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Towards an Effective Architectural Form: The Composition of Squareness and Roundness Based on Scale Proportion—Evidence from the Yingxian Wooden Pagoda

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Abstract: Investigating the mathematical and geometric principles embedded in ancient classic architecture is a significant tradition in the history of architectural development. Drawing inspiration from the modular design and creative ideology based on the geometric proportions of squareness and roundness in ancient Chinese architecture, we propose a new mode of squareness and roundness composition based on scale proportion specifically for the design of multi-story buildings. Taking Yingxian Wooden Pagoda as the case study, we not only re-evaluate the modular system and proportional rules followed in the design of the entire pagoda, but also reveal the technical approaches and geometric rules for effectively controlling the form of multi-story buildings. In particular, the mode of squareness and roundness composition based on scale proportion, utilizing a modular grid combined with squareness and roundness drawings as decision-making tools, can control the scale and proportion of buildings across different design dimensions and organically coordinate the design of multi-story buildings' plans and elevations. Thus, it can achieve an effective balance of multi-story architectural forms. This study has potential applications in the creation of traditional multi-story buildings and heritage restoration projects, and offers valuable insights for future research on ancient multi-story buildings.

Keywords: ancient Chinese architecture; squareness and roundness composition; scale proportion; modular system; Yingxian Wooden Pagoda

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1. Introduction

Chinese ancient architecture, which is considered one of the oldest and longest-lasting architectural systems in the world, has matured through thousands of years of uninterrupted development and evolution. It has formed a rich tradition of engineering techniques and cultural arts, leaving behind a legacy of ancient architectural heritage. Starting from historical traditions, exploring the engineering and artistic ideas, principles, and methods contained in classic buildings and exploring their modern value not only aids in the scientific preservation of heritage but also inspires architectural creators. Liang (2001) once observed that architectural art creation cannot be completely independent from the foundation of traditions. If architects who master scientific and technological knowledge can draw from local materials and wisdom, their artistic creativity will naturally become more robust unconsciously [1].

Therefore, embarking on systematic research into the design of ancient architecture and turning our attention to fundamental architectural questions such as scale proportion and geometric composition continues to be a long-lasting academic proposition with dual significance. First, under the influence of proportional rules, ancient architecture often utilizes a universal unit system (i.e., modulus) to control the relative sizes from details to the whole. This approach not only creates an architectural scale rich in expressiveness but also exhibits a strict mathematical order [2]. Conversely, examining the scale constitution, modular system, and proportional rules of ancient architecture can allow us to better summarize the mathematical laws of the beauty of architectural forms, revealing the original design intentions and methods. Moreover, scale proportion is crucial for architectural composition, as it establishes a series of visual relationships. Although it may be challenging for observers to notice the specific proportions of a building, geometry, as a visual language, assists in capturing the harmony and rhythm produced by proportions, and provides corresponding intuitive feedback [3]. Thus, through geometric drawing analysis, elucidating the principles of architectural composition, structure, and proportion can enable a deeper insight into the architectural design process. This approach uncovers the orderliness of formal aesthetics and the hierarchy of structural organization, and allows for a rational explanation of the architect's design logic [4].

The exploration of scale proportion in ancient Chinese architecture began in the 1930s with the Society for Research in Chinese Architecture's emergency surveys and research on several forms of ancient architectural heritage [5]. As the research deepened, Liang (1945), a member of the society, deciphered two ancient Chinese architectural official texts—the Song Dynasty's "Yingzao Fashi" and the Qing Dynasty's "Gongcheng Zuofa". He clearly stated that ancient Chinese architecture used the standardized dimensions of dougong (bracket sets) as the unit of measurement for all architectural proportions. This is similar in principle to Greek and Roman architecture, using the diameter of columns as the modulus to balance the proportions of various parts, such as the Song Dynasty's *cai fen* modulus system and the Qing Dynasty's doukou modulus system [6]. Taking the *cai fen* modulus system as an example, it is a standardized and stereotyped design method used in building construction and architecture. The system uses *fen* as the basic module, which has a fixed number but variable value in different cases. According to the "Yingzao Fashi" published in 1103 A.D. during the Song Dynasty, "*cai*" referred to the cross-section of a standard wooden component, with the cross-section size divided into eight grades. This book specifies the cross-section size and application range of each grade of *cai*, with the grade of *cai* matching the size of the building. The height-to-width ratio of *cai* is 3:2, with the *cai* height divided into 15 equal parts and the *cai* width into 10 equal parts, each equal part being 1 *fen*. The proportions of each part of the building and the length of each component are all measured in multiples of *fen* [7]. The *cai fen* modulus system was only a basic system of official architectural modularization practice in the Song Dynasty. In different eras and regions, there were often design tendencies directly based on the modulus of *dan cai* height (15 *fens*), *zu cai* height (21 or 20 *fens*), integer feet and inches, etc., to meet the requirements of the construction activities of ancient Chinese wooden buildings characterized by prefabricated assembling. Following this academic interest, Chen (1981) made significant breakthroughs by inspiring mutual enlightenment between the "Yingzao Fashi" and the study of actual architectural objects. He comprehensively explained the system standards and structural technical logic of the *cai fen* modulus system for various component specifications and sizes. This also involved the inherent proportional relationships of the three fundamental dimensions of architecture—length, width, and height—confirming that the *cai fen* modulus system was the basic principle of structure and architectural design at that time [7]. Long (2010) used the ancient pagoda design documents recorded in the "Yingzao Fayuan" as clues with which to explore the proportional relationships affecting the formal beauty of various pagodas, summarizing the mathematical methods of ancient pagoda design [8].

Meanwhile, considering Chinese architecture as the source of East Asian architecture, scholars have further expanded the broad landscape of transnational comparative architectural history research. Fu (1992) discovered that by the mid-6th century AD at the latest, in addition to using the *cai* height as the modulus, Chinese and Japanese architectural designs also used column height to establish proportional relationships in facades and sections [9]. Zhang (2004), after comparing the nature, laws, and evolution of scale constitution in ancient Chinese and Japanese architecture, concluded that the overall scale of architecture during the Tang, Liao, and Song periods was not directly constrained by the "Yingzao Fashi" *cai fen* modulus system. He proposed using the datum length as an expanded modulus with which to grasp the overall scale and proportional design, which is widely applicable in multi-storied buildings in both China and Japan [10]. Cha and Kim (2023) found that wooden architecture from the Joseon era utilized the length and width of brackets (*dougong*) as moduli to control the proportional design of facades, showing similarities with ancient Chinese architectural design techniques [11]. This indicates that the ancient East Asian region commonly employed modular methods to control architectural scale proportions, including three types of moduli: the *cai* modulus, expanded modulus, and integral size dimensions [12].

The exploration of geometric composition in ancient Chinese architecture mainly revolves around the relational properties of shapes and numbers among basic geometrical figures such as squares, circles, and triangles. Utilizing these three basic shapes for simple drafting can enable the accurate generation and visual understanding of all important irrational number proportions in architecture [13], such as $\sqrt{2}$, $\sqrt{3}/2$, $\left(\sqrt{5}+1\right)/2$, etc. These have been gradually verified by the academic community as fundamental principles of ancient Chinese architectural design. Wang (1984), through statistical analysis of timber architecture from the Tang, Liao, Song, and Jin dynasties, was among the first to discover the square and circle compositional proportion $\sqrt{2}$. This is not only a common ratio of eave height to column height but is also widely used to balance the scale design of architectural elevations and sections [14]. Zhang (2013) explored the ratio $\sqrt{2}$ in the modular woodwork standards of the Song Dynasty's "*cai fen*" system and the Qing Dynasty's "*doukou*" system [15]. Wang (2009) first discovered the compositional technique of embedding a series of equilateral triangles within a $\sqrt{3}/2$ rectangle, thereby controlling the inherent structure of the capital's spatial form [16]. Wang (2018), selecting over 400 cases including cities, architectural complexes, and individual buildings, systematically demonstrated the important tradition of ancient Chinese craftsmen using square and circle composition ratios for urban planning and architectural design [17].

In the East Asian cultural sphere, Japan and Korea have also traditionally used $\sqrt{2}$ proportions, with modern Japanese scholars referring to them as the "silver ratio", on par with the Western "golden ratio" [15]. Korean scholars Cha and Kim (2019a, 2019b, 2022c), based on the historical spread of ancient Chinese mathematical thought to the Korean Peninsula, have shown that the $\sqrt{2}$ square and circle compositional proportion was also a design principle followed by early Korean stupas and timber structures [18–20]. More complex geometric compositional proportions have also been discovered in the architecture of the Seokguram Grotto in Korea, echoing Buddhist order and philosophical implications [21]. Using basic geometric shapes to precisely control architectural scale and proportion seems to be a universal principle in the pursuit of architectural aesthetics, appearing in the architectural history of different regions around the world. Padovan (2002), Wittkower (1998), Buthayna (2012), and Cohen (2013, 2014) have discussed the proportion systems and mathematical principles of European classical architecture [13,22–25]. Oikonomou (2011), Eltrapolsi (2022), and Xavier (2022) have revealed the compositional proportions and geometric principles of traditional architecture in northern Greece, Tripoli, Libya, and Cuenca, Ecuador [26–28]. Arnold (2018) indicated that tenth-century geometry marked a shift to a new method distinct from the Roman era, guiding the compositional and spatial perception design of Islamic architecture in Córdoba [29].

Throughout architectural history, architects have always integrated geometric analysis into their creations, aiming to design aesthetically pleasing and comfortable forms and structures, whether it is the golden section, the modular grid, square and circle drawing, etc., all of which are more beneficial than detrimental to the architectural creation. Classical geometric analysis provides architects with auxiliary lines that are powerful tools to help establish order and efficiency while avoiding arbitrariness. Geometric auxiliary lines provide reliable guidance for design and construction work and can organically relate and integrate various architectural elements, and each element maintains its original character while creating a harmonious relationship with the whole [30]. The various auxiliary lines also provide architectural creativity with compositional processes, modeling relationships, and the means to achieve visual balance. Le Corbusier emphasized that architects select various auxiliary lines for compositional needs, to establish order and tidiness in the geometric balance, or rather, expect to achieve true purity [31].

In addition to the above-recognized advantages, there are also some criticisms: firstly, geometric auxiliary lines stifle imagination and create a jack-of-all-trades; secondly, strict adherence to classical geometric compositions will lead to conflicts with architectural structures; thirdly, blindly pursuing formal aesthetics through geometric analysis will increase unnecessary construction costs. To eliminate these drawbacks, we are required to familiarize the evidence of the real existence and flexible use of various geometric compositions from the historical classics, uphold the attitude of structural rationalism and comprehensive balance, and avoid excessive dependence and abuse. Indeed, the various geometric compositions are not secret recipes but happen to establish design discipline and freedom for architectural creation [32].

