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On Plane Wave Solutions in Lorentz-Violating Extensions of Gravity

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Abstract: In this paper, we obtain dispersion relations corresponding to plane wave solutions in Lorentz-breaking extensions of gravity with dimension 3, 4, 5 and 6 operators. We demonstrate that these dispersion relations display a usual Lorentz-invariant mode when the corresponding additive term involves higher derivatives.

Keywords: gravitational waves; Lorentz and charge-parity-time (CPT) violation

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1. Introduction

The observations of gravitational waves performed within the LIGO experiment [1] certainly represent one of the most important experimental confirmations of general relativity. However, various modifications and extensions of gravity are now being discussed. The main motivations for these extensions are as follows: first, the need for development of a perturbatively consistent gravity model, which is expected to be both renormalizable and ghost-free; second, the necessity to explain the cosmic acceleration originally reported in [2]. At the same time, the concept of Lorentz symmetry breaking, possessing various motivations—string theory, minimal length, quantum fluctuations of geometry, loop quantum gravity, etc.—can clearly be implemented within the context of gravity, and the Lorentz-violating (LV) standard model extension (LV SME) [3,4] was generalized to include gravity in [5]. All these studies clearly establish questions about possible gravitational wave solutions in LV extended gravity models. It is well known that in Lorentz-breaking extensions of other theories, e.g., electrodynamics plane wave solutions displaying nontrivial behavior, such as birefringence and rotation of the polarization plane in the vacuum (see e.g., [3,4]), it is natural to search for such phenomena also in the gravitational wave case.

The first study of plane wave solutions in LV gravity was performed in [6], where the four-dimensional Chern–Simons (CS) modified gravity presenting Lorentz-breaking behavior for the special form of the CS coefficient $\vartheta = k_\mu x^\mu$ was considered. However, it turns out that the only consistent plane wave solution in this theory displays only usual, Lorentz-invariant dispersion relations, where the intensities of two polarizations for gravitational waves are different. A more interesting situation takes place in [7], where the additive one-derivative LV term breaks the gauge symmetry (for a detailed discussion of gauge symmetry breaking in gravity see [8])—in this case two polarizations propagate with distinct phase velocities depending on the Lorentz-breaking parameter and these are different from the speed of light.

Therefore, the natural problem consists of studying plane wave solutions in gravity theories with various recently proposed LV additive terms [9,10]. This issue will be discussed in the present paper.

The structure of the paper is as follows. In Section 2, we consider the dispersion relations in modified gravity models represented as a sum of the usual Einstein–Hilbert action and terms introduced in [9]. In Section 3, we obtain the dispersion relations in theories whose action is given by a sum of the Einstein–Hilbert term and some new linearized gauge-invariant terms. Finally, in Section 4, we summarize our results.

2. Dispersion Relations for Full-Fledged LV Terms in Gravitational Sector

In this section we consider the dispersion relations generated by additive full-fledged LV terms in the gravitational sector, as proposed in [9]. Our starting point is the following decomposition of the $h_{\mu\nu}$ tensor into its irreducible components originally introduced in [6] (see also [7]):

$$h^{00} = n, \quad h^{0i} = n^i_T + \partial^i n_L;$$
$$h^{ij} = (\delta^{ij} - \frac{\partial^i\partial^j}{\nabla^2})\phi + \frac{\partial^i\partial^j}{\nabla^2}\chi + (\partial^i\xi^j_T + \partial^j\xi^i_T) + h^{ij}_{TT}, \tag{1}$$

where $h^{ij}_{TT} \equiv \tilde{h}_{ij}$ is transverse and traceless and ξ^i_T is transverse. Our signature is $(+,-,-,-)$. In this case, the Lagrangian for a spin-2 field of the linearized gravity (see, e.g., [11])

$$\mathcal{L}_0 = \frac{1}{4}\partial_\mu h^\alpha_\alpha \partial^\mu h^\beta_\beta - \frac{1}{2}\partial_\beta h^\alpha_\alpha \partial^\mu h^\beta_\mu - \frac{1}{4}\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} + \frac{1}{2}\partial_\alpha h_{\nu\beta}\partial^\nu h^{\alpha\beta}, \tag{2}$$

which is nothing more than the well-known Einstein–Hilbert Lagrangian for the weak field, takes the form (see, e.g., [6]):

$$\mathcal{L}_{FP} = -\frac{1}{4}\tilde{h}^{ij}\Box\tilde{h}^{ij} + \frac{1}{2}\phi\Box\phi + \frac{1}{2}(\partial^i\sigma^j_T)^2 + \phi\Lambda, \tag{3}$$

where $\sigma^i_T = n^i_T + \dot{\xi}^i_T$ is transverse and $\Lambda = \nabla^2(n + 2\dot{n}_L) + \ddot{\chi}$ is the Lagrange multiplier. These quantities show that $\tilde{h}_{ij}$ is the only propagating field [6].

