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Geodetic Applications and Improvement of the X- and L-Method of Deformation Analysis

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Abstract: Monitoring displacements of the object can be performed using geodetic methods by selecting reference points on the surrounding terrain and points on the object that discretely describe the object's behavior. The measurements are repeated in several epochs. By analyzing the geodetic network we can determine the status of a single point, i.e., whether the point has moved or not. The article discusses the testing of congruence, the testing of transformation of a single triangle, and the calculation of other deformation parameters in 2D networks resulting from the changes of points coordinates between two epochs. This is essentially the content of the Munich deformation method presented by W.M. Welsch, which includes the X- and L-method. The article also proposes some corrections to the original Munich approach. Finally, the applicability of the method is shown on a well-known practical example.

Keywords: deformation analysis; Munich approach; strain parameters; geodetic network

1. Introduction

In the field of geoscience, deformation analysis is a cornerstone for understanding the dynamic nature of our planet's surface. Measured changes in the position and shape of natural and man-made elements/objects in the environment over time can provide important information about the deformations and damages of structures due to seismic activity, tectonic shifts, and other geologic processes. Geodetic methods of deformation analysis play a central role in detecting the changes with the order of precision of few mm. Their unique feature is that they describe the state of the object based on the movements of discrete control points placed on the observed object.

The methods used to determine whether a geodetic point moved or not are called deformation analyses. Several methods of deformation analysis are well known: The essence of the Delft approach, introduced by J. van Mierlo and J. J. Kok [1] is the independent adjustment of individual measurement epochs and the transformation of the displacement solution from one geodetic datum to another for the appropriate statistical testing of point displacements. In the Hannover method the basis is the global congruence test. It consists of the test of homogeneity of precision of observations in two measurements, a global test of stability of network points in two measurements, the test of stability of reference points, a procedure for determining moving reference points and the test of point movements on the object [2,3]. The basic idea of the Fredericton method is to select the most appropriate deformation model for the displacement field and, using the least squares method, to determine the deformation parameters (except for the rotation) that are considered independent of the determination of the geodetic datum [4–6]. The core of the Karlsruhe method is the independent adjustment of individual measurements and then the overall adjustment of observations of both measurements simultaneously, where the reference points must be stationary. In both measurements the network scale must be the same, and in both measurements the accuracy of the observations must be homogeneous [1,7,8].

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In this article we discuss the Munich approach. This method was introduced by Welsch [9]. The essence of this method is to connect the points of the geodetic network into a triangular grid and explore the resulting strain state in each triangle. Before the deformation analysis, we must:

- ensure that the accuracy of the measurements in each epoch is not statistically significantly different from the accuracy of the measurements in other epochs,
- remove gross errors from the measurements,
- perform least squares (LS) adjustment of measurements in the geodetic network of two epochs as a free network [10] with the same approximate values of the unknowns,
- reduce the orientation unknowns, scale factor of the network,
- transform results into the same geodetic datum,
- check the homogeneity of the accuracy of the considered measurements and calculate

$$s^2 = \frac{\mathbf{v}_1^{\mathrm{T}}\mathbf{P}_{\mathrm{ll}_1}\mathbf{v}_1 + \mathbf{v}_2^{\mathrm{T}}\mathbf{P}_{\mathrm{ll}_2}\mathbf{v}_2}{f_1 + f_2} = \frac{f_1 s_1^2 + f_2 s_2^2}{f}, \quad (1)$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ represent vectors of measurements' residuals, $\mathbf{P}_{\mathrm{ll}_1}$ and $\mathbf{P}_{\mathrm{ll}_2}$ matrices of weights, f_1 and f_2 numbers of redundant measurements (redundancy), s_1^2 in s_2^2 reference variances a posteriori after the adjustment of previous and actual epoch, respectively,

- compute the displacement vector

$$\mathbf{u} = \hat{\mathbf{x}}_2 - \hat{\mathbf{x}}_1, \quad (2)$$

where $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ are two vectors of adjusted point coordinates after the LS adjustment of 1st and 2nd epochs in t_1 and t_2,