Surrounding the scale proportions and geometric composition of ancient architectural creations, scholars have obtained rich results, inspiring this article in terms of theoretical understanding and research methods. However, there are still several shortcomings. First, detailed architectural survey drawings that have been recently completed are not usually published promptly, and existing research often cites data from early publications, making it difficult to reach a consensus on the laws of architectural scale constitution. Second, past studies, limited by survey materials, have rarely examined the design connection between large timber structures, the building itself, and the group environment, leading to an insufficient systemic understanding of moduli and proportions. Third, current research on square and circle geometric composition mostly focuses on exploring architectural proportions, but lacks summarization of design patterns and methods from a practical design perspective. These shortcomings are common issues in research related to ancient Chinese architecture and are challenges faced in the case study of the Yingxian Wooden Pagoda.

In this paper, we aim to redefine the moduli and proportional rules of architectural scale as a starting point, searching for a new squareness and roundness geometric composition model to control the design of multi-story architectural forms. By combining actual measurement data and geometric drawing analysis of the Yingxian Wooden Pagoda, we not only verify the effectiveness of this model but also provide novel approaches to interpreting the mathematical mysteries of the design of the Yingxian Wooden Pagoda. Therefore, this study will contribute to the existing knowledge in the following ways:

(1) Utilizing innovative technology, we complete a relatively comprehensive set of digital survey drawings of the Yingxian Wooden Pagoda, and based on deducing the length of the original construction ruler, re-summarize the scale constituent characteristics of the Yingxian Wooden Pagoda.

(2) Coordinating an analysis of the Yingxian Wooden Pagoda's structural woodwork, the building itself, and the temple environment scales, we induct the modulus system and proportional rules of the entire pagoda's scale design, supplementing the understanding of the implied mathematical laws in the pagoda's design.

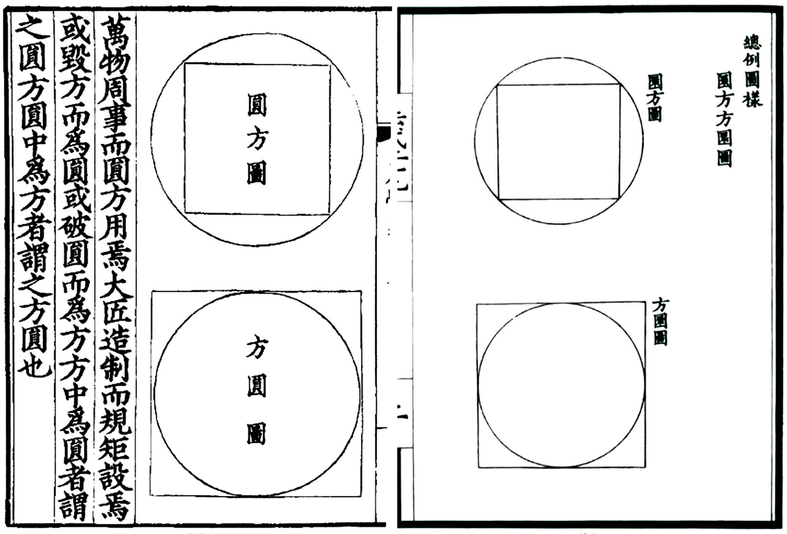
(3) Unlike authors of previous studies on square and circle geometric compositional proportions, we consider the symbolic meaning and geometric relationships of squareness and roundness composition, organically linking them with the elevations and plans, establishing a method to effectively control the design of ancient multi-story architectural forms.

2. Theoretical Framework

2.1. Theoretical Basis

For ancient Chinese timber architecture, characterized by prefabrication and assembly, modularization serves as a fundamental principle of structure and architectural design. It not only helps control architectural scale proportions but also facilitates the standardized processing of structural components, providing the intrinsic driving force for creating architectural forms' internal mathematical order. According to historical records, the use of moduli by the ancient Chinese has a long history, traceable back to the land planning model of the Well-Field System during the Xia, Shang, and Zhou dynasties, and back to the cartographic method of "measuring distance and drawing squares". In the fields of architecture and planning, the layout of the Zhongshan Royal Tomb's "Zhaoyu Tu" (Plan of the Tomb Area), an archaeological discovery from as early as 2300 years ago, implied the existence of a modular grid. The "*Pingge*" method in the Qing Dynasty's "*Yangshi Lei*" architectural drawings comprises a set of mature modular design concepts and methods. By utilizing moduli to control the scale proportions of both architectural environments and individual buildings, this method has been used to create places with a sense of warmth and intimacy [33]. The application of moduli in ancient Chinese architecture almost spans the entire process from group planning to individual design, to prefabricated construction, playing multiple roles in coordinating design processes, simplifying engineering estimates, and enhancing construction efficiency.

Additionally, influenced by the cosmological concept of "round heaven and square earth", the Chinese often use circles and squares to symbolize heaven and Earth. Coupled with the agricultural nation's supreme reverence for nature, this endows circles and squares with unique cultural consensus and national sentiment. The ancient astronomical and mathematical text "Zhou Bi Suan Jing" states "The square belongs to earth, the circle to heaven; heaven is round, and the earth is square. Square numbers are standard, from which circles are derived". This text also illustrates "Rounded-square and Squared-circle Diagrams" (Figure 1), explaining the proportions arising from the circumscribed and inscribed geometric relationships between squares and circles, namely "square to circle ratio of five to seven" ($\sqrt{2}$) and "square and circle with the same diameter" (1), becoming abstract expressions of the "round heaven and square earth" ideology. It explicitly states that "This method of squares and circles is applied to all things and affairs, with the great craftsmen creating and establishing rules and compasses accordingly". In respect of the Northern Song Dynasty's "Yingzao Fashi", a symbol of Chinese architecture's standardization, authors Li Jie referenced the aforementioned diagrams and discussions at the beginning of the book, subtly indicating that the symbolic significance and geometric rules inherent in square and circle compositions were fundamental principles adhered to by ancient craftsmen, becoming an important clue for the study of compositional proportions in ancient Chinese architecture today.



(a) (b)

Figure 1. (**a**) "Rounded-square and Squared-circle Diagrams" in the ancient Chinese astronomical and mathematical text "Zhou Bi Suan Jing"; (**b**) "Rounded-square and Squared-circle Diagrams" in the Northern Song Dynasty architectural official book "Yingzao Fashi".

2.2. Technical Path

According to the tradition of Chinese ancient architecture's modular design and squared-circle composition, we propose a squareness and roundness composition model based on scale proportion. The discovery of this model started from the study of two typical ancient multi-story wooden towers [34,35], and was later applied to the morphological analysis of Tang Dynasty brick pagoda architecture [36], preliminarily verifying its reliability in controlling the forms of multi-story buildings. To form a theoretical model of applicative value, we summarize the main contents and technical paths of the model.

The prime content is the analysis of architectural scale, modulus, and proportion. Because of the differences in the times and regions of the ancient construction ruler, if we use the metric units to explore the architectural scale, although we can compare the absolute size of and changes in different architectural scales, we cannot reveal the original scale design rules and intentions. Therefore, deducing the length of the original construction ruler becomes the premise of scale analysis. Then, metric-measured data can be converted into traditional Chinese units to analyze the laws of architectural scale constitution and find the modulus system controlling different levels of architectural scale. On this basis, drawing a modulus grid in conjunction with summarizing the architectural proportion reveals the mathematical laws of formal beauty.

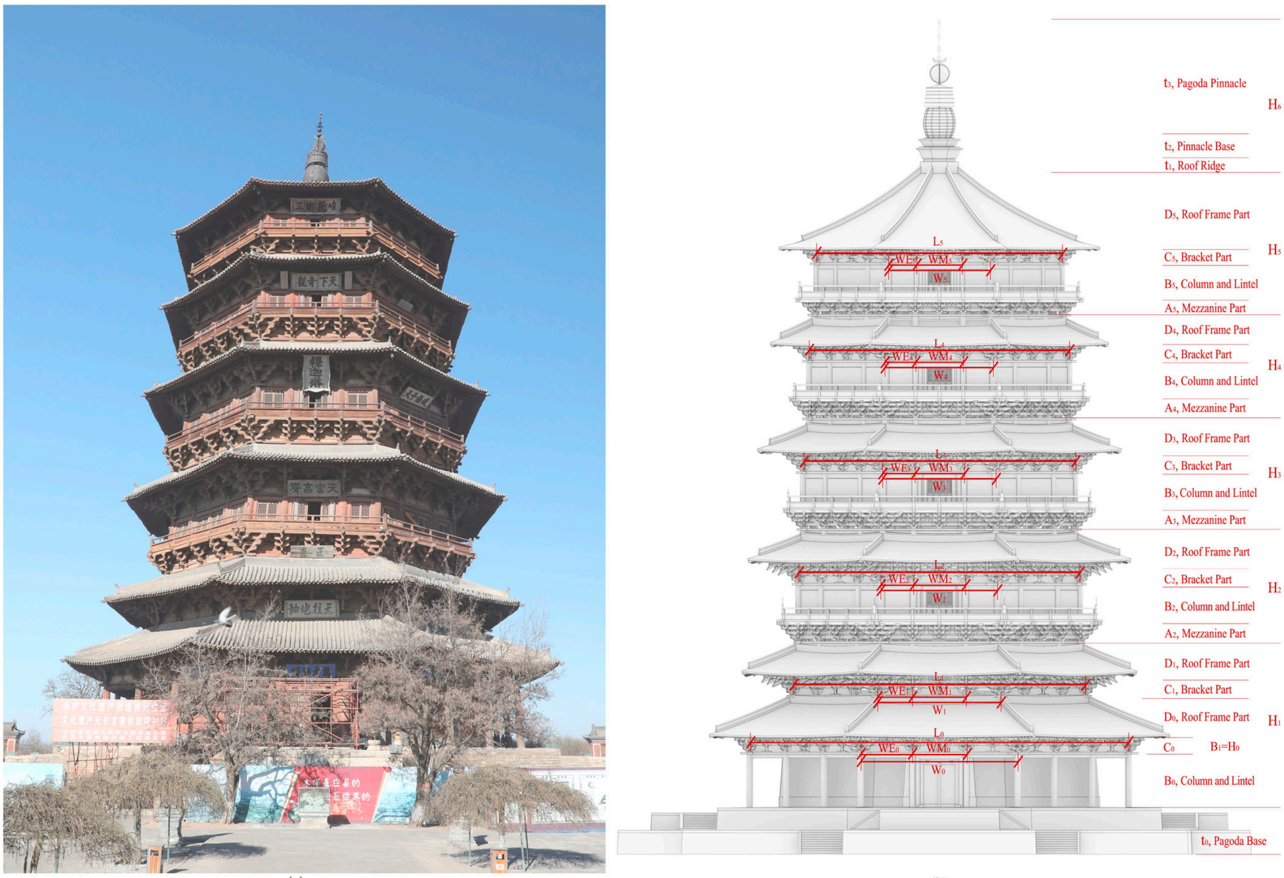
The core of the model is the analysis of controlling the form of multi-story buildings through squareness and roundness composition. Starting from the cultural significance and mutually inventive mathematical relationships of squares and circles as symbols of heaven and Earth, the cosmological view of "heaven covers, earth supports" is projected into architecture, establishing an organic connection between squares, circles, and the plan and elevation. Based on the geometric relationships of "Rounded-square and Squared-circle Diagrams", using special squares in the plan of each story to generate a series of circular auxiliary lines controls key points in the elevation forms of multi-story buildings, especially complex roof shapes. The analytical approach aligns with the relevant explanations in "Zhou Bi Suan Jing" and can be summarized into three key steps: "derive squares from the plan, generate circles from squares, and balance the facade forms".