For this Lagrangian, the dispersion relations for the only physical modes presented by $\tilde{h}_{ij}$ are the usual ones, $E^2 = \vec{p}^2$, as it must be.

So, let us perform a similar decomposition for various additive LV terms introduced in table VI given in [9] in the linearized case, with dimensions of these terms up to 6. Since we are interested in the dynamics of $\tilde{h}_{ij}$, which is traceless, we can assume $\sqrt{|g|} = 1$. In addition, in the linearized case, we can require the external vectors (tensors) $(\overset{\smile}{k}{}^{(n)})^{\mu_1\cdots\mu_{n-2}} \equiv (k^{(n)})^{\mu_1\cdots\mu_{n-2}}$ to be (approximately) constant in order to avoid non-constant free parameters in dispersion relations (i.e., to require that only gravitational fields can propagate), so all derivatives of external vectors (tensors) are disregarded. From the physical viewpoint, this condition is consistent with the conservation of the energy–momentum tensor since it corresponds to the homogeneity of space–time.

From now, our methodology is as follows. For any additive Lorentz-breaking term, we will keep only its observable (transverse-traceless) components, obtaining thus extra contributions to the Lagrangians of these components, and we will study the propagation of the plane waves described by these physical components and corresponding dispersion relations. For the sake of simplicity, we assume that the Lorentz-breaking tensor parameters can be completely characterized by one Lorentz-breaking vector (or pseudovector), similar to aether terms [12,13].

The simplest example of the LV parameter in gravity given in [9] is $(\overset{\smile}{k}{}^{(3)}_{\Gamma})^\mu \equiv (k^{(3)})^\mu$ defining the dimension-3 operator $(k^{(3)})^\mu\Gamma^\alpha_{\mu\alpha}$. In the linearized case, we can write

$$\Gamma^\alpha_{\mu\alpha} = -\frac{1}{2}h^{\alpha\gamma}\partial_\mu h_{\alpha\gamma} + O(h^3).$$

Let us assume that our plane gravitational wave propagates along the $x_3 = z$ axis, i.e., $h_{\mu\nu} = \tilde{h}_{\mu\nu}e^{i(Et-pz)}$. In this case, there will be no second derivatives acting on any components of decomposition of $h_{\mu\nu}$ except of the usual transverse-traceless $\tilde{h}_{ij}$. Similarly to [6,7], we can define two polarizations of $\tilde{h}_{ij}$ as follows: $\tilde{h}_{11} = -\tilde{h}_{22} = T$, $h_{12} = h_{21} = S$; all other components of $\tilde{h}_{ij}$ are zero.

First of all, in this case we have (with $(k^{(3)})^\mu \equiv k^\mu$)

$$\mathcal{L}^{(3)} = -\frac{1}{2}(k^{(3)})^\mu \tilde{h}^{ij}\partial_\mu \tilde{h}_{ij} = -\frac{1}{2}(k^0\tilde{h}^{ij}\partial_0\tilde{h}_{ij} + k^3\tilde{h}^{ij}\partial_3\tilde{h}_{ij}), \tag{4}$$

where we disregarded all other components of $h_{\mu\nu}$. We immediately see that this term is evidently a total derivative, hence its impact on the modified linearized equations of motion is trivial; thus, the adding of $\mathcal{L}^{(3)}$ will not affect plane wave solutions independently of the direction of the vector $(k^{(3)})^\mu$. Unlike (4), the term $\epsilon^{\mu\nu\lambda\rho}b_\mu h_{\nu\alpha}\partial_\lambda h^\alpha_\rho$ discussed in [7], which is also a dimension-3 term, is described by a pseudo-vector b_μ, and in this case the dispersion relations are different, so that for $b^\mu = -\frac{b}{2}\hat{z}$, i.e., the LV vector is parallel to the wave direction; they look like $(E \pm b)^2 - (p+b)^2 = 0$, which implies a group velocity less than the speed of light [7]. Moreover, this term cannot be expressed in terms of usual geometric objects, such as a connection or a curvature, hence it is apparently well-defined only within a linearized gravity but not in a full-fledged one. We note that this term breaks the gauge symmetry, and this fact establishes a natural question about the impact of breaking the gauge symmetry on the dispersion relations (see [8] for a discussion of violating the general covariance in gravity). In this section and in the next one, we will work both in gauge-breaking and gauge-invariant scenarios in order to see if the breaking of gauge invariance implies unusual dispersion relations.

