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Admissible Control for Non-Linear Singular Systems Subject to Time-Varying Delay and Actuator Saturation: An Interval Type-2 Fuzzy Approach

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Abstract: Applied in many fields, nonlinear systems involving delay and algebraic equations are referred to as singular systems. These systems remain challenging due to saturation constraints that affect actuators and cause harm to their operation. Furthermore, the complexity of the problem will increase when uncertainty also simultaneously affects the system under consideration. To address this issue, this paper investigated a feasible control strategy for nonlinear singular systems with time-varying delay that are subject to uncertainty and actuator saturation. The IT-2 fuzzy model was adopted to describe the dynamic of the non-linear delayed systems using lower and upper membership functions to deal with the uncertainty. Moreover, the polyhedron model was applied to characterize the saturation function. The goal of the control approach was to design a relevant IT2 fuzzy state feedback controller with mismatched membership functions so that the closed-loop system is admissible. On the basis of an appropriate Lyapunov–Krasovskii functional, sufficient delay-dependent conditions were established and an optimization problem was formulated in terms of linear matrix inequality constraints to optimize the attraction domain. Simulation examples are provided to verify the effectiveness of the proposed method.

Keywords: singular system; IT-2 fuzzy model; time varying delay; actuator saturation



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1. Introduction

This section includes the literature review, notations, and acronyms used in the document, as well as an outline of the publication and its goals.

1.1. Literature Review

Singular systems that are described by a couple of algebraic and differential equations are characterized by their different modes, namely finite dynamic modes, infinite nondynamic modes, and infinite dynamic modes, respectively. The infinite dynamic modes have the feature to destroy the stability and performance of the system. Thus, the admissibility, which includes stability, regularity, and non-impulsiveness/causality, should be verified when dealing with this class of systems. As a consequence, the investigation of singular systems is both theoretically and practically important [1,2]. It is worth noting that time delays are common in many physical plants, and they can have a substantial negative impact on the performance and even the stability of practical systems [3–7]. Singular models and time-delay phenomena are general enough to enable some fundamental results

from the theory of state-space systems to be extended to this class of systems (see, for instance, [8–12]). Alternatively, it is widely known that all physical plants are affected by the actuator and/or sensor saturation. The saturation constraint has a nonlinear impact on the performance of the controlled system unless the controller is built to account for it. In addition, ignorance of the consequences of actuator saturation may destroy system stability. Control with input/output saturation has been subjected to a great deal of research and development over the last decade [13–17].

In general, the research on nonlinear systems [18] is an extremely hard issue due to their inherent complexity. Due to its rigorous mathematical structure, the T-S fuzzy model has recently been applied to handle nonlinear complex systems, since this model is known for its powerful approximation of smoothly nonlinear systems [19–21]. In the research on nonlinear singular systems against input saturation, the T-S fuzzy method has thus far proven useful, including the finite-time output feedback control for T-S fuzzy singular system with actuator saturation [22], the fault-tolerant sliding mode control for T-S fuzzy systems with time-varying delay and actuator saturation [23], the stabilization of the T-S descriptor systems against input saturation [24], and the fuzzy dissipative controller design for non-linear descriptor systems subject to time-delay and actuator saturation [25].

In many cases, uncertainty arises, such as partially unknown parameters, unpredictable disturbances in the system, varying interpretations of linguistic variables of the fuzzy models, etc. It is worth noting, however, that the membership functions of type-1 fuzzy sets are well-known, and the control problem cannot be handled directly for nonlinear plants with parameter uncertainties. As type-1 fuzzy sets have limitations in dealing with non-linear plants subject to parameter uncertainties, the IT-2 fuzzy model was proposed and used in [26,27]. Research on IT-2 fuzzy systems has attracted attention due to the advantages of IT-2 fuzzy sets over type-1 fuzzy sets in dealing with parameter uncertainties, and many control design results have been developed [28–30]. To mention a few, in [31], fuzzy decentralized output feedback control was investigated using the event-triggered approach for interval type-2 fuzzy systems against input saturation. The study in [32] discussed the issue of filter design for IT-2 fuzzy systems with D-stability constraints. The authors in [33] investigated discrete-time interval-valued fuzzy systems with actuator faults to study the reliable non-fragile control problem with H_∞ performance. When the system is subject to stochastic network delays and actuator saturation, a distributed-delay-dependent method was proposed in [34] for stabilizing interval type-2 Takagi–Sugeno fuzzy systems. Under saturated state feedback, robust stabilization was investigated in [35] for uncertain linear systems with discrete and distributed delays.

Very recently, the IT-2 fuzzy approach was extended to non-linear singular systems with uncertainties. To mention a few, admissibilization for IT-2 fuzzy singular systems was studied in [36]. Based on the LMI approach, the dynamic output-feedback control design issue was investigated in [37] for singular interval-valued fuzzy systems.

There is a need to emphasize that the previous papers discussed various fuzzy stabilization problems for IT-2 fuzzy singular systems without simultaneously considering the phenomenon of input saturation. In light of this, we undertook this study. Furthermore, although some related results have recently been published for delayed nonlinear systems characterized by IT-2 fuzzy models [38,39], they cannot be applied to nonlinear singular systems. This paper attempts to overcome the deficiencies of the aforementioned studies by dealing with the stabilization problem for input-saturated IT-2 fuzzy singular systems while simultaneously taking into account that uncertainty and delay may affect the process.

1.2. Objective and Outline

This study has the following main objectives:

- (i) Instead of existing control schemes developed for type-1 fuzzy singular systems with delay and actuator saturation [40], this study explored state feedback controllers based on IT-2 fuzzy rules in order to handle uncertain non-linear singular systems.

- (ii) The delay property and actuator saturation for the IT-2 fuzzy singular system under consideration were simultaneously considered in this study. Moreover, compared with the results suggested in [36,41], a more realistic problem was investigated in this paper that cannot be solved by the methods in the previous references.
- (iii) A new Lyapunov–Krasovskii functional candidate was constructed, and the delay-range-dependent approach was adopted to derive an admissibilization criterion via LMI formulation. Furthermore, the domain of attraction of the origin can be estimated for the underlying system.

After outlining the introduction and the objectives of our study, the paper is organized as follows: Section 2 presents the model and assumptions, as well as a description of the problem under study. In Section 3, we present and discuss the main findings of the paper. Specifically, this section is dedicated to developing a new delay-dependent admissibility criterion using the IT-2 fuzzy model from (3) and selecting a suitable Lyapunov–Krasovskii functional. To further ensure the usage of this scheme, we developed an LMI criterion to establish that the closed-loop system is admissible and to optimize the attraction domain. To demonstrate the potential applications of the proposed scheme and validate its effectiveness, numerical simulations on mass–spring–damper and inverted pendulum systems are presented in Section 4. Lastly, we conclude with some conclusions regarding the obtained results, as well as some suggestions for future research in Section 5.

1.3. Notations

Table 1 lists the notations and acronyms that should be used in this study.

Table 1. List of notations and acronyms used in the paper.

Symbol	Acronym/Notation
\mathbb{R}	set of the real numbers
$\mathcal{X} \in \mathbb{R}^n$	n -dimensional Euclidean space
$\mathbf{X} \in \mathbb{R}^{n \times m}$	$n \times m$ real matrix
$\mathbf{X} > 0$	real symmetric positive definite matrix \mathbf{X}
$\ \mathbf{X}\ $	norm of the matrix \mathbf{X}
\mathbf{X}^\top	transpose of the matrix \mathbf{X}
$\text{sym}(\mathbf{X})$	$\mathbf{X} + \mathbf{X}^\top$
$\lambda()$	eigenvalue of a matrix
$\ \phi\ _c$	$\max_{\alpha \in [-d_2, 0]} \ \phi(\alpha)\ $
$\ \dot{\phi}\ _c$	$\max_{\alpha \in [-d_2, 0]} \ \dot{\phi}(\alpha)\ $
*	term that is induced by symmetry
r	number of if-then rules
LMI	linear matrix inequalities
IT-2	Interval type-2 fuzzy model
TS	Takagi–Sugeno

2. Preliminaries and Problem Statement

The aim of this section is to introduce some preliminaries that facilitate the understanding of our proposal and state the problem that we are investigating.

2.1. IT-2 TS Fuzzy Model

Consider a class of non-linear singular systems that can be described by the following IT-2 TS fuzzy model:

\mathbf{R}_i : If $\theta_1(x(t))$ is \mathcal{M}_i^1 and If $\theta_2(x(t))$ is $\mathcal{M}_i^2 \cdots$ If $\theta_s(x(t))$ is \mathcal{M}_i^s , Then

$$\begin{cases} E\dot{x}(t) = A_i x(t) + A_{di} x(t - d(t)) + B_i \sigma(u(t)) \\ x(t) = \phi(t), \forall t \in [-d_2, 0], \end{cases} \quad (1)$$

where \mathcal{M}_i^k is an IT-2 fuzzy set of rule i corresponding to the premise variable $\theta_i(\mathbf{x}(t))$, $k = 1, 2, \dots, s$, k is the number of premise variables, and $i \in \mathbb{S} \triangleq \{1, 2, \dots, r\}$ is the number of rules. $\mathbf{x}(t) \in \mathbb{R}^n$ and $\boldsymbol{\sigma}(\mathbf{u}(t)) \in \mathbb{R}^m$ define, respectively, the state and saturated input vectors. Matrices \mathbf{A}_i , \mathbf{A}_{di} and \mathbf{B}_i in model (1) are known with appropriate dimensions. $d(t)$ stands for the time-varying delay, and $\boldsymbol{\phi}(t)$ defines the initial state for all $t \in [-d_2, 0]$.

2.2. Assumptions and Resulting Model

A1 $d(t)$ is a continuous function such that

$$0 < d_1 \leq d(t) \leq d_2, \quad 0 \leq \dot{d}(t) \leq d_r \tag{2}$$

where d_1 represents the lower delay bound, d_2 stands for the upper delay bound, and d_r is the delay variation rate.

A2 Singular matrix \mathbf{E} satisfies $\text{rank}(\mathbf{E}) = q < n$.

A3 $\boldsymbol{\sigma}(\mathbf{u}(t))$ is the saturation that affects the actuator according to the following model:

$$\boldsymbol{\sigma}(\mathbf{u}(t)) = [\sigma(u_1(t)) \quad \sigma(u_2(t)) \quad \dots \quad \sigma(u_m(t))]^\top$$

$$\sigma(u_l(t)) = \begin{cases} \bar{u}_l & \text{if } u_l(t) > \bar{u}_l \\ u_l(t) & \text{if } -\bar{u}_l \leq u_l(t) \leq \bar{u}_l \\ -\bar{u}_l & \text{if } u_l(t) < -\bar{u}_l \end{cases}$$

where $u_l(t)$ is the l th value of $\mathbf{u}(t)$, and \bar{u}_l is the l th saturated level, $l = 1, \dots, m$.