- compute cofactor matrix of coordinate differences

$$\mathbf{Q}_{\mathbf{u}} = \mathbf{Q}_{\hat{\mathbf{x}}_1\hat{\mathbf{x}}_1} + \mathbf{Q}_{\hat{\mathbf{x}}_2\hat{\mathbf{x}}_2}, \quad (3)$$

where $\mathbf{Q}_{\hat{\mathbf{x}}_1\hat{\mathbf{x}}_1} = \left(\mathbf{A}_1^{\mathrm{T}}\mathbf{P}_{\mathrm{ll}_1}\mathbf{A}_1\right)^+$ and $\mathbf{Q}_{\hat{\mathbf{x}}_2\hat{\mathbf{x}}_2} = \left(\mathbf{A}_2^{\mathrm{T}}\mathbf{P}_{\mathrm{ll}_2}\mathbf{A}_2\right)^+$ are cofactor matrices of coordinate differences after adjustment, $\mathbf{A}_1$ and $\mathbf{A}_2$ are the design matrices, $\mathbf{P}_{\mathrm{ll}_1}$ and $\mathbf{P}_{\mathrm{ll}_2}$ weight matrix of observations of 1st and 2nd epoch in time t_1 and t_2 [10].

Deformation analysis can be performed in two different ways:

- by X-method, which is based on the comparison of the coordinates of the points in two epochs, which depends on the geodetic datum. If two epochs correspond to different geodetic datum, the problem can be solved by S-transformation [11],
- by L-method, which is based on the comparison of the measured quantities, which are not biased by the geodetic datum, i.e., angles and distances in geodetic network.

2. Methods/Theory

2.1. Congruence Test of Geodetic Network

2.1.1. X-Method

The decision whether statistically significant displacements and deformations have occurred in the geodetic network is derived by congruence test. Firstly, the null and alternative hypothesis [12,13] are proposed:

H_0: $E(\hat{\mathbf{x}}_1) = E(\hat{\mathbf{x}}_2)$ or $E(\mathbf{u}) = \mathbf{0}$; the coordinates of all points in the network have not changed between two epochs;

H_a: $E(\hat{\mathbf{x}}_1) \neq E(\hat{\mathbf{x}}_2)$ or $E(\mathbf{u}) \neq \mathbf{0}$; the coordinates of at least one point in the network have changed between two epochs.

We use the test statistic [12,13]:

$$T_{11}^2 = \frac{s_{\mathbf{u}}^2}{s^2}, \quad (4)$$

where

$$s_{\mathbf{u}}^2 = \frac{q_{\mathbf{u}}}{f_{\mathbf{u}}} = \frac{\mathbf{u}^{\mathrm{T}}\mathbf{Q}_{\mathbf{u}}^{-}\mathbf{u}}{\mathrm{rank}\,\mathbf{Q}_{\mathbf{u}}} = \frac{\mathbf{u}^{\mathrm{T}}\mathbf{Q}_{\mathbf{u}}^{-}\mathbf{u}}{u-d}, \tag{5}$$

is the weighted variance of coordinate differences of points in the geodetic network, where $f_{\mathbf{u}} = \mathrm{rank}\,\mathbf{Q}_{\mathbf{u}} = u - d$ represents the number of degrees of freedom, $u = 2m$ is the number of coordinate unknowns (m is the number of points in geodetic network), and d is the datum defect which is equal to the defect of rank of matrix $\mathbf{Q}_{\mathbf{u}}$.

The test statistic T_{11}^2 is distributed according to the Fischer distribution $F_{f_{\mathbf{u}},f,1-\alpha}$ [12,13] with $f_{\mathbf{u}}$ and f numbers of degrees of freedom:

- if $T_{11}^2 \leq F_{f_{\mathbf{u}},f,1-\alpha}$ the null hypothesis cannot be rejected and with the significance level α we cannot claim that deformations have occurred in the network,
- if $T_{11}^2 > F_{f_{\mathbf{u}},f,1-\alpha}$ the null hypothesis is rejected and with significance level of less than α we can claim that deformations have occurred in the network.

2.1.2. L-Method

To test the congruence by the L-method, we state the null and alternative hypothesis [13] as follows:

H_0: $E(\Delta\mathbf{l}) = \mathbf{0}$; observations values in network have not changed between two epochs;

H_a: $E(\Delta\mathbf{l}) \neq \mathbf{0}$; observations values in network have changed between two epochs.

