclude that

$$\dot{v}(\boldsymbol{e}_d(t)) = \sum_{i=1}^{s} h_i^{\sigma}(\boldsymbol{\vartheta}(t))\boldsymbol{e}_d^{\mathrm{T}}(t)\boldsymbol{\Omega}_i^{\sigma}\boldsymbol{e}_d(t) < 0\,, \tag{19}$$

where, for the covered systematization,

$$\boldsymbol{\Omega}_i^{\sigma} = \begin{bmatrix}\boldsymbol{A}_{ei}^{\sigma\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_{ei}^{\sigma} + \kappa\boldsymbol{I}_n + \mu^{-1}\boldsymbol{C}^{\sigma\mathrm{T}}\boldsymbol{C}^{\sigma} + \boldsymbol{P}\boldsymbol{I}_n\boldsymbol{P} & * \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P} & -\mu\boldsymbol{I}_d\end{bmatrix} \prec 0\,. \tag{20}$$

Thus, it follows from the above that

$$\begin{bmatrix}\boldsymbol{A}_{ei}^{\sigma\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_{ei}^{\sigma} + \kappa\boldsymbol{I}_n + \boldsymbol{P}\boldsymbol{I}_n\boldsymbol{P} & * \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P} & -\mu\boldsymbol{I}_d\end{bmatrix} + \begin{bmatrix}\boldsymbol{C}^{\sigma\mathrm{T}} \\ \boldsymbol{0}\end{bmatrix}\mu^{-1}\boldsymbol{I}_m\begin{bmatrix}\boldsymbol{C}^{\sigma} & \boldsymbol{0},\end{bmatrix} \prec 0 \tag{21}$$

and using the Schur complement property [30] the condition can be rewritten as

$$\begin{bmatrix}\boldsymbol{P}\boldsymbol{A}_{ei}^{\sigma} + \boldsymbol{A}_{ei}^{\sigma\mathrm{T}}\boldsymbol{P} + \kappa\boldsymbol{I}_n + \boldsymbol{P}\boldsymbol{I}_n\boldsymbol{P} & * & * \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P} & -\mu\boldsymbol{I}_d & * \\ \boldsymbol{C}^{\sigma} & \boldsymbol{0} & -\mu\boldsymbol{I}_m\end{bmatrix} \prec 0\,. \tag{22}$$

Analogously it can be straightforward to write

$$\begin{bmatrix}\boldsymbol{P}\boldsymbol{A}_{ei}^{\sigma} + \boldsymbol{A}_{ei}^{\sigma\mathrm{T}}\boldsymbol{P} + \kappa\boldsymbol{I}_n & * & * \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P} & -\mu\boldsymbol{I}_d & * \\ \boldsymbol{C}^{\sigma} & \boldsymbol{0} & -\mu\boldsymbol{I}_m\end{bmatrix} + \begin{bmatrix}\boldsymbol{P} \\ \boldsymbol{0} \\ \boldsymbol{0}\end{bmatrix}\boldsymbol{I}_n\begin{bmatrix}\boldsymbol{P} & \boldsymbol{0} & \boldsymbol{0}\end{bmatrix} \prec 0, \tag{23}$$

and the final form of LMI after applying again the property of Schur complement is

$$\begin{bmatrix} \boldsymbol{P}\boldsymbol{A}_{ei}^{\sigma} + \boldsymbol{A}_{ei}^{\sigma\mathrm{T}}\boldsymbol{P} + \kappa\boldsymbol{I}_n & * & * & * \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P} & -\mu\boldsymbol{I}_d & * & * \\ \boldsymbol{C}^{\sigma} & \boldsymbol{0} & -\mu\boldsymbol{I}_m & * \\ \boldsymbol{P} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I}_n \end{bmatrix} \prec 0\,. \tag{24}$$

Respecting $\boldsymbol{A}_{ei}^{\sigma}$ introduced in (9), the basic rule in including the system matrix parameters into the LMI structure means that

$$\boldsymbol{P}\boldsymbol{A}_{ei}^{\sigma} = \boldsymbol{P}\boldsymbol{A}_{i}^{\sigma} - \boldsymbol{P}\boldsymbol{J}_{i}^{\sigma}\boldsymbol{C}^{\sigma} = \boldsymbol{P}\boldsymbol{A}_{i}^{\sigma} - \boldsymbol{V}_{i}^{\sigma}\boldsymbol{C}^{\sigma}\,, \tag{25}$$

where $\boldsymbol{V}_{i}^{\sigma} = \boldsymbol{P}\boldsymbol{J}_{i}^{\sigma}$ for $i \in \{1,\ldots,s\}, \sigma \in \{1,\ldots,n_w\}$.

Thus, one can conclude from this that (24), (25) imply (12) of the standard structure, and so it is possible to close the proof. □

3. Metzler–Takagi–Sugeno Fuzzy Switching Observer

In this case the Lipschitz function, $f(\boldsymbol{q}(t)) \in \mathbb{R}_+$ is positive, $\boldsymbol{B}_i^{\sigma} \in \mathbb{R}_+^{n\times r}$ $\boldsymbol{C}^{\sigma} \in \mathbb{R}_+^{m\times n}$, $\boldsymbol{D} \in \mathbb{R}_+^{n\times d}$ are nonnegative, and $\boldsymbol{A}_i^{\sigma} \in \mathbb{R}_{-+}^{n\times n}$ are strictly Metzler. Such redefined system is noted as the M-T-S fuzzy switching positive system and its state variable vector is nonnegative.

To respect these assumptions, a matrix $\boldsymbol{A}_i^{\sigma}$ with only negative diagonal elements and with strictly positive (greater then zero) off-diagonal elements is considered the strictly Metzler matrix. Consequently, a strictly Metzler matrix $\boldsymbol{A}_i^{\sigma}$ is so defined under n^2 constraints (structural parametric constraints)

$$a_{lh}^{\sigma} < 0,\ l = h, \quad a_{lh}^{\sigma} > 0, l \neq h,\ \forall l,h \in \langle 1,n\rangle\,. \tag{26}$$

To guarantee the parametric constraints fulfillment in design of closed-loop systems, or in observer system matrices, the diagonal stabilization principle has to be used [31].

Remark 2. *If a strictly Metzler $\boldsymbol{A} \in \mathbb{M}_{-+}^{n\times n}$ is represented with relation to the observer design task in the following rhombic form, where the diagonal exactness are constructed by the column index defined multiple circular shifts of elements of the columns of $\boldsymbol{A}$ as follows [11],*

$$\boldsymbol{A}_{\ominus} = \begin{bmatrix} a_{11} & & & & \\ a_{21} & a_{22} & & & \\ a_{31} & a_{32} & a_{33} & & \\ \vdots & \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \\ & a_{12} & a_{13} & \cdots & a_{1n} \\ & & a_{23} & \cdots & a_{2n} \\ & & & \ddots & \vdots \\ & & & & a_{n-1,n} \end{bmatrix}, \tag{27}$$

then the diagonal matrix structures, related to $\boldsymbol{A}_{\ominus}$ with the index $h = 0,\ldots,n-1$,

$$\boldsymbol{A}(\nu,\nu) = \mathrm{diag}\begin{bmatrix} a_{11} & a_{22} & \cdots & a_{nn} \end{bmatrix} \prec 0\,, \tag{28}$$

$$\boldsymbol{A}(\nu+h,\nu) = \mathrm{diag}\begin{bmatrix} a_{1+h,1} & \cdots & a_{n,n-h} & a_{1,n-h+1} & \cdots & a_{h,n} \end{bmatrix} \succ 0\,, \tag{29}$$

represent the set of Metzler parametric constraints (26).