For studying of higher-order terms, it is useful to write down lower-order contributions to the Riemann and Ricci tensors. For the Riemann tensor we have (cf. [11])

$$\begin{aligned} R_{\mu\nu\alpha\beta} &= \frac{1}{2}(\partial_\nu\partial_\alpha h_{\mu\beta} - \partial_\mu\partial_\alpha h_{\nu\beta} - \partial_\nu\partial_\beta h_{\mu\alpha} + \partial_\mu\partial_\beta h_{\nu\alpha}) + \\ &+ \partial_\alpha\Gamma^{(2)}_{\mu,\nu\beta} - \partial_\beta\Gamma^{(2)}_{\mu,\nu\alpha} + \Gamma^{(1)\gamma}_{\beta\nu}\Gamma^{(1)}_{\mu,\gamma\alpha} - \Gamma^{(1)\gamma}_{\alpha\nu}\Gamma^{(1)}_{\mu,\gamma\beta}, \end{aligned} \tag{5}$$

where $\Gamma^{(1,2)\gamma}_{\beta\nu}$ are first- and second-orders in expansions of Christoffel symbols in $h_{\alpha\beta}$, explicitly,

$$\begin{aligned} \Gamma^{(1)\alpha}_{\beta\gamma} &= \frac{1}{2}(\partial_\beta h^\alpha_\gamma + \partial_\gamma h^\alpha_\beta - \partial^\alpha h_{\beta\gamma}); \\ \Gamma^{(2)\alpha}_{\beta\gamma} &= -\frac{1}{2}h^{\alpha\delta}(\partial_\beta h_{\gamma\delta} + \partial_\gamma h_{\beta\delta} - \partial_\delta h_{\beta\gamma}). \end{aligned} \tag{6}$$

Our next example is the dimension-4 term $(\breve{k}^{(4)}_R)^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$. The importance of this term relates to the fact that this is the simplest CPT-even LV term in gravity, which for a special "aether-like" form of $(\breve{k}^{(4)}_R)^{\mu\nu\rho\sigma}$ given by $(\breve{k}^{(4)}_R)^{\mu\nu\rho\sigma} = u^\mu u^\rho \eta^{\nu\sigma} - u^\mu u^\sigma \eta^{\nu\rho} + u^\nu u^\sigma \eta^{\mu\rho} - u^\nu u^\rho \eta^{\mu\sigma}$ is reduced to the gravitational aether term introduced in [12]. Some studies of dispersion relations in this theory have been performed in [14], where causality and unitarity are analyzed within the context of bumblebee gravity for space-like and time-like backgrounds of the bumblebee field. Explicitly, it is demonstrated that there are two graviton dispersion relations, $p^2 + \xi(b\cdot p)^2 = 0$ and $(b\cdot p)^2 - b^2p^2 = 0$, where b_μ is the LV constant vector (actually, it is the v.e.v. of the bumblebee field), and ξ is the known bumblebee-gravity coupling (see [5]). We note that the first dispersion relation is a rather standard one for massless CPT-even LV theories; it arises, for example, in aether-like CPT-even models of scalar and gauge fields [12,13]. As for the second relation, it corresponds to breaking the unitarity, and the energy strongly depends on the direction of propagation [14]. For the vector field, such a relation has been obtained in [15] for a non-canonical gauge theory where the aether term is not suppressed in comparison with the Maxwell term. It is natural to expect that for a generic form of $(\breve{k}^{(4)}_R)^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ dispersion relations do not essentially differ.

Therefore, it is especially interesting to study the higher-derivative LV extensions of gravity. Higher derivatives reveal information about the whole theory and are one of the ways to attain renormalizability [16]. It is well known [16] that in higher-derivative Lorentz-invariant theories, ghost states arise. Their presence makes the theory unstable. However, Lorentz-breaking higher-derivative terms in certain cases, for example when higher derivatives are purely spatial, do not display ghost states [17]. Therefore, the cases where the higher time derivatives are ruled out due to the appropriate choice of Lorentz-breaking parameters are certainly of special interest. In addition, four derivative terms are considered in cosmological models to explain cosmic acceleration [18]. The most well-known example of the dimension-5 terms is the gravitational CS term, whose dispersion relations have been discussed in [6] and proved to be usual ones $E^2 = \vec{p}^2$, although the CPT-breaking manifested itself through difference in intensities for two circular polarizations (various issues related to the linearized gravitational CS term are also discussed in [19–21]). One more CS-like term from table VI given in [9] is proportional to two Levi–Civita symbols, but it vanishes within the metric formalism since contractions like $\epsilon_{abcd}\omega_\mu^{ab}$, which are present within this term, are equal to zero for a Riemannian connection.