3. Methodology

3.1. Case Study

The Yingxian Wooden Pagoda, also known as the Sakyamuni Pagoda of Fogong Temple, is located in the northwest of Ying County, Shanxi Province. It was built in the second year of the Qingning era of the Liao Dynasty (AD 1056) and has a history spanning nearly a thousand years. The pagoda has an octagonal plan, with five stories and six eaves, containing four mezzanine stories inside, making a total of nine levels, and a measured height of 65.77 m. The Yingxian Wooden Pagoda is the oldest, tallest, climbable wooden Buddhist pagoda still in existence, representing one of the great creations in the history of human architecture (Figure 2).

The pagoda in China is a new architectural type autonomously created under the influence of the foreign Buddhist culture. The most common form of early Buddhist pagodas in China was the wooden pagoda. As the oldest extant pavilion-style wooden pagoda in China, the Yingxian Wooden Pagoda is an important physical testimony to the integration and innovation in the sinicization process of Buddhist architecture. Particularly in breaking through the limitations of the traditional core pillar structure technology in Chinese wooden high-rise Buddhist pagodas and effectively utilizing the new advantages of the encircling columnar structure system to meet the developmental needs of religious space art, the Yingxian Wooden Pagoda stands as a successful example, reflecting a significant transformative breakthrough. Therefore, it has been praised as a diachronic summary of the evolution and development of high-rise wooden pagodas over the centuries before the Liao and Song dynasties [37].



(a) (b)

Figure 2. (**a**) Realistic view of Yingxian Wooden Pagoda; (**b**) Schematic diagram of main data related to this paper.

Since the beginning of the last century, when the Yingxian Wooden Pagoda entered the modern academic field, it has been a classic subject in the field of Eastern architectural history. It has significant value, especially in research regarding ancient pavilion architecture's structural technology and formal aesthetics. Its unique octagonal and five-story shape has typical significance in the study of scale design technology [38]. As a piece of Liao Dynasty architecture, the Yingxian Wooden Pagoda continues the legacy of the Tang Dynasty and initiates the construction style of the Song Dynasty. It is not only a key physical object for studying the architectural style and compositional methods of the Liao period but was also an important example for exploring architectural design rules during the Tang and Song periods [39].

Considering the prominent historical and academic value of the Yingxian Wooden Pagoda mentioned above, we have chosen it as the subject of the study. This was not only to evaluate the applicability of the theory but also to understand the mathematical laws behind the design of the Yingxian Wooden Pagoda.

3.2. Data

Between 2017 and 2018, our team conducted on-site data collection concerning the Yingxian Wooden Pagoda, employing various innovative technologies such as 3D laser scanning and drone oblique photography measurements. The model of the 3D laser scanner was Leica-ScanStation C10, the movable target was arranged in the measurement change station, the precision of the project file was set to high quality, and the splicing processing was carried out by software Leica Cyclone 7.4. The drone model was DJI-PHANTOM 4 PRO, collecting 1402 photos on site, importing them into the software Agisoft Metashape Professional 2.0, and selecting the highest precision or quality parameters while aligning the photos, generating meshes, and generating textures. The 3D point cloud model and external orthophoto of the pagoda were finally acquired, with an overall data error of ±1 cm. After post-calculation processing of the measured data and image resources, a complete set of survey drawing files was drawn in the software Autocad 2016. The survey data related to this paper are introduced as follows (Figure 2).

Plan Data: Each floor of the wooden pagoda has an octagonal plan with an inner and outer ring of columns, one column at each angle of the inner groove, and one space per face; two additional columns are added between the two corner columns of the outer groove on each face, making three spaces per face, totaling thirty-two columns used. The planar column grid data involved in this paper follow previous conventions, which are based on the average values of the eight sides for each column's top plan (Because the plan of the columns' top of ancient wooden building is less affected by the structural deformation under the pulling knots of the Pu Pai Fang and lintel), including the side length, W_i; the middle bay span, WM_i; the bilateral bay span, WE_i; and the planar width, L_i (Figure 3).

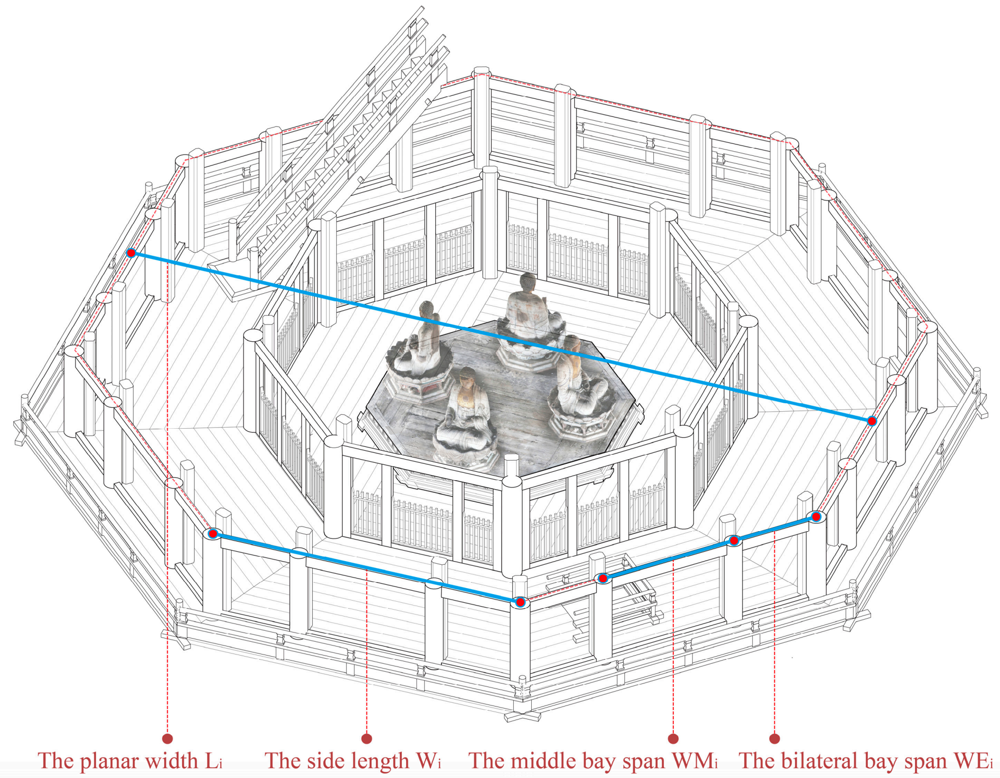


Figure 3. Schematic diagram of the main planar data of Yingxian Wooden Pagoda.

Vertical Data: These consist of storied heights and segmented heights. First, the area from the bottom of the first-floor eave column to just under the fifth-floor roof ridge is divided into five stories based on the upper edge of the Pu Pai Fang on the outer eave of each mezzanine story, with storied heights, H_i. Secondly, each story includes four sections: the mezzanine part, A_i, the column and lintel part, B_i, the bracket part, C_i, and the roof frame part, D_i, with the first floor lacking a mezzanine part. Additionally, the base height is t_0, the top height of the pagoda is H_6, and the total height of the pagoda is H (Figure 2).

(1) Mezzanine part, A_i: the height from the upper skin of the Pu Bai Fang in the mezzanine story to the bottom of the ground beam in the exposed story, i.e., the height of the brackets' layer of the mezzanine (excluding Pu Pai Fang);

(2) The column and lintel part, B_i: the height from the bottom of the outer eave column (or the bottom of ground beam) to the top of the column, i.e., the column height in the exposed story, with only the bottom porch including the height of the Pu Pai Fang;

(3) The bracket part, C_i: the height from the top of the eave column to the eave, i.e., the total height of the visible parts of the Pu Bai Fang and brackets' layer on the façade;

(4) The roof frame part, D_i: the height from the eave to the upper edge of the Pu Pai Fang in the mezzanine story, for the fifth floor, from the eave to the bottom of roof ridge (or the upper edge of the ridge purlin).

All data are the average values of the eight sides, with subscript i ranging from 0 to 5, corresponding to the bottom porch and five exposed stories. Our consideration of the measurement data and analyzed results can be impacted by structural deformation, measurement errors, and original construction logic. Firstly, new technology has improved the accuracy and scope of data collection, leading to analyzed results that are closer to the original truth; secondly, multi-source image data such as three-dimensional point clouds and aerial images have become available, which eliminates the high risk and low accuracy of the previous manual high-altitude collection of the roof data of each story and enhances the reliability of squareness and roundness composition verification analysis. Furthermore, the vertical measurement division of the pagoda differs significantly from previous studies. According to the division method presented in this paper, each story's column and lintel part and bracket part should be regarded as the body of the building. The mezzanine part and roof frame part should be treated as the base and roof, respectively. This division aligns with the traditional architectural three-part facade composition, with each story forming similar yet independent architectural units. This division more closely matches the vertical construction logic and facade features of the wooden pagoda.

3.3. Methods

3.3.1. Analysis of Architectural Scale

Estimating the values of the original construction ruler and then converting the architectural metric measured data into traditional Chinese units is a prerequisite for scale analysis. The first step is necessary due to the lack of physical evidence for the construction ruler used during the Liao Dynasty, to which the Yingxian Wooden Pagoda belongs. Therefore, we adhere to the academic consensus that "the length of Liao ruler is equivalent to the length of Tang ruler", taking the length of forty commonly used Tang Dynasty rulers, ranging from 290 to 318 mm, as the range of experimental values. The second step is to determine a standard for rounding and balancing; to improve the fit of the measurement samples, the rounding error is expanded to ± 0.02 feet [40]. In addition to whole and half feet, 0.25 feet, 0.75 feet, and whole inches are included as rounding reference values. The third step is to use the measured data of the column top plan on each exposed story (W_i, WM_i, WE_i, and WB_i) as the main object for conversion and rounding, with column height, B_i, as a secondary evaluation object. This is because vertical data's reliability is affected by the cumulative effects of different components' bending, tilting, and displacement, unlike the planar data of column heads, but has a limited impact on the column height of individual components. The fourth step is to take values within the 290–318 mm range, convert metric-measured values into traditional units, and determine the rounded values (i.e., restored values) according to the rounding standard, calculating the conformity (conformity = $100\% - |\text{restored value} - \text{measured value}| / \text{restored value}$) between restored values and actual measurements. The higher the number of rounded values and the fit, the closer the experimental value is to the original construction ruler value.

Finally, using the original construction ruler lengths obtained from the above analysis, all planar and vertical metric measurements were converted to foot units. Then, we examined the numerical size and constituent features of various scales, such as the planar side length, bay span, and planar width, as well as the vertical storied heights and segmented heights. We summarized the change rules of the various scales story by story, and explore the original design intent behind the rules.