Moreover, utilization of this principle leads to the Metzler matrix $\boldsymbol{A}$ parameterizations as [11]

$$\boldsymbol{A} = \sum_{h=0}^{n-1} \boldsymbol{A}(\nu+h,\nu)\boldsymbol{L}^{h\mathrm{T}}, \quad \boldsymbol{L} = \begin{bmatrix} \boldsymbol{0}^{\mathrm{T}} & 1 \\ \boldsymbol{I}_{n-1} & \boldsymbol{0} \end{bmatrix}, \tag{30}$$

where $\boldsymbol{L} \in \mathbb{R}^{n\times n}$ is the circulant permutation matrix. Implementing (30) for any $\boldsymbol{A}_e = \boldsymbol{A} - \boldsymbol{J}\boldsymbol{C} \in \mathbb{R}^{n\times n}_{-+}$ address the needs of the following parametrization (see, for example, [32])

$$\boldsymbol{A}_e = \sum_{h=0}^{n-1} \big(\boldsymbol{A}_i(\nu+h,\nu) - \sum_{k=1}^{m} \boldsymbol{J}_{kh}\boldsymbol{C}_{dk}\big)\boldsymbol{L}^{h\mathrm{T}}, \tag{31}$$

where, with relation to (28), (29), the diagonal matrices $\boldsymbol{J}_{kh}, \boldsymbol{C}_{dk} \in \mathbb{R}^{n\times n}_{+}$ are defined as follows:

$$\boldsymbol{C}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{c}_1 & \cdots & \boldsymbol{c}_m \end{bmatrix}, \quad \boldsymbol{C}_{dk} = \mathrm{diag}\big[\boldsymbol{c}_k^{\mathrm{T}}\big], \tag{32}$$

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{j}_1 & \cdots & \boldsymbol{j}_m \end{bmatrix}, \quad \boldsymbol{J}_k = \mathrm{diag}\big[\boldsymbol{j}_{ik}\big], \quad \boldsymbol{J}_{kh} = \boldsymbol{L}^{h\mathrm{T}}\boldsymbol{J}_k\boldsymbol{L}^h. \tag{33}$$

To apply this parametrization principle the following corollary is objective.

Corollary 1. *Using Remark 2 and the error Equations (8) and (9) entails the parametrizations*

$$\boldsymbol{A}_i^{\sigma}(\nu,\nu) = \mathrm{diag}\big[a^{\sigma}_{i11} \; a^{\sigma}_{i22} \; \cdots \; a^{\sigma}_{inn}\big] \tag{34}$$

$$\boldsymbol{A}_i^{\sigma}(\nu+h,\nu) = \mathrm{diag}\Big[a^{\sigma}_{i,1+h,1} \; \cdots \; a^{\sigma}_{i,n,n-h} \; a^{\sigma}_{i,1,n-h+1} \; \cdots \; a^{\sigma}_{ihn}\Big] \tag{35}$$

$$\boldsymbol{C}^{\sigma\mathrm{T}} = \begin{bmatrix} \boldsymbol{c}^{\sigma}_1 & \cdots & \boldsymbol{c}^{\sigma}_m \end{bmatrix}, \quad \boldsymbol{C}^{\sigma}_{dk} = \mathrm{diag}\big[\boldsymbol{c}_k^{\sigma\mathrm{T}}\big], \tag{36}$$

$$\boldsymbol{J}^{\sigma}_i = \begin{bmatrix} \boldsymbol{j}^{\sigma}_{i1} & \cdots & \boldsymbol{j}^{\sigma}_{im} \end{bmatrix}, \quad \boldsymbol{J}^{\sigma}_{ik} = \mathrm{diag}\big[\boldsymbol{j}^{\sigma}_{ik}\big], \quad \boldsymbol{J}^{\sigma}_{ikh} = \boldsymbol{L}^{h\mathrm{T}}\boldsymbol{J}^{\sigma}_{ik}\boldsymbol{L}^h. \tag{37}$$

With Lyapunov function (14) and Corollary 1, it is possible to give the design criteria.

Theorem 2. *The matrices $\boldsymbol{A}^{\sigma}_{ei} \in \mathbb{R}^{n\times n}_{-+}$ for all $i \in \langle 1, s\rangle$, $\sigma \in \{1,\ldots,n_w\}$ are strictly Metzler and Hurwitz if for by the system (1), (2) defined strictly Metzler matrices $\boldsymbol{A}^{\sigma}_i \in \mathbb{R}^{n\times n}_{-+}$ and nonnegative matrices $\boldsymbol{C}^{\sigma} \in \mathbb{R}^{m\times n}_{+}$, $\boldsymbol{D} \in \mathbb{R}^{n\times d}_{+}$ there exist positive definite diagonal matrices $\boldsymbol{P}, \boldsymbol{V}^{\sigma}_{ik} \in \mathbb{R}^{n\times n}_{+}$ and positive scalars $\kappa, \mu \in \mathbb{R}_+$, such that for $i = 1,\ldots,s$, $\sigma \in \{1,\ldots,n_w\}$, $h = 1,\ldots,n-1$*

$$\boldsymbol{P} \succ 0, \quad \boldsymbol{V}^{\sigma}_{ik} \succ 0, \tag{38}$$

$$\boldsymbol{P}\boldsymbol{A}^{\sigma}_i(\nu,\nu) - \sum_{k=1}^{m} \boldsymbol{V}^{\sigma}_{ik}\boldsymbol{C}^{\sigma}_{dk} \prec 0, \tag{39}$$

$$\boldsymbol{P}\boldsymbol{L}^h\boldsymbol{A}^{\sigma}_i(\nu+h,\nu)\boldsymbol{L}^{h\mathrm{T}} - \sum_{k=1}^{m} \boldsymbol{V}^{\sigma}_{ik}\boldsymbol{L}^h\boldsymbol{C}^{\sigma}_{dk}\boldsymbol{L}^{h\mathrm{T}} \succ 0, \tag{40}$$

$$\begin{bmatrix} \boldsymbol{P}\boldsymbol{A}^{\sigma}_i + \boldsymbol{A}^{\sigma\mathrm{T}}_i\boldsymbol{P} - \sum_{k=1}^{m} \boldsymbol{V}^{\sigma}_{ik}\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{C}^{\sigma}_{dk} - \sum_{k=1}^{m} \boldsymbol{C}^{\sigma}_{dk}\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{V}^{\sigma}_{ik} + \kappa\boldsymbol{I}_n & * & * & * \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{P} & -\mu\boldsymbol{I}_d & * & * \\ \boldsymbol{C}^{\sigma} & \boldsymbol{0} & -\mu\boldsymbol{I}_m & * \\ \boldsymbol{P} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I}_n \end{bmatrix} \prec 0. \tag{41}$$

Confirming the feasible task, the gains for $i \in \{1, \ldots, s\}$, $\sigma \in \{1, \ldots, n_w\}$, are defined as

$$\boldsymbol{J}_{ik}^{\sigma} = \boldsymbol{P}^{-1}\boldsymbol{V}_{ik}^{\sigma}, \quad \boldsymbol{j}_{ik}^{\sigma} = \boldsymbol{J}_{ik}^{\sigma}\boldsymbol{l}, \quad \boldsymbol{J}_{i}^{\sigma} = \begin{bmatrix} \boldsymbol{j}_{i1}^{\sigma} & \cdots & \boldsymbol{j}_{im}^{\sigma} \end{bmatrix}, \quad \boldsymbol{l}^{\mathrm{T}} = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}. \tag{42}$$

Proof. According to the parametrization of $\boldsymbol{A}_{ei}^{\sigma}$ (34)–(37) it has to yield for all i, σ and the related h

$$\boldsymbol{A}_{i}^{\sigma}(\nu,\nu) - \sum_{k=1}^{m} \boldsymbol{J}_{ik}^{\sigma}\boldsymbol{C}_{dk}^{\sigma} \prec 0\,, \tag{43}$$

$$\boldsymbol{A}_{i}^{\sigma}(\nu+h,\nu)\boldsymbol{L}^{h\mathrm{T}} - \sum_{k=1}^{m} \boldsymbol{J}_{ikh}^{\sigma}\boldsymbol{C}_{dk}^{\sigma}\boldsymbol{L}^{h\mathrm{T}} \succ 0\,. \tag{44}$$

Multiplying this by the positive definite diagonal matrix $\boldsymbol{P}$, the left side of the inequality (43) yields

$$\boldsymbol{P}\boldsymbol{A}_{i}^{\sigma}(\nu,\nu) - \sum_{k=1}^{m} \boldsymbol{P}\boldsymbol{J}_{ik}^{\sigma}\boldsymbol{C}_{dk}^{\sigma} \prec 0, \tag{45}$$

and using the notation $\boldsymbol{V}_{ik}^{\sigma} = \boldsymbol{P}\boldsymbol{J}_{ik}^{\sigma}$ then (45) implies (39).