We use the test statistic [12,13]:

$$T_{12}^2 = \frac{s_{\Delta\mathbf{l}}^2}{s^2}, \tag{6}$$

where

$$s_{\Delta\mathbf{l}}^2 = \frac{q_{\Delta\mathbf{l}}}{f_{\Delta\mathbf{l}}} = \frac{\Delta\mathbf{l}^{\mathrm{T}}\mathbf{Q}_{\Delta\mathbf{l}}^{-}\Delta\mathbf{l}}{\mathrm{rank}\,\mathbf{Q}_{\Delta\mathbf{l}}} = \frac{\Delta\mathbf{l}^{\mathrm{T}}\mathbf{Q}_{\Delta\mathbf{l}}^{-}\Delta\mathbf{l}}{u-d}, \tag{7}$$

is the weighted variance of observation differences,

$$\Delta\mathbf{l} = \mathbf{l}_2 - \mathbf{l}_1, \tag{8}$$

is the change in the values of the same type of measurements $\mathbf{l}_1$ in $\mathbf{l}_2$, which are computed from the adjusted coordinates in epochs t_1 in t_2, and

$$\mathbf{Q}_{\Delta\mathbf{l}} = \mathbf{L}\mathbf{Q}_{\mathbf{u}}\mathbf{L}^{\mathrm{T}}, \tag{9}$$

is the cofactor matrix of measurement differences, where $\mathrm{rank}\,\mathbf{Q}_{\Delta\mathbf{l}} = \mathrm{rank}\,\mathbf{Q}_{\mathbf{u}}$, as it is $\mathbf{L}^{\mathrm{T}}\left(\mathbf{L}\mathbf{Q}_{\mathbf{u}}\mathbf{L}^{\mathrm{T}}\right)^{-}\mathbf{L}$ the g-inversion [14] of $\mathbf{Q}_{\mathbf{u}}$ [13].

The differences $\Delta\mathbf{l}$ can be written as a function of point displacements

$$\Delta\mathbf{l} = \mathbf{L}\mathbf{u}, \tag{10}$$

with the corresponding cofactor matrix

$$\mathbf{Q}_{\Delta\mathbf{l}} = \mathbf{L}\mathbf{Q}_{\mathbf{u}}\mathbf{L}^{\mathrm{T}}. \tag{11}$$

The elements of the matrix of partial derivatives of measurements with respect to the coordinate unknowns $\mathbf{L} = \left[\frac{\partial\mathbf{l}}{\partial\hat{\mathbf{x}}}\right]$ depend on the type of measurements:

(a) If the distances in the network are considered, the elements of matrix $\mathbf{L}$ and vector $\Delta\mathbf{l}$ are of the form:

$$\mathbf{L} = \mathbf{L}_{\Delta D} = \begin{bmatrix} \frac{\partial D_{ij}}{\partial\hat{y}_i} & \frac{\partial D_{ij}}{\partial\hat{x}_i} & \frac{\partial D_{ij}}{\partial\hat{y}_j} & \frac{\partial D_{ij}}{\partial\hat{x}_j} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} -\sin\nu_{ij} & -\cos\nu_{ij} & \sin\nu_{ij} & \cos\nu_{ij} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{12}$$

and

$$\Delta\mathbf{l} = \Delta\mathbf{l}_{\Delta D} = \begin{bmatrix} D_{ij_2} - D_{ij_1} \\ \vdots \end{bmatrix}, \quad (13)$$

where $\nu_{ij} = \left(\nu_{ij_1} + \nu_{ij_2}\right)/2$ is the average value of bearing angles, $\nu_{ij_1} = \arctan\frac{y_{j_1}-y_{i_1}}{x_{j_1}-x_{i_1}}$ and $\nu_{ij_2} = \arctan\frac{y_{j_2}-y_{i_2}}{x_{j_2}-x_{i_2}}$, which are computed from the adjusted coordinates between points P_i and P_j in the previous and current time epochs t_1 in t_2, $D_{ij_1} = \sqrt{\left(y_{j_1} - y_{i_1}\right)^2 + \left(x_{j_1} - x_{i_1}\right)^2}$ and $D_{ij_2} = \sqrt{\left(y_{j_2} - y_{i_2}\right)^2 + \left(x_{j_2} - x_{i_2}\right)^2}$ are the distances which are computed from the adjusted coordinates between points P_i and P_j in the previous and current time epoch.

