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Mass Ratio and Spot Parameter Estimation from Eclipsing Binary Star Light Curves

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Abstract: Eclipsing binary stars have a rich history of contributing to the field of stellar astrophysics. Most of the available information on the fundamental properties of stars has come from the analysis of observations of binaries. The availability of powerful computers and sophisticated codes that apply physical models has resulted in determinations of masses and radii of sufficient accuracy to provide critical tests of theories of stellar structure and evolution. Despite their sophistication, these codes still require the guiding hand of trained scientists to extract reliable information. The computer code will produce results, but it is still imperative for the analyst to ensure that those results make astrophysical sense, and to ascertain their reliability. Care must be taken to ensure that we are asking the codes for parameters for which there is information in the data. The analysis of synthetic observations with simulated observational errors of typical size can provide valuable insight to the analysis process because the parameters used to generate the observations are known. Such observations are herein analyzed to guide the process of determining mass ratios and spot parameters from eclipsing binary light curves. The goal of this paper is to illustrate some of the subtleties that need to be recognized and treated properly when analyzing binary star data.

Keywords: eclipsing binaries; light curves; photometric mass ratios; spots

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1. Introduction

Binary (and higher multiple) systems play an important role in our understanding of the nature of astronomical objects. From solar system objects to clusters of galaxies, binary and multiple systems are ubiquitous and serve as astrophysical laboratories for measuring the fundamental properties of their constituent objects, such as mass and size. A smaller number of such objects have mutual orbits that cause them to pass in front of one another as seen from Earth, leading to "mutual events" in planetary science terminology (e.g., the Pluto-Charon system), "transits" in the case of an exoplanet and its host star, or "eclipses" in the case of two stars. Of these, eclipsing binary stars have the most extensive history of providing fundamental information from their light curves, going back to the early 20th century.

In many ways, the analysis of observations of binary stars is a mature field, with sophisticated models available for use by anyone with even the most modest computational resources. We live in a time where the pace of observational technology has greatly increased both the number of known eclipsing binaries and the volume of data on them. Our analysis methods have evolved to deal with this explosion of data. There is a temptation, however, to process large volumes of data and put too much trust in the results without giving due thought to what they mean. Artificial intelligence promises to improve such analyses, but getting there still requires skilled scientists to extract knowledge reliably. There are many subtleties to treat correctly when it comes to binary star data analysis, and this paper is an attempt to illustrate some of them.

In its most basic form, science is a fairly simple process of using models to explain observations. Binary star data analysis fits that description. There are numerous models that have been developed (e.g., stars as spheres, ellipsoids, or equipotential surfaces) and various implementations of those models, these days as computer programs (e.g., WINK [1], EBOP [2], WD [3]). One must obviously choose a model that is capable of accurately representing the binary system to be analyzed. Using a model based on spherical stars to analyze an overcontact binary is doomed to failure, for example. The use of a sophisticated model does not necessarily mean that a better result will be achieved. The most sophisticated binary star model using equipotential surfaces for the two stars, detailed treatment of the reflection effect, accurate stellar atmospheres, etc. will give useless results when applied to light curve observations of β Lyrae because the system contains an accretion disk which the model cannot treat. To achieve results that we can have some confidence in, we must understand what we can expect to estimate from the application of a model and what we cannot accurately estimate, no matter how fancy our analysis tools are. Wilson [4] provides an introduction to the analysis of binary star light curves.