Analogously, multiplying the left side of (44) by $\boldsymbol{P}\boldsymbol{L}^{h}$ yields

$$\boldsymbol{P}\boldsymbol{L}^{h}\boldsymbol{A}_{i}^{\sigma}(\nu+h,\nu)\boldsymbol{L}^{h\mathrm{T}} - \sum_{k=1}^{m} \boldsymbol{P}\boldsymbol{L}^{h}\boldsymbol{L}^{h\mathrm{T}}\boldsymbol{J}_{ik}^{\sigma}\boldsymbol{L}^{h}\boldsymbol{C}_{dk}^{\sigma}\boldsymbol{L}^{h\mathrm{T}} \succ 0, \tag{46}$$

and with the notation $\boldsymbol{V}_{ik}^{\sigma} = \boldsymbol{P}\boldsymbol{J}_{ik}^{\sigma}$, then (46) implies (40), because $\boldsymbol{L}^{h}\boldsymbol{L}^{h\mathrm{T}} = \boldsymbol{I}_{n}$.

In the given sense, (39), (40) force the Metzler parametric constraints in the design task. Writing now for $\boldsymbol{A}_{ei}^{\sigma} = \boldsymbol{A}_{i}^{\sigma} - \boldsymbol{J}_{i}^{\sigma}\boldsymbol{C}^{\sigma} \in \mathbb{R}_{-+}^{n \times n}$, then

$$\begin{aligned} &\boldsymbol{P}(\boldsymbol{A}_{i}^{\sigma} - \boldsymbol{J}_{i}^{\sigma}\boldsymbol{C}^{\sigma}) + (\boldsymbol{A}_{i}^{\sigma} - \boldsymbol{J}_{i}^{\sigma}\boldsymbol{C}^{\sigma})^{\mathrm{T}}\boldsymbol{P} \\ =&\boldsymbol{P}\Big(\boldsymbol{A}_{i}^{\sigma} - \sum_{k=1}^{m} \boldsymbol{j}_{ik}^{\sigma}\boldsymbol{c}_{k}^{\sigma\mathrm{T}}\Big) + \Big(\boldsymbol{A}_{i}^{\sigma} - \sum_{k=1}^{m} \boldsymbol{j}_{ik}^{\sigma}\boldsymbol{c}_{k}^{\sigma\mathrm{T}}\Big)^{\mathrm{T}}\boldsymbol{P} \\ =&\boldsymbol{P}\Big(\boldsymbol{A}_{i}^{\sigma} - \sum_{k=1}^{m} \boldsymbol{J}_{ik}^{\sigma}\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{C}_{dk}^{\sigma}\Big) + \Big(\boldsymbol{A}_{i}^{\sigma} - \sum_{k=1}^{m} \boldsymbol{J}_{ik}^{\sigma}\boldsymbol{l}\boldsymbol{l}^{\mathrm{T}}\boldsymbol{C}_{dk}^{\sigma}\Big)^{\mathrm{T}}\boldsymbol{P}, \end{aligned} \tag{47}$$

and using the analogy with (25) when applying (47), then (41) results. This concludes the proof. □

The obtained design conditions are convenient for the M-T-S fuzzy switching observer synthesis, assuming that the premise variables are directly measurable. Because $\boldsymbol{A}_{ei}^{\sigma} \in \mathbb{R}_{-+}^{n \times n}$ for all $i \in \langle 1, s \rangle$, $\sigma \in \{1, \ldots, n_w\}$ are strictly Metzler and Hurwitz, the observer asymptotic stability is guarantied.

4. Metzler–Takagi–Sugeno Fuzzy Switching Interval Observer

In this case, there are also considered nonnegative disturbance $\boldsymbol{d}(t) \in \mathbb{R}_{+}^{d}$, Lipschitz function $f(\boldsymbol{q}(t)) \in \mathbb{R}_{+}^{n}$, the output matrices $\boldsymbol{C}^{\sigma} \in \mathbb{R}_{+}^{m \times n}$, and the disturbance input matrix $\boldsymbol{D} \in \mathbb{R}_{+}^{n \times d}$, but $\boldsymbol{q}(0)$ as well as the strictly Metzler system parameters $\boldsymbol{A}_{i}^{\sigma}$ are unknown but bounded by known constant bounding vectors and known constant bounding matrices of appropriate dimensions in such a way that for all $i \in \langle 1, s \rangle$, $\sigma \in \{1, \ldots, n_w\}$ (these inequalities are understood elementwise):

$$0 \leq \underline{\boldsymbol{q}}(0) \leq \boldsymbol{q}(0) \leq \overline{\boldsymbol{q}}(0)\,, \quad \underline{\boldsymbol{A}}_{i}^{\sigma} \leq \boldsymbol{A}_{i}^{\sigma} \leq \overline{\boldsymbol{A}}_{i}^{\sigma}\,, \tag{48}$$

$$0 \leq \underline{\boldsymbol{\vartheta}}(t) \leq \boldsymbol{\vartheta}(t) \leq \overline{\boldsymbol{\vartheta}}(t)\,. \tag{49}$$

Due to the comprehensiveness of the interval defined system parameters and measurable premise variables, then it can be defined the following:

$$\begin{aligned}\dot{\overline{\boldsymbol{q}}}_e(t) &= \sum_{i=1}^{s} h_i^{\sigma}(\overline{\boldsymbol{\vartheta}}(t))\big(\overline{\boldsymbol{A}}_i^{\sigma}\overline{\boldsymbol{q}}_e(t) + \boldsymbol{B}_i^{\sigma}\boldsymbol{u}(t) + \boldsymbol{J}_i^{\sigma}(\boldsymbol{y}(t) - \overline{\boldsymbol{y}}_e(t))\big) + f^{\sigma}(\overline{\boldsymbol{q}}_e(t)) \\ &= \sum_{i=1}^{s} h_i^{\sigma}(\overline{\boldsymbol{\vartheta}}(t))\big((\overline{\boldsymbol{A}}_i^{\sigma} - \boldsymbol{J}_i^{\sigma}\boldsymbol{C}^{\sigma})\overline{\boldsymbol{q}}_e(t) + \boldsymbol{B}_i^{\sigma}\boldsymbol{u}(t)\big) + \sum_{i=1}^{s} h_i^{\sigma}(\boldsymbol{\vartheta}(t))\boldsymbol{J}_i^{\sigma}\boldsymbol{C}^{\sigma}\boldsymbol{q}(t) + f^{\sigma}(\overline{\boldsymbol{q}}_e(t)),\end{aligned} \tag{50}$$