Based on the IT-2 fuzzy approach, the following interval defines the firing strength of the i th rule:

$$\mathbb{M}_i = \left[\prod_{k=1}^s \underline{\omega}_{\mathcal{M}_i^k(\theta(\mathbf{x}(t)))} \quad \prod_{k=1}^s \bar{\omega}_{\mathcal{M}_i^k(\theta(\mathbf{x}(t)))} \right] = \left[\underline{\mu}_i(\mathbf{x}(t)) \quad \bar{\mu}_i(\mathbf{x}(t)) \right]$$

where $\underline{\mu}_i(\mathbf{x}(t)) \geq 0$ and $\bar{\mu}_i(\mathbf{x}(t)) \geq 0$ are, respectively, the the lower and upper membership functions, and $\underline{\omega}_{\mathcal{M}_i^k(\theta(\mathbf{x}(t)))} \geq 0$, and $\bar{\omega}_{\mathcal{M}_i^k(\theta(\mathbf{x}(t)))} \geq 0$ stand, respectively, for the lower and upper grades of membership. Therefore, the non-linear singular system can be described as

$$\begin{cases} \mathbf{E}\dot{\mathbf{x}}(t) = \sum_{i=1}^r \mu_i(\mathbf{x}(t)) (\mathbf{A}_i \mathbf{x}(t) + \mathbf{A}_{di} \mathbf{x}(t - d(t)) + \mathbf{B}_i \boldsymbol{\sigma}(\mathbf{u}(t))) \\ \mathbf{x}(t) = \boldsymbol{\phi}(t), \quad t \in [-d_2, 0] \end{cases} \tag{3}$$

$\mu_i(\mathbf{x}(t))$ denotes the grade of the membership of the i th local system defined as

$$\mu_i(\mathbf{x}(t)) = \underline{\alpha}_i(\mathbf{x}(t)) \underline{\mu}_i(\mathbf{x}(t)) + \bar{\alpha}_i(\mathbf{x}(t)) \bar{\mu}_i(\mathbf{x}(t)), \quad \sum_{i=1}^r \mu_i(\mathbf{x}(t)) = 1$$

where $\underline{\alpha}_i(\mathbf{x}(t))$ and $\bar{\alpha}_i(\mathbf{x}(t))$ are two weighting coefficient functions satisfying

$$0 \leq \underline{\alpha}_i(\mathbf{x}(t)), \bar{\alpha}_i(\mathbf{x}(t)) \leq 1, \quad \underline{\alpha}_i(\mathbf{x}(t)) + \bar{\alpha}_i(\mathbf{x}(t)) = 1 \tag{4}$$

Note that, by introducing weighting coefficient functions, we can represent any time-variant or time-invariant unmeasured parameters of the general non-linear system. Moreover, these functions are not necessarily known but exist and satisfy (4).

As a matter of convenience, $\mu_i(\mathbf{x}(t))$ will be referred to as μ_i in the sequel.

2.3. IT-2 Fuzzy State Feedback Controller Design

Here, the subsequent IT-2 fuzzy state-feedback controller structure was adopted to admissibilize the system under consideration:

Controller Rule j : If $\vartheta_1(x(t))$ is \mathcal{N}_j^1 and If $\vartheta_2(x(t))$ is $\mathcal{N}_j^2 \cdots$ If $\vartheta_s(x(t))$ is \mathcal{N}_j^s , Then

$$u(t) = K_j x(t)$$

where K_j is the gain matrix to be designed. Similarly, $\vartheta(x(t)) = [\vartheta_1(x(t)), \vartheta_2(x(t)), \dots, \vartheta_s(x(t))]$ defines the premise vector, and $\mathcal{N}_j^{k_c}$, ($k_c = 1, 2, \dots, s$) represents the type-2 fuzzy sets of the j -th controller rule.

The following is the firing interval for the j th rule:

$$\mathbb{N}_j = [\underline{v}_j(x(t)) \quad \bar{v}_j(x(t))], j \in \mathbb{S}$$

where

$$\underline{v}_j(x(t)) = \prod_{k_c=1}^s \underline{\omega}_{\mathcal{N}_j^{k_c}(\vartheta(x(t)))} \geq 0, \quad \bar{v}_j(x(t)) = \prod_{k_c=1}^s \bar{\omega}_{\mathcal{N}_j^{k_c}(\vartheta(x(t)))} \geq 0,$$

$\underline{v}_j(x(t))$ and $\bar{v}_j(x(t))$ define, respectively, the lower and upper membership functions. $\underline{\omega}_{\mathcal{N}_j^{k_c}(\vartheta(x(t)))} \geq 0$ and $\bar{\omega}_{\mathcal{N}_j^{k_c}(\vartheta(x(t)))} \geq 0$ stand for the lower and upper grades of the membership of $\vartheta(x(t))$ in $\mathcal{N}_j^{k_c}$, respectively. The global fuzzy model can be inferred as follows:

$$u(t) = \sum_{j=1}^r v_j(x(t)) (K_j x(t)) \tag{5}$$

$$v_j(x(t)) = \frac{\underline{\beta}_j(x(t)) \underline{v}_j(x(t)) + \bar{\beta}_j(x(t)) \bar{v}_j(x(t))}{\sum_{l=1}^r (\underline{\beta}_l(x(t)) \underline{v}_l(x(t)) + \bar{\beta}_l(x(t)) \bar{v}_l(x(t)))}, v_j(x(t)) \geq 0, \sum_{j=1}^r v_j(x(t)) = 1 \tag{6}$$

Remark 1. Over the past several decades, type-1 T-S fuzzy systems have been extensively investigated. It is interesting to note that all of these studies are founded on the PDC approach, in which the controller and the plant both have the same membership functions. Nevertheless, this assumption is not always valid since membership functions may be uncertain in practice. As proposed in [36,41], we aim to address this issue using the interval-valued type-2 fuzzy controller (5).

From (5), the fuzzy-model-based actuator saturation control input is expressed as:

$$\sigma(u(t)) = \sigma\left(\sum_{j=1}^r v_j(x(t)) K_j x(t)\right) \tag{7}$$

To deal with the saturation function, the following lemmas should be provided for further development.

Definition 1.

- For a positive scalar ρ , an ellipsoid set is defined as

$$\varepsilon(E^\top P E, \rho) = \{x \in \mathbf{R}^n; x^\top E^\top P E x \leq \rho\},$$

where $E^\top P E$ defines a positive definite matrix.

- For a given matrix $\mathbf{H} \in \mathbb{R}^{m \times n}$, a polyhedral set is given by

$$\mathcal{L}(\mathbf{H}, \bar{u}) = \{x \in \mathbb{R}^n; |H_l x| \leq \bar{u}_l, l = 1, \dots, m\},$$

where H_l represents the l th row of \mathbf{H} , and \bar{u}_l is a positive given scalar.

Lemma 1 ([40]). For given matrices $\mathbf{K}, \mathbf{H} \in \mathbb{R}^{m \times n}$, if $x \in \mathcal{L}(\mathbf{H}, \bar{u})$, then

$$\sigma(\mathbf{K}x, \bar{u}) = \text{co}\{\mathbf{M}_s \mathbf{K} + \mathbf{M}_s^- \mathbf{H}, s \in [1, \zeta]\}; \zeta = 2^m$$

where $x \in \mathbb{R}^n$ is a vector and co stands for the convex hull. In addition, we have

$$\sigma(\mathbf{K}x, \bar{u}) = \sum_{s=1}^{\zeta} \delta_s (\mathbf{M}_s \mathbf{K} + \mathbf{M}_s^- \mathbf{H}) x(t)$$

where \mathbf{M}_s is an appropriate dimensional diagonal matrix with either elements 1 or 0, and $\delta_1, \dots, \delta_{\zeta}$ are positives scalars such that $\sum_{s=1}^{\zeta} \delta_s = 1$.

By combining (3) and (7), the following closed-loop system is obtained according to the above lemma:

$$E \dot{x}(t) = \sum_{s=1}^{\zeta} \sum_{i=1}^r \sum_{j=1}^r \mu_i \nu_j \delta_s (A_{ijs} x(t) + A_{di} x(t - d(t))) \tag{8}$$

where $A_{ijs} = A_i + B_i (\mathbf{M}_s \mathbf{K}_j + \mathbf{M}_s^- \mathbf{H}_j)$, $s \in [1, \zeta]$.

2.4. Problem Statement

The main objective of this paper is to develop an IT-2 fuzzy controller that maintains the closed-loop system admissible in the presence of actuator saturation for non-linear singular systems expressed by an IT-2 fuzzy model as defined in (3).

3. Main Results

3.1. Admissibility Analysis

In this section, the admissibility of the closed-loop system (8) will be addressed using the following lemmas.

Lemma 2 ([42]). For a given vector $x(t) : [u, v] \mapsto \mathbb{R}^n$ where the derivative $\dot{x}(t)$ is a piecewise continuous function on the interval $[u, v]$, the following inequality holds for any given matrix $\mathbf{R} > 0$:

$$\int_u^v \dot{x}(s)^\top \mathbf{R} \dot{x}(s) ds \geq \frac{1}{v-u} (\mathbf{Y}_1^\top(x) \mathbf{R} \mathbf{Y}_1(x) + 3\mathbf{Y}_2^\top(x) \mathbf{R} \mathbf{Y}_2(x) + 5\mathbf{Y}_3^\top(x) \mathbf{R} \mathbf{Y}_3(x))$$

where

$$\begin{aligned} \mathbf{Y}_1(x) &= x(v) - x(u), \mathbf{Y}_2(x) = x(v) + x(u) - \frac{2}{v-u} \int_u^v x(s) ds \\ \mathbf{Y}_3(x) &= x(v) - x(u) + \frac{6}{v-u} \int_u^v x(s) ds - \frac{12}{(v-u)^2} \int_u^v \int_s^v x(w) dw ds \end{aligned}$$

Lemma 3 ([43]). For any singular matrix E with $\text{rank}(E) = q$, and decomposed as $E = E_L E_R^\top$, a full row rank matrix \mathbf{U} and a full column rank matrix \mathbf{V} can be found such that $\mathbf{U}E = 0$ and $E\mathbf{V} = 0$.

For any symmetric matrix $\mathbf{P} \in \mathbb{R}^{n \times n}$ and non-singular matrix $\mathbf{X} \in \mathbb{R}^{(n-q) \times (n-q)}$, we define a non-singular matrix $\mathbf{PE} + \mathbf{U}^\top \mathbf{XV}^\top$ so that

$$(\mathbf{PE} + \mathbf{U}^\top \mathbf{XV}^\top)^{-1} = \bar{\mathbf{P}}\mathbf{E}^\top + \mathbf{V}\bar{\mathbf{X}}\mathbf{U}$$

where $\bar{\mathbf{P}} = \mathbf{E}_L^\top \mathbf{PE}_L > 0$ and $\bar{\mathbf{X}} = (\mathbf{V}^\top \mathbf{X})^{-1} \mathbf{V}^{-1} (\mathbf{UU}^\top)^{-1}$.