In the most favorable situation for analyzing eclipsing binaries, both light and radial velocity curves from spectroscopy will be available. With an astrophysically appropriate model there are many things we can reliably estimate about binary stars from these data. Parameters like masses, radii, shapes, rotation rates, and luminosities are commonly found in papers on binary stars. However, frequently there is carelessness that leads to overconfidence in results when models are pushed too far. Often this situation results when analyses of eclipsing binaries are based solely on light curves and parameters that have little to no influence on the light curves are estimated. Light curves are relatively easy to obtain with modern robotic telescopes and cameras. Obtaining good radial velocity curves requires much more advanced instrumentation, and that fact explains why radial velocity observations for eclipsing binaries are significantly less numerous. While there are many systems that do have radial velocity data available, most known eclipsing binaries have no useful radial velocity data. There is a temptation to publish light curve analyses and to make assumptions to derive absolute parameters, with subsequently overconfident musings on things like the evolutionary history of the system. A particularly bad example is the analysis of the light curve of an overcontact binary with partial eclipses. These binaries have short orbital periods and it is thus easy to churn out papers on them, but for the most part, they contribute very little to our understanding of stars. In contrast, analyses involving multiple types of observables, such as, most commonly, light and radial velocity curves (e.g., Sekaran, et al. [5]) or in more exotic situations, light curves, radial velocity curves and X-ray pulse arrival times (e.g., Wilson & Terrell [6]), often produce a deeper understanding of the system. A good example of how more complete observations can give a clear picture of a previously confusing system is TT Herculis. As Terrell & Nelson [7] explain, new radial velocity data of the secondary star showed that the system has a double contact morphology that illustrates why previous analyses were so disparate. Before the Terrell & Nelson [7] spectroscopy, only radial velocities of the primary were available and given that the system has partial eclipses, the photometric mass ratio was unreliable (see Section 2). Once the mass ratio was reliably determined, a clear picture of the system's morphology appeared. So, while multiple type of observables often give us a more complete picture of an individual binary, we most often find ourselves in the situation of having only photometric data and wish to ascertain as much information as we can from those data, but we must be careful not to draw conclusions that the data cannot support.

This paper discusses what can be deduced about mass ratios and spot parameters when only light curves are available for an eclipsing binary. Photometric mass ratios have been discussed and estimated for various types of eclipsing binaries for decades now, but there seems to be lingering confusion about why and when they can be accurately estimated. Spot parameters, namely their locations on the surface of a star, their size and their temperature, have a long history of application in light curve analyses but have received relatively little attention in terms of how accurately they can be determined. Anyone who has attempted to model light curve asymmetries with spots has discovered the inherent difficulties in the process.

2. Photometric Mass Ratios

The ideas underlying photometric mass ratios go back many decades to early attempts to model eclipsing binary light curves. The genesis of the idea that the lobe-filling condition can be used to estimate a photometric mass ratio appears in Wood [8] where he argues that it provides a limit on the mass ratio for R Canis Majoris, a system now known [9,10] to be semidetached. Wilson [11] gives a brief discussion of the early work that led to the use of photometric mass ratios, and explains why they can be very accurate in certain situations, namely semidetached and overcontact binaries that exhibit complete (i.e., total/annular) eclipses. The chain of logic behind photometric mass ratios is fairly simple, but many manuscripts are still submitted wherein it is clear that it is not understood.

Terrell & Wilson [12] gave a slightly more detailed explanation of the logic, and attempted to clarify the situation by analyzing synthetic observations of semidetached and overcontact binaries. Their Figure 1 for overcontact binaries and Figure 2 for semidetached binaries show why accurate mass ratios can, in principle, be determined. For overcontact binaries, the important quantity estimated from the light curve is the ratio of the radii of the two stars. For semidetached binaries, it is the radius of the lobe-filling star. However, accurate mass ratios require more than the lobe-filling constraints. Terrell & Wilson [12] showed that the nature of the eclipses (namely, complete versus partial) was crucial to accurately determining the mass ratio. The accuracy of the mass ratio estimate degrades very rapidly when the eclipses go from complete to partial, as shown in their Figure 4 for observational scatter of the typical 1% precision. The same held true for even higher precision observations easily achieved by space-based observatories like TESS (see their Figure 5). More precise observations do not change the situation. Accurate photometric mass ratio determinations require overcontact or semidetached configurations and complete eclipses. The bottom line is that there simply is not enough information in the light curves of systems with partial eclipses to accurately determine the mass ratio.

In papers on semidetached or overcontact systems with partial eclipses where there are no radial velocity data, a frequent approach is to perform light curve solutions at a discrete set of fixed values for the mass ratio q and find the value that gives the best fit to the light curve, the so-called q-search method. The usefulness of this approach is questionable because of the arguments given above. Because the information on the mass ratio in a light curve is limited, *any* method of analysis is going to be limited in its ability to extract an accurate mass ratio. The analysis may provide some broad estimate, but it will be limited.

To explore these limitations, we analyzed synthetic observations with random errors of 1% for an overcontact binary with $q = 0.3$ and a modified surface potential [13] $\Omega_1 = 2.37$. The Wilson–Devinney (WD) program was used to generate these observations, following the same approach of Terrell & Wilson [12]. Four values of the inclination i were used to illustrate a nearly central eclipse ($i = 88^\circ$), a just barely complete eclipse ($i = 80^\circ$), a just barely partial eclipse ($i = 79^\circ$) and a noticeably partial eclipse ($i = 75^\circ$). The differential corrections (DC) program was then used to solve the light curves at fixed values of q from 0.1 to 0.6 in 0.01 steps. DC was run until the corrections were ten times smaller than the errors, or the solution diverged.