$$\begin{aligned}\dot{\underline{\boldsymbol{q}}}_e(t) &= \sum_{i=1}^{s} h_i^{\sigma}(\underline{\boldsymbol{\vartheta}}(t))\big(\underline{\boldsymbol{A}}_i^{\sigma}\underline{\boldsymbol{q}}_e(t) + \boldsymbol{B}_i^{\sigma}\boldsymbol{u}(t) + \boldsymbol{J}_i^{\sigma}(\boldsymbol{y}(t) - \underline{\boldsymbol{y}}(t))\big) + f^{\sigma}(\underline{\boldsymbol{q}}_e(t)) \\ &= \sum_{i=1}^{s} h_i^{\sigma}(\underline{\boldsymbol{\vartheta}}(t))\big((\underline{\boldsymbol{A}}_i^{\sigma} - \boldsymbol{J}_i^{\sigma}\boldsymbol{C}^{\sigma})\underline{\boldsymbol{q}}_e(t) + \boldsymbol{B}_i^{\sigma}\boldsymbol{u}(t)\big) + \sum_{i=1}^{s} h_i^{\sigma}(\boldsymbol{\vartheta}(t))\boldsymbol{J}_i^{\sigma}\boldsymbol{C}^{\sigma}\boldsymbol{q}(t) + f^{\sigma}(\underline{\boldsymbol{q}}_e(t)),\end{aligned} \tag{51}$$

where (2) yields together with

$$\overline{\boldsymbol{y}}_e(t) = \boldsymbol{C}^{\sigma}\overline{\boldsymbol{q}}_e(t), \quad \underline{\boldsymbol{y}}_e(t) = \boldsymbol{C}^{\sigma}\underline{\boldsymbol{q}}_e(t) \tag{52}$$

and with the more specific scope for $t \geq 0$ if $\overline{\boldsymbol{q}}_e(0) = \overline{\boldsymbol{q}}(0)$, $\underline{\boldsymbol{q}}_e(0) = \underline{\boldsymbol{q}}(0)$ it is

$$\boldsymbol{0} \leq \underline{\boldsymbol{q}}_e(t) \leq \boldsymbol{q}(t) \leq \overline{\boldsymbol{q}}_e(t) \tag{53}$$

$$\overline{\boldsymbol{A}}_{ei}^{\sigma} = \overline{\boldsymbol{A}}_i^{\sigma} - \boldsymbol{J}_i^{\sigma}\boldsymbol{C}^{\sigma}, \quad \underline{\boldsymbol{A}}_{ei}^{\sigma} = \underline{\boldsymbol{A}}_i^{\sigma} - \boldsymbol{J}_i^{\sigma}\boldsymbol{C}^{\sigma}. \tag{54}$$

By using the observation errors

$$\overline{\boldsymbol{e}}(t) = \boldsymbol{q}(t) - \overline{\boldsymbol{q}}_e(t), \quad \underline{\boldsymbol{e}}(t) = \boldsymbol{q}(t) - \underline{\boldsymbol{q}}_e(t), \tag{55}$$

it follows from (1), (53), (54) that

$$\begin{aligned}\dot{\overline{\boldsymbol{e}}}(t) &= \dot{\boldsymbol{q}}(t) - \dot{\overline{\boldsymbol{q}}}_e(t) \\ &= \sum_{i=1}^{s} h_i^{\sigma}(\overline{\boldsymbol{\vartheta}}(t))\overline{\boldsymbol{A}}_{ei}^{\sigma}\overline{\boldsymbol{e}}(t) - \sum_{i=1}^{s}\Big(h_i^{\sigma}(\overline{\boldsymbol{\vartheta}}(t))\overline{\boldsymbol{A}}_{ei}^{\sigma} - h_i^{\sigma}(\boldsymbol{\vartheta}(t))\boldsymbol{A}_{ei}^{\sigma}\Big)\boldsymbol{q}(t) + \\ &+ \boldsymbol{f}^{\sigma}(\boldsymbol{q}(t)) - \boldsymbol{f}^{\sigma}(\overline{\boldsymbol{q}}_e(t)) + \boldsymbol{D}\boldsymbol{d}(t) \\ &= \sum_{i=1}^{s} h_i^{\sigma}(\overline{\boldsymbol{\vartheta}}(t))\overline{\boldsymbol{A}}_{ei}^{\sigma}\overline{\boldsymbol{e}}(t) + \overline{\Delta f^{\sigma}}(\boldsymbol{q}(t), \overline{\boldsymbol{q}}_e(t)) - \overline{\boldsymbol{\Lambda}^{\sigma}}(\overline{\boldsymbol{\vartheta}}(t), \boldsymbol{\vartheta}(t))\boldsymbol{q}(t) + \boldsymbol{D}\boldsymbol{d}(t),\end{aligned} \tag{56}$$

$$\begin{aligned}\dot{\underline{\boldsymbol{e}}}(t) &= \dot{\boldsymbol{q}}(t) - \dot{\underline{\boldsymbol{q}}}_e(t) \\ &= \sum_{i=1}^{s} h_i^{\sigma}(\underline{\boldsymbol{\vartheta}}(t))\underline{\boldsymbol{A}}_{ei}^{\sigma}\underline{\boldsymbol{e}}(t) - \sum_{i=1}^{s}\Big(h_i^{\sigma}(\underline{\boldsymbol{\vartheta}}(t))\underline{\boldsymbol{A}}_{ei}^{\sigma} - h_i^{\sigma}(\boldsymbol{\vartheta}(t))\boldsymbol{A}_{ei}^{\sigma}\Big)\boldsymbol{q}(t) + \\ &+ \boldsymbol{f}^{\sigma}(\boldsymbol{q}(t)) - \boldsymbol{f}^{\sigma}(\underline{\boldsymbol{q}}_e(t)) + \boldsymbol{D}\boldsymbol{d}(t) \\ &= \sum_{i=1}^{s} h_i^{\sigma}(\overline{\boldsymbol{\vartheta}}(t))\underline{\boldsymbol{A}}_{ei}^{\sigma}\underline{\boldsymbol{e}}(t) + \underline{\Delta f^{\sigma}}(\boldsymbol{q}(t), \underline{\boldsymbol{q}}_e(t)) - \underline{\boldsymbol{\Lambda}^{\sigma}}(\underline{\boldsymbol{\vartheta}}(t), \boldsymbol{\vartheta}(t))\boldsymbol{q}(t) + \boldsymbol{D}\boldsymbol{d}(t),\end{aligned} \tag{57}$$

where

$$\overline{\Delta f^{\sigma}}(\boldsymbol{q}(t), \overline{\boldsymbol{q}}_e(t)) = \boldsymbol{f}^{\sigma}(\boldsymbol{q}(t)) - \boldsymbol{f}^{\sigma}(\overline{\boldsymbol{q}}_e(t)), \tag{58}$$

$$\underline{\Delta f^{\sigma}}(\boldsymbol{q}(t), \underline{\boldsymbol{q}}_e(t)) = \boldsymbol{f}^{\sigma}(\boldsymbol{q}(t)) - \boldsymbol{f}^{\sigma}(\underline{\boldsymbol{q}}_e(t)), \tag{59}$$

$$\overline{\boldsymbol{\Lambda}^{\sigma}}(\overline{\boldsymbol{\vartheta}}(t), \boldsymbol{\vartheta}(t)) = \sum_{i=1}^{s}\Big(h_i^{\sigma}(\overline{\boldsymbol{\vartheta}}(t))\overline{\boldsymbol{A}}_{ei}^{\sigma} - h_i^{\sigma}(\boldsymbol{\vartheta}(t))\boldsymbol{A}_{ei}^{\sigma}\Big), \tag{60}$$

$$\underline{\boldsymbol{\Lambda}^{\sigma}}(\underline{\boldsymbol{\vartheta}}(t), \boldsymbol{\vartheta}(t)) = \sum_{i=1}^{s}\Big(h_i^{\sigma}(\underline{\boldsymbol{\vartheta}}(t))\underline{\boldsymbol{A}}_{ei}^{\sigma} - h_i^{\sigma}(\boldsymbol{\vartheta}(t))\boldsymbol{A}_{ei}^{\sigma}\Big). \tag{61}$$