3.3.2. Analysis of Architectural Modulus and Proportion

Exploring the modulus and proportion of ancient architecture is crucial for interpreting aesthetic order, summarizing design principles, and understanding construction logic. During the on-site surveying of the Yingxian Wooden Pagoda, traces of modularization in components such as brackets and roof elements were already noted. With subsequent data organization and scale analysis, vertically segmented heights with approximate values emerged, indicating the potential existence of a multi-level modulus system in the pagoda. We combined existing research viewpoints on the modular design of ancient pagodas and adhered to the analytical approach of coordinating the scale modulus of structural components, single building, and temple pagoda environments.

Firstly, we calculated and analyzed the ratio between the segmented height, the storied height, and the total height of the pagoda, counted the frequency of the integer ratio, and presumed the basic modulus and the expanded modulus. Secondly, assuming the that height of standardized structural components is the minimum modulus, we examined whether integer ratios exist among the three types of moduli, and weighed the possibility of the minimum modulus. Finally, we used the surveying drawings to establish a grid diagram with the modulus as auxiliary lines, evaluating the control effect of the three types of moduli on the different design dimensions, and then combining intuitive feedback with mathematical calculations to summarize the proportional rules of the entire pagoda design.

3.3.3. Analysis Combined with Composition of Squareness and Roundness

To validate the universality of the squareness and roundness composition model in controlling the morphology of ancient multi-story buildings, the following steps will clarify our drawing methods and processes (see Figure 4).

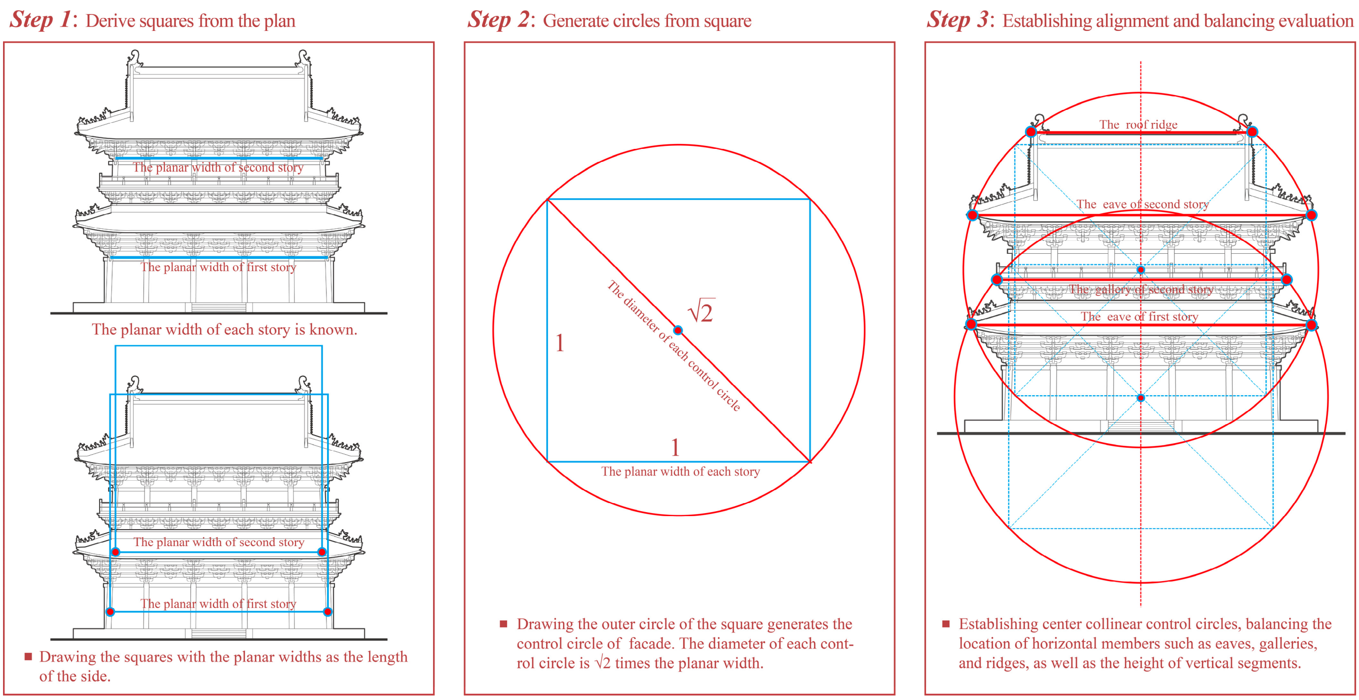


Figure 4. Schematic diagram of the basic principle and operation steps for squareness and roundness composition controlling the form of multi-story buildings.

(1) Deriving squares from the plan: This involved extracting a square from each story plan, with the square side length being the planar width of each story. As the first step in squared-circle drawing, this step established a correlation with the planar scale.

(2) Generating circles from squares: Following the methods of "Rounded-square and Squared-circle Diagrams" from the ancient literature, we circumscribed the squares

obtained from each story plan, and generated the control circles of the facade, each of which has a diameter $\sqrt{2}$ times the planar width.

(3) Establishing alignment and balancing evaluation: control circles are distributed longitudinally along the central axis of the facade, forming a series of center collinear control circles. These circles constrain the spatial relationships of the roof eaves on the facade, with various patterns such as the nth circle controlling the nth and $(n-1)$th floors, the nth and $(n+1)$th floors, or the $(n-1)$th and $(n+1)$th floors.

In practical implementation, the diameter of each control circle is $\sqrt{2}$ times the planar width of each story. Therefore, the essence of the squareness and roundness composition is to balance the position and width of the parallel chords of eaves, ridges, and galleries in the control circle of a fixed size, and then grasp the height of vertical segments such as the distance between eaves. The correct establishment of the squareness and roundness composition must meet two conditions: one is the control circle, where there are at least two axisymmetric parallel control chords, that is, four control points; the second is the control circle, where the chords intersect each other to form at least a common chord, with the eaves' width coinciding. These two conditions make the squareness and roundness composition stable. The mode of squareness and roundness composition is not an immutable design rule but possesses a certain degree of flexibility. This flexibility is manifested in the diversified rules of center collinear control circles controlling the facade, which precisely corresponds to the rich and varied aesthetic forms of ancient multi-story buildings.

4. Results

This section progresses based on the research methods proposed above. Initially, the length of the construction ruler used for the wooden pagoda is calculated, which redefines the constitution of planar and vertical dimensions. Starting from this point, the modular system of the entire pagoda's dimensional composition is analyzed. Based on this, modular grid maps are drawn to verify the control over architectural dimensions from different design perspectives and to assist in exploring the proportional rules of the entire pagoda's vertical design. Finally, the effective approaches and fundamental principles of squareness and roundness composition in controlling the form of the entire pagoda are analyzed.

4.1. Architectural Scale Constituent Characteristics

4.1.1. Construction Ruler of Yingxian Wooden Pagoda

Taking the metric measurements of the column top plan and column height on each exposed story as the main and auxiliary verification objects, respectively, and converting values within the range 290–318 mm, it was found that when the construction ruler length is 292 mm, not only are the quantity and accuracy of rounded data high, but they also satisfy two important phenomena of planar and vertical scale constitution simultaneously:

(1) From the first to the fifth floor of the exposed stories, the side length and the bay span are close to whole feet, half feet, and quarter feet, and exhibit a simple relationship of scale constitution and change law. The data of the second floor are affected by structural deformation, but the restoration values and actual measurements of the other floors have a conformity of more than 99.8%.

(2) The heights of the columns from the first to the fifth floor are close to ten whole feet (*Zhang*), with the fifth floor reduced by a whole half foot, displaying a characteristic of conciseness using the ruler. The restoration values of the eave column heights were calculated with a ruler length of 292 mm, and the actual measured values have a rounding error of no more than 1.5 inches, with the lowest accuracy being about 99.5%.

4.1.2. Planar Scale

Using the calculated construction ruler length of 292 mm to convert the actual measured data of this exposed story (Table 1), the following planar scale constituent characteristics were discovered:

Table 1. Main scale constitution of the exposed story plan of Yingxian Wooden Pagoda (1 foot = 292 mm).

No.	Position	Data	WM_i	$2WE_i$	W_i	L_i
01	Bottom Porch	Actual Value/mm	4452	8034	12,486	29,965
		Converted Value/feet	15.246	27.514	42.760	102.619
		Restored Value/feet	15.25	27.5	42.75	102.6
		Conformity	99.98%	99.95%	99.97%	99.98%
02	First floor	Actual Value/mm	4228	5336	9564	23,075
		Converted Value/feet	14.479	18.273	32.753	79.024
		Restored Value/feet	14.5	18.25	32.75	79
		Conformity	99.86%	99.87%	99.99%	99.97%
03	Second floor	Actual Value/mm	4162	5020	9182	22,074
		Converted Value/feet	14.253	17.192	31.445	75.596
		Restored Value/feet	14.25	17.25	31.5	76
		Conformity	99.98%	99.66%	99.83%	99.47%
04	Third floor	Actual Value/mm	3798	5034	8832	21,317
		Converted Value/feet	13.007	17.240	30.247	73.003
		Restored Value/feet	13	17.25	30.25	73
		Conformity	99.95%	99.94%	99.99%	99.99%
05	Fourth floor	Actual Value/mm	3725	4738	8463	20,451
		Converted Value/feet	12.757	16.226	28.983	70.038
		Restored Value/feet	12.75	16.25	29	70
		Conformity	99.95%	99.85%	99.94%	99.95%
06	Fifth floor	Actual Value/mm	3649	4349	7998	19,276
		Converted Value/feet	12.503	14.894	27.390	66.014
		Restored Value/feet	12.5	14.9	27.4	66
		Conformity	99.98%	99.96%	99.97%	99.98%

The side length, W_i: From the first to the fifth floor, the lengths are 32.75 feet, 31.5 feet, 30.25 feet, 29 feet, and 27.4 feet, respectively, with each story decreasing by 1.25 feet, 1.25 feet, 1.25 feet, and 1.6 feet, respectively. From the first to the fourth floor, each story decreases by 1.25 feet, exactly equal to the height of the exposed story's *zu cai* (actual measurement: 365 mm). The bottom porch is 42.75 feet (actual measurement: 12,497 mm), increasing by 10 feet (1 *Zhang*) compared with the first floor.

The bay span WM_i & WE_i: The calculated values of WM_i from the first to the fifth floor are, respectively, 14.5 feet, 14.25 feet, 13 feet, 12.75 feet, and 12.5 feet. Only the third-floor deviates from the gradation rule, decreasing by 1.25 feet, while the other floors decrease by 0.25 feet each. Scholars have explained the significant decrease in the third-floor bay span from the perspectives of visual transition and construction logic [41,42] (Xiao (2004) suggests that "a careful analysis of each bay span of the third floor reveals that the middle bay span is close to the upper story, while the bilateral bay span is close to the lower story. It can be inferred that this is an intentional adjustment made by the ancient architect for visual effects (gentle transitions)". Additionally, Li (2021) observes the following: "As the side length of the external and interior cao gradually decreases from bottom to top, if the middle bay span of the external cao is not adjusted, it will result in two issues: firstly, the caudal end of the beam and the bracket of the interior cao would be not installed together; secondly, there will be a significant disparity between the middle and the bilateral bay span in the upper story of the pagoda, causing a dissonance in the facade effect").