It remains to study the dimension-5 term proportional to $D_\kappa R_{\rho\sigma\mu\nu}$. There are a number of ways to decompose tensor $(\overset{\smile(5)}{k}_D)^{\rho\sigma\mu\nu\kappa}$. If it is completely symmetric, this term is evidently ruled out due to the antisymmetry of the Riemann tensor with respect to some indices; thus, the equations of motion are reduced to the Einstein ones, hence the dispersion relations again have the usual form $E^2 = \vec{p}^2$. To obtain a nontrivial impact of this term within the dispersion relations context, we can decompose the dimension-5 coefficient as $(\overset{\smile(5)}{k}_D)^{\rho\sigma\mu\nu\kappa} = k^\rho(k^\sigma k^\nu \eta^{\mu\kappa} - k^\sigma k^\kappa \eta^{\mu\nu} + k^\kappa k^\mu \eta^{\sigma\nu} - k^\mu k^\nu \eta^{\sigma\kappa})$, so it has an aether-like structure completely characterized by one vector. After making the contraction, we find that the following additional term comprises the linearized Einstein equations:

$$\begin{aligned} G^{(5)}_{\alpha\beta} = \frac{1}{4}\Big(&\frac{1}{2}\eta_{\alpha\beta}k^\kappa k^\rho (k\cdot\partial)\Box h_{\rho\kappa} + k_\alpha k_\beta \partial^\kappa\partial^\rho (k\cdot\partial) h_{\kappa\rho} + k^\mu \partial_\alpha (k\cdot\partial)^2 h_{\beta\mu} - \\ &- k^\kappa k^\rho \partial_\beta \partial_\alpha (k\cdot\partial) h_{\rho\kappa} - k_\beta \partial^\rho (k\cdot\partial)^2 h_{\alpha\rho} + (\alpha \leftrightarrow \beta)\Big). \end{aligned} \tag{7}$$

Considering the decomposition of the metric perturbation in Equations (1) and replacing it in Equation (7), we find the additional terms in the equation of motion derived from the Lagrangian in Equation (2), which, in the sector of the physical components $\tilde{h}_{ij}$, assumes the form

$$\begin{aligned} &\Box\tilde{h}_{ij} + \frac{1}{2}\eta_{ij}k^a k^b (k\cdot\partial)\Box\tilde{h}_{ab} + k_i k_j (k\cdot\partial)\partial^a\partial^b \tilde{h}_{ab} + \\ &+ \frac{1}{2}\partial_i (k\cdot\partial)^2 k^a \tilde{h}_{ja} + \frac{1}{2}\partial_j (k\cdot\partial)^2 k^a \tilde{h}_{ia} - k^a k^b \partial_i \partial_j (k\cdot\partial)\tilde{h}_{ab} - \\ &- \frac{1}{2}k_j \partial^a (k\cdot\partial)^2 \tilde{h}_{ia} - \frac{1}{2}k_i \partial^a (k\cdot\partial)^2 \tilde{h}_{ja} + (\ldots) = 0. \end{aligned} \tag{8}$$

Here, dots are for the terms that do not depend on $\tilde{h}_{ik}$. For further study, it is important to note that, first, all such terms are accompanied by Lorentz-breaking constant vectors (tensors) known to be small, which can be treated effectively as small sources in the corresponding wave equations for $\tilde{h}_{ij}$, and thus affect only higher-order contributions to the plane wave solutions; second, they do not influence the equations and dispersion relations for relevant, transverse-traceless components of h_{ij}. Again, we consider the plane wave solutions, $h_{ij} = \tilde{h}_{ij} e^{ipx}$. As we already have done throughout this text, let us now disregard the terms proportional to $\partial^a \tilde{h}_{ab}$ and its derivatives, which vanish in our case. We have

$$\begin{aligned} &\Box\tilde{h}_{ij} + \frac{1}{2}\eta_{ij}k^a k^b (k\cdot\partial)\Box\tilde{h}_{ab} + \\ &+ \frac{1}{2}\partial_i (k\cdot\partial)^2 k^a \tilde{h}_{ja} + \frac{1}{2}\partial_j (k\cdot\partial)^2 k^a \tilde{h}_{ia} - k^a k^b \partial_i \partial_j (k\cdot\partial)\tilde{h}_{ab} + (\ldots) = 0. \end{aligned} \tag{9}$$

Here, as well as in the following equations, the dots denote contributions to the effective equations of motion, which do not depend on $\tilde{h}_{ij}$ and hence do not affect the dispersion relations. Just as for consideration of the dimension-3 term (see the discussion above and in [7]), we can assume $\tilde{h}_{ij}$ to have two polarization states, given by $\tilde{h}_{11} = -\tilde{h}_{22} = T$ and $\tilde{h}_{12} = \tilde{h}_{21} = S$. In this case, the dispersion relations are again the usual ones $E^2 = \vec{p}^2$, for $\vec{k}$ either parallel or orthogonal to the wave vector. The same conclusion is valid for a generic direction of $\vec{k}$, since our plane wave depends on t and $z = x_3$ only, and the terms in the second line of the equation above will not modify the dispersion relations for physical components $h_{11,12,22}$; here we are reminded that all other components of $h_{\mu\nu}$ do not describe physical degrees of freedom and hence can be set to zero. We conclude that the presence of higher-derivative LV terms implies the arising of the unique dispersion relation $E^2 = \vec{p}^2$.