Figure 1 shows results that support the idea that the eclipse circumstances determine how strongly the mass ratio affects the light curve. The top panels show that there is very little variation in the mean residual of the solutions for a broad range of mass ratios when the eclipses are partial. The bottom panels show that the fits are noticeably improved when the eclipses are complete. At $i = 80^\circ$ (left panel), the eclipses are just barely total but there is a reasonably well-defined minimum in the mean residual at the correct value of q. When the eclipses are nearly central (right panel), there is a very pronounced minimum.

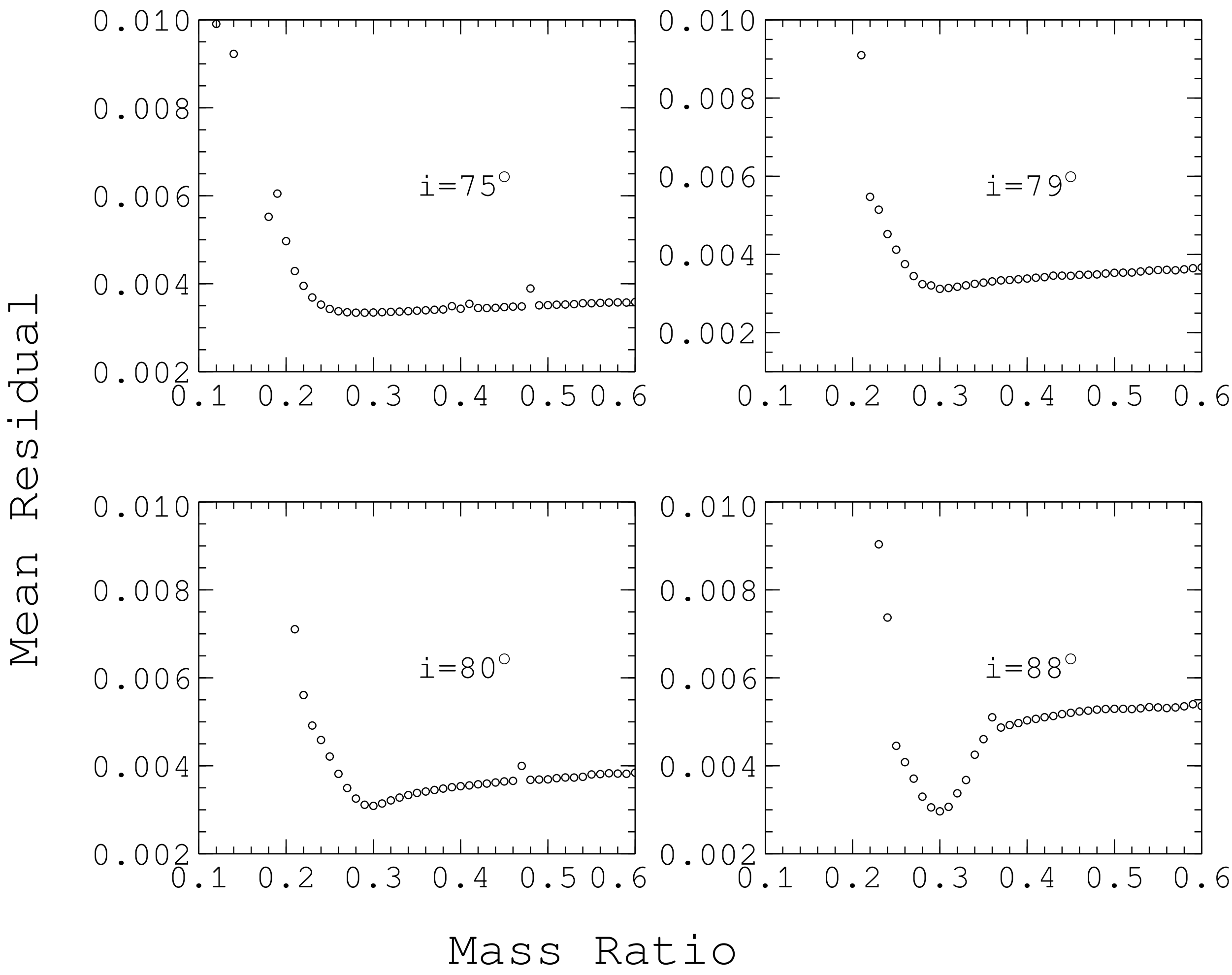

Figure 1. The q-search results for an overcontact binary with $q = 0.3$. The top panels are for inclination values that result in partial eclipses and the bottom panels are for inclinations that produce complete eclipses. The ordinate values are the mean residual from the fit to the synthetic light curve data and are in units of normalized flux where the flux at phase 0.25 is around 1.4.