The planar width, L_i: This width from the first to the fifth floor of the exposed stories is 79 feet, 76 feet, 73 feet, 70 feet, and 66 feet, respectively, decreasing by 3 feet, 3 feet, 3 feet, and 4 feet, respectively, totaling a decrease of 13 feet.

4.1.3. Vertical Scale

The vertical actual measurements of the wooden pagoda are converted using a construction ruler length of 292 mm (Table 2), where most segmented and storied dimensions are close to whole feet or *Zhang*, revealing many rules of dimensional composition.

Table 2. Main scale constitution of the elevation of Yingxian Wooden Pagoda (1 foot = 292 mm).

No.	Story	Segment	Height Value					
			Actual Value/mm	Converted Value/Feet	Story Height/Feet	Basic Module M = 5 Feet		
01	Bottom porchH_0	Column and Lintel PartB_0	4377	14.99	$H_0 \approx 30$ (Conformity 99.94%)	3 M	3 M	6 M
		Bracket PartC_0	1023	3.5		0.7 M	3 M	
		Roof Frame PartD_0	3365	11.52		2.3 M		
02	First floorH_1	Columnand Lintel PartB_1	8726	29.88	$H_1 \approx 45$ (Conformity 99.58%)	6 M	6 M	9 M
		Bracket PartC_1	1463	5.01		1 M	3 M	
		Roof Frame PartD_1	2896	9.92		2 M		
03	Second floorH_2	Mezzanine PartA_2	1475	5.05	$H_2 \approx 30$ (Conformity 99.77%)	1 M	3 M	6 M
		Column and Lintel PartB_2	2907	9.96		2 M		
		Bracket PartC_2	1468	5.03		1 M	3 M	
		Roof Frame PartD_2	2930	10.03		2 M		
04	Third floorH_3	Mezzanine PartA_3	1473	5.04	$H_3 \approx 30$ (Conformity 99.43%)	1 M	3 M	6 M
		Column and Lintel PartB_3	2891	9.90		2 M		
		Bracket PartC_3	1457	4.99		1 M	3 M	
		Roof Frame PartD_3	2889	9.89		2 M		
05	Fourth floorH_4	Mezzanine PartA_4	1489	5.10	$H_4 \approx 28$ (Conformity 99.87%)	1 M	3 M	5.6 M
		Column and Lintel PartB_4	2876	9.85		2 M		
		Bracket PartC_4	1028	3.52		0.7 M	2.6 M	
		Roof Frame PartD_4	2772	9.49		1.9 M		
06	Fifth floor H_5	Mezzanine PartA_5	1175	4.02	$H_5 \approx 37$ (Conformity 99.92%)	0.8 M	2.7 M	7.4 M
		Column and Lintel PartB_5	2763	9.46		1.9 M		
		Bracket PartC_5	892	3.05		0.6 M	4.7 M	
		Roof Frame PartD_5	5965	20.43		4.1 M		
07	Top of the pagoda H_6	Roof Ridget_1	1170	4.01	$H_6 \approx 10.5$ (Conformity 99.54%)	0.8 M	2.1 M	2 M
		Pinnacle Baset_2	1910	6.54		1.3 M		
		Pagoda Pinnaclet_3	9520	$t_3 \approx 32.5$ (Conformity 99.68%)		6.5 M		9 M
08	Others	Pagoda Baset_0	3635	$h_0 \approx 12.5$ (Conformity 99.59%)		2.5 M		
09	Total Height of the Pagoda H		65,770	$H \approx 225$ (Conformity 99.89%)		45 M		

Segmented Height: The first floor column height, B_1 (30 feet), is twice that of the bottom porch column, height B_0 (15 feet), and three times that of the second to fourth floor column heights, $B_{2–4}$ (about 10 feet) and $B_5 \approx 9.5$ feet; the mezzanine part heights of the second to fourth floors, $A_{2–4}$, and the bracket part heights of the first to third floors, $C_{1–3}$, are both approximately 5 feet, $A_5 \approx 4$ feet, $C_4 \approx 3.5$ feet, and $C_5 \approx 3$ feet; the roof frame part heights of the first to third floors are $D_{1–3} \approx 10$ feet, $D_4 = 9.5$ feet, and $D_5 = 20.5$ feet.

Storied Height: The first floor is $H_1 \approx 45$ feet and $H_1{:}B_1{:}B_0 \approx 3{:}2{:}1$; the second and third floors are $H_2 \approx H_3 \approx 30$ feet and $H_2{:}\ B_2 \approx H_3{:}\ B_3 \approx 3{:}1$; the fourth is floor $H_4 \approx 28$ feet; the fifth floor (to the upper edge of the ridge purlin) is $H_5 \approx 37$ feet; the height from the fourth floor to the fifth story eave is $H_4 + H_5 - D_5 \approx 44.5$ feet, the base height is $t_0 \approx 33$ feet, and the pinnacle height is $t_3 \approx 32.5$ feet.

There are notable changes in the vertical segmented and storied dimensions on the fourth and fifth floors. What could be the original design intent behind this? One conjecture is that, as the building is a multi-story pavilion-style pagoda, ancient craftsmen considered the second and third floors the standard stories for vertical storying design, aiming to establish a unified formal order and visual rhythm by repeating the segmented dimensions of the second and third floors. Meanwhile, expanding the first floor column height, B_1, by three times increased spatial height for constructing the bottom porch, with the aim to create a sense of stability at the base of the pagoda, and moderately reducing the bracket part, C_4, and roof frame part, D_4, as well as the segmented heights of the fifth floor excluding the roof frame part (the fifth-floor roof frame part, as a typical octagonal pyramidal roof, is markedly different from the roof structures of the first to fourth floors; its significant volume and complex beam structure make its design and construction distinctly particular), in response to the artistic modeling needs of gradually reducing planar dimensions with each story.

4.2. Modular System and Proportional Rules

In the vertical scaling, the repetitive appearance of a 5-foot measure and its integer multiples cannot be simply regarded as a coincidental pattern, but more like a primitive, intentionally modular design pattern. When converting vertical scales using 5 feet and 15 feet (Table 2), it was discovered that numerous integral ratios were generated, which were organically associated with various parts of the wooden pagoda. For instance, 5 feet corresponds to the repetitive height of the bracket part, $C_{1–3}$, or the mezzanine part, $A_{2–4}$; 10 feet corresponds to the repetitive height of the exposed story columns, $B_{2–4}$, or the roof frame part, $D_{1–3}$; 15 feet corresponds to the height of the bottom porch columns, and the mezzanine part combined with the column and lintel part, or the bracket part combined with the roof frame part; 30 feet represents both the height of the first-floor columns and the height of the second and third floors; 45 feet corresponds to the height of the first floor, etc. From this, it can be understood that the entire pagoda's design is based on a basic modulus of M = 5 feet (i.e., half a *Zhang*), with an expanded module of 3 M = 15 feet. Simultaneously, M is also four times the height of the bracket's *zu cai*, 1.25 feet, indicating the existence of a minimum modulus of m = 1.25 feet, which controls the standardized design of the wooden structural components' bracket. Other scholars have also found a similar phenomenon of modalization using the 1962 surveying data of Chen Mingda [43]. It can be envisaged that, for ancient craftsmen, facing the design and construction tasks of such a large-scale wooden pagoda, whether for the pursuit of mathematical proportional beauty or for the enhancement of standardized construction efficiency, compared with solely using the construction ruler as a unit of measure, the auxiliary use of a modular system like this is obviously more convenient and feasible. This viewpoint is confirmed through the verification analysis of the modular grid based on the constructed drawings.

(1) In the detailed diagrams of the brackets on each exposed story (Figure 5), auxiliary lines spaced at the height of the *zu cai* m coincide perfectly with the vertical stratification lines of the brackets, and closely align with the boundaries of each bracket arm or the edges of the bearing blocks in the horizontal direction. The minimum modulus m controls the overall height and the horizontal projection distance of the brackets to a certain extent, also reflecting the high standardization of bracket design and processing at that time.

(2) In the general layout of the temple pagoda, auxiliary lines spaced at 3 M intervals overlap with parts of the boundaries of the entire temple and historical buildings (Figure 6). The distance from the center of the wooden pagoda to the north and south boundaries of the temple is approximately 17 × 3 M, and 40 × 3 M from the entrance archway; the southern platform of the pagoda base is 10 × 3 M from the temple gate and the northern platform of the pagoda base is 2 × 3 M from the rear courtyard of the pagoda, which has dimensions of 14 × 3 M and 9 × 3 M. This shows that the expanded modulus of 3 M is also used in controlling the external environment scale of the temple pagoda architectural complex and has been followed in successive heritage renewals.

(3) In the frontal orthographic images, auxiliary lines spaced at M and 3 M intervals mostly match the vertical segmentation and stratification boundaries, especially the 3 M auxiliary lines, which are closely aligned with the tops of columns at each exposed story, the upper edges of Pu Pai Fang of the bottom porch and mezzanine story, the eave of the fifth story, the upper edge of the brick pinnacle base, and the pagoda pinnacle, dividing the entire pagoda into 15 parts (Figure 7).

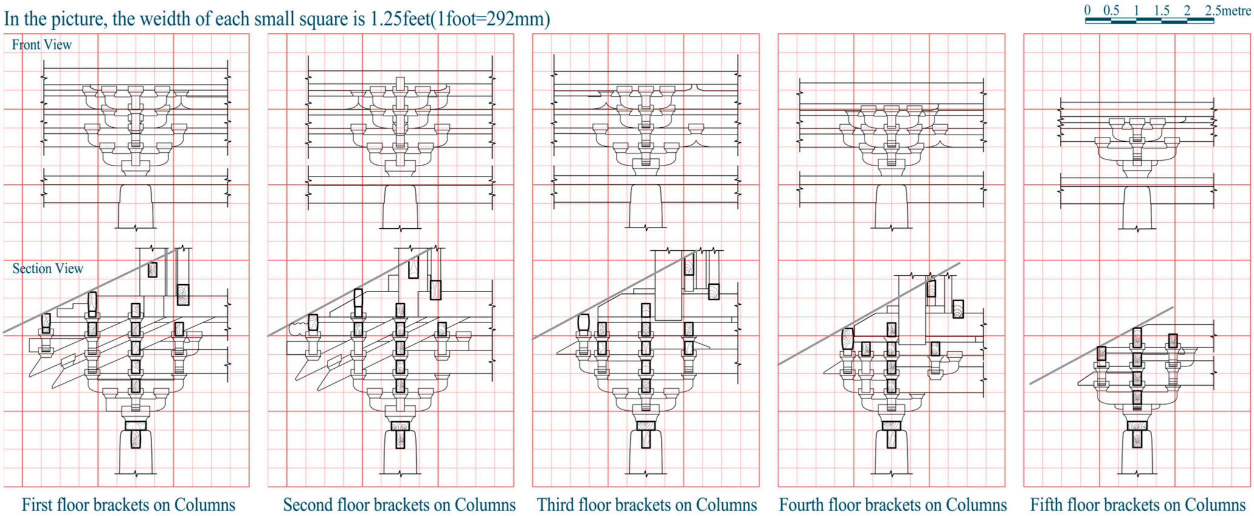


Figure 5. Modular grid analysis of the bracket set on columns of each exposed story of Yingxian Wooden Pagoda.