We can continue by studying the remaining terms from table VI in [9]. The next operator to study is the dimension-6 one $D_\kappa D_\lambda R_{\rho\sigma\mu\nu}$, i.e., there is one more partial derivative compared with the previous term, and we can use the relation $\delta(D_\kappa D_\lambda R^\rho{}_{\sigma\mu\nu}) = \delta\Gamma^\rho_{\kappa\tau}\partial_\lambda R^\tau{}_{\sigma\mu\nu} + \Gamma^\rho_{\kappa\tau}\partial_\lambda \delta R^\tau{}_{\sigma\mu\nu} + O(h^3)$. Similar to the above calculations, we can also decompose the dimension-6 coefficient in the aether-like form $(\overset{\smile(6)}{k}_D)^{\rho\sigma\mu\nu\kappa\lambda} = k^\mu k^\lambda (k^\rho k^\kappa \eta^{\nu\sigma} - k^\sigma k^\kappa \eta^{\kappa\rho} + k^\kappa k^\sigma \eta^{\nu\rho} - k^\kappa k^\rho \eta^{\nu\sigma})$. In this case, as we would expect, the equation of motion contains an additive term involving one more derivative and one more degree of the momentum. Explicitly, this fourth-derivative term looks like:

$$G^{(6)}_{\alpha\beta} = \frac{1}{4}\Big(\frac{1}{2}\eta_{\alpha\beta}k^\kappa k^\rho (k\cdot\partial)^2\Box h_{\rho\kappa} + k_\alpha k_\beta \partial^\kappa\partial^\rho (k\cdot\partial)^2 h_{\kappa\rho} + k^\mu \partial_\alpha (k\cdot\partial)^3 h_{\beta\mu} - \\ -k^\kappa k^\rho \partial_\beta\partial_\alpha (k\cdot\partial)^2 h_{\rho\kappa} - k_\beta \partial^\rho (k\cdot\partial)^3 h_{\alpha\rho} + (\alpha \leftrightarrow \beta)\Big). \quad (10)$$

Equation (10) demonstrates the arising of additional terms in the equation of motion derived from the Lagrangian in Equation (2), and in the sector of the physical components it takes the form:

$$\Box\tilde{h}_{ij} + \frac{1}{2}\eta_{ij}k^a k^b (k\cdot\partial)^2 \Box\tilde{h}_{ab} + \\ +\frac{1}{2}k^a \partial_i (k\cdot\partial)^3 \tilde{h}_{ja} - k^a k^b \partial_i\partial_j (k\cdot\partial)^2 \tilde{h}_{ab} + \frac{1}{2}k^a \partial_j (k\cdot\partial)^3 \tilde{h}_{ia} + (\ldots) = 0. \quad (11)$$

In the same way as above, we substitute plane wave solutions into the above equation. As in the dimension-5 term, if $\tilde{h}_{ij}$ has two polarization states, we do not find any additional term in the dispersion relation and again have $E^2 = \vec{p}^2$. So, either for $\vec{k}$ parallel or orthogonal to the wave vector $\vec{p}$, we arrive at the usual dispersion relation.

It remains for us to study the last dimension-6 term $(\overset{\smile(6)}{k}_R)^{\alpha\beta\gamma\delta\mu\nu\zeta\lambda} R_{\alpha\beta\gamma\delta} R_{\mu\nu\zeta\lambda}$ from table VI of [9]. It is also possible to decompose this coefficient in the aether-like form $(\overset{\smile(6)}{k}_R)^{\alpha\beta\gamma\delta\mu\nu\zeta\lambda} = k^\alpha k^\beta k^\gamma (k^\lambda k^\nu k^\sigma \eta^{\zeta\mu} - k^\lambda k^\mu k^\sigma \eta^{\zeta\nu} + k^\zeta k^\mu k^\sigma \eta^{\lambda\nu} - k^\zeta k^\nu k^\sigma \eta^{\lambda\mu})$. This term leads to the following additive term to the modified Einstein tensor:

$$G^{(6)}_{\alpha\beta} = (k\cdot\partial)^3 k^\lambda k^2 (\partial_\alpha h_{\lambda\beta}) - (k\cdot\partial)^2 k^\lambda k^\zeta k^2 \partial_\alpha\partial_\beta h_{\lambda\zeta} - (k\cdot\partial)^2 k^\lambda k_\alpha k^2 \Box h_{\lambda\beta} + \\ +(k\cdot\partial)k^2 k^\lambda k^\zeta k_\alpha \partial_\beta \Box h_{\lambda\zeta} + (\alpha \leftrightarrow \beta) \quad (12)$$