Theorem 1. Let \bar{u}_l be a positive scalar, and d_1, d_2, d_r be scalars satisfying assumption A1 if matrices $\mathbf{P} > 0, \mathbf{X}, \mathbf{Q}_1 > 0, \mathbf{Q}_2 > 0, \mathbf{Q}_3 > 0, \mathbf{W}_1 > 0, \mathbf{W}_2 > 0, \mathbf{V}_1 > 0, \mathbf{V}_2 > 0, \mathbf{R} > 0,$ and $\mathbf{Z} > 0$ exist and verify the following conditions:

$$\tilde{\Psi}(\mathbf{A}_i, \mathbf{A}_{di}) = \begin{bmatrix} \tilde{\Psi}_{11ijs} & \tilde{\Psi}_{12ijs}\mathbf{R} & \tilde{\Psi}_{13ijs}\mathbf{Z} \\ * & -\mathbf{R} & 0 \\ * & * & -\mathbf{Z} \end{bmatrix} < 0, \quad i, j \in \mathbb{S}, s = 1, 2, \dots, \zeta \tag{9}$$

$$\begin{bmatrix} \mathbf{E}^\top \mathbf{\Pi} & (\mathbf{H}_j)_l^\top \\ * & \frac{1}{\rho} \bar{u}_l^2 \end{bmatrix} \geq 0, \quad l = 1, \dots, m \tag{10}$$

Then, the controller in (5) exists, and for any compatible initial condition satisfying (11), the closed-loop system (8) is admissible within the set $\varepsilon(\mathbf{E}^\top \mathbf{PE}, \rho)$.

$$\Omega(\zeta_1, \zeta_2) = \left\{ \phi, : \frac{\|\phi\|_c^2}{\zeta_1} + \frac{\|\phi\|_c^2}{\zeta_2} \leq 1 \right\} \tag{11}$$

$$\zeta_1 = \frac{\rho}{\chi_1}, \quad \zeta_2 = \frac{\rho}{\chi_2},$$

$$\begin{aligned} \chi_1 &= \lambda_{\max}(\mathbf{E}^\top \mathbf{\Pi}) + d_1 \lambda_{\max}(\mathbf{Q}_1) + d_2 (\lambda_{\max}(\mathbf{Q}_2) + \lambda_{\max}(\mathbf{Q}_3)) + \frac{1}{2} d_1^3 \lambda_{\max}(\mathbf{W}_1) \\ &+ \frac{1}{2} d_{12}^2 (d_2 + d_1) \lambda_{\max}(\mathbf{W}_2) + \frac{d_1^5}{12} \lambda_{\max}(\mathbf{V}_1) + \frac{1}{12} d_{12}^2 (d_2^3 - d_1^3) \lambda_{\max}(\mathbf{V}_2) \end{aligned} \tag{12}$$

$$\chi_2 = \frac{1}{2} d_1^3 \lambda_{\max}(\mathbf{E}^\top \mathbf{RE}) + \frac{1}{2} d_{12}^2 (d_2 + d_1) \lambda_{\max}(\mathbf{E}^\top \mathbf{ZE})$$

$$\tilde{\Psi}_{11ijs} = \begin{bmatrix} \Psi_{11ijs} & \Psi_{12i} & \mathbf{\Pi}^\top \mathbf{A}_{di} & 0 & \Psi_{15} & 0 & \Psi_{17} & 0 \\ * & \Psi_{22} & 0 & 3\mathbf{E}^\top \mathbf{ZE} & \Psi_{25} & -24\mathbf{E}^\top \mathbf{ZE} & \Psi_{27} & 30\mathbf{E}^\top \mathbf{ZE} \\ * & * & \Psi_{33} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Psi_{44} & 0 & 36\mathbf{E}^\top \mathbf{ZE} & 0 & -30\mathbf{E}^\top \mathbf{ZE} \\ * & * & * & * & \Psi_{55} & 0 & \Psi_{57} & 0 \\ * & * & * & * & * & \Psi_{66} & 0 & 180\mathbf{E}^\top \mathbf{ZE} \\ * & * & * & * & * & * & \Psi_{77} & 0 \\ * & * & * & * & * & * & * & \Psi_{88} \end{bmatrix},$$

$$\tilde{\Psi}_{12ijs} = \text{col} \{ d_1 \mathbf{A}_{ijs}^\top, 0, d_1 \mathbf{A}_{di}^\top, 0, 0, 0, 0, 0 \},$$

$$\tilde{\Psi}_{13ijs} = \text{col} \{ d_{12} \mathbf{A}_{ijs}^\top, 0, d_{12} \mathbf{A}_{di}^\top, 0, 0, 0, 0, 0 \}$$

$$\begin{aligned} \Psi_{11ijs} &= \mathbf{Q}_1 + \mathbf{Q}_2 + d_1^2 \mathbf{W}_1 + d_{12}^2 \mathbf{W}_2 + \frac{d_1^4}{4} \mathbf{V}_1 + \frac{d_{12}^4}{4} \mathbf{V}_2 \\ &- 9\mathbf{E}^\top \mathbf{RE} + \text{sym}(\mathbf{\Pi}^\top \mathbf{A}_{ijs}) \end{aligned}$$

$$\Psi_{12} = 3\mathbf{E}^\top \mathbf{RE},$$

$$\Psi_{15} = -24\mathbf{E}^\top \mathbf{RE}$$

$$\Psi_{17} = 30\mathbf{E}^\top \mathbf{RE}$$

$$\Psi_{22} = \mathbf{Q}_3 - \mathbf{Q}_1 - 9\mathbf{E}^\top \mathbf{RE} - 9\mathbf{E}^\top \mathbf{ZE}$$

$$\Psi_{25} = 36\mathbf{E}^\top \mathbf{RE}$$

$$\Psi_{27} = -30\mathbf{E}^\top \mathbf{RE}$$

$$\Psi_{33} = -(1 - d_r) \mathbf{Q}_2 - 9\mathbf{E}^\top \mathbf{ZE},$$

$$\Psi_{44} = -\mathbf{Q}_3 - 9\mathbf{E}^\top \mathbf{ZE}$$

$$\Psi_{55} = -192\mathbf{E}^\top \mathbf{RE} - d_1^2 \mathbf{W}_1$$

$$\Psi_{57} = 180\mathbf{E}^\top \mathbf{RE}$$

$$\Psi_{66} = -192\mathbf{E}^\top \mathbf{ZE} - d_{12}^2 \mathbf{W}_2$$

$$\Psi_{77} = -180\mathbf{E}^\top \mathbf{RE} - \frac{d_1^4}{4} \mathbf{V}_1$$

$$\Psi_{88} = -180\mathbf{E}^\top \mathbf{ZE} - \frac{d_{12}^4}{4} \mathbf{V}_2$$

$$\mathbf{\Pi} = (\mathbf{PE} + \mathbf{U}^\top \mathbf{XV}^\top)$$

Matrices $\mathbf{U} \in \mathbb{R}^{(n-q) \times n}$ and $\mathbf{V} \in \mathbb{R}^{n \times (n-q)}$ are as defined in Lemma 3.

Proof. First, we choose the following Lyapunov–Krasovky functional to address the stability of system (8):

$$V(\mathbf{x}(t)) = V_1(\mathbf{x}(t)) + V_2(\mathbf{x}(t)) + V_3(\mathbf{x}(t)) + V_4(\mathbf{x}(t))$$

where

$$\begin{aligned} V_1(\mathbf{x}(t)) &= \mathbf{x}(t)^\top \mathbf{E}^\top \mathbf{P} \mathbf{E} \mathbf{x}(t) \\ V_2(\mathbf{x}(t)) &= \int_{t-d_1}^t \mathbf{x}(s)^\top \mathbf{Q}_1 \mathbf{x}(s) ds + \int_{t-d(t)}^t \mathbf{x}(s)^\top \mathbf{Q}_2 \mathbf{x}(s) ds + \int_{t-d_2}^{t-d_1} \mathbf{x}(s)^\top \mathbf{Q}_3 \mathbf{x}(s) ds \\ V_3(\mathbf{x}(t)) &= d_1 \int_{-d_1}^0 \int_{t+\theta}^t \dot{\mathbf{x}}(s)^\top \mathbf{E}^\top \mathbf{R} \mathbf{E} \dot{\mathbf{x}}(s) ds d\theta + d_{12} \int_{-d_2}^{d_1} \int_{t+\theta}^t \dot{\mathbf{x}}(s)^\top \mathbf{E}^\top \mathbf{Z} \mathbf{E} \dot{\mathbf{x}}(s) ds d\theta \\ V_4(\mathbf{x}(t)) &= d_1 \int_{-d_1}^0 \int_{t+\theta}^t \mathbf{x}(s)^\top \mathbf{W}_1 \mathbf{x}(s) ds d\theta + d_{12} \int_{-d_2}^{d_1} \int_{t+\theta}^t \mathbf{x}(s)^\top \mathbf{W}_2 \mathbf{x}(s) ds d\theta \\ V_5(\mathbf{x}(t)) &= \frac{d_1^2}{2} \int_{-d_1}^0 \int_u^0 \int_{t+\theta}^t \mathbf{x}(s)^\top \mathbf{V}_1 \mathbf{x}(s) ds d\theta du + \frac{d_{12}^2}{2} \int_{-d_2}^{d_1} \int_u^0 \int_{t+\theta}^t \mathbf{x}(s)^\top \mathbf{V}_2 \mathbf{x}(s) ds d\theta du \end{aligned}$$