(b) If the angles in the network are considered, the elements of matrix **L** and vector $\Delta\mathbf{l}$ are of the form:

$$\mathbf{L} = \mathbf{L}_{\Delta\alpha} = \begin{bmatrix} \frac{\partial\alpha_{ij}}{\partial\hat{y}_i} & \frac{\partial\alpha_{ij}}{\partial\hat{x}_i} & \frac{\partial\alpha_{ij}}{\partial\hat{y}_j} & \frac{\partial\alpha_{ij}}{\partial\hat{x}_j} & \frac{\partial\alpha_{ij}}{\partial\hat{y}_k} & \frac{\partial\alpha_{ij}}{\partial\hat{x}_k} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$
$$= \begin{bmatrix} \left(-\frac{\cos\nu_{ik}}{D_{ik}} + \frac{\cos\nu_{ij}}{D_{ij}}\right) & \left(\frac{\sin\nu_{ik}}{D_{ik}} - \frac{\sin\nu_{ij}}{D_{ij}}\right) & \left(-\frac{\cos\nu_{ij}}{D_{ij}}\right) & \left(\frac{\sin\nu_{ij}}{D_{ij}}\right) & \left(\frac{\cos\nu_{ik}}{D_{ik}}\right) & \left(-\frac{\sin\nu_{ik}}{D_{ik}}\right) & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \quad (14)$$

and

$$\Delta\mathbf{l} = \Delta\mathbf{l}_{\Delta\alpha} = \begin{bmatrix} \alpha_{ijk_2} - \alpha_{ijk_1} \\ \vdots \end{bmatrix}, \quad (15)$$

where $D_{ij} = \left(D_{ij_1} + D_{ij_2}\right)/2$ and $D_{ik} = \left(D_{ik_1} + D_{ik_2}\right)/2$ are the average values of the distances computed from the adjusted coordinates between points P_i and P_j or P_i and P_k in the previous and current time epochs, and $\alpha_{ijk_1} = \nu_{ik_1} - \nu_{ij_1}$ and $\alpha_{ijk_2} = \nu_{ik_2} - \nu_{ij_2}$ are differences in bearing angles on point P_i computed from the adjusted coordinates between points P_i and P_j or P_i and P_k, in the previous and current epochs.

(c) If the angles and the distances in the network are considered, the elements of matrix **L** and vector $\Delta\mathbf{l}$ are obtained by joining the matrices $\mathbf{L}_{\Delta D}$ and $\mathbf{L}_{\Delta\alpha}$ for **L** and $\Delta\mathbf{l}_{\Delta D}$ and $\Delta\mathbf{l}_{\Delta\alpha}$ for $\Delta\mathbf{l}$:

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_{\Delta D} \\ \mathbf{L}_{\Delta\alpha} \end{bmatrix} \text{ and } \Delta\mathbf{l} = \begin{bmatrix} \Delta\mathbf{l}_{\Delta D} \\ \Delta\mathbf{l}_{\Delta\alpha} \end{bmatrix}. \quad (16)$$

The test statistic T_{12}^2 is distributed according to the Fischer distribution $F_{f_{\Delta\mathbf{l}},f,1-\alpha}$ [12,13]:

- if $T_{12}^2 \leq F_{f_{\Delta\mathbf{l}},f,1-\alpha}$ the null hypothesis cannot be rejected and with the significance level α we cannot claim that deformations have occurred in the network,
- if $T_{12}^2 > F_{f_{\Delta\mathbf{l}},f,1-\alpha}$ the null hypothesis is rejected and with the significance level of less than α we can claim that deformations have occurred in the network.

2.2. Strain Testing

2.2.1. X-Method

We test the resulting deformations of the geodetic network by dividing it into triangles and test the transformation (change of shape or size) of each triangle. We write the null and alternative hypothesis:

H_0: $E(\mathbf{u} - \mathbf{H_u p}) = \mathbf{0}$; the shape or size of the triangle has not changed between two epochs;
H_a: $E(\mathbf{u} - \mathbf{H_u p}) \neq \mathbf{0}$; the shape or size of the triangle has changed between two epochs.