As Wilson [14] points out, for an overcontact system with complete eclipses, the amplitude of variation of the light curve is very sensitive to the mass ratio. For equal masses the fractional areas eclipsed are large, leading to deep eclipses. When the masses are very different, the fractional areas eclipsed are small, leading to shallow eclipses.

The observational evidence is clear that most (and perhaps all) overcontact binaries are actually hierarchical triple systems [15]. Since third light affects light curves by reducing the amplitude of variation, the determination of an accurate mass ratio is dependent upon an accurate determination of any third light in the light curve. To illustrate the effect of third light on the determination of the mass ratio, q-search runs were done for a binary with the same parameters as described above but with varying amounts of third light added to the observations. The inclination was set to 88°, so that q would be strongly determined, but third light was set to zero and not adjusted. Since third light reduces the light curve amplitude, we would expect the estimated mass ratio to be smaller than the actual value since DC was not allowed to adjust the amount of third light. Figure 2 shows that is exactly what happens. The run illustrated in leftmost panel has a small amount of third light added and it has very little effect on the mass ratio that is determined. The run in the middle panel has a moderate amount of third light and the estimated mass ratio is 0.27, about 10% smaller than the correct value. The run in the rightmost panel has a large amount of third light and the mass ratio is only very weakly determined at $q = 0.14$, far from the correct value.

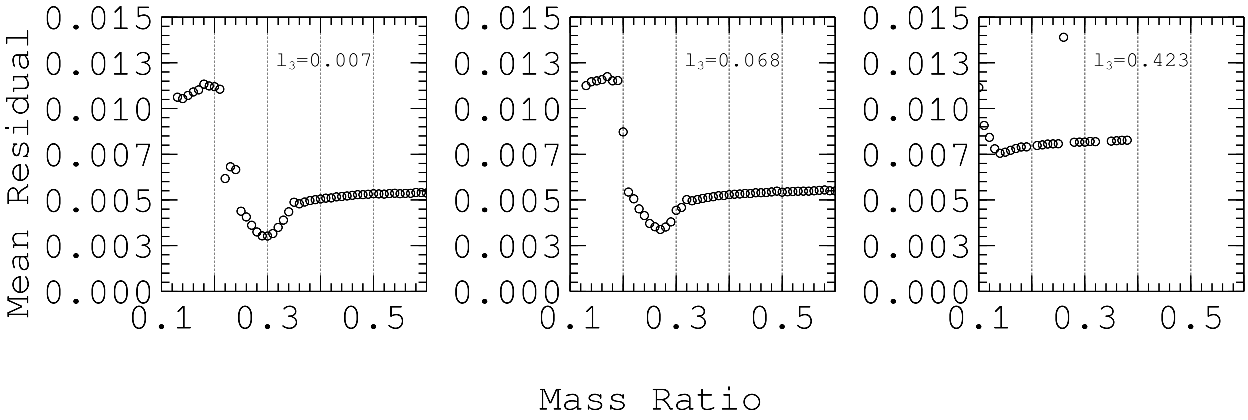

Figure 2. The q-search simulations for the same binary used in Figure 1 with $i = 88°$ and various values of third light. The third light unit is the ratio of the third light to the total system light at phase 0.25. Ordinate units are the same as Figure 1.

Another set of q-search runs was done to see if DC would be able to estimate an accurate mass ratio if it was allowed to adjust third light. As above, the inclination was set to 88° to get nearly central eclipses and the third light amount was set to the moderate value used above of about 7% of the total system light at phase 0.25 for generating the observations. DC was then run with a starting value of no third light which was then allowed to adjust. Figure 3 shows the results. From higher trial mass ratios, the fit quickly improves as the mass ratio is reduced. At the correct q value of 0.3, the mean residual curve is fairly flat until increasing again at $q = 0.23$. The reason for this flat region is the correlation between the mass ratio and third light. At $q = 0.3$, DC correctly recovers the amount of third light that was added to the observations. As the mass ratio decreases the recovered amount of third light also decreases until it becomes so small as to be immeasurable, even trending to unphysical negative values. Note that this is consistent with the mass ratio value recovered when third light is set to zero and not adjusted.