In evaluating the derivative of $V(\mathbf{x}(t))$ along the trajectories of system (8), and noting that $\mathbf{U} \mathbf{E} = 0$, we obtain

$$\begin{aligned} \dot{V}_1(\mathbf{x}(t)) &= \mathbf{x}(t)^\top \mathbf{E}^\top \mathbf{P} \mathbf{E} \dot{\mathbf{x}}(t) = \mathbf{x}(t)^\top \mathbf{\Pi}^\top \sum_{s=1}^{\xi} \sum_{i=1}^r \sum_{j=1}^r \mu_i \nu_j \delta_s \left(\mathbf{A}_{ijs} \mathbf{x}(t) + \mathbf{A}_{di} \mathbf{x}(t-d(t)) \right) \\ \dot{V}_2(\mathbf{x}(t)) &\leq \mathbf{x}(t)^\top (\mathbf{Q}_1 + \mathbf{Q}_2) \mathbf{x}(t) + \mathbf{x}(t-d_1)^\top (\mathbf{Q}_3 - \mathbf{Q}_1) \mathbf{x}(t-d_1) \\ &\quad - (1-d_r) \mathbf{x}(t-d(t))^\top \mathbf{Q}_2 \mathbf{x}(t-d(t)) + \mathbf{x}(t-d_2)^\top \mathbf{Q}_3 \mathbf{x}(t-d_2) \\ \dot{V}_3(\mathbf{x}(t)) &= \dot{\mathbf{x}}(t)^\top (d_1^2 \mathbf{E}^\top \mathbf{R} \mathbf{E} + d_{12}^2 \mathbf{E}^\top \mathbf{Z} \mathbf{E}) \dot{\mathbf{x}}(t) - d_1 \int_{t-d_1}^t \dot{\mathbf{x}}(s)^\top \mathbf{E}^\top \mathbf{R} \mathbf{E} \dot{\mathbf{x}}(s) ds \\ &\quad - d_{12} \int_{t-d_2}^{t-d_1} \dot{\mathbf{x}}(s)^\top \mathbf{E}^\top \mathbf{Z} \mathbf{E} \dot{\mathbf{x}}(s) ds \tag{13} \\ \dot{V}_4(\mathbf{x}(t)) &= \mathbf{x}(t)^\top (d_1^2 \mathbf{W}_1 + d_{12}^2 \mathbf{W}_2) \mathbf{x}(t) - d_1 \int_{t-d_1}^t \mathbf{x}(s)^\top \mathbf{W}_1 \mathbf{x}(s) ds - d_{12} \int_{t-d_2}^{t-d_1} \mathbf{x}(s)^\top \mathbf{W}_2 \mathbf{x}(s) ds \\ \dot{V}_5(\mathbf{x}(t)) &= \mathbf{x}(t)^\top \left(\frac{d_1^4}{4} \mathbf{V}_1 + \frac{d_{12}^4}{4} \mathbf{V}_2 \right) \mathbf{x}(t) - \frac{d_1^2}{2} \int_{-d_1}^0 \int_{t+\theta}^t \mathbf{x}(s)^\top \mathbf{V}_1 \mathbf{x}(s) ds d\theta \\ &\quad - \frac{d_{12}^2}{2} \int_{-d_2}^{d_1} \int_{t+\theta}^t \mathbf{x}(s)^\top \mathbf{V}_2 \mathbf{x}(s) ds d\theta \end{aligned}$$

According to Lemma 2, we know that:

$$\begin{aligned} -d_1 \int_{t-d_1}^t \dot{\mathbf{x}}(s)^\top \mathbf{E}^\top \mathbf{R} \mathbf{E} \dot{\mathbf{x}}(s) ds &\leq \zeta_1(t)^\top \mathbf{\Lambda}(\mathbf{R}) \zeta_1(t) \\ -d_{12} \int_{t-d_2}^{t-d_1} \dot{\mathbf{x}}(s)^\top \mathbf{E}^\top \mathbf{Z} \mathbf{E} \dot{\mathbf{x}}(s) ds &\leq \zeta_2(t)^\top \mathbf{\Lambda}(\mathbf{Z}) \zeta_2(t) \end{aligned} \tag{14}$$

where $\zeta_1(t) = \left[\mathbf{x}(t)^\top \quad \mathbf{x}(t-d_1)^\top \quad \frac{1}{d_1} \int_{t-d_1}^t \mathbf{x}(s)^\top ds \quad \frac{2}{d_1^2} \int_{-d_1}^0 \int_{t+\theta}^t \mathbf{x}(s)^\top ds d\theta \right]^\top$,
 $\zeta_2(t) = \left[\mathbf{x}(t-d_1)^\top \quad \mathbf{x}(t-d_2)^\top \quad \frac{1}{d_{12}} \int_{t-d_2}^{t-d_1} \mathbf{x}(s)^\top ds \quad \frac{2}{d_{12}^2} \int_{-d_2}^{-d_1} \int_{t+\theta}^t \mathbf{x}(s)^\top ds d\theta \right]^\top$, and

$$\mathbf{\Lambda}(\mathbf{X}) = \begin{bmatrix} -9\mathbf{E}^\top \mathbf{X} \mathbf{E} & 3\mathbf{E}^\top \mathbf{X} \mathbf{E} & -24\mathbf{E}^\top \mathbf{X} \mathbf{E} & 30\mathbf{E}^\top \mathbf{X} \mathbf{E} \\ * & -9\mathbf{E}^\top \mathbf{X} \mathbf{E} & 36\mathbf{E}^\top \mathbf{X} \mathbf{E} & -30\mathbf{E}^\top \mathbf{X} \mathbf{E} \\ * & * & -192\mathbf{E}^\top \mathbf{X} \mathbf{E} & 180\mathbf{E}^\top \mathbf{X} \mathbf{E} \\ * & * & * & -180\mathbf{E}^\top \mathbf{X} \mathbf{E} \end{bmatrix}$$

Jensen’s inequality leads to

$$\begin{aligned}
 & -d_1 \int_{t-d_1}^t \mathbf{x}(s)^\top \mathbf{W}_1 \mathbf{x}(s) ds \leq -\left(\int_{t-d_1}^t \mathbf{x}(s) ds\right)^\top \mathbf{W}_1 \left(\int_{t-d_1}^t \mathbf{x}(s) ds\right) \\
 & -d_{12} \int_{t-d_2}^{t-d_1} \mathbf{x}(s)^\top \mathbf{W}_2 \mathbf{x}(s) ds \leq -\left(\int_{t-d_2}^{t-d_1} \mathbf{x}(s) ds\right)^\top \mathbf{W}_2 \left(\int_{t-d_2}^{t-d_1} \mathbf{x}(s) ds\right) \\
 & -\frac{d_1^2}{2} \int_{-d_1}^0 \int_{t+\theta}^t \mathbf{x}(s)^\top \mathbf{V}_1 \mathbf{x}(s) ds d\theta \leq -\left(\int_{-d_1}^0 \int_{t+\theta}^t \mathbf{x}(s) ds d\theta\right)^\top \mathbf{V}_1 \left(\int_{-d_1}^0 \int_{t+\theta}^t \mathbf{x}(s) ds d\theta\right) \\
 & -\frac{d_{12}^2}{2} \int_{-d_2}^{d_1} \int_{t+\theta}^t \mathbf{x}(s)^\top \mathbf{V}_2 \mathbf{x}(s) ds d\theta \leq -\left(\int_{-d_2}^{d_1} \int_{t+\theta}^t \mathbf{x}(s) ds d\theta\right)^\top \mathbf{V}_2 \left(\int_{-d_2}^{d_1} \int_{t+\theta}^t \mathbf{x}(s) ds d\theta\right)
 \end{aligned} \tag{15}$$

Let

$$\begin{aligned}
 \zeta(t) = & \left[\mathbf{x}^\top(t) \quad \mathbf{x}^\top(t-d_1) \quad \mathbf{x}^\top(t-d(t)) \quad \mathbf{x}^\top(t-d_2) \quad \frac{1}{d_1} \int_{t-d_1}^t \mathbf{x}^\top(s) ds \right. \\
 & \left. \frac{1}{d_{12}} \int_{t-d_2}^{t-d_1} \mathbf{x}^\top(s) ds \quad \frac{2}{d_1^2} \int_{-d_1}^0 \int_{t+\theta}^t \mathbf{x}^\top(s) ds d\theta \quad \frac{2}{d_{12}^2} \int_{-d_2}^{d_1} \int_{t+\theta}^t \mathbf{x}^\top(s) ds d\theta \right].
 \end{aligned}$$

Combining (13)–(15) yields

$$\dot{V}(\mathbf{x}(t)) \leq \sum_{s=1}^{\zeta} \sum_{i=1}^r \sum_{j=1}^r \mu_i \nu_j \delta_s \left(\zeta(t)^\top (\tilde{\Psi}_{11ijs} + \tilde{\Psi}_{12ijs}^\top \mathbf{R} \tilde{\Psi}_{12ijs} + \tilde{\Psi}_{13ijs}^\top \mathbf{Z} \tilde{\Psi}_{13ijs}) \zeta(t) \right)$$

Performing the Schur complement lemma to (9) yields $\tilde{\Psi}_{11ijs} + \tilde{\Psi}_{12ijs}^\top \mathbf{R} \tilde{\Psi}_{12ijs} + \tilde{\Psi}_{13ijs}^\top \mathbf{Z} \tilde{\Psi}_{13ijs} < 0$, which implies that $\dot{V}(\mathbf{x}(t)) \leq 0$ and the closed-loop system in (8) is stable.

Next, we prove the regularity and impulse-free properties of system (8). From (9), we know that $\Psi_{11ijs} < 0$, which implies that

$$\text{sym}(\Pi^\top \mathbf{A}_{ijs}) - 9\mathbf{E}^\top \mathbf{R} \mathbf{E} < 0 \tag{16}$$

For matrix \mathbf{E} , there exist two non-singular matrices \mathbf{M} and \mathbf{N} , such that

$$\begin{aligned}
 \hat{\mathbf{E}} = \mathbf{M} \mathbf{E} \mathbf{N} &= \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{\mathbf{A}}_{ijs} = \mathbf{M} \mathbf{A}_{ijs} \mathbf{N} = \begin{bmatrix} \hat{\mathbf{A}}_{ijs11} & \hat{\mathbf{A}}_{ijs12} \\ \hat{\mathbf{A}}_{ijs21} & \hat{\mathbf{A}}_{ijs22} \end{bmatrix}, \\
 \hat{\Pi} = \mathbf{M}^{-T} \Pi \mathbf{N} &= \begin{bmatrix} \hat{\Pi}_{11} & \hat{\Pi}_{12} \\ \hat{\Pi}_{21} & \hat{\Pi}_{22} \end{bmatrix}, \quad \hat{\mathbf{R}} = \mathbf{M}^{-T} \mathbf{R} \mathbf{M}^{-1} = \begin{bmatrix} \hat{\mathbf{R}}_{11} & \hat{\mathbf{R}}_{12} \\ \hat{\mathbf{R}}_{21} & \hat{\mathbf{R}}_{22} \end{bmatrix}
 \end{aligned} \tag{17}$$

Based on Lemma 3, we know that $\mathbf{E}^\top \Pi = \Pi^\top \mathbf{E}$. Pre- and post-multiplying $\Pi^\top \mathbf{E}$ and (16) by \mathbf{N}^\top and \mathbf{N} , respectively, using (17), we know that $\hat{\Pi}_{12} = 0$ and

$$\text{sym}(\hat{\Pi}_{22}^\top \hat{\mathbf{A}}_{ijs22}) < 0$$

Thus, this would mean that $\hat{\mathbf{A}}_{ijs22}$ is non-singular; then, it can be concluded, given the definition suggested in [1], that system (8) is regular and impulse-free.