Remark 3. *By using (4) and (55) and adopting the standard manipulations defined by (10), then (58), (59) can be rewritten in the inequality forms as*

$$\|\overline{\Delta f^{\sigma}}(\boldsymbol{q}(t),\overline{\boldsymbol{q}}_e(t))\|^2 \leq \overline{\kappa}\,\overline{\boldsymbol{e}}^{\mathrm{T}}(t)\overline{\boldsymbol{e}}(t)\,, \quad \|\underline{\Delta f^{\sigma}}(\boldsymbol{q}(t),\underline{\boldsymbol{q}}_e(t))\|^2 \leq \underline{\kappa}\,\underline{\boldsymbol{e}}^{\mathrm{T}}(t)\underline{\boldsymbol{e}}(t)\,. \tag{62}$$

Keeping in mind that $\overline{\boldsymbol{A}}^{\sigma}_{ei}$, $\underline{\boldsymbol{A}}^{\sigma}_{ei}$ have to be strictly Metzler and Hurwitz and the upper and lower bounds of the measurable premise variables are also measurable, then

$$\lim_{\boldsymbol{\vartheta}(t)\to\overline{\boldsymbol{\vartheta}}(t)} \boldsymbol{\Lambda}^{\sigma}(\overline{\boldsymbol{\vartheta}}(t),\boldsymbol{\vartheta}(t)) = \sum_{i=1}^{s}\left(h_i^{\sigma}(\overline{\boldsymbol{\vartheta}}(t))\overline{\boldsymbol{A}}^{\sigma}_{ei} - h_i^{\sigma}(\overline{\boldsymbol{\vartheta}}(t))\overline{\boldsymbol{A}}^{\sigma}_{ei}\right) = \boldsymbol{0}\,, \tag{63}$$

$$\lim_{\boldsymbol{\vartheta}(t)\to\underline{\boldsymbol{\vartheta}}(t)} \boldsymbol{\Lambda}^{\sigma}(\underline{\boldsymbol{\vartheta}}(t),\boldsymbol{\vartheta}(t)) = \sum_{i=1}^{s}\left(h_i^{\sigma}(\underline{\boldsymbol{\vartheta}}(t))\overline{\boldsymbol{A}}^{\sigma}_{ei} - h_i^{\sigma}(\underline{\boldsymbol{\vartheta}}(t))\underline{\boldsymbol{A}}^{\sigma}_{ei}\right) = \boldsymbol{0} \tag{64}$$

and, evidently, it is satisfied by

$$\underline{\boldsymbol{A}}^{\sigma}_{i} - \boldsymbol{J}^{\sigma}_{i}\boldsymbol{C}^{\sigma} \leq \boldsymbol{A}^{\sigma}_{i} - \boldsymbol{J}^{\sigma}_{i}\boldsymbol{C}^{\sigma} \leq \overline{\boldsymbol{A}}^{\sigma}_{i} - \boldsymbol{J}^{\sigma}_{i}\boldsymbol{C}^{\sigma}\,, \tag{65}$$

because

$$\underline{\boldsymbol{A}}^{\sigma}_{i} \leq \boldsymbol{A}^{\sigma}_{i} \leq \overline{\boldsymbol{A}}^{\sigma}_{i}\,. \tag{66}$$

In addition, considering for useful analyses and comparative results the interval bounded output matrices $\overline{\boldsymbol{C}}^{\sigma}$, $\underline{\boldsymbol{C}}^{\sigma}$, where

$$\underline{\boldsymbol{C}}^{\sigma} \leq \boldsymbol{C}^{\sigma} \leq \overline{\boldsymbol{C}}^{\sigma}\,, \tag{67}$$

then the generalization of the standard Metzler interval observer matrix structures cannot be obtained, because

$$\underline{\boldsymbol{A}}^{\sigma}_{i} - \boldsymbol{J}^{\sigma}_{i}\overline{\boldsymbol{C}}^{\sigma} \nleq \boldsymbol{A}^{\sigma}_{i} - \boldsymbol{J}^{\sigma}_{i}\boldsymbol{C}^{\sigma} \nleq \overline{\boldsymbol{A}}^{\sigma}_{i} - \boldsymbol{J}^{\sigma}_{i}\underline{\boldsymbol{C}}^{\sigma}\,. \tag{68}$$

Because $\overline{\boldsymbol{\Lambda}^{\sigma}}(\overline{\boldsymbol{\vartheta}}(t),\boldsymbol{\vartheta}(t))\boldsymbol{q}(t)$, $\underline{\boldsymbol{\Lambda}^{\sigma}}(\underline{\boldsymbol{\vartheta}}(t),\boldsymbol{\vartheta}(t))\boldsymbol{q}(t)$, are bounded and $\boldsymbol{q}(t)$ has to be positive (nonnegative), considering this formulation of the design task as an polytopic convex problem, then

$$\dot{\overline{\boldsymbol{e}}}(t) = \sum_{i=1}^{s} h_i^{\sigma}(\overline{\boldsymbol{\vartheta}}(t))\overline{\boldsymbol{A}}^{\sigma}_{ei}\overline{\boldsymbol{e}}(t) + \overline{\Delta f^{\sigma}}(\boldsymbol{q}(t),\overline{\boldsymbol{q}}_e(t)) + \boldsymbol{D}\boldsymbol{d}(t)\,, \tag{69}$$

$$\dot{\underline{\boldsymbol{e}}}(t) = \sum_{i=1}^{s} h_i^{\sigma}(\overline{\boldsymbol{\vartheta}}(t))\underline{\boldsymbol{A}}^{\sigma}_{ei}\underline{\boldsymbol{e}}(t) + \underline{\Delta f^{\sigma}}(\boldsymbol{q}(t),\underline{\boldsymbol{q}}_e(t)) + \boldsymbol{D}\boldsymbol{d}(t)\,. \tag{70}$$

To apply this parametrization principle in design of this class of M-T-S observers, the following corollary is objective.

Corollary 2. *By using Remark 2 and the state observation error Equations (69), (69) entails the system matrix parametrizations*

$$\overline{\boldsymbol{A}}^{\sigma}_{i}(\nu,\nu) = \mathrm{diag}\left[\overline{a}^{\sigma}_{i11}\ \overline{a}^{\sigma}_{i22}\ \cdots\ \overline{a}^{\sigma}_{inn}\right], \tag{71}$$

$$\underline{\boldsymbol{A}}^{\sigma}_{i}(\nu,\nu) = \mathrm{diag}\left[\underline{a}^{\sigma}_{i11}\ \underline{a}^{\sigma}_{i22}\ \cdots\ \underline{a}^{\sigma}_{inn}\right], \tag{72}$$

$$\overline{\boldsymbol{A}}^{\sigma}_{i}(\nu+h,\nu) = \mathrm{diag}\left[\overline{a}^{\sigma}_{i,1+h,1}\ \cdots\ \overline{a}^{\sigma}_{i,n,n-h}\ \ \overline{a}^{\sigma}_{i,1,n-h+1}\ \cdots\ \overline{a}^{\sigma}_{ihn}\right], \tag{73}$$

$$\underline{\boldsymbol{A}}^{\sigma}_{i}(\nu+h,\nu) = \mathrm{diag}\left[\underline{a}^{\sigma}_{i,1+h,1}\ \cdots\ \underline{a}^{\sigma}_{i,n,n-h}\ \ \underline{a}^{\sigma}_{i,1,n-h+1}\ \cdots\ \underline{a}^{\sigma}_{ihn}\right], \tag{74}$$

whereas the parameterizations (36), (37) stay unchanged.