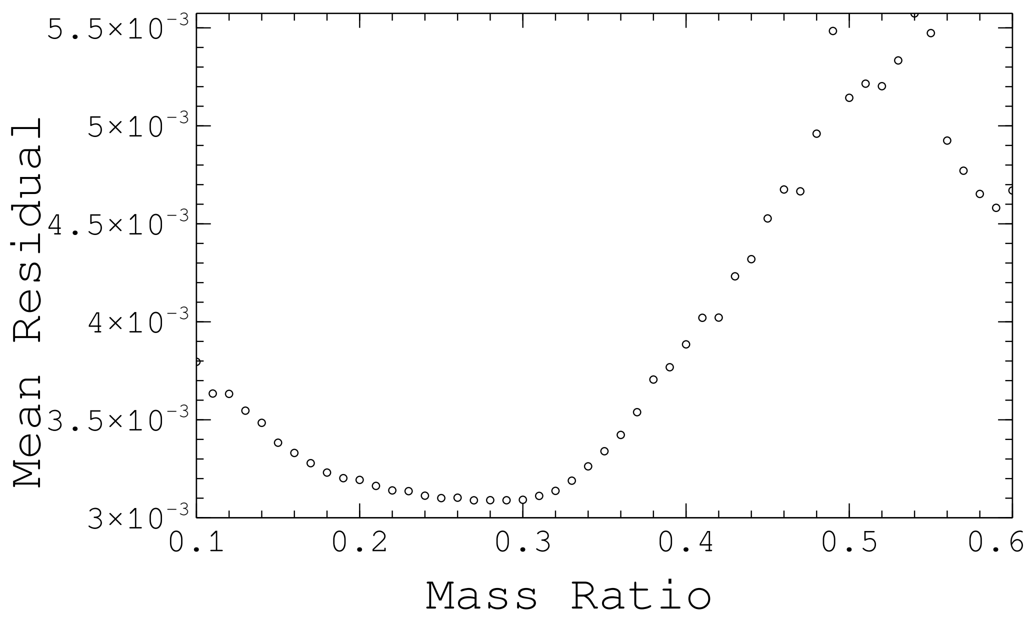

Figure 3. The q-search simulations for the same binary used in Figure 1 with $i = 88°$ and third light set to about 7% of the total system light at phase 0.25. Ordinate units are the same as Figure 1.

These results suggest caution when determining photometric mass ratios where there is an unknown amount of third light, even in the best case scenario of having nearly central eclipses. If the steep slope of the curve to the right of the true value of the mass ratio in Figure 3 holds for other cases, it may prove to be a saving grace. Does the transition from high slope to the flat part of the curve always occur at the correct pair of q/l_3 values? Future work will have to be done to assess that possibility.

3. Spot Parameters

Asymmetries are frequently see in eclipsing binary light curves. Often these are thought to be caused by spots on the stellar surfaces and have been modeled as such for some time now [16]. Spots can be hotter or cooler than the surrounding photosphere, with V361 Lyr [17] being a spectacular example of the former. W UMa stars frequently show time-variable asymmetries that are usually modeled as cool spots that move across the surface with time, although some systems like CE Leonis [18] have been modeled with hot spots. Experience with spot solutions has shown that correlations among the parameters can be severe, making the estimation of spot parameters from light curves a task that requires much caution. When doing light curve analysis, one must always be cautious of local minimum issues, and especially so when spots are involved.

The big question for spot solutions concerns the uniqueness of the solution. Each spot requires four additional parameters: two for the position on the stellar surface, one for the spot size, and one for the spot temperature. Of course, one could use many spots to achieve a near-perfect fit to the light curves but that would result in a meaningless solution in terms of uniqueness. Normally, one or two spots are used to model the major asymmetries in the light curve(s) to avoid this uniqueness problem, at the risk, perhaps, of ending up with a model that might lead to incorrect conclusions. In the absence of other types of data, such as line profiles (see Vogt & Penrod [19]), one must be very careful to assess the uniqueness problem when modeling light curve asymmetries of light curves and draw any conclusions with appropriate uncertainty. It is insufficient to guess some initial parameters and hope that a local minimizer, like the differential corrections algorithm employed in WD, will give the best set of parameters. Only solutions that explore a broad region of parameter space can give reasonable confidence that a global minimum of the residuals has been found, or if solutions with very different parameters can produce statistically identical light curves.