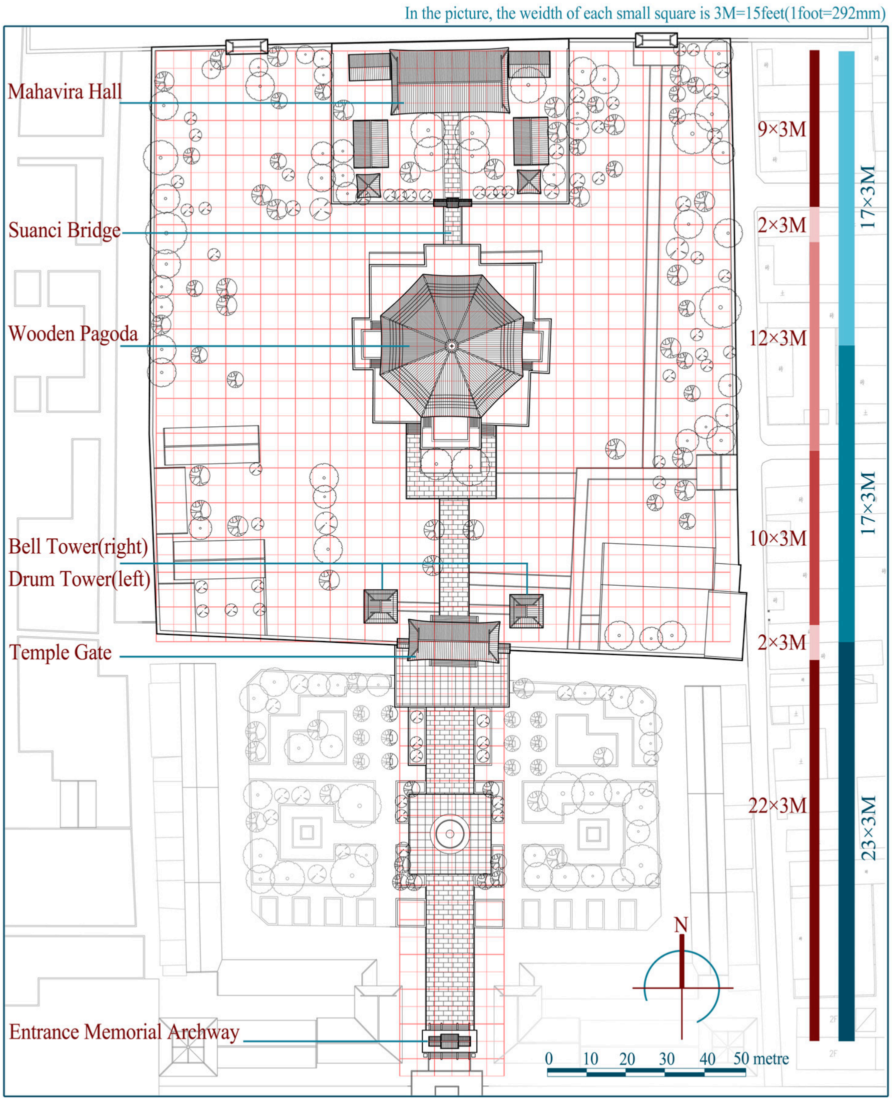


Figure 6. Modular grid analysis of architectural group layout in the general plane of the temple pagoda.

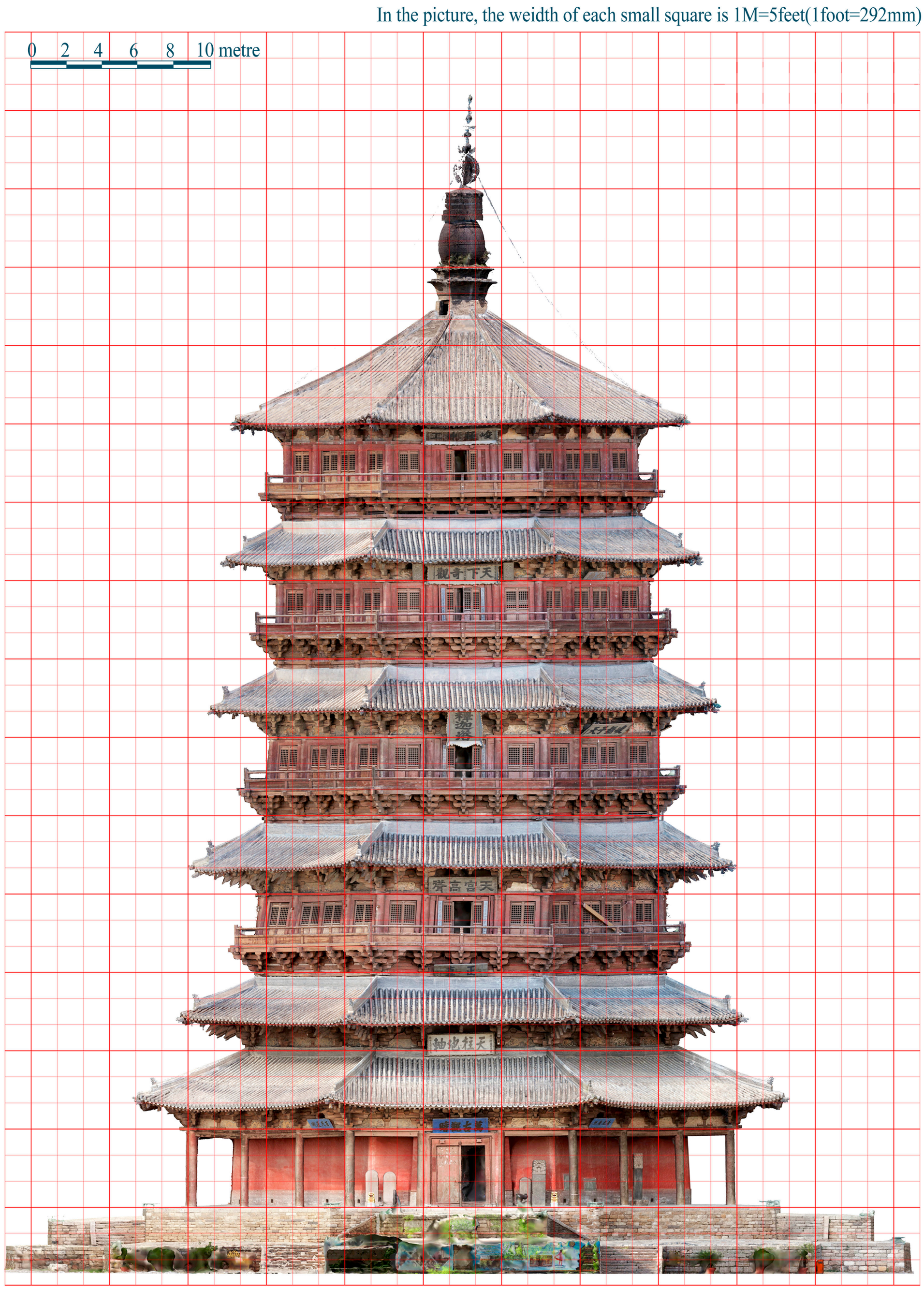


Figure 7. Modular grid analysis of the orthographic projection of Yingxian Wooden Pagoda's facade.

Before the analysis of architectural proportions, it is necessary to clarify the main body heights, $H - t_0 - t_3$ and $H - t_0 - t_1 - t_2 - t_3$ (setting $H - t_0 - t_3$ and $H - t_0 - t_1 - t_2 - t_3$ as the measures of pagoda height serves two main purposes: Firstly, the measured height of the base has been significantly influenced by natural sedimentation and human interventions over nearly a millennium, while the pagoda spire is heavily inclined, inevitably leading to considerable errors in comparison with the original appearance. Secondly, in ancient Chinese wooden architecture characterized by prefabrication and assembly, the overall wooden framework design forms a batch of its own, and is the origin of design and construction). The former is from the base platform surface to the upper edge of the brick spire base, and the latter is from the base platform surface to the height of the fifth story's ridge, which is also the height of the main wooden structure of the entire pagoda. Through the ratio analysis of the facade and sectional grid diagrams, the main proportions within the formal beauty of the wooden pagoda can be obtained:

(4) The total height H and the main body height, $H - t_0 - t_3$, of the entire pagoda establish two sets of numerical ratio relationships with the high-frequency local scales 3 M ($A_{2–4} + B_{2–4}$, $C_{1–3} + D_{1–3}$), 6 M (B_1, H_2, H_3), and 9 M (H_1), respectively, with 5:1, 7.5:1, 15:1, and 4:1, 6:1, 12:1, indicating the importance of the first-floor height (9 M) and column height (6 M) in the vertical scale design of the entire pagoda.

(5) Dividing the height, $H - t_0 - t_3$, into three parts (the first-floor height, H_1; the second and third floor height, $H_2 + H_3$; and the height from the fourth floor to the upper edge of the spire base, $H_4 + H_5 + t_1 + t_2$), the ratio of these three parts is approximately 3:4:5 . If the high-frequency local scales 3 M and 6 M are combined, a continuous proportion of 1:2:3:4:5 in the facade form can be obtained (Figure 8).