Pre and post-multiplying (10) by $\text{diag}(\sqrt{\frac{\bar{u}_l}{\rho}}, \sqrt{\frac{\rho}{\bar{u}_l}})$, we obtain

$$\begin{bmatrix} \frac{\bar{u}_l}{\rho} \mathbf{E}^\top \Pi & (\mathbf{H}_j)_l^\top \\ * & \bar{u}_l \end{bmatrix} \geq 0 \tag{18}$$

Using (18) yields:

$$[\pm \mathbf{x}(t)^\top \quad 1] \begin{bmatrix} \frac{\bar{u}_l}{\rho} \mathbf{E}^\top \Pi & (\mathbf{H}_j)_l^\top \\ * & \bar{u}_l \end{bmatrix} \begin{bmatrix} \pm \mathbf{x}(t) \\ 1 \end{bmatrix} \geq 0$$

Equivalently, we obtain

$$2|\mathbf{H}_j \mathbf{x}(t)| \leq \bar{u}_l \left(1 + \frac{1}{\rho} \mathbf{x}(t)^\top \mathbf{E}^\top \mathbf{P} \mathbf{E} \mathbf{x}(t)\right) \leq 2\bar{u}_l, \forall j = 1, \dots, r$$

Thus, it can be verified that $\varepsilon(\mathbf{E}^\top \mathbf{P} \mathbf{E}, \rho) \subset \bigcap_{j=1}^r \mathcal{L}(\mathbf{H}_j, \bar{\mathbf{u}})$, and using the fact that $\dot{V}(\mathbf{x}(t)) \leq 0$, it is easy to verify that $V(\mathbf{x}(t)) \leq V(\mathbf{x}(0))$; this results in

$$\begin{aligned} \mathbf{x}(t)^\top \mathbf{E}^\top \mathbf{P} \mathbf{E} \mathbf{x}(t) &\leq V(\mathbf{x}(t)) \leq V(\mathbf{x}(0)) \leq \|\phi\|_c^2 \left\{ \lambda_{\max}(\mathbf{E}^\top \mathbf{\Pi}) + d_1 \lambda_{\max}(\mathbf{Q}_1) \right. \\ &+ d_2 (\lambda_{\max}(\mathbf{Q}_2) + \lambda_{\max}(\mathbf{Q}_3)) + \frac{1}{2} d_1^3 \lambda_{\max}(\mathbf{W}_1) + \frac{1}{2} d_{12}^2 (d_2 + d_1) \lambda_{\max}(\mathbf{W}_2) \\ &+ \left. \frac{1}{12} d_1^5 \lambda_{\max}(\mathbf{V}_1) + \frac{1}{12} d_{12}^2 (d_2^3 - d_1^3) \lambda_{\max}(\mathbf{V}_2) \right\} \\ &+ \|\phi\|_c^2 \left\{ \frac{1}{2} d_1^3 \lambda_{\max}(\mathbf{E}^\top \mathbf{R} \mathbf{E}) + \frac{1}{2} d_{12}^2 (d_2 + d_1) \lambda_{\max}(\mathbf{E}^\top \mathbf{Z} \mathbf{E}) \right\} \\ &\leq \chi_1 \|\phi\|_c^2 + \chi_2 \|\phi\|_c^2 \leq \rho \end{aligned}$$

Thus, the constraint in (11) is verified for any compatible initial condition, and this completes the proof.

□

3.2. Fuzzy Controller Design

Our task here is to translate the conditions in Theorem 1 into LMI terms that can be solved with the existing solvers.

Theorem 2. For positive scalars d_1, d_2, d_r , and $\bar{u}_l > 0$, if there exist scalars $\gamma > 0, \lambda_1$, and λ_2 , and matrices $\bar{\mathbf{P}} > 0, \bar{\mathbf{X}}, \bar{\mathbf{Q}}_1 > 0, \bar{\mathbf{Q}}_2 > 0, \bar{\mathbf{Q}}_3 > 0, \bar{\mathbf{W}}_1 > 0, \bar{\mathbf{W}}_2 > 0, \bar{\mathbf{V}}_1 > 0, \bar{\mathbf{V}}_2 > 0, \bar{\mathbf{R}} > 0, \bar{\mathbf{Z}} > 0, \bar{\mathbf{H}}_j, \mathbf{F}_j$, and $\mathbf{\Gamma}_i^\top = \mathbf{\Gamma}_i$ such that, under the condition $\mu_j - \kappa_j v_j \geq 0, (0 < \kappa_j < 1)$, the following LMIs hold:

$$\begin{aligned} \bar{\Psi}_{ijs} - \mathbf{\Gamma}_i &< 0 \\ \kappa_i (\bar{\Psi}_{iis} - \mathbf{\Gamma}_i) - \mathbf{\Gamma}_i &< 0 \\ \kappa_j (\bar{\Psi}_{ijs} - \mathbf{\Gamma}_i) + \kappa_i (\bar{\Psi}_{jis} - \mathbf{\Gamma}_j) + \mathbf{\Gamma}_i + \mathbf{\Gamma}_j &< 0 \\ \begin{bmatrix} \bar{\mathbf{\Pi}}^\top \mathbf{E}^\top & (\bar{\mathbf{H}}_j)_l^\top \\ * & \gamma \bar{u}_l^2 \end{bmatrix} &\geq 0, \gamma = \frac{1}{\rho}, \quad i, j \in \mathbb{S}, s = 1, 2 \dots \varsigma, l = 1, \dots, m \end{aligned} \tag{19}$$

$$\tag{20}$$

then the closed-loop singular system (8) is admissible within the set $\varepsilon(\mathbf{E}^\top \mathbf{P} \mathbf{E}, \rho)$ for any compatible initial condition satisfying (11). Moreover, the gains \mathbf{H}_j and \mathbf{K}_j are computed by $\mathbf{H}_j = \bar{\mathbf{H}}_j \bar{\mathbf{\Pi}}^{-1}$, $\mathbf{K}_j = \mathbf{F}_j \bar{\mathbf{\Pi}}^{-1}$, and the parameters in (12) are defined as

$$\begin{aligned} \chi_1 &= \lambda_{\max}(\mathbf{E}^\top \bar{\mathbf{\Pi}}^{-1}) + d_1 \lambda_{\max}(\bar{\mathbf{\Pi}}^{-T} \bar{\mathbf{Q}}_1 \bar{\mathbf{\Pi}}^{-1}) \\ &+ d_2 (\lambda_{\max}(\bar{\mathbf{\Pi}}^{-T} \bar{\mathbf{Q}}_2 \bar{\mathbf{\Pi}}^{-1}) + \lambda_{\max}(\bar{\mathbf{\Pi}}^{-T} \bar{\mathbf{Q}}_3 \bar{\mathbf{\Pi}}^{-1})) \\ &+ \frac{1}{2} d_1^3 \lambda_{\max}(\bar{\mathbf{\Pi}}^{-T} \bar{\mathbf{W}}_1 \bar{\mathbf{\Pi}}^{-1}) + \frac{1}{2} d_{12}^2 (d_2 + d_1) \lambda_{\max}(\bar{\mathbf{\Pi}}^{-T} \bar{\mathbf{W}}_2 \bar{\mathbf{\Pi}}^{-1}) \\ &+ \frac{d_1^5}{12} \lambda_{\max}(\bar{\mathbf{\Pi}}^{-T} \bar{\mathbf{V}}_1 \bar{\mathbf{\Pi}}^{-1}) + \frac{1}{12} d_{12}^2 (d_2^3 - d_1^3) \lambda_{\max}(\bar{\mathbf{\Pi}}^{-T} \bar{\mathbf{V}}_2 \bar{\mathbf{\Pi}}^{-1}) \\ \chi_2 &= \frac{1}{2} d_1^3 \lambda_{\max}(\mathbf{E} \bar{\mathbf{\Pi}}^{-1} \bar{\mathbf{R}} \bar{\mathbf{\Pi}}^{-T} \mathbf{E}^\top) + \frac{1}{2} d_{12}^2 (d_2 + d_1) \lambda_{\max}(\mathbf{E} \bar{\mathbf{\Pi}}^{-1} \bar{\mathbf{Z}} \bar{\mathbf{\Pi}}^{-T} \mathbf{E}^\top) \end{aligned}$$

where

$$\Psi_{ijs} = \begin{bmatrix} \bar{\Psi}_{11ijs} & \bar{\Psi}_{12ijs} & \bar{\Psi}_{13ijs} \\ * & -\lambda_1 \text{sym}(\bar{\Pi}) + \lambda_1^2 \bar{R} & 0 \\ * & * & -\lambda_2 \text{sym}(\bar{\Pi}) + \lambda_2^2 \bar{Z} \end{bmatrix}$$

$$\bar{\Psi}_{11ijs} = \begin{bmatrix} \hat{\Psi}_{11ijs} & \hat{\Psi}_{12i} & A_{di} \bar{\Pi} & 0 & \hat{\Psi}_{15i} & 0 & \hat{\Psi}_{17i} & 0 \\ * & \hat{\Psi}_{22} & 0 & 3E\bar{Z}E^\top & \hat{\Psi}_{25i} & -24E\bar{Z}E^\top & \hat{\Psi}_{27i} & 30E\bar{Z}E^\top \\ * & * & \hat{\Psi}_{33} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\bar{Q}_3 - 9E\bar{Z}E^\top & 0 & 36E\bar{Z}E^\top & 0 & -30E\bar{Z}E^\top \\ * & * & * & * & \hat{\Psi}_{55} & 0 & \hat{\Psi}_{57} & 0 \\ * & * & * & * & * & \hat{\Psi}_{66} & 0 & 180E\bar{Z}E^\top \\ * & * & * & * & * & * & \hat{\Psi}_{77} & 0 \\ * & * & * & * & * & * & * & \hat{\Psi}_{88i} \end{bmatrix}$$

$$\begin{aligned} \bar{\Psi}_{12ijs} &= \text{col} \{ \bar{A}_{ijs}, 0, \bar{\Pi}^\top A_{di}^\top, 0, 0, 0, 0, 0 \}, \\ \bar{\Psi}_{13ijs} &= \text{col} \{ \bar{A}_{ijs}, 0, \bar{\Pi}^\top A_{di}^\top, 0, 0, 0, 0, 0 \} \\ \hat{\Psi}_{11i} &= \bar{Q}_1 + \bar{Q}_2 + d_1^2 \bar{W}_1 + d_{12}^2 \bar{W}_2 + \frac{d_1^4}{4} \bar{V}_1 + \frac{d_{12}^4}{4} \bar{V}_2 \\ &\quad - 9\bar{E}\bar{R}\bar{E}^\top + \text{sym}(\bar{A}_{ijs}^\top) \\ \hat{\Psi}_{12i} &= 3E\bar{R}E^\top, \\ \hat{\Psi}_{15i} &= -24E\bar{R}E^\top \\ \hat{\Psi}_{17i} &= 30E\bar{R}E^\top, \\ \hat{\Psi}_{22i} &= \bar{Q}_3 - \bar{Q}_1 - 9E\bar{R}E^\top - 9E\bar{Z}E^\top \\ \hat{\Psi}_{25i} &= 36E\bar{R}E^\top \\ \hat{\Psi}_{27i} &= -30E\bar{R}E^\top \\ \hat{\Psi}_{33i} &= -(1 - d_r)\bar{Q}_2, \\ \hat{\Psi}_{55i} &= -192E\bar{R}E^\top - d_1^2 \bar{W}_1 \\ \hat{\Psi}_{57i} &= 180E\bar{R}E^\top, \\ \hat{\Psi}_{66i} &= -192E\bar{Z}E^\top - d_1^2 \bar{W}_2 \\ \hat{\Psi}_{77i} &= -180E\bar{R}E^\top - \frac{d_1^4}{4} \bar{V}_1, \\ \hat{\Psi}_{88i} &= -180E\bar{Z}E^\top - \frac{d_{12}^4}{4} \bar{V}_2 \\ A_{ijs} &= A_i \bar{\Pi} + B_i (M_s F_j + M_s^- \bar{H}_j) \\ \bar{\Pi} &= \bar{P}E^\top + V\bar{X}U \end{aligned}$$