Now the equation of motion is given by

$$\Box\tilde{h}_{ij} - 2(k\cdot\partial)^2 k^2 k^l k_i \Box\tilde{h}_{lj} + 2(k\cdot\partial)k^2 k^l k^m k_i \partial_j \Box\tilde{h}_{lm} - \\ -2(k\cdot\partial)^2 k^2 k^l k^m \partial_j\partial_i \tilde{h}_{lm} + 2(k\cdot\partial)^3 k^2 k^l \partial_i \tilde{h}_{lj} + (i \leftrightarrow j) + (\ldots) = 0 \quad (13)$$

In this case, if k is a space-like vector, parallel or orthogonal to the wave vector, the dispersion relations are again the usual ones $E^2 = \vec{p}^2$.

3. Dispersion Relations for Linearized Gauge-Invariant LV Terms

Now, let us present another approach to the study of dispersion relations in linearized gravity. In this case we start with the quadratic action instead of the full-fledged one, but assume its invariance under the gauge transformations of the metric fluctuation $h_{\mu\nu}$ of the usual form

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \tag{14}$$

where ξ_m is a parameter of transformations.

Let us find fourth-order linearized gauge-invariant terms that are, at the same time, being constructed on the base of the Einstein tensors in order to guarantee the gauge invariance, and can be expressed in terms of the Ricci tensor and the scalar curvature.

To do this, we note that the linearized Einstein equations look like

$$\begin{aligned} Q_{\mu\nu} &= \frac{\delta S_{FP}}{\delta h_{\mu\nu}} = -\frac{1}{2}(\partial^\lambda \partial_\mu h_{\lambda\nu} + \partial^\lambda \partial_\nu h_{\lambda\mu}) + \frac{1}{2}\eta_{\mu\nu}\partial_\lambda\partial_\rho h^{\lambda\rho} + \frac{1}{2}\partial_\mu\partial_\nu h + \frac{1}{2}\Box h_{\mu\nu} - \\ &- \frac{1}{2}\eta_{\mu\nu}\Box h = 0. \end{aligned} \tag{15}$$

It is evident, and easy to check, that these equations are gauge invariant, $\delta Q_{\mu\nu} = 0$, under (14). Hence, we can define the CPT-even gauge-invariant action with only fourth derivatives:

$$\mathcal{L}_{four} = \frac{1}{2} b_\mu Q^{\mu\nu} b^\lambda Q_{\lambda\nu}, \tag{16}$$

where $Q_{\mu\nu} = -(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu})$ and this implies

$$\mathcal{L}_{four} = \frac{1}{2} b_\mu (R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}) b^\lambda (R_{\lambda\nu} - \frac{1}{2} R g_{\lambda\nu}), \tag{17}$$

where we take only linear terms in h as in Equation (15).

We note that this action differs from the one considered in [22], which involved contraction of Riemann tensors rather than Ricci tensors used in our case. A four-derivative Lorentz-breaking term like this was considered in [23] in the context of electrodynamics.

Similarly, in the CPT-odd case, we have

$$\mathcal{L}_{odd} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} b_\alpha b^\mu Q_{\mu\beta} \partial_\gamma b^\nu Q_{\nu\delta}. \tag{18}$$

These modified terms should be added to the usual Lagrangian in Equation (2) for the linearized gravity (2).

As in the previous section, our aim consists of searching for unusual dispersion relations. Let us first calculate the ingredients of $Q_{\mu\nu}$ with the use of the decomposition (1). We have $h = h^{00} - h^{ij}\delta_{ij} = n - 2\phi - \chi$ and $\partial_\lambda\partial_\rho h^{\lambda\rho} = -2\nabla^2 \dot{n}_L + \nabla^2\chi + \ddot{n}$. Then, we define $\partial^\lambda\partial_\mu h_{\lambda\nu} + \partial^\lambda \partial_\nu h_{\lambda\mu} \equiv P_{\mu\nu}$. We have

$$\begin{aligned} P_{00} &= 2\ddot{h}_{00} - 2\partial_i \dot{h}_{i0} = 2(\ddot{n} - \nabla^2 \dot{n}_L); \\ P_{0i} &= \ddot{h}_{0i} - \partial_j \dot{h}_{ji} + \partial_i \dot{h}_{00} - \partial_i\partial_j h_{j0} = \\ &= \ddot{n}_{iT} + \partial_i \ddot{n}_L - \partial_i \dot{\chi} + \partial_i \dot{n} - \partial_i \nabla^2 n_L; \\ P_{ij} &= \partial_i \dot{h}_{0j} + \partial_j \dot{h}_{0i} - \partial_i \partial_k h_{kj} - \partial_j \partial_k h_{ki} = \\ &= \partial_i \dot{n}_{jT} + \partial_j \dot{n}_{iT} + 2\partial_i\partial_j n_L - 2\partial_i \partial_j \chi + \nabla^2(\partial_i \xi_{jT} + \partial_j \xi_{iT}). \end{aligned} \tag{19}$$