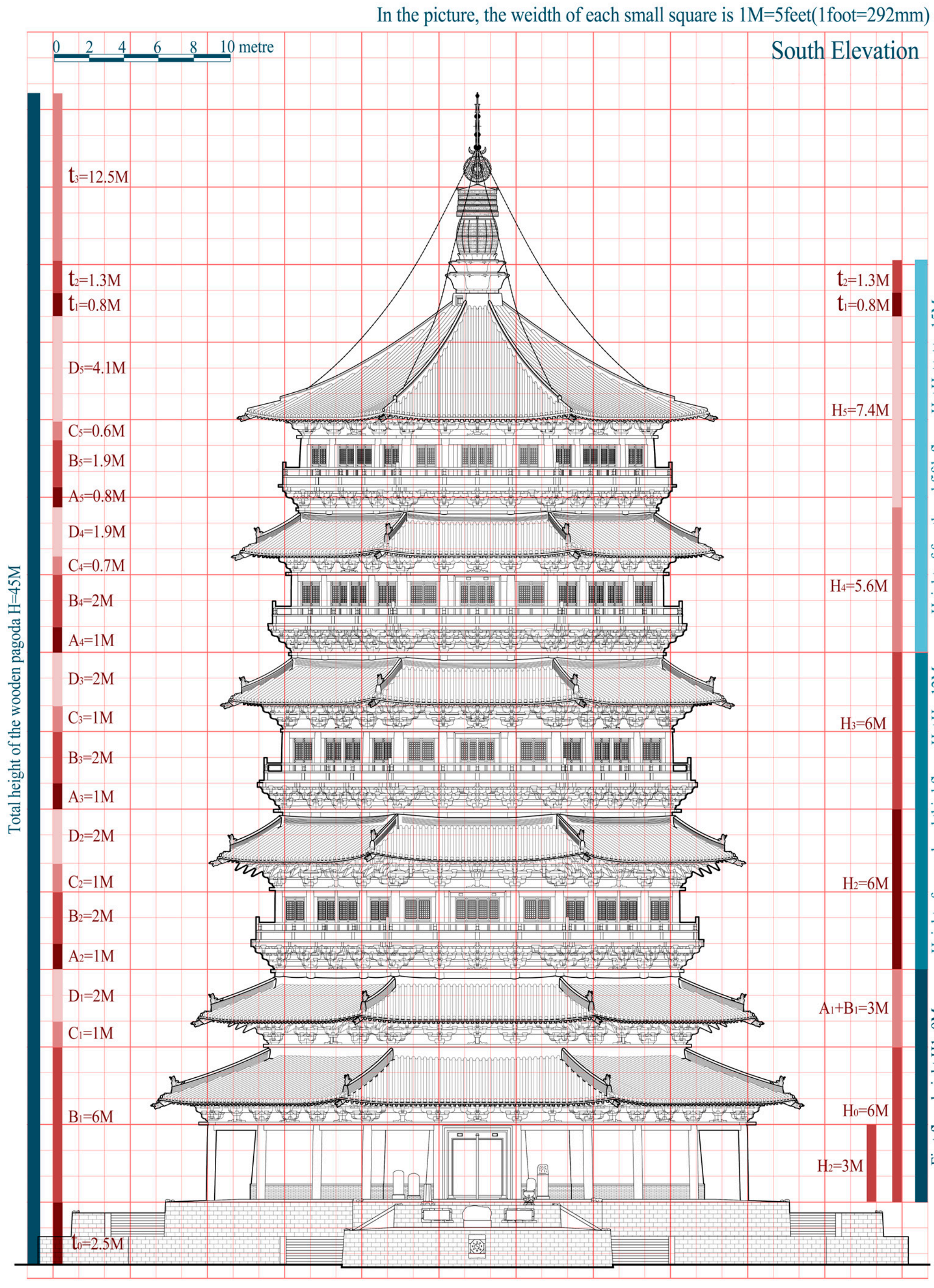


Figure 8. Proportional analysis of the external form of Yingxian Wooden Pagoda.

(6) Referencing the sectional diagram, dividing the total wooden structure height, $H - t_0 - t_1 - t_2 - t_3$, into three parts based on the indoor floors of the second and fourth stories (the indoor height of the first floor, $H_1 + A_2$ = 10 M; the indoor height of the second and third floors, $H_2 + H_3 - A_2 + A_4$ = 12 M; and the height from the fourth story to the upper edge of the ridge purlin, $H_4 + H_5 - A_4$ = 12 M), the ratio of these three is approximately 5:6:6 (Figure 9).

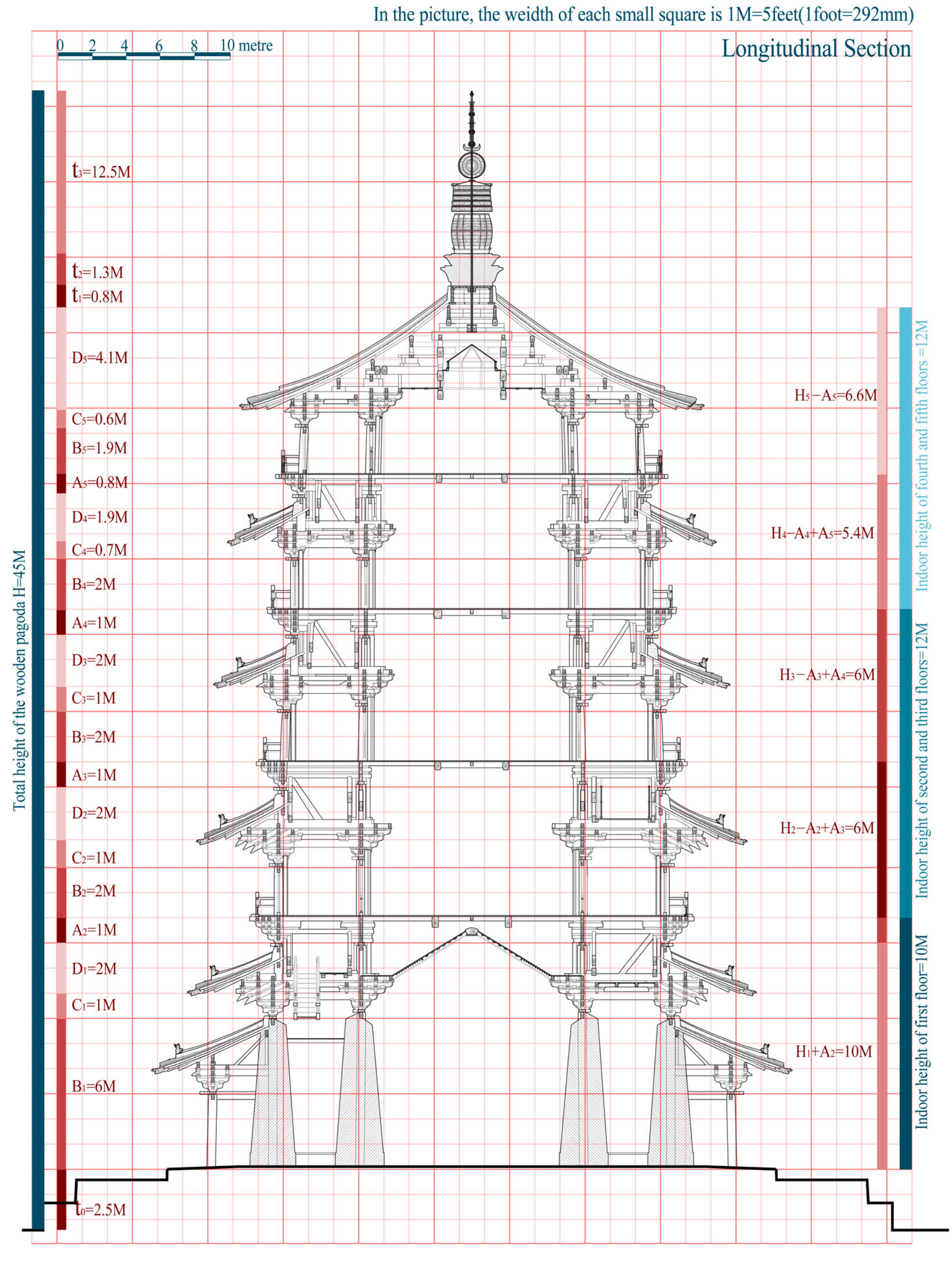


Figure 9. Proportional analysis of the interior space of Yingxian Wooden Pagoda.

The proportions mentioned confirm that ancient architects, in adherence to the pagoda's form of gradual tapering with each ascending story, considered the fourth and fifth floors as a unified entity. Although the heights of the fourth and fifth floors did not form an exact multiple of the basic modulus M, the comprehensive design of these two levels ensured that their total height was equal to that of the second and third floors. Furthermore, this approach resulted in the creation of two sets of minimalist proportions, both externally and internally, establishing dual orderliness in form and space. This method underscores the architectural mastery in balancing visual aesthetics and structural functionality, demonstrating a sophisticated understanding of modular design principles. By integrating these proportions, architects were able to achieve a harmonious blend of the pagoda's external silhouette and its internal spatial organization, thereby crafting a legacy of architectural elegance that resonates with both the past and the present.

4.3. Control Principles of Squareness and Roundness Composition

Although modular grids can control the scale and proportion of architectural forms, they comprise a control blind spot when facing the complex forms of ancient Chinese multi-story buildings, especially the critical nodes of curved roof shapes. Combining squareness and roundness compositions not only compensates for this shortcoming but also establishes an organic connection with the plan and elevation. Through repeated squareness and roundness drawing experiments on the plan and elevation of the Yingxian Wooden Pagoda, the applicability of the squareness and roundness composition model in the Yingxian Wooden Pagoda was confirmed. It was found that two sets of series of center collinear control circles effectively control the overall form of the pagoda from both the overall appearance and the single facade perspective.

4.3.1. Analysis Results of the Overall Appearance

In the analysis of the overall appearance outline, it was found that squares, the sides of which are the octagonal planar width of each exposed story, circumscribe circles. These series of circular control lines coincide with key morphological nodes on the elevation with a high degree of conformity and exhibit strong control (Figure 10). Specifically, the circular control lines O1 to O5, centered on the geometric center of the plan, have diameters $\sqrt{2}$ times that of the planar width. Since the planar width of a regular octagon is $\sqrt{2}$+ 1 times its side length, the diameter of each control circle is also $2 + \sqrt{2}$ times the corresponding story's side length. The circular control lines are arranged from bottom to top along the central axis of the elevation, forming a sequence of center collinear control circles. These not only define the eaves of each story but also intersect with each other to form common chords, aligning with the location and total width of the eaves and galleries of each story. For example, control circle O3 of the middle story controls the eaves of its own and the next lower story, the galleries of the upper and lower story, and the ridge of the first story. Additionally, the apex of O5 on the fifth story is close to the upper edge of the brick pinnacle base.

4.3.2. Analysis Results of the Single Facade

Given the regular octagonal plan of the Yingxian Wooden Pagoda, the facade of each side maintains a high degree of uniformity in form and scale, each with similarity and independence, which are fundamental modules affecting the overall facade design of the wooden pagoda. This leads to the following question: does the façade of each side also follow the squareness and roundness composition model? Experiments confirmed that the answer is positive, and the control law is even simpler (Figure 11). In summary, for each exposed story (including the bottom porch), squares with the side length of the octagonal plan as their side length circumscribe circles O′0, O′1, O′2, O′3, O′4, and O′5, with each circle's diameter being $\sqrt{2}$ times the corresponding story's side length. Similarly, they are arranged in sequence along the facade axis to form a series of center collinear control circles, controlling the ridges and eaves of each story. Moreover, the center of each story's control circle is near the bottom of the lintel of each exposed story, with adjacent circles intersecting to form common chords, aligning with the width and position of the ridges of each story.