According to the theory of homogeneous deformations, the linear functional relationship between the coordinates of the network points from two epochs can be written as follows:

$$\hat{\mathbf{x}}_2 = \mathbf{F}\cdot\hat{\mathbf{x}}_1 + \mathbf{t}, \quad (17)$$

where $\mathbf{F} = \left[\frac{\partial \hat{\mathbf{x}}_2}{\partial \hat{\mathbf{x}}_1}\right] = \begin{bmatrix} \frac{\partial \hat{\mathbf{x}}_2}{\partial \hat{\mathbf{x}}_1} & \frac{\partial \hat{\mathbf{x}}_2}{\partial \hat{\mathbf{y}}_1} \\ \frac{\partial \hat{\mathbf{y}}_2}{\partial \hat{\mathbf{x}}_1} & \frac{\partial \hat{\mathbf{y}}_2}{\partial \hat{\mathbf{y}}_1} \end{bmatrix}$ are the derivatives of point coordinates of actual measuring campaign with respect to previous one, $\mathbf{t} = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$ is the vector of the components of the displacement of a rigid body (object) in the directions of the coordinate axes.

If we subtract vector $\hat{\mathbf{x}}_1$ from the Equation (17) we get the vector of point displacements:

$$\mathbf{u} = \hat{\mathbf{x}}_2 - \hat{\mathbf{x}}_1 = (\mathbf{F} - \mathbf{I})\cdot\hat{\mathbf{x}}_1 + \mathbf{t} = \mathbf{G}\cdot\hat{\mathbf{x}}_1 + \mathbf{t} = (\mathbf{E} + \mathbf{R})\cdot\hat{\mathbf{x}}_1 + \mathbf{t} \tag{18}$$

or

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \left(\begin{bmatrix} e_{xx} & e_{xy} \\ e_{xy} & e_{yy} \end{bmatrix} + \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \right) \begin{bmatrix} \hat{x}_1 \\ \hat{y}_1 \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \tag{19}$$

or by components

$$u_x = \hat{x}_1\cdot e_{xx} + \hat{y}_1\cdot e_{xy} \qquad - \hat{y}_1\cdot\omega + t_x \tag{20}$$

$$u_y = \qquad \hat{x}_1\cdot e_{xy} + \hat{y}_1\cdot e_{yy} + \hat{x}_1\cdot\omega + t_y \tag{21}$$

or in matrix form

$$\mathbf{u} = \mathbf{H_u}\cdot\mathbf{p}, \tag{22}$$

where $\mathbf{H_u} = \begin{bmatrix} \hat{x}_1 & \hat{y}_1 & 0 & -\hat{y}_1 & 1 & 0 \\ 0 & \hat{x}_1 & \hat{y}_1 & \hat{x}_1 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$ is the matrix of the deformation model, which relates the deformation parameters to point displacements, and $\mathbf{p}^{\mathrm{T}} = \begin{bmatrix} e_{xx} & e_{xy} & e_{yy} & \omega & t_x & t_y \end{bmatrix}$ is the vector of kinematic quantities, where e_{xx} and e_{yy} are normal strains, e_{xy} is shear strain, ω is the rotation and t_x and t_y are displacements, i.e., rigid body movements.

When we consider the system (22) for a single triangle, we have three possibilities:

(a) If we consider one point in triangle, we have two Equations (20) and (21), therefore two lines in $\mathbf{H_u}$, six unknowns in $\mathbf{p}$ and system (22) has no unique solution.

(b) If we consider all points in triangle, we have six equations, three for (20) and three for (21), therefore six lines in $\mathbf{H_u}$, six unknowns in $\mathbf{p}$ and the solution for system is $\mathbf{p} = \mathbf{H_u}^{-1}\cdot\mathbf{u}$. We use the test statistic:

$$T_{21}^2 = \frac{s_{\mathbf{p}}^2}{s^2}, \tag{23}$$

where $s_{\mathbf{p}}^2 = \frac{q_{\mathbf{p}}}{f_{\mathbf{p}}} = \frac{\mathbf{u}^{\mathrm{T}}\mathbf{Q_u}^{-1}\mathbf{u}}{n}$, $\mathbf{Q_u}$ is the cofactor matrix of coordinate differences contains only the corresponding elements that refer to the points that form the considered triangle, so we must compute it as