Proof. Consider $\sum_{i=1}^r \sum_{j=1}^r \mu_i(\mu_j - \nu_j)\Gamma_i = 0$, where $\Gamma_i = \Gamma_i^\top$ is any arbitrary matrix with suitable dimensions. Then, we have

$$\begin{aligned} \sum_{s=1}^{\zeta} \delta_s \left(\sum_{i=1}^r \sum_{j=1}^r \mu_i \nu_j \bar{\Psi}_{ijs} \right) &= \sum_{s=1}^{\zeta} \delta_s \left(\sum_{i=1}^r \sum_{j=1}^r \mu_i \nu_j \bar{\Psi}_{ijs} + \sum_{i=1}^r \sum_{j=1}^r \mu_i(\mu_j - \nu_j)\Gamma_i \right) \\ &= \sum_{s=1}^{\zeta} \delta_s \left(\sum_{i=1}^r \sum_{j=1}^r \mu_i(\nu_j + \kappa_j \mu_j - \kappa_j \mu_j) \bar{\Psi}_{ijs} + \sum_{i=1}^r \sum_{j=1}^r \mu_i(\mu_j - \nu_j + \kappa_j \mu_j - \kappa_j \mu_j)\Gamma_i \right) \\ &= \sum_{s=1}^{\zeta} \delta_s \left(\sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j (\kappa_j \bar{\Psi}_{ijs} - \kappa_j \Gamma_i + \Gamma_i) + \sum_{i=1}^r \sum_{j=1}^r \mu_i(\nu_j - \kappa_j \mu_j)(\bar{\Psi}_{ijs} - \Gamma_i) \right) \\ &= \sum_{s=1}^{\zeta} \delta_s \left(\sum_{i=1}^r \mu_i^2 (\kappa_i \Psi_{iis} - \kappa_i \Gamma_i + \Gamma_i) + \sum_{i=1}^{r-1} \sum_{j=i+1}^r \mu_i \mu_j (\kappa_j \bar{\Psi}_{ijs} - \kappa_j \Gamma_i + \Gamma_i + \kappa_i \Psi_{jis} - \kappa_i \Gamma_j + \Gamma_j) \right) \\ &\quad + \sum_{i=1}^r \sum_{j=1}^r \mu_i(\nu_j - \kappa_j \mu_j)(\bar{\Psi}_{ijs} - \Gamma_i) \end{aligned} \tag{21}$$

It can be concluded from (19) that

$$\sum_{s=1}^{\zeta} \delta_s \left(\sum_{i=1}^r \sum_{j=1}^r \mu_i \nu_j \bar{\Psi}_{ijs} \right) < 0 \tag{22}$$

The theorem implies that $-\lambda_2 \text{sym}(\bar{\Pi}) + \lambda_2^2 \bar{Z} < 0$ and that $\bar{\Pi}$ is non-singular.

Since $\bar{P} > 0$, it is easy to verify that $E_L^T \bar{P} E_L > 0$. Thus, according to Lemma 3, we have

$$\bar{\Pi}^{-1} = (PE + U^T X V^T) = \Pi$$

Using the fact that, for any $\lambda_1 > 0$ and $\lambda_2 > 0$, the following condition holds:

$$\begin{aligned} 0 &\leq (\lambda_1 \bar{\Pi} - R^{-1})^T R (\lambda_1 \bar{\Pi} - R^{-1}) = R^{-1} - \lambda_1 \text{sym}(\bar{\Pi}) + \lambda_1^2 \bar{\Pi}^T R \bar{\Pi} \\ 0 &\leq (\lambda_2 \bar{\Pi} - Z^{-1})^T R (\lambda_2 \bar{\Pi} - Z^{-1}) = R^{-1} - \lambda_2 \text{sym}(\bar{\Pi}) + \lambda_2^2 \bar{\Pi}^T Z \bar{\Pi} \end{aligned}$$

Equivalently, we obtain

$$-R^{-1} \leq -\lambda_1 \text{sym}(\bar{\Pi}) + \lambda_1^2 \bar{R}, \quad -Z^{-1} \leq -\lambda_2 \text{sym}(\bar{\Pi}) + \lambda_2^2 \bar{Z} \quad (23)$$

Let $F_j = K_j \bar{\Pi}$ and $\bar{H}_j = H_j \bar{\Pi}$. Using (23), and performing the congruence transformation to (22) by $\text{diag}(\Pi, \Pi, \Pi, \Pi, \Pi, \Pi, \Pi, \Pi, I, I)$, its transpose

$$\tilde{\Psi}(A_{ijs}, A_{di}) < 0$$

holds using the following expressions:

$$\begin{aligned} Q_i &= \bar{\Pi}^{-T} \bar{Q}_i \bar{\Pi}^{-1}, \quad (i = 1, 2, 3), & W_1 &= \bar{\Pi}^{-T} \bar{W}_1 \bar{\Pi}^{-1}, & W_2 &= \bar{\Pi}^{-T} \bar{W}_2 \bar{\Pi}^{-1}, & V_1 &= \bar{\Pi}^{-T} \bar{V}_1 \bar{\Pi}^{-1}, \\ V_2 &= \bar{\Pi}^{-T} \bar{V}_2 \bar{\Pi}^{-1}, & R &= \bar{\Pi}^{-1} \bar{R} \bar{\Pi}^{-T}, & Z &= \bar{\Pi}^{-1} \bar{Z} \bar{\Pi}^{-T}. \end{aligned}$$

Then, by convexity and using the fact that $\delta_s \geq 0$, and $\sum_{s=1}^c \delta_s = 1$, it can be deduced that system (8) is admissible according to Theorem 1. Pre and post-multiplying (20) by $\text{diag}(\Pi, I)$, (10) holds. \square

Remark 2. For the purpose of maximizing the the set of initial conditions on (11), the following optimization problem can be solved:

$$\begin{aligned} \text{OP:} \quad & \min_{\substack{\bar{\Pi}, \bar{Q}_1 > 0, \bar{Q}_2 > 0, \bar{Q}_3 > 0, \bar{W}_1 > 0, \bar{W}_2 > 0, \bar{V}_1 > 0, \bar{V}_2 > 0, \bar{R} > 0, \bar{Z} > 0, \gamma > 0}} \Theta \\ & \text{subject to} \\ & \text{a) (19)-(20)} \\ & \text{b) } \begin{bmatrix} w_1 I & E_R \\ * & E_R^T \bar{P} E_R \end{bmatrix} \geq 0 \quad (24) \\ & \text{c) } w_2 I - \bar{Q}_1 > 0, w_3 I - \bar{Q}_2 > 0, w_4 I - \bar{Q}_3 > 0, \\ & w_5 I - \bar{W}_1 > 0, w_6 I - \bar{W}_2 > 0, w_7 I - \bar{V}_1 > 0, w_8 I - \bar{V}_2 > 0, w_9 I - \bar{Z} > 0, w_{10} I - \bar{R} > 0 \\ & \text{d) } \bar{w}_1 = \lambda_{\max}^{-1}(E) w_1 \geq w_k, k = 2, 3, \dots, 10 \end{aligned}$$

where

$$\begin{aligned} \Theta &= \tau_1 \left\{ w_1 + w_2 d_1 + (w_3 + w_4) d_2 + \frac{1}{2} w_5 d_1^3 + \frac{1}{2} w_6 d_{12}^2 (d_2 + d_1) \right. \\ & \left. + \frac{d_1^5}{12} w_7 + \frac{1}{12} d_{12}^2 (d_2^3 - d_1^3) w_8 \right\} + \tau_2 \left\{ \frac{1}{2} d_1^3 w_9 + \frac{1}{2} d_{12}^2 (d_2 + d_1) w_{10} \right\} + \tau_3 \gamma \end{aligned}$$

where variables w_i , ($i = 1, 2, \dots, 10$) are introduced for the optimization procedure, and τ_c , ($c = 1, 2, 3$) represents the weighting relative to the objective function.

From LMI b), we have

$$E_R (E_R^T \bar{P} E_R)^{-1} E_R^T = E_R (E_L^T P E_L) E_R^T = E^T \Pi = E^T \bar{\Pi}^{-1} < w_1 I$$

The satisfaction of LMI b) implies that

$$\lambda_{max}(E^T \bar{\Pi}^{-1}) \leq w_1, \text{ and } \lambda_{max}(\bar{\Pi}^{-1}) \leq \bar{w}_1$$

Additionally, the satisfaction of the LMIs in c) implies that

$$\begin{aligned} \chi_1 &\leq w_1 + (w_2 d_1 + (w_3 + w_4) d_2 + \frac{1}{2} w_5 d_1^3 + \frac{1}{2} w_6 d_{12}^2 (d_2 + d_1) + \frac{d_1^5}{12} w_7 + \frac{1}{12} d_{12}^2 (d_2^3 - d_1^3) w_8) \bar{w}_1^2 \\ \chi_2 &\leq \left(\frac{1}{2} d_1^3 w_9 + \frac{1}{2} d_{12}^2 (d_2 + d_1) w_{10} \right) \lambda_{max}(E^T E) \bar{w}_1^2 \end{aligned}$$

We know that $\zeta_i = \frac{\rho}{\chi_i}$. If we minimize the criterion as defined in (24), then the bounds on $\|\phi\|$ and $\|\dot{\phi}\|$ tend to be greater.

4. Numerical Applications

As part of this section, we present two examples illustrating the proposed control strategy.