To indicate related conditions and potentially strictly positive parameter solutions of M-T-S fuzzy switching interval positive observers, the following presents a systematized method.

Theorem 3. *The matrices $\overline{A}_{ei}^{\sigma}$, $\underline{A}_{ei}^{\sigma} \in \mathbb{R}_{-+}^{n\times n}$ for all $i \in \langle 1, s\rangle$, $\sigma \in \{1, \ldots, n_w\}$ are strictly Metzler and Hurwitz if for given strictly Metzler matrices $\overline{A}_i^{\sigma}$, $\underline{A}_i^{\sigma} \in \mathbb{R}_{-+}^{n\times n}$, and nonnegative matrices $C^{\sigma} \in \mathbb{R}_{+}^{m\times n}$ there exist positive definite diagonal matrices $P, V_{ik}^{\sigma} \in \mathbb{R}_{+}^{n\times n}$ and positive scalars $\overline{\kappa}, \underline{\kappa}, \mu \in \mathbb{R}_+$, such that for $i = 1, \ldots, s$, $\sigma \in \{1, \ldots, n_w\}$, $h = 1, \ldots, n-1$, $l^{\mathrm{T}} = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}$ and the design parameters defined in Corollary 2*

$$P \succ 0, \quad V_{ik} \succ 0, \tag{75}$$

$$P\overline{A}_i^{\sigma}(\nu,\nu) - \sum_{k=1}^{m} V_{ik}^{\sigma} C_{dk}^{\sigma} \prec 0, \quad P\underline{A}_i^{\sigma}(\nu,\nu) - \sum_{k=1}^{m} V_{ik}^{\sigma} C_{dk}^{\sigma} \prec 0, \tag{76}$$

$$PL^h\overline{A}_i^{\sigma}(\nu+h,\nu)L^{h\mathrm{T}} - \sum_{k=1}^{m} V_{ik}^{\sigma} L^h C_{dk}^{\sigma} L^{h\mathrm{T}} \succ 0, \quad PL^h\underline{A}_i^{\sigma}(\nu+h,p)L^{h\mathrm{T}} - \sum_{k=1}^{m} V_{ik}^{\sigma} L^h C_{dk}^{\sigma} L^{h\mathrm{T}} \succ 0 \tag{77}$$

$$\begin{cases} \begin{bmatrix} P\overline{A}_i^{\sigma} + \overline{A}_i^{\sigma\mathrm{T}}P + \overline{\kappa}I_n - \sum\limits_{k=1}^{m} V_{ik}^{\sigma} l l^{\mathrm{T}} C_{dk} - \sum\limits_{k=1}^{m} C_{dk}^{\sigma} l l^{\mathrm{T}} V_{ik}^{\sigma} & * & * & * \\ D^{\mathrm{T}}P & -\mu I_d & * & * \\ C^{\sigma} & 0 & -\mu I_m & * \\ P & 0 & 0 & -I_n \end{bmatrix} \prec 0, \\ \begin{bmatrix} P\underline{A}_i^{\sigma} + \underline{A}_i^{\sigma\mathrm{T}}P + \underline{\kappa}I_n - \sum\limits_{k=1}^{m} V_{ik}^{\sigma} l l^{\mathrm{T}} C_{dk} - \sum\limits_{k=1}^{m} C_{dk}^{\sigma} l l^{\mathrm{T}} V_{ik}^{\sigma} & * & * & * \\ D^{\mathrm{T}}P & -\mu I_d & * & * \\ C^{\sigma} & 0 & -\mu I_m & * \\ P & 0 & 0 & -I_n \end{bmatrix} \prec 0. \end{cases} \tag{78}$$

If the task is feasible, the rules to compute $J_i^{\sigma} \in \mathbb{R}_{+}^{n\times m}$ for $i \in \{1, \ldots, s\}$, $\sigma \in \{1, \ldots, n_w\}$, are given by (42).

The proof is omitted being similar to the proof of Theorem 2.

Note that in the given sense (76) and (77) force the Metzler parametric constraints in the design, and the LMIs in (78) prescribe the fuzzy interval observer asymptotic stability and the solving task offers attenuation of disturbance effects by μ as an upper bound of the H_∞ norm of the disturbance transform function matrix with relation to the estimated system output.

5. Illustrative Example

To illustrate the proposed interval observer design for M-T-S fuzzy switching model (1), (2), all interval bounds on the system matrices and the required supporting design parameters are given as

$$\underline{A}_1^1 = \begin{bmatrix} -0.2720 & 1.9380 & 1.4540 \\ 0.0580 & -3.9610 & 0.0650 \\ 0.1100 & 0.0580 & -2.9080 \end{bmatrix}, \quad \underline{A}_2^1 = \begin{bmatrix} -0.2730 & 1.9440 & 1.4510 \\ 0.0590 & -3.9610 & 0.1070 \\ 0.1090 & 0.0510 & -2.9180 \end{bmatrix},$$

$$\underline{A}_1^2 = \begin{bmatrix} -0.2760 & 2.0940 & 1.4450 \\ 0.0520 & -3.9510 & 0.0920 \\ 0.1250 & 0.0840 & -2.9380 \end{bmatrix}, \quad \underline{A}_2^2 = \begin{bmatrix} -0.2720 & 2.1020 & 1.4150 \\ 0.0570 & -3.9510 & 0.1200 \\ 0.1000 & 0.0770 & -2.9420 \end{bmatrix},$$

$$\overline{A}_1^1 = \begin{bmatrix} -0.2580 & 2.0160 & 1.5570 \\ 0.1420 & -3.6480 & 0.0720 \\ 0.2060 & 0.0730 & -2.5540 \end{bmatrix}, \quad \overline{A}_2^1 = \begin{bmatrix} -0.2580 & 2.0660 & 1.5530 \\ 0.1420 & -3.6450 & 0.2010 \\ 0.2120 & 0.0510 & -2.5560 \end{bmatrix},$$

$$\overline{A}_1^2 = \begin{bmatrix} -0.2410 & 2.1600 & 1.4450 \\ 0.1450 & -3.6420 & 0.1170 \\ 0.1830 & 0.0970 & -2.5950 \end{bmatrix}, \quad \overline{A}_2^2 = \begin{bmatrix} -0.2680 & 2.1640 & 1.5560 \\ 0.1570 & -3.6390 & 0.1720 \\ 0.2020 & 0.0810 & -2.5750 \end{bmatrix},$$

where

$$\rho(\underline{A}_1^1) = \{-0.1817\ -2.9680\ -3.9912\}, \quad \rho(\underline{A}_2^1) = \{-0.1822\ -2.9785\ -3.9914\},$$
$$\rho(\underline{A}_1^2) = \{-0.1787\ -3.0050\ -3.9814\}, \quad \rho(\underline{A}_2^2) = \{-0.1857\ -2.9940\ -3.9853\},$$
$$\rho(\overline{A}_1^1) = \{-0.0455\ -2.6845\ -3.7301\}, \quad \rho(\overline{A}_2^1) = \{-0.0350\ -2.7040\ -3.7200\},$$
$$\rho(\overline{A}_1^2) = \{-0.0429\ -2.7043\ -3.7309\}, \quad \rho(\overline{A}_2^2) = \{-0.0389\ -2.7103\ -3.7328\},$$

and

$$C^1 = C^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0.045 \\ 0.080 \\ 0.053 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad l = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$f^1(t) = f^2(t) = \begin{bmatrix} q_2(t)q_3(t) \\ 0 \\ 0 \end{bmatrix}.$$

The straightforward calculation shows that the prescribed needs on the $\overline{A}_i^\sigma$, $\underline{A}_i^\sigma$ are strictly Metzler and Hurwitz and $\underline{A}_i^\sigma \le \overline{A}_i^\sigma$ for all i, σ, and that C, D are nonnegative matrices, and all are satisfied.