$$\mathbf{Q_u} = \mathbf{Q}_{\hat{\mathbf{x}}_1\hat{\mathbf{x}}_1} + \mathbf{Q}_{\hat{\mathbf{x}}_2\hat{\mathbf{x}}_2} = \left(\mathbf{A}_1^{\mathrm{T}}\mathbf{P}_{\mathbf{ll}_1}\mathbf{A}_1 + \mathbf{D}_1^{\mathrm{T}}\mathbf{D}_1\right)^{-1} + \left(\mathbf{A}_2^{\mathrm{T}}\mathbf{P}_{\mathbf{ll}_2}\mathbf{A}_2 + \mathbf{D}_2^{\mathrm{T}}\mathbf{D}_2\right)^{-1}, \tag{24}$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are configuration (design) matrices (coefficients of equations of observation residuals), $\mathbf{P}_{\mathbf{ll}_1}$ and $\mathbf{P}_{\mathbf{ll}_2}$ weight matrices of observations and $\mathbf{D}_1$ and $\mathbf{D}_2$ are datum matrices of previous and current time epochs t_1 and t_2 [10], $f_{\mathbf{p}} = \text{rank}\ \mathbf{Q_u} = n = 3$ is the number of degrees of freedom.

The test statistic is distributed according to the Fisher distribution $F_{n,f,1-\alpha}$:

- if $T_{21}^2 \le F_{n,f,1-\alpha}$ the null hypothesis cannot be rejected and with the significance level α we cannot claim that deformations have occurred in the triangle,
- if $T_{21}^2 > F_{n,f,1-\alpha}$ the null hypothesis is rejected and with the significance level of less than α we can claim that deformations have occurred in the triangle.

(c) If in the system (22) we consider more equations than unknowns the solution is obtained by the method of LS adjustment. System (22) is transformed into

$$\mathbf{u}+\mathbf{v_u}=\mathbf{H_u}\cdot\mathbf{p}, \tag{25}$$

where $\mathbf{v_u}$ is the correction vector of point coordinate differences and $\mathbf{P_u}=\mathbf{Q_u^{-1}}$ is the weight matrix of point coordinates differences. The solution of the system is then

$$\mathbf{p}=\left(\mathbf{H_u^T P_u H_u}\right)^{-1}\cdot\mathbf{H_u^T P_u u}. \tag{26}$$

We use the test statistic:

$$T_{21}^2=\frac{s_\mathbf{p}^2}{s^2}, \tag{27}$$

where:

$$s_\mathbf{p}^2=\frac{q_{\mathbf{u}+\mathbf{v_u}}}{f_\mathbf{p}}=\frac{(\mathbf{u}+\mathbf{v_u})^\mathrm{T}\mathbf{Q_u^{-1}}(\mathbf{u}+\mathbf{v_u})}{n}=\frac{(\mathbf{H_u}\cdot\mathbf{p})^\mathrm{T}\mathbf{Q_u^{-1}}(\mathbf{H_u}\cdot\mathbf{p})}{n}, \tag{28}$$

which is not the same as the equation

$$s_\mathbf{p}^2=\frac{q_{\mathbf{v_u}}}{f_\mathbf{p}}=\frac{\mathbf{v_u^T P_u v_u}}{f_\mathbf{u}-\mathrm{rank}\left(\mathbf{H_u^T P_u H_u}\right)}=\frac{\mathbf{v_u^T P_u v_u}}{f_\mathbf{u}-n_p}, \tag{29}$$

written by Welsch [12]—Equation (19), Welsch and Zhang [13]—Equations (3)–(8); where $n_p=6$ is the number of kinematic parameters.

The test statistic is distributed according to the Fischer distribution $F_{n,f.1-\alpha}$:

- if $T_{21}^2 \leq F_{n,f,1-\alpha}$ the null hypothesis cannot be rejected and with the significance level α we cannot claim that deformations have occurred in the triangle,
- if $T_{21}^2 > F_{n,f,1-\alpha}$ the null hypothesis is rejected and with the significance level of less than α we can claim that deformations have occurred in the triangle.

2.2.2. L-Method

To test the transformation of an individual triangle (strain analysis) in the L-method, we write zero and alternative hypothesis [13]:

H_0: $E(\Delta\mathbf{l}-\mathbf{H}_{\Delta\mathbf{l}}\cdot\mathbf{e_x})=\mathbf{0}$; the shape or size of the triangle has not changed between two epochs;

H_a: $E(\Delta\mathbf{l}-\mathbf{H}_{\Delta\mathbf{l}}\cdot\mathbf{e_x})\neq\mathbf{0}$; the shape or size of the triangle has changed between two epochs.