4.1. Mass-Spring-Damper System

With the help of the example of a mass–spring–damper system shown in Figure 1 and borrowed from [44], the efficiency and correctness of the proposed control scheme can be demonstrated. Define $x_1(t)$, $x_2(t)$, and $x_3(t)$ as the displacement, velocity, and acceleration of the system, respectively, and $u(t)$ is the applied force. Newton’s law can be used to describe the mechanical system as follows:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_3(t) \\ 0 = -m x_3(t) - x_2(t) - f(x_1(t)) + h(x_2(t)) u(t) \end{cases} \tag{25}$$

where m is the mass, and the non-linear functions in the model are defined as follows:

$$f(x_1(t)) = 1.13x_1(t) + 0.13x_1(t)(t - d(t)), \quad h(x_2(t)) = 1 + 0.13(x_2(t) + \Delta x_2(t))^3$$

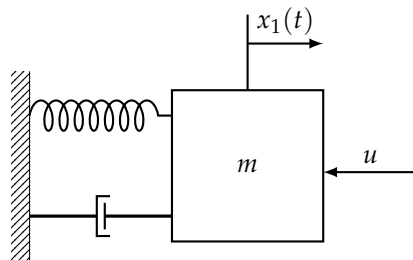


Figure 1. Mass–spring–damper system.

Assume that $m = 1$, $x_1(t) \in [-1.5 \ 1.5]$, $x_2(t) \in [-1.5 \ 1.5]$, and $0 \leq \Delta x_2(t) \leq 0.1$. Given the uncertainty associated with the parameter $\Delta x_2(t)$, it is evident that the IT-2 T-S fuzzy system should be adopted to model the non-linear system (25). The lower and upper bounds of membership functions of the corresponding IT-2 TS fuzzy model are listed in Table 2.

Table 2. Lower and upper membership functions of the plant.

Lower Membership Functions	Upper Membership Functions
$\underline{\mu}_1 = \frac{3.375 + x_2(t)^3}{7.471}$	$\bar{\mu}_1 = \frac{3.375 + (x_2(t) + 0.1)^3}{7.471}$
$\underline{\mu}_2 = \frac{4.096 - (x_2(t) + 0.1)^3}{7.471}$	$\bar{\mu}_2 = \frac{4.096 - x_2(t)^3}{7.471}$

The weighting functions are chosen as $\underline{\alpha}_i = \sin^2(x_2(t))$ and $\bar{\alpha}_i = 1 - \underline{\alpha}_i^2$ for $i = 1, 2$. The interval-valued fuzzy system (1) is defined by the following matrices:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_1 = A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1.13 & -1 & -1 \end{bmatrix}, A_{d1} = A_{d2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.13 & 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 1.43875 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0 \\ 0.56125 \end{bmatrix}$$

This example aims to design a fuzzy controller (5) that guarantees the admissibility of closed-loop systems. To accomplish this goal, Table 3 lists the lower and upper bounds of membership functions to characterize the interval-valued fuzzy controller.

Table 3. Lower and upper membership functions of the controller.

Lower Membership Functions	Upper Membership Functions
$\underline{\nu}_1(x_2) = 0.3e^{-(x_2^2/0.35)}$	$\bar{\nu}_1(x_2) = \underline{\nu}_1(x_2)$
$\underline{\nu}_2(x_2) = -0.3 + 0.3e^{-(x_2^2/0.35)}$	$\bar{\nu}_2(x_2) = \underline{\nu}_2(x_2)$

To well-determine the membership functions, the non-linear weight functions are selected as $\underline{\beta}_j(x_2(t)) = \cos^2(x_2(t))$ and $\bar{\beta}_j(x_2(t)) = 1 - \underline{\beta}_j(x_2(t))$. Define $d_1 = 0.1, d_2 = 0.25, d_r = 0.6, \kappa_1 = 0.9,$ and $\kappa_2 = 0.8$.

Then, by solving the problem formulated in (24), a feasible solution can be obtained with minimum values of $\lambda_1 = 0.35$ and $\lambda_2 = 0.4,$ and the design parameters are given by

$$H_1 = [-1.1115 \quad -0.033098 \quad 0], \quad H_2 = [-1.0735 \quad -0.26308 \quad 0],$$

$$K_1 = [-1.4242 \quad -0.14551 \quad 0.83455], \quad K_2 = [-1.7949 \quad -0.27625 \quad 0.77306],$$

$$P = \begin{bmatrix} 6.4385 & -0.11536 & 0 \\ -0.11536 & 5.9703 & 0 \\ 2.7418 & 4.7886 & 8.0895 \end{bmatrix}$$

The simulation results are presented in Figure 2 using the above-mentioned fuzzy control gains. In particular, Figure 2a depicts the state responses of the saturated closed-loop for the initial condition $\phi(t) = [1.25, 1, -0.7]^T, t \in [-0.25, 0]$. Based on Figure 2c, it is evident that the closed system is well-controlled. Figure 2b illustrates the estimated domain of attraction for various initial conditions when actuator saturation is present and time-varying delay exists.

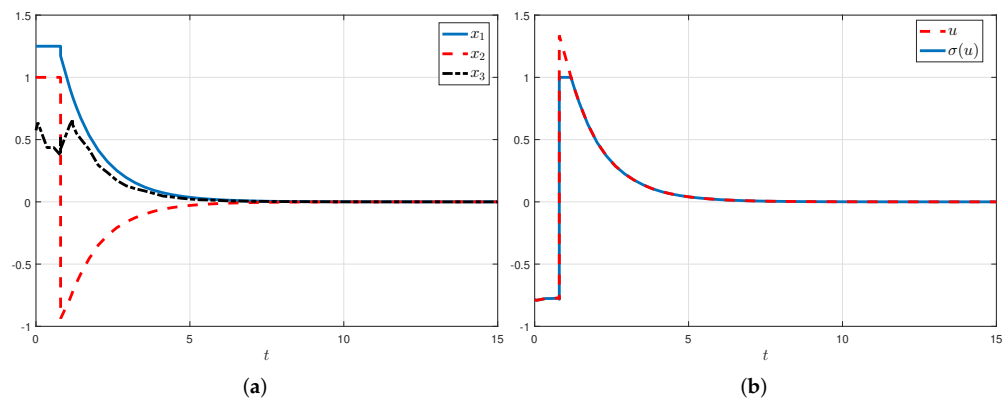


Figure 2. Cont.

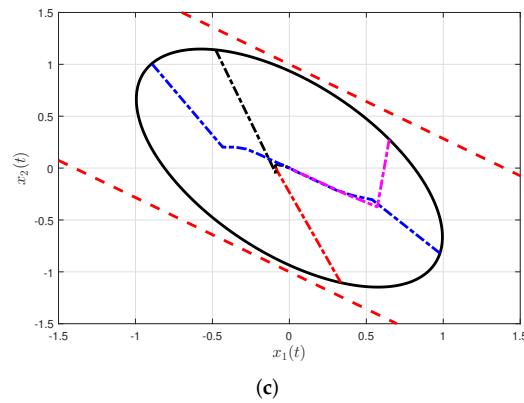


Figure 2. Simulation plots for mass–spring–damper system. (a) Trajectories of $x_1(t)$, $x_2(t)$, and $x_3(t)$. (b) Ideal and saturated control input signals. (c) Estimated domains of attraction and state trajectories under various initial conditions.

4.2. Inverted Pendulum System

Our goal in this section is to illustrate the effectiveness of the proposed control scheme by comparing it with the relevant work proposed in [45] for an inverted pendulum system described by the interval type-2 fuzzy model without delay and saturation. The following is a system that describes the process:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = f(x(t))x_1(t) + h(x(t))u(t) \\ 0 = L \sin(x_1(t)) - x_3(t) \end{cases} \quad (26)$$

Non-linear functions $f(x(t))$ and $h(x(t))$ are defined as

$$f(x(t)) = \frac{(g - am_p L x_2^2(t) \cos(x_1(t))) \sin(x_1(t))}{(4L/3 - am_p L x_2^2(t) \cos^2(x_1(t)))x_1(t)}, \quad h(x(t)) = \frac{-a \cos(x_1(t))}{4L/3 - am_p L \cos^2(x_1(t))}$$

$x_1(t)$ represents the angle between the pendulum and vertical, $x_2(t)$ represents the angular velocity, and $x_3(t)$ is the relative horizontal distance between the pendulum center and cart. The force applied to the cart is given by $u(t)$. The numerical values of the model are: $2L = 1$ m is the length of the pendulum, $g = 9.8$ m/s² is the gravity acceleration, $m_c \in [2 \ 3]$ kg is the mass of the cart, $m_p \in [8 \ 16]$ kg is the mass of the pendulum, and $a = 1/(m_c + m_p)$. For the sake of this study, the inverted pendulum was taken to operate in a domain fixed by $x_1(t) \in [-\frac{5\pi}{12} \ \frac{5\pi}{12}]$, and $x_2(t) \in [-5 \ 5]$.

Next, we describe the inverted pendulum system (26) as follows:

$$E\dot{x}(t) = \sum_{i=1}^4 \mu_i(x(t))((1 - \beta)A_i x(t)(t) + \beta A_i x(t)(t - d(t)) + B_i u(t)) \quad (27)$$

where $\beta = 0.1$, and

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1 = A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 10.0078 & 0 & -1 \end{bmatrix}, \quad A_3 = A_4 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 18.4800 & 0 & -1 \end{bmatrix}$$

$$B_1 = B_3 = \begin{bmatrix} 0 \\ 0 \\ -0.1765 \end{bmatrix}, \quad B_2 = B_4 = \begin{bmatrix} 0 \\ 0 \\ -0.0206 \end{bmatrix}$$

Table 4 defines the lower and upper membership functions used for this example.

Table 4. Lower and upper membership functions of the inverted pendulum.