Ideally one would like to explore all parts of parameter space, guided by astrophysical intuition of course, but that is computationally intractable. There are many optimization algorithms that have been employed in light curve codes to search broad regions of parameter space (e.g., SIMPLEX [20] and simulated annealing [21]), but one that has heretofore seen little application in the light curve analysis field, probably because of the computational cost, is genetic algorithms. Metcalfe [22] used a genetic algorithm optimizer to analyze light curves of the W UMa star BH Cas, and Terrell & Nelson [23] applied a genetic algorithm optimizer to determine spot parameters for GSC 3870-01172. Recent work (e.g., Cunningham, et al. [24] and Csizmadia, S. [25]) indicates that interest in genetic algorithms may be increasing.

To explore light curve solutions for spotted stars, the Wilson–Devinney (WD) program [3,13,26] was interfaced with the Distributed Evolutionary Algorithms in Python (DEAP) package [27]. DEAP is a very flexible package that has several built-in evolutionary algorithms but also allows the user to develop and use others. Genetic algorithms mimic the evolutionary process where "genes" are a representation of the parameters of the model being fit and individuals in a population are defined by their "genes". As in biological evolution, processes such as crossover and mutation alter the population's gene pool. For each generation of the population, the mutation and crossover processes are simulated and then a fitness for each individual is computed. The best individuals are then selected to produce the next generation.

In this work, the eaSimple algorithm in DEAP was used to explore the parameter space for an overcontact binary with $q = 0.2$ and complete eclipses. A dark spot with a radius of 0.3 radians and a temperature factor of 0.9 was placed on the primary star at a co-latitude of 1.5 radians and a longitude of 1.9 radians. WD was used to generate synthetic observations, as described above, with a scatter of 0.01 mag. The crossover probability was set to 0.65 and the mutation probability was set to 0.03 to ensure that a broad volume of parameter space would be sampled. The population size was set to 10,000 individuals and at least 70 generations were computed. DEAP keeps a record of the best individuals discovered through all generations, the so-called “hall of fame”, and the best individual from this list is used to get the best-fit parameters for the simulation. Each simulation entailed the computation of about 500,000 to 1,000,000 light curves, and dozens of simulations were run for each scenario. Clearly this level of computation requires significant hardware, but such hardware is now available at reasonable expense. These simulations were performed on a computer with 64 CPU cores that cost about $10,000 US.

Bounds for each of the adjusted parameters are specified to control the volume of parameter space to search. The spot parameter bounds were set to allow for any location on the star’s surface, spot temperature factor from 0.8 to 1.0 and spot radius from 0.1 to 1.0 radians. Other parameters were set to wide but reasonable values to ensure good coverage of parameter space. For example the inclination was sampled between 65° and 89.9°. In these simulations, the mass ratio was fixed at the true value.

Three scenarios were explored in this work: a single light light curve matching the TESS passband, and two scenarios with two light curves each, one with BV light curves and one with BI_C light curves. The goal was to see how well spot parameters could be recovered from a single light curve, as well as from a simultaneous analysis of two light curves. For the two-curve scenario, BV and BI_C were chosen to see how the filter choice might affect the reliability of the derived parameters, since spot phenomena are temperature-dependent.

Figure 4 shows the estimates of spot parameters for the simulations of each scenario. The left panel shows the spot location parameters and the right panel shows the intrinsic spot parameters. Black circles indicate the parameters for the single light curve simulations, red squares denote the BV simulations and the green triangles denote the BI_C simulations. The plus signs are located at the correct values for the parameters.

The plot of spot location parameters shows that the spot longitude is much more accurately determined than the spot latitude (note the very different scales on the axes for those quantities), but that is hardly surprising given that the eclipsing star moves across the spot mainly in longitude. The plot of spot temperature versus spot radius shows the strong correlation between those two quantities, especially for the single light curve solutions. The single light curve solutions clearly do a poor job of recovering the true spot parameters. The BV solutions show signs of improving the parameter estimation, but the BI_C solutions, with their wider wavelength coverage, obviously provide better leverage to break the radius-temperature correlation. This is also unsurprising. The additional temperature information allows the optimizer to give better estimates of the intrinsic spot parameters. The spot latitude still remains uncertain but there is no doubt that multiple light curves with wide wavelength coverage are required when using spots to model light curve asymmetries.