The relationship between distance deformations e, angles deformations $\Delta\alpha$ and deformation parameters can be written as [12,13,15]:

$$e_{ij}=e_{xx}\cos^2\nu_{ij}+e_{xy}\sin 2\nu_{ij}+e_{yy}\sin^2\nu_{ij} \text{ and} \tag{30}$$

$$\Delta\alpha_{ijk}=e_{xy}\left(\cos 2\nu_{ik}-\cos 2\nu_{ij}\right)+\frac{1}{2}\left(e_{yy}-e_{xx}\right)\left(\sin 2\nu_{ik}-\sin 2\nu_{ij}\right), \tag{31}$$

where:

$$e_{ij}=\frac{D_{ij_2}-D_{ij_1}}{D_{ij_1}} \tag{32}$$

is the specific normal extension and

$$\Delta\alpha_{ijk}=\alpha_{ijk_2}-\alpha_{ijk_1} \tag{33}$$

is the change of the right angle.

Equations (30) and (31) can be written in matrix form:

$$\Delta\mathbf{l}=\mathbf{H}_{\Delta\mathbf{l}}\cdot\mathbf{e_x}, \tag{34}$$

$$\mathbf{Q}_{\Delta\mathbf{l}} = \mathbf{L}\mathbf{Q}_{\mathbf{u}}\mathbf{L}^{\mathrm{T}}, \tag{35}$$

where $\mathbf{Q}_{\Delta\mathbf{l}}$ is the corresponding cofactor matrix, as the normal deformations e or angle changes $\Delta\alpha$ are considered as observations, where

$$\Delta\mathbf{l} = \Delta\mathbf{l}_e = \begin{bmatrix} e_{ij} \\ \vdots \end{bmatrix}, \tag{36}$$

is the vector of specific normal deformations if distances are considered and

$$\Delta\mathbf{l} = \Delta\mathbf{l}_{\Delta\alpha} = \begin{bmatrix} \Delta\alpha_{ijk} \\ \vdots \end{bmatrix}, \tag{37}$$

is the vector of right angle changes if angles are considered,

$$\mathbf{H}_{\Delta\mathbf{l}} = \mathbf{H}_{\Delta\mathbf{l}_e} = \begin{bmatrix} \cos^2\nu_{ij} & \sin 2\nu_{ij} & \sin^2\nu_{ij} \\ \vdots & \vdots & \vdots \end{bmatrix}, \tag{38}$$

is one line in deformation model matrix if distances are considered,

$$\mathbf{H}_{\Delta\mathbf{l}} = \mathbf{H}_{\Delta\mathbf{l}_{\Delta\alpha}} = \begin{bmatrix} \frac{1}{2}\left(-\sin 2\nu_{ik} + \sin 2\nu_{ij}\right) & \left(\cos 2\nu_{ik} - \cos 2\nu_{ij}\right) & \frac{1}{2}\left(\sin 2\nu_{ik} - \sin 2\nu_{ij}\right) \\ \vdots & \vdots & \vdots \end{bmatrix}, \tag{39}$$

is one line in deformation model matrix if angles are considered,

$$\mathbf{e}_{\mathbf{x}}^{\mathrm{T}} = \begin{bmatrix} e_{xx} & e_{xy} & e_{yy} \end{bmatrix}, \tag{40}$$

is the vector of deformation parameters.

When system (34) is under consideration, there are three possibilities:

(a) If we consider one normal deformation (32) or one angle change (33) we have one line in matrix of the deformation model $\mathbf{H}_{\Delta\mathbf{l}} = \mathbf{H}_{\Delta\mathbf{l}_e}$ or $\mathbf{H}_{\Delta\mathbf{l}} = \mathbf{H}_{\Delta\mathbf{l}_{\Delta\alpha}}$. Such a system (34) has no unique solution.