The system investigation for bounded $q_2(t) \le 1.5$, $q_3(t) \le 1.3$ offers the bound on Lipschitz parameter κ as [33]

$$\left\| \frac{\partial f(q(t))}{\partial q(t)} \right\|_2 = \left\| \begin{matrix} 0 & q_3 & q_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\|_2 \le \left\| \begin{matrix} 0 & 1.3 & 1.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right\|_2 = 1.9849 = \lambda\,, \quad \kappa_m = \lambda^2 = 3.94\,,$$

which is the upper constraint κ_m, because $q(t)$ is positive.

Respecting the requirement (36), then C is diagonally represented as

$$C_{d1}^1 = C_{d1}^2 = \text{diag}\big[1\ 0\ 0\big], \quad C_{d2}^1 = C_{d2}^2 = \text{diag}\big[0\ 0\ 1\big],$$

and, for example, the parametrization by the set of diagonal matrices of the matrix $\underline{A}_2^2$, respectively, $\overline{A}_2^2$ of the switched fuzzy model is

$$\underline{A}_2^2(\nu,\nu) = \text{diag}\big[-0.2720\ -3.9510\ -2.9420\big],$$
$$\underline{A}_2^2(\nu+1,\nu) = \text{diag}\big[0.0570\ 0.0770\ 1.4150\big], \quad \underline{A}_2^2(\nu+2,\nu) = \text{diag}\big[0.1000\ 2.1020\ 0.1200\big],$$
$$\overline{A}_2^2(\nu,\nu) = \text{diag}\big[-0.2680\ -3.6390\ -2.5750\big],$$
$$\overline{A}_2^2(\nu+1,\nu) = \text{diag}\big[0.1570\ 0.0810\ 1.5560\big], \quad \overline{A}_2^2(\nu+2,\nu) = \text{diag}\big[0.2020\ 2.1640\ 0.1720\big].$$

At this level of description, one can also simply make sure that

$$\underline{A}_2^2(\nu,\nu) \le \overline{A}_2^2(\nu,\nu)\,, \quad \underline{A}_2^2(\nu+1,\nu) \le \overline{A}_2^2(\nu+1,\nu)\,, \quad \underline{A}_2^2(\nu+2,\nu) \le \overline{A}_2^2(\nu+2,\nu)\,.$$

All others system matrices must be parameterized in the same way to respect the desired diagonal principle.

The SeDuMi toolbox [34] is applied in a standard way to construct and solve the prescribed set of $N = 44$ LMIs, defined by Theorem 3. It is feasible that searching provides the matrix and scalar variables

$$P = \text{diag}\big[0.4978\ 0.7549\ 0.8280\big], \quad \mu = 1.7629,\ \overline{\kappa} = 1.9236,\ \underline{\kappa} = 2.2568,$$

$$\boldsymbol{V}_{11}^{1} = \mathrm{diag}\big[2.1118\ 0.0141\ 0.0335\big], \quad \boldsymbol{V}_{12}^{1} = \mathrm{diag}\big[0.3236\ 0.0165\ 0.4872\big]$$
$$\boldsymbol{V}_{21}^{1} = \mathrm{diag}\big[2.1195\ 0.0142\ 0.0336\big], \quad \boldsymbol{V}_{22}^{1} = \mathrm{diag}\big[0.3240\ 0.0328\ 0.4886\big]$$
$$\boldsymbol{V}_{11}^{2} = \mathrm{diag}\big[2.1394\ 0.0127\ 0.0353\big], \quad \boldsymbol{V}_{12}^{2} = \mathrm{diag}\big[0.3072\ 0.0246\ 0.4767\big]$$
$$\boldsymbol{V}_{21}^{2} = \mathrm{diag}\big[2.1412\ 0.0139\ 0.0310\big], \quad \boldsymbol{V}_{22}^{2} = \mathrm{diag}\big[0.3184\ 0.0337\ 0.4830\big]$$

These results fulfil the diagonal positiveness criterion on the LMI variables and enforce the strictly positive gains

$$\boldsymbol{J}_{1}^{1} = \begin{bmatrix} 4.2418 & 0.6499 \\ 0.0187 & 0.0218 \\ 0.0404 & 0.5885 \end{bmatrix}, \quad \boldsymbol{J}_{2}^{1} = \begin{bmatrix} 4.2574 & 0.6508 \\ 0.0188 & 0.0435 \\ 0.0406 & 0.5902 \end{bmatrix},$$
$$\boldsymbol{J}_{1}^{2} = \begin{bmatrix} 4.2973 & 0.6170 \\ 0.0168 & 0.0326 \\ 0.0426 & 0.5757 \end{bmatrix}, \quad \boldsymbol{J}_{2}^{1} = \begin{bmatrix} 4.3009 & 0.6395 \\ 0.0184 & 0.0446 \\ 0.0375 & 0.5834 \end{bmatrix},$$

which are necessary to solve the following range of matrices with Metzler features, while retaining the observers stability,

$$\underline{\boldsymbol{A}}_{e1}^{1} = \begin{bmatrix} -4.5138 & 1.9380 & 0.8041 \\ 0.0393 & -3.9610 & 0.0432 \\ 0.0696 & 0.0580 & -3.4965 \end{bmatrix}, \quad \underline{\boldsymbol{A}}_{e2}^{1} = \begin{bmatrix} -4.5304 & 1.9440 & 0.8002 \\ 0.0402 & -3.9610 & 0.0635 \\ 0.0684 & 0.0510 & -3.5082 \end{bmatrix},$$
$$\underline{\boldsymbol{A}}_{e1}^{2} = \begin{bmatrix} -4.5733 & 2.0940 & 0.8280 \\ 0.0352 & -3.9510 & 0.0594 \\ 0.0824 & 0.0840 & -3.5137 \end{bmatrix}, \quad \underline{\boldsymbol{A}}_{e2}^{2} = \begin{bmatrix} -4.5729 & 2.1020 & 0.7755 \\ 0.0386 & -3.9510 & 0.0754 \\ 0.0625 & 0.0770 & -3.5254 \end{bmatrix},$$
$$\overline{\boldsymbol{A}}_{e1}^{1} = \begin{bmatrix} -4.4998 & 2.0160 & 0.9071 \\ 0.1233 & -3.6480 & 0.0502 \\ 0.1656 & 0.0730 & -3.1425 \end{bmatrix}, \quad \overline{\boldsymbol{A}}_{e2}^{1} = \begin{bmatrix} -4.5154 & 2.0660 & 0.9022 \\ 0.1232 & -3.6450 & 0.1575 \\ 0.1714 & 0.0510 & -3.1462 \end{bmatrix},$$
$$\overline{\boldsymbol{A}}_{e1}^{2} = \begin{bmatrix} -4.5383 & 2.1600 & 0.8280 \\ 0.1282 & -3.6420 & 0.0844 \\ 0.1404 & 0.0970 & -3.1707 \end{bmatrix}, \quad \overline{\boldsymbol{A}}_{e2}^{2} = \begin{bmatrix} -4.5689 & 2.1640 & 0.9165 \\ 0.1386 & -3.6390 & 0.1274 \\ 0.1645 & 0.0810 & -3.1584 \end{bmatrix},$$