Lower Membership Functions	Upper Membership Functions
$\underline{\omega}_{\mathcal{M}_1^1}(x_1(t)) = \underline{\omega}_{\mathcal{M}_1^2}(x_1(t)) = \frac{f_{1max}-f_1}{f_{1max}-f_{1min}}$ $\underline{\omega}_{\mathcal{M}_1^3}(x_1(t)) = \underline{\omega}_{\mathcal{M}_1^4}(x_1(t)) = 1 - \underline{\omega}_{\mathcal{M}_1^1}(x_1(t))$ with $x_2(t) = 0, m_p = m_{pmax}$ and $m_c = m_{cmin}$	$\bar{\omega}_{\mathcal{M}_1^1}(x_1(t)) = \bar{\omega}_{\mathcal{M}_1^2}(x_1(t)) = \frac{f_{1max}-f_1}{f_{1max}-f_{1min}}$ $\bar{\omega}_{\mathcal{M}_1^3}(x_1(t)) = \bar{\omega}_{\mathcal{M}_1^4}(x_1(t)) = 1 - \bar{\omega}_{\mathcal{M}_1^1}(x_1(t))$ with $x_2(t) = x_{2max}, m_p = m_{pmax}$ and $m_c = m_{cmin}$
$\underline{\omega}_{\mathcal{M}_2^1}(x_2(t)) = \underline{\omega}_{\mathcal{M}_2^3}(x_2(t)) = \frac{f_{2max}-f_2}{f_{2max}-f_{2min}}$ $\underline{\omega}_{\mathcal{M}_2^2}(x_2(t)) = \underline{\omega}_{\mathcal{M}_2^4}(x_2(t)) = 1 - \underline{\omega}_{\mathcal{M}_2^1}(x_2(t))$ with $m_p = m_{pmin}$ and $m_c = m_{cmax}$	$\bar{\omega}_{\mathcal{M}_2^1}(x_2(t)) = \bar{\omega}_{\mathcal{M}_2^3}(x_2(t)) = \frac{f_{2max}-f_2}{f_{2max}-f_{2min}}$ $\bar{\omega}_{\mathcal{M}_2^2}(x_2(t)) = \bar{\omega}_{\mathcal{M}_2^4}(x_2(t)) = 1 - \bar{\omega}_{\mathcal{M}_2^1}(x_2(t))$ with $m_p = m_{pmin}$ and $m_c = m_{cmax}$

We choose the controller membership functions as $\underline{\nu}_1(x_1(t)) = \bar{\nu}_1(x_1(t)) = 0.3e^{(-x_1^2(t)/0.35)}$, $\underline{\nu}_2(x_1(t)) = \bar{\nu}_2(x_1(t)) = 0.3 - \underline{\nu}_1(x_1(t))$, $\underline{\nu}_3(x_1(t)) = \bar{\nu}_3(x_1(t)) = 0.7e^{(-x_1^2(t)/0.35)}$, and $\underline{\nu}_4(x_1(t)) = \bar{\nu}_4(x_1(t)) = 0.7 - \underline{\nu}_3(x_1(t))$.

Let $d_1 = 0, d_2 = 0.05, \bar{u} = 300, \kappa_1 = 0.9$, and $\kappa_2 = 0.8, \kappa_3 = 0.9$, and $\kappa_4 = 0.8$. Problem (24) produces a feasible solution for $\lambda_1 = 122$ and $\lambda_2 = 122$. The controller gains are listed below.

$$\begin{aligned}
 H_1 &= [485.99 \quad 150.97 \quad 0], \\
 K_1 &= [7949.1 \quad 2472.3 \quad 338.89], \quad K_2 = [11109 \quad 3457.4 \quad 469.51], \\
 K_3 &= [7965.8 \quad 2469.4 \quad 338.69], \quad K_4 = [11170 \quad 3457.7 \quad 469.7],
 \end{aligned}
 \tag{28}$$

To perform some simulations, based on the parameters mentioned above and the controller in [45] with the following gains:

$$K_1 = [-1174.4396 \quad -189.5425 \quad -96.3306], \quad K_2 = [-256.0090 \quad 2.4789 \quad -61.9688]
 \tag{29}$$

different cases are considered:

Case 1: Delay does not affect system. Here, we set $\beta = 0$, and both controllers are applied to the system under $\phi(t) = [-\frac{\pi}{6}, 3, 0.5]^T$. The evolution of the state signals is plotted in Figure 3.

Case 2: A delay affects the system. By using the gains in (28) and (29), respectively, for $\beta = 0.1$ and the above delay parameters, the simulation results are shown in Figures 3 and 4 for the initial condition $\phi(t) = [-\frac{\pi}{20}, 0.5, 0.5]^T, t \in [-0.05, 0]$.

Case 3: A delay and saturation affect the system For this case, Figure 5 shows the states and saturated input signals of the pendulum system when the control law calculated using (28) is implemented for different delays and different initial conditions.

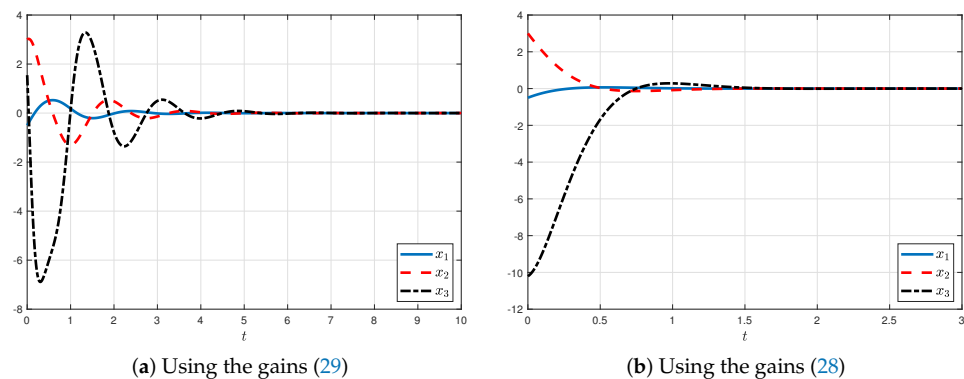


Figure 3. State trajectories of the system without delay.

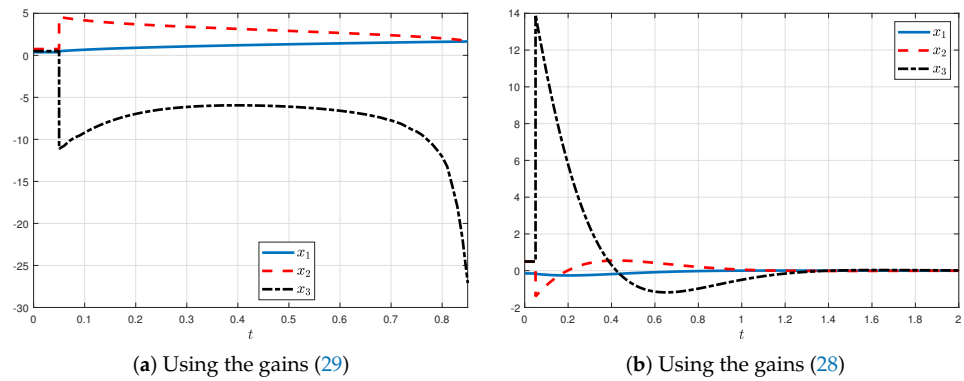


Figure 4. State trajectories of the system with delay.

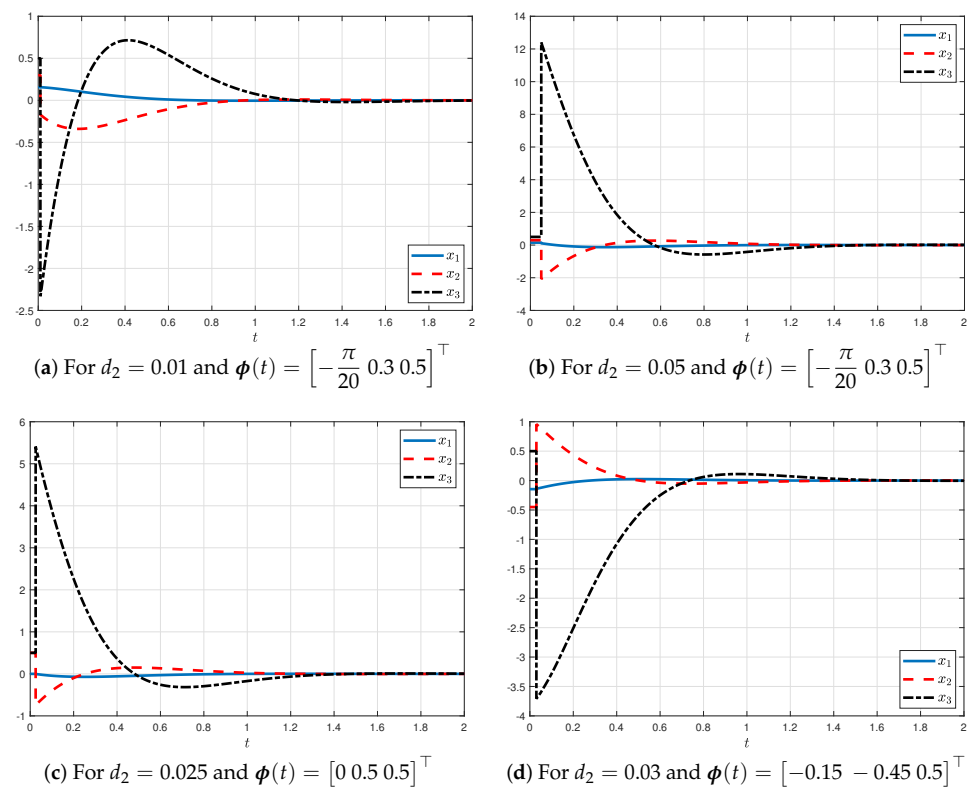


Figure 5. State trajectories using the proposed controller for different delays and initial conditions.

Figure 3 indicates that the two implemented controllers are capable of guaranteeing a convergence of the system’s states when the latter is not affected by delay and saturation. However, in contrast, as can be seen from Figure 4, the controller proposed by [45] is not able to stabilize the system when the delay occurs. Upon examination of the plotted figure in Figure 5, it is evident that the presented control law stabilizes the system despite the time-varying delay and saturation of inputs.

It should be noted that the simulation is conducted assuming measurement errors in m_p and m_c , which appear in membership functions such that $m_c = 2.5 + 0.5 \sin(t)$ and $m_p = 12 + 4 \sin(t)$. This means that the stability conditions based on the type-1 fuzzy system cannot be applied. In light of these results, the synthesized control law is effective in stabilizing the underlying system as well as remaining robust despite input saturation and uncertainty.

4.3. Comparative Explanations

The suggested strategy in this article can effectively solve the problem of admissibilization for mass–spring–damper and inverted pendulum mechanical systems based on the IT-2 fuzzy singular model subject to the time-varying delay and actuator saturation constraints. When compared to previous findings, the following are the primary advantages of the suggested method:

- (i) Compared to existing findings in [22,24,46], the outcome developed in this paper is more realistic and general, since the IT-2 fuzzy model incorporates the system uncertainties. In addition, the premise membership functions of fuzzy controllers and fuzzy systems are not the same.
- (ii) Though further results for interval-valued fuzzy singular systems have been published, such as [36,45], none of these results will be applicable when the system under examination exhibits actuator saturation.
- (iii) For this class of systems, considering the effects of dynamic quantization, using a dynamic/static output feedback controller or an observer-based controller [16] can be a significant issue.

5. Conclusions

An attempt was made in this study to provide solutions to the main challenges that come up when dealing with non-linear singular systems, such as uncertainty, time-varying delay, and saturation. The proposed control scheme emphasizes the use of a state feedback controller based on an IT-2 fuzzy model that exploits both the lower and upper membership functions to adequately characterize uncertainties. By employing an appropriate Lyapunov–Krasovskii functional with convex optimization techniques, the controller existence was analyzed. The proposed control scheme was validated by numerical simulations considering mass–spring–damper and inverted pendulum systems. Research areas that need to be pursued in the near future include quantized output feedback stabilization problems for Markovian jump singular IT-2 fuzzy systems with sensor and actuator saturation.

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