We see that none of these terms involve the physical $\tilde{h}^{ij}$ components, they only enter the $\frac{1}{2}\Box h_{\mu\nu}$ term of $Q_{\mu\nu}$. Hence we see that one has

$$b_\mu Q^{\mu\nu} = \frac{1}{2} b_i \Box \tilde{h}^{ij} \delta^\nu_j + (\dots), \tag{20}$$

where the dots denote the physically irrelevant components; that is, those other than $\tilde{h}^{ij}$. In this case some of them can acquire dynamics but it is common for higher-derivative theories, see, e.g., [24]. It is important to emphasize that, to have a nontrivial impact, the Lorentz-breaking vector b_μ should have an essential space-like part that is only contracted to $\tilde{h}^{ij}$. As a result, the Lorentz-breaking term (16) after integration by parts takes the form

$$\mathcal{L}_{four} = \frac{1}{2} b_i b^k \tilde{h}^{ij} \Box^2 \tilde{h}_{kj} + (\dots). \tag{21}$$

It remains for us to study the dispersion relation for the Lagrangian given by the sum of (3) and (21), which, in the relevant sector, yields

$$\mathcal{L}_{free} = -\frac{1}{4} \tilde{h}^{ij} \Box \tilde{h}^{ij} + \frac{1}{2} b_i b^k \tilde{h}^{ij} \Box^2 \tilde{h}_{kj} + (\dots). \tag{22}$$

The corresponding equation of motion is

$$-\frac{1}{2} \Box \tilde{h}_{ij} + b_i b^k \Box^2 \tilde{h}_{kj} = (-\frac{1}{2} \Box \delta^k_i + \Box^2 b^k b_i) \tilde{h}_{kj} + (\dots) = 0. \tag{23}$$

As done previously, we can consider that there are only two polarization states. We see that here there are two situations: (i) one has simply $\Box \tilde{h}_{ij} = 0$, which is the usual Lorentz-invariant situation, where the dispersion relations are the usual ones $E^2 = \vec{p}^2$; (ii) $(\delta^k_i - 2\Box b^k b_i)\tilde{h}_{kj} = 0$, which either requires the b_i vector to be directed along the wave propagation direction or, in the Fourier representation, requires $\det(\delta^a_i + 2p^2 b^a b_i) = 0$, which enforces b_a to be related with the wave vector p, which is clearly senseless except for degenerated cases.

It is interesting to compare this situation with the explicitly CPT-breaking case where the quadratic Lagrangian is a sum of the usual Lorentz-invariant expression (2) and the CPT-odd term (18), which involves five derivatives. In the same way, we concentrate on studying the dynamics of $\tilde{h}_{ij}$. Therefore, we have a sum of the second-order term (3) and the fifth-order term

$$\mathcal{L}_5 = \frac{1}{8} \epsilon_{\alpha\beta\gamma\delta} b^\alpha b_i \Box \tilde{h}^{ij} \delta^\beta_j \partial^\gamma b_k \Box \tilde{h}^{kl} \delta^\delta_l = \frac{1}{8} \epsilon_{\alpha j \gamma l} b^\alpha b_i \Box \tilde{h}^{ij} \partial^\gamma b_k \Box \tilde{h}^{kl} + (\dots), \tag{24}$$

arising from (18). We see that the Lorentz-breaking vector should have a nontrivial space-like part. If it is purely space-like, we have after integration by parts

$$\mathcal{L}_5 = -\frac{1}{8} \epsilon_{mjl} b^m b_i b_k \tilde{h}^{ij} \Box^2 \dot{\tilde{h}}^{kl} + (\dots), \tag{25}$$

whose corresponding equation of motion is

$$-\frac{1}{2} \Box \tilde{h}_{ij} - \frac{1}{4} \epsilon_{mjl} b^m b_i b_k \Box^2 \dot{\tilde{h}}^{kl} + (\dots) = 0. \tag{26}$$

It is clear that this equation can be rewritten in the form $\Box \Pi^{kl}_{ij} \tilde{h}_{kl} = 0$, hence it is compatible with the usual Lorentz-invariant plane wave solutions satisfying the usual equation $\Box \tilde{h}_{ij} = 0$. As in the previous case, one can have only $b_3 \neq 0$ due to there being only two polarization states. In this case, the equation above will be identically satisfied. Therefore, we see that due to the higher-derivative modes, there is no essential difference between the propagation of waves in CPT-even and CPT-odd cases.