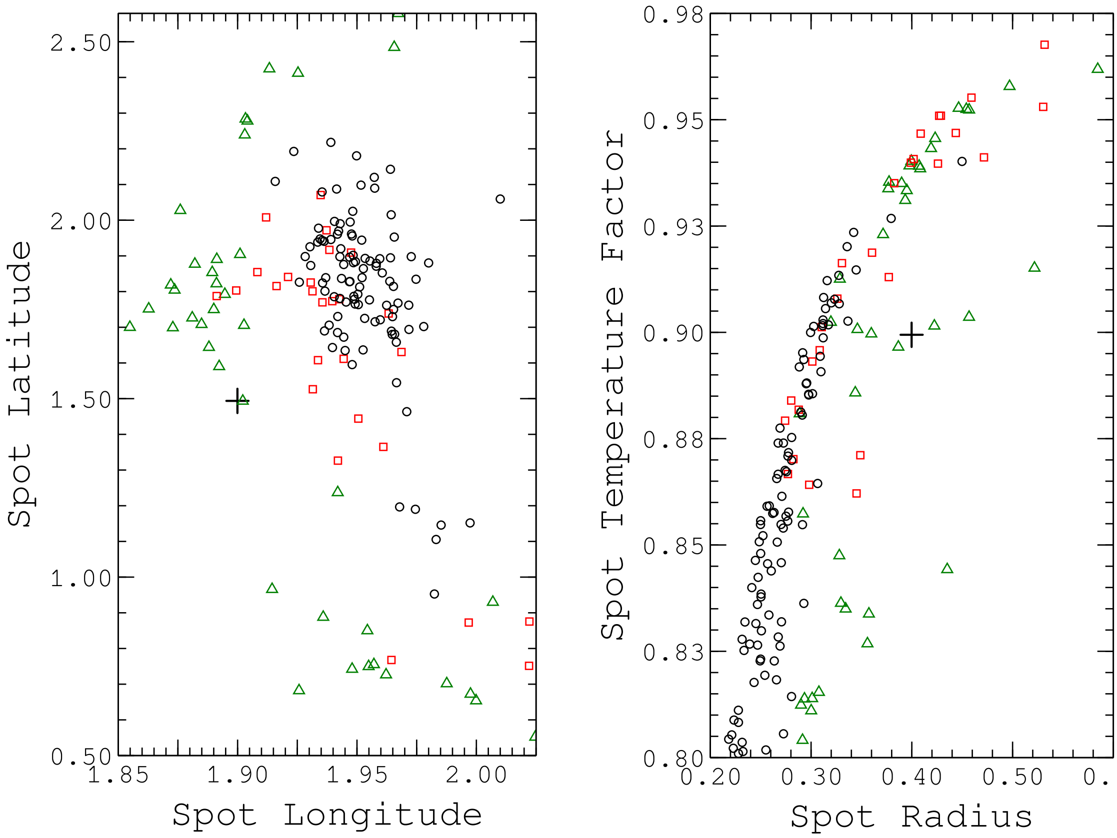

Figure 4. The spot parameters for the best individuals in each simulation The single light curve solutions are represented by black circles. The two light curve solutions for BV are represented by red squares and the BI_C solutions are represented by green triangles. The plus signs indicate the true values.

4. Summary

Solving synthetic light curves of eclipsing binaries produced by an appropriate physical model can provide insight that improves the modeling of observations of real binaries. When measuring photometric mass ratios, accuracy demands that the eclipses be complete. Light curves of systems with partial eclipses simply do not have the necessary information to measure an accurate mass ratio. Attempts to do so by the q-search method, or any method, are futile. Such approaches may provide some information on the mass ratio, but the derived values should be used with appropriate caution. Third light, if not properly modeled, will lead to mass ratios that are too low, a problem that cannot be ignored given that many, if not most, close binaries ($p < 3.0$ days) appear to be triple systems [15,28,29]. Genetic algorithm optimizers show promise for determining spot parameters from light curves in certain situations, namely the simultaneous solution of light curves with broad wavelength coverage. Broader wavelength coverage allows the optimizer to reduce the correlation between the size and temperature of a spot. Spot latitudes only weakly affect light curves while the spot longitude can be accurately measured. However, one should remain cautious about uniqueness problems where spot solutions are concerned.

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