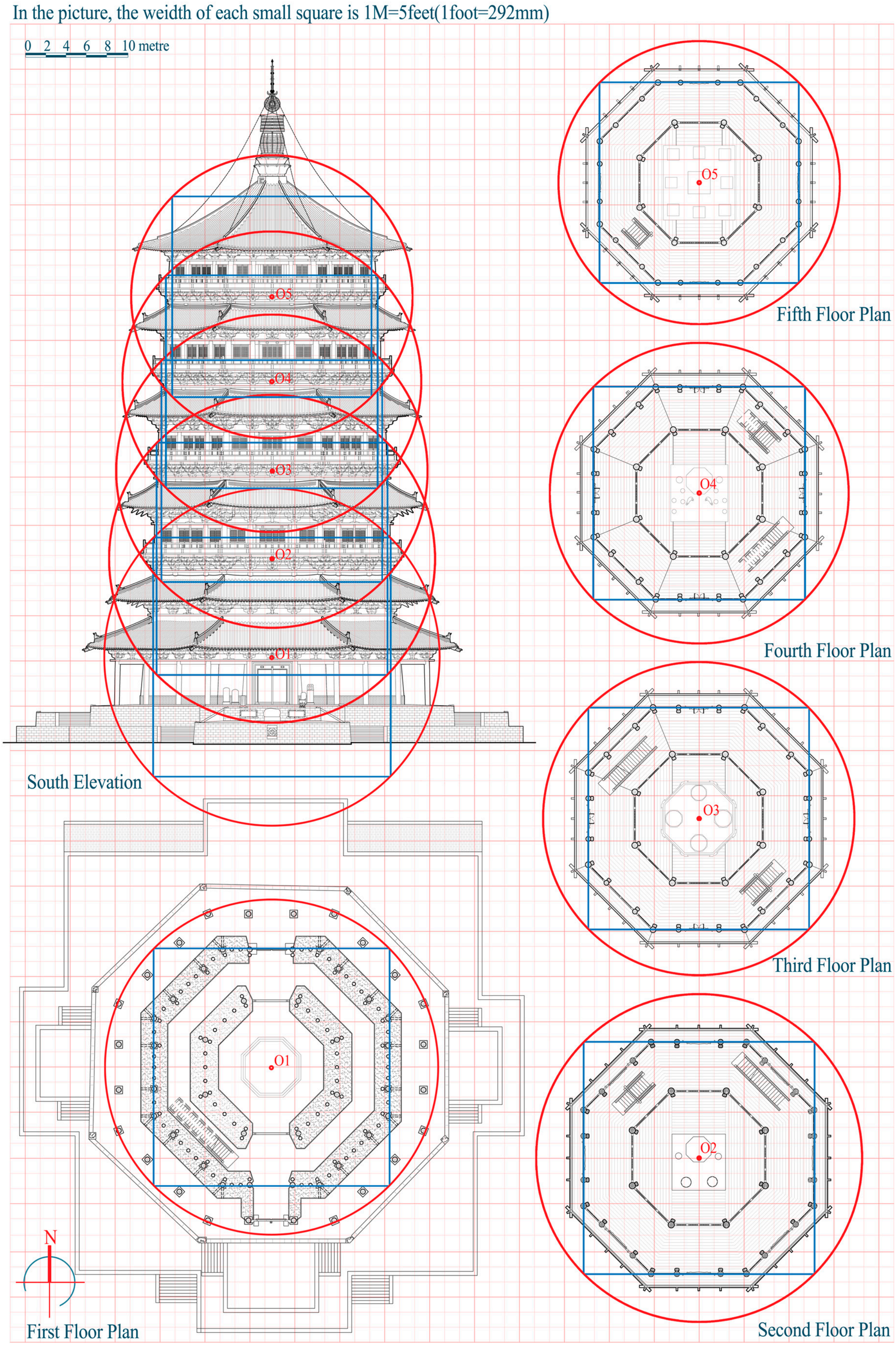


Figure 10. Squareness and roundness composition analysis of Yingxian Wooden Pagoda—overall appearance form.

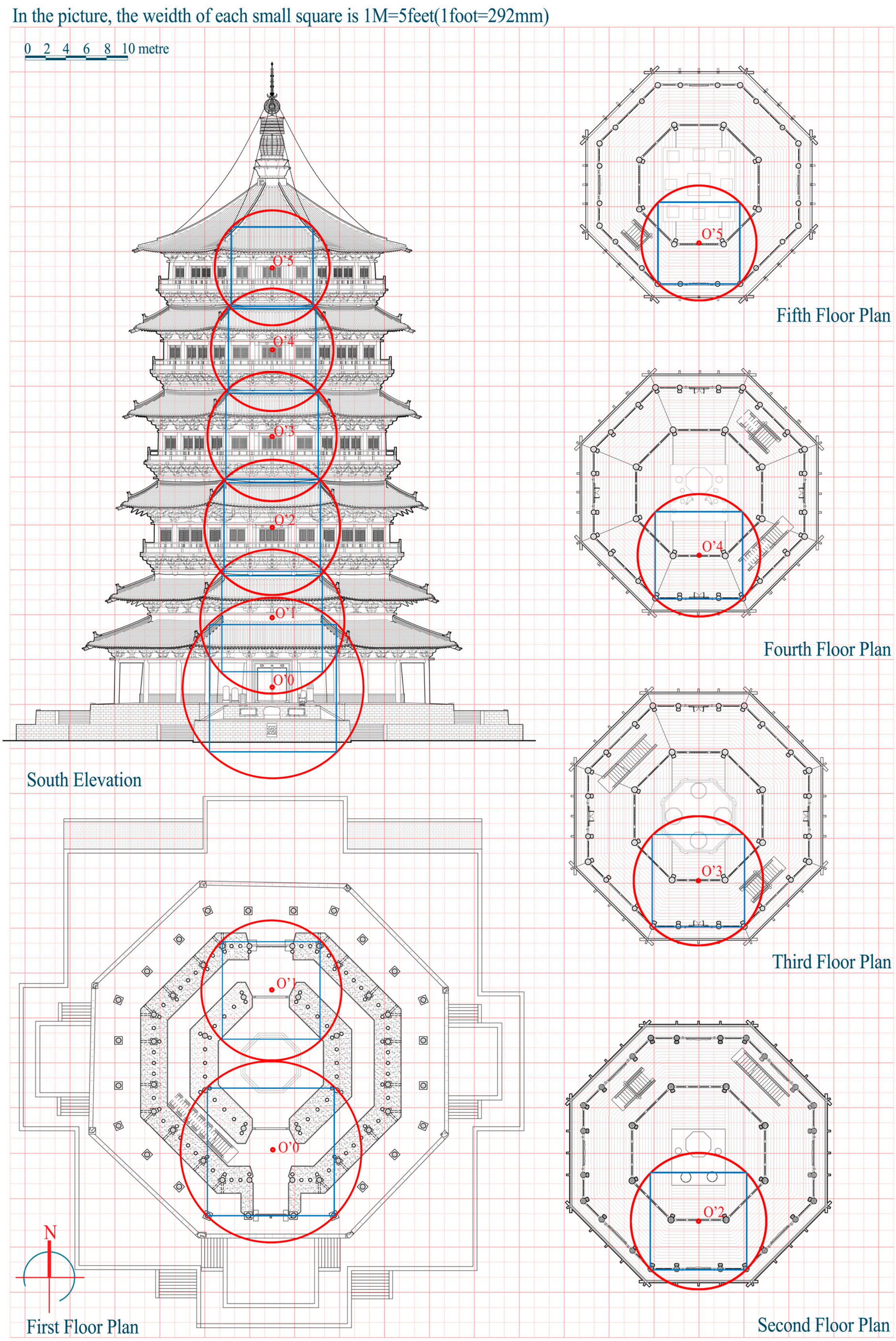


Figure 11. Squareness and roundness composition analysis of Yingxian Wooden Pagoda—single facade form.

4.3.3. Summary of Squareness and Roundness Composition Principles

Integrating the above analysis of squareness and roundness drawings, two sets of concentric circle sequences generated based on different planar scales exhibit continuous control patterns across different stories, summarized in the following aspects:

(1) Circles circumscribing squares based on the planar width of the nth story control the eaves of the nth and $n-1$st stories, the galleries of the $n+1$st and $n-1$st stories, and the ridge of the $n-2$nd story. The control principle of the eaves is similar to that of some Tang Dynasty multi-storied brick pagodas.

(2) Circles circumscribing squares based on the side length of the nth story control each facade's eaves of the nth story, and the ridges of the nth and $n-1$st stories. This complements the control content of the previous set of center collinear control circles on the opposite facade outlines.

(3) The squareness and roundness composition model controlling the appearance of the Yingxian Wooden Pagoda still fits the design logic of coordinating plan and elevation. By alternately controlling the width of the upper and lower eaves that are parallel to each other, control circles based on the planar width set the eave spacing and achieve control over the vertical segmented height of the facade.

(4) Building on previous foundations, the intersecting control circles have created numerous parallel common chords, enhancing repetitive alternating control over the horizontal distances of features such as galleries and ridges. This expands the constraints on the vertical segmented heights, making the squareness and roundness composition of the entire pagoda's facade more stable.

5. Conclusions

In this paper, we focused on the design forms of ancient Chinese multi-story buildings and highlighted a squareness and roundness composition model based on scale and proportion. Through the verification analysis of the Yingxian Wooden Pagoda with this model, we not only reacquainted ourselves with the mathematical and geometric laws followed by the entire pagoda's design but also positively reflected on the basic approaches and principles of effective control over the forms of multi-story buildings using this model. The research conclusions are as follows:

(1) Utilizing 3D laser scanning and UAV aerial photogrammetry, we obtained data on the Yingxian Wooden Pagoda and produced a complete set of digital surveying and mapping archives. By calculating the length of the construction ruler (292 mm), we redefined the main scale constitution and its variation rules in the architectural plan and elevation, and analyzed the underlying design intentions.

(2) The design of the entire pagoda follows a modular system consisting of the smallest modulus, $m = 1.25$ feet; the basic modulus, $M = 5$ feet; and the expanded modulus $3\ M = 15$ feet. This coordinated the layout of the architectural group, the design of individual buildings, and the standardized processing of brackets efficiently, aiding in the design collaboration between architectural structure and aesthetic form.

(3) Under the governance of the modular system, the vertical scale design of the entire pagoda established numerous numerical ratio associations between total height, storied height, and segmented height. We also set two sets of simple proportional rules, 3:4:5 and 5:6:6, in the external form and internal space, imbuing the entire pagoda with a rigorous mathematical order and design logic from the inside out.

(4) The modular grid map, based on different design dimensions, which is similar to that of a three-dimensional design reference coordinate system, can serve as an analytical tool. Compared with mere mathematical calculations, it captures the mathematical rules followed by the above pagoda design more comprehensively. It also aligns with the Chinese traditional architectural modular design concept, offering a method with which to coordinate architectural scale design.

(5) Integrating squareness and roundness compositions can control the overall form of the pagoda more effectively, revealing the structural hierarchy and aesthetic rhythm of

the entire pagoda's facade organization. This can establish an overall approach to using squareness and roundness drawings to connect plan width with eave width and other horizontal variables, thereby defining the vertical storied and segmented height of the facade, and establishing a co-control mechanism for plan and elevation design.

In summary, the squareness and roundness composition model based on scale and proportion integrates the philosophical thoughts that square–circle symbolizes heaven–earth, and mathematical relationships. It can fully mobilize the aesthetic dynamic properties of mathematics and geometry, possessing both theoretical analysis and practical design value. From the perspective of architectural analysis, starting with scale and proportion, the comprehensive use of modular grids and squareness and roundness compositions can reveal the mathematical rules of classic architectural design and intuitively feedback the geometric composition rules and cultural connotations of architectural forms. For architects, the modular grid and squareness and roundness composition become effective tools for design decision-making, balancing the scale and proportion of multi-story buildings and strengthening the synergistic design considerations for plans and elevations, contributing to creating harmonious order and place significance between the multi-story building forms and spatial environment.

However, the internal mechanism of the squareness and roundness composition model in controlling the form variables of multi-story buildings remains to be elucidated through quantitative mathematical analysis. Future research will select historical precedents from different times and regions for typological comparison and validation, refine mathematical models capable of controlling the form variables of different styles of buildings, and establish an inclusive theoretical model to adapt to the diversified demands of different design scenarios.

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