We can introduce more gauge-invariant terms considering the projection-like operator

$$\Pi^{\mu\nu} = \eta^{\mu\nu}\Box - \partial^\mu\partial^\nu, \tag{27}$$

so that $\Pi^{\mu\nu}\Pi_{\nu\lambda} = \Box\Pi^\mu_\alpha$. Then, we consider $\Pi^{\mu\nu}h_{\nu\alpha}$. Its gauge transformation is

$$\delta\Pi^{\mu\nu}h_{\nu\alpha} = \partial_\alpha\Box\xi^\mu - \partial_\alpha\partial^\mu(\partial\cdot\xi). \tag{28}$$

Afterwards, we construct the vector

$$K_\alpha = b_\mu\Pi^{\mu\nu}h_{\nu\alpha}, \tag{29}$$

whose gauge transformation is

$$\delta K_\alpha = \partial_\alpha[\Box(b\cdot\xi) - (b\cdot\partial)(\partial\cdot\xi)] = \partial_\alpha\Sigma[\xi]. \tag{30}$$

Therefore, the Lagrangian

$$\mathcal{L}_{even} = \frac{1}{2}K_\alpha\Pi^{\alpha\beta}K_\beta, \tag{31}$$

will be gauge-invariant since its variation is proportional to $\Pi^{\alpha\beta}\delta K_\beta = 0$. So, we proceed to constructing the higher-derivative aether-like Lorentz-breaking gauge-invariant action for linearized gravity.

We note that one can construct a CPT-odd gauge-invariant contribution within this prescription as well; this looks like

$$\mathcal{L}'_{odd} = \epsilon^{\alpha\beta\gamma\delta}b_\alpha K_\beta\partial_\gamma K_\delta. \tag{32}$$

We note that $\mathcal{L}_{even}$ is of the sixth-order in derivatives and $\mathcal{L}_{odd}$, $\mathcal{L}'_{odd}$ is of the fifth-order. Actually, $\mathcal{L}_{odd}$ (18) and $\mathcal{L}'_{odd}$ (32) differ only by irrelevant additive terms that vanish if we set all non-physical fields (i.e., all fields other than the transverse-traceless $\tilde{h}_{ij}$) equal to zero. In principle, this is possible due to the gauge symmetry of these Lagrangians, which restricts physical degrees of freedom to $\tilde{h}_{ij}$. Therefore, (32) and (18) are physically equivalent. We note that these orders in derivatives are very high, corresponding to dimension-7 and -8 operators (to the best of our knowledge, such orders, except of essentially nonlocal models, have been studied only within the very specific context of Rashba coupling [25,26]; we note that in [9], the table includes only operators with dimensions up to 6) and, moreover, they cannot be decreased without introduction of undesired nonlocal terms involving negative degrees of $\Box$, which are rather dangerous from the unitarity/causality viewpoint. However, the corresponding full-fledged contributions to the action expressed in terms of the Riemann curvature tensor and its covariant derivatives are not known, and searching for them is a nontrivial problem.

4. Summary

We have discussed the modification of dispersion relations for various LV extensions of gravity and corresponding changes in the plane wave solutions. We demonstrated explicitly that only in certain cases do the dispersion relations turn out to be essentially different from Lorentz invariant ones. We showed that the dispersion relations continue to be the usual ones for a specific class of Lorentz-breaking extensions of the gravity; namely, the aether-like ones characterized by one constant vector. Clearly, this establishes questions about the form of dispersion relations in more involved cases. Certainly, gauge-breaking LV extensions of gravity, as discussed in [8], require more detailed studies. In particular, it is interesting to construct more involved LV extensions of gravity, which could display unusual dispersion relations whose possibility has been demonstrated in [27].

Another result of our study is the formulation of a prescription allowing for generating gauge-invariant LV extensions of the Einstein–Hilbert Lagrangian for a weak field with any arbitrary number of derivatives. We expect that such extensions will be useful for studying certain physical phenomena.

A possible extension of this paper could involve studies of plane wave solutions on a nontrivial curved background. Another possible development of this study could involve detailed consideration of massive LV gravity, while up to now most studies of massive gravity have been devoted to the Lorentz-invariant case, see [28] and references therein. In addition, it is natural to expect that nontrivial phenomena taking place within wave propagation as discussed in this paper can be used in future gravitational wave observations in order to find LV extensions of gravity, which could be more appropriate from the experimental viewpoint.

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