where

$$\rho(\underline{\boldsymbol{A}}_{e1}^{1}) = \big\{-3.4179\ -3.8917\ -4.6616\big\}, \quad \rho(\underline{\boldsymbol{A}}_{e2}^{1}) = \big\{-3.4246\ -3.8998\ -4.6751\big\},$$
$$\rho(\underline{\boldsymbol{A}}_{e1}^{2}) = \big\{-3.4134\ -3.9109\ -4.7138\big\}, \quad \rho(\underline{\boldsymbol{A}}_{e2}^{2}) = \big\{-3.4382\ -3.9028\ -4.7084\big\},$$
$$\rho(\overline{\boldsymbol{A}}_{e1}^{1}) = \big\{-2.9737\ -3.5217\ -4.7949\big\}, \quad \rho(\overline{\boldsymbol{A}}_{e2}^{1}) = \big\{-2.9344\ -3.5736\ -4.7985\big\},$$
$$\rho(\overline{\boldsymbol{A}}_{e1}^{2}) = \big\{-2.9981\ -3.5275\ -4.8255\big\}, \quad \rho(\overline{\boldsymbol{A}}_{e2}^{2}) = \big\{-2.9445\ -3.5470\ -4.8748\big\}.$$

By comparing all solutions found by the algorithm, it is safe to say that for all $i = 1,2$, $\sigma = 1,2$ constructs a set of stable Metzler observer matrices, satisfying the desired conditions $\underline{\boldsymbol{A}}_{ei}^{\sigma} \le \overline{\boldsymbol{A}}_{ei}^{\sigma}$ and

$$\max_{i,\sigma} \rho(\underline{\boldsymbol{A}}_{i}^{\sigma}) > \max_{i,\sigma} \rho(\underline{\boldsymbol{A}}_{ei}^{\sigma}), \quad \max_{i,\sigma} \rho(\overline{\boldsymbol{A}}_{i}^{\sigma}) > \max_{i,\sigma} \rho(\overline{\boldsymbol{A}}_{ei}^{\sigma}),$$

which means that the dynamics of the fuzzy switching interval observer is faster than the dynamics of the fuzzy switching system.

Moreover, it can be seen that due to the influence of negative state feedback acting on the structures of estimators, these matrices have a tendency to become diagonally dominant [35,36] because the confirmation of one of the characteristic features of systems with dynamics is defined by strictly using Metzler matrices.

This generally leads to negative real eigenvalues of the estimator dynamics matrices and ultimately to stable aperiodic trajectories of the estimated state variables.

The proposed conditions in terms of LMIs and diagonal matrix variables for the design of M-T-S fuzzy switched interval observers are illustrated, if strictly Metzler system matrix interval constraints are defined and the premise variables are measurable. If the switching M-T-S fuzzy observer is built on purely Metzler system matrix interval constraints, the structured nonnegative diagonal matrix variables $\boldsymbol{V}_{ik}^{\sigma}$ must be used, when adapting the principle presented in [23]. This will be manifested by the fact that the stable switched observer's matrices $\underline{\boldsymbol{A}}_{ei}^{\sigma}, \overline{\boldsymbol{A}}_{ei}^{\sigma}$ will be only purely Metzler, and, in view of this, $\boldsymbol{J}_{i}^{\sigma}$ will be nonnegative, but will possess the tendency to become diagonally dominant.

Unfortunately, such a choice description of the diagonal structured matrix variables may no longer be trivial and unambiguous.

6. Concluding Remarks

The main goals in this article are the creation of LMI-based structures for the interval state observer design in coincidence with the Metzler parametric constraints of positive systems, T-S fuzzy models of nonlinear switching systems, the Lipschitz form of a state-dependent nonlinearities, the interval system matrix representations, and bounded system disturbances.

It is proven that the diagonal stabilisation principle is implementable by a common set of LMIs in the construction of strictly positive interval observer gains, guaranteeing Metzler and Hurwitz system matrices of switched interval observers based on the M-T-S switched fuzzy models, as well as boundedness and positiveness of the estimated system states.

The existence of a common quadratic Lyapunov function for all linear modes assures the quadratic stability of the switched system [37]. Because quadratic stability is a special class of exponential stability, which implies asymptotic stability for linear systems, the existence of a common quadratic Lyapunov function can be expressed in terms of LMIs. The disadvantage is that such existence is only sufficient for the asymptotic stability of switched linear systems and the results are rather conservative. If every subsystem is stable, there can be found positive definite symmetric matrices that solve the Lyapunov equation for each mode, which are patchable together under the switch to construct a global Lyapunov function [38], but the switched quadratic Lyapunov function method also gives only sufficient conditions. Because the parametric bounds and the principle of diagonal stabilization make the conditions for the synthesis of positive systems only sufficient, the proposed design conditions are formed for a given class of switched observers with respect to a common quadratic Lyapunov function. Computing real dwell time for the Hurwitz stability of the switched linear systems [39], the dynamics of a switched interval observer mode can be also reforced interactively by using the D-stability approach (see for example [40]).

The presented approach seems to be a suitable common algorithmic chosen to realize the design task having the complexity of the interval M-T-S fuzzy switching positive observer, being naturally parameterized for the positive M-T-S switched fuzzy models. Considering the similarity within the kindred class of LMI algorithms, the computational complexity is easily interpretable for the presented particular case. The proposed algorithm structures are informal and easily creatable, which is why it can be the most common structure chosen. As a special case, the LMI-based design condition structures of the present paper gives the possibility to reflect the existence of such interval observers when the M-T-S fuzzy system is not subject to a Lipschitz nonlinearity.

The method can be used as a basis for immediate future work toward the solution of the positive stochastic switched systems and in principle, when using a suitable Lyapunov–Krasovskii functional, the presented methodology can be adapted for certain classes of M-T-S fuzzy systems with time delays. Because the model parameters are related to positive switching dynamics, therefore they propose a basis regarding the fractional M-T-S fuzzy switched systems.

The Metzler system matrix parametric constraints give rise to substantially complex design methods, when applied to positive systems with interval-defined model parameters. Utilization of the interval observers for interconnected schemes with relation to distributed interval estimation and distributed feedback control [41,42] has to reflect, moreover, that their application for continuous linear large-scale systems is limited due to the system's complexity. These problems are still open in these applications, because it is difficult to ensure that the system state will be enclosed by cooperative estimated upper and lower bounds of the observed system state [43]. The same must be taken into account in interval estimation strategies for antidisturbance control of drones [44]. The scientific activities of the authors will be focused on these areas in the long term.

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Abbreviations

The following abbreviations are used in this manuscript:

LMI	Linear Matrix Inequality
MIMO	Multiple-Input Multiple-Output
SeDuMi	Self Dual Minimization
M-T-S	Metzler-Takagi-Sugeno fuzzy model

Notations

The following basic notations are used in this manuscript:

$\boldsymbol{q}(t)$, $\boldsymbol{u}(t)$, $\boldsymbol{y}(t)$, $\boldsymbol{d}(t)$, $\boldsymbol{\vartheta}(t)$	state, input, output, disturbance and premise variables vectors
$\boldsymbol{A}_i^{\sigma}$, $\boldsymbol{B}_i^{\sigma}$, $\boldsymbol{C}^{\sigma}$, $\boldsymbol{D}$	switched system matrix parameters
$\boldsymbol{A}_{ei}^{\sigma}$, $\boldsymbol{J}^{\sigma}$	switched observer system matrices, switched observer gains
$\boldsymbol{P}$, $\boldsymbol{V}_{ik}^{\sigma}$, κ	matrix variables of LMIs
$\boldsymbol{f}(s)$, $\boldsymbol{I}_n$, γ	Liptschitz function, identity matrix, scalar tuning parameter

All other notations are defined in the given context fluently.

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