(b1) If we consider normal deformations (32) e_{ij} in triangle we have three lines in matrix of the deformation model (since we have three independent distances in triangle) and the solution of the system (34) is

$$\mathbf{e}_{\mathbf{x}} = \mathbf{H}_{\Delta\mathbf{l}}^{-1} \cdot \Delta\mathbf{l}, \tag{41}$$

We use the test statistic:

$$T_{22}^2 = \frac{s_{\Delta\mathbf{l}}^2}{s^2}, \tag{42}$$

where

$$s_{\Delta\mathbf{l}}^2 = \frac{q_{\Delta\mathbf{l}}}{f_{\Delta\mathbf{l}}} = \frac{\Delta\mathbf{l}^{\mathrm{T}}\mathbf{Q}_{\Delta\mathbf{l}}^{-1}\Delta\mathbf{l}}{\operatorname{rank}\mathbf{Q}_{\Delta\mathbf{l}}} = \frac{\Delta\mathbf{l}^{\mathrm{T}}\mathbf{Q}_{\Delta\mathbf{l}}^{-1}\Delta\mathbf{l}}{n}, \tag{43}$$

and $n = 3$ is the number of degrees of freedom.

The test statistic is distributed according to the Fischer distribution $F_{n,f,1-\alpha}$:

- if $T_{22}^2 \leq F_{n,f,1-\alpha}$ the null hypothesis cannot be rejected and with the significance level α we cannot claim that deformations have occurred in the triangle,
- if $T_{22}^2 > F_{n,f,1-\alpha}$ the null hypothesis is rejected and with the significance level of less than α we can claim that deformations have occurred in the triangle.

(b2) If we consider the angular changes (33) $\Delta\alpha_{ijk}$ in the triangle, we have only two independent angles—the third depends on the other two, then we can write only two independent equations (31) for a single triangle and the system (34) still has no unique solution. The matrix $\mathbf{H}_{\Delta l}$ is singular, its rank is 2.

(c) If we consider more normal deformations (32) e_{ij} and angular changes (33) $\Delta\alpha_{ijk}$ than are necessary (e.g., three normal deformations and two/three angular changes in the triangle), then in system (34) we have more equations than unknowns. The solution $\mathbf{e_x}$ is obtained by the least squares method [12,13].

(c1) When we compute the solution $\mathbf{e_x}$, the system (21) is transformed into

$$\Delta\mathbf{l} + \mathbf{v}_{\Delta l} = \mathbf{H}_{\Delta l} \cdot \mathbf{e_x}, \quad (44)$$

where

$$\Delta\mathbf{l} = \begin{bmatrix} \Delta\mathbf{l}_e \\ \Delta\mathbf{l}_{\Delta\alpha} \end{bmatrix} = \begin{bmatrix} e_{ij} \\ \Delta\alpha_{ijk} \end{bmatrix} = \begin{bmatrix} \frac{D_{ij_2} - D_{ij_1}}{D_{ij_1}} \\ \alpha_{ijk_2} - \alpha_{ijk_1} \end{bmatrix} = \begin{bmatrix} \text{relative dist. change} \\ \text{angle change} \end{bmatrix}, \quad (45)$$

with the corresponding $\mathbf{P}_{\Delta l}$, where $\mathbf{v}_{\Delta l}$ is the residual vector of relative normal deformations and angular changes,

$$\mathbf{P}_{\Delta l} = \mathbf{Q}_{\Delta l}^{-1}, \quad (46)$$

is the weight matrix of normal deformations and angular changes and

$$\mathbf{H}_{\Delta l} = \begin{bmatrix} \mathbf{H}_{\Delta l_e} \\ \mathbf{H}_{\Delta l_{\Delta\alpha}} \end{bmatrix}, \quad (47)$$

matrix of the deformation model when we consider normal deformations and angular changes. We get the solution of the system

$$\mathbf{e_x} = \left(\mathbf{H}_{\Delta l}^{\mathrm{T}} \mathbf{P}_{\Delta l} \mathbf{H}_{\Delta l}\right)^{-1} \cdot \mathbf{H}_{\Delta l}^{\mathrm{T}} \mathbf{P}_{\Delta l} \Delta\mathbf{l}. \quad (48)$$

(c2) When we compute the test statistic, the system (34) can be written as

$$\Delta\mathbf{l} + \mathbf{v}_{\Delta l} = \mathbf{H}_{\Delta l} \cdot \mathbf{e}_2, \quad (49)$$

and we use the test statistic:

$$T_{22}^2 = \frac{s_{\mathbf{e}_2}^2}{s^2}, \quad (50)$$

where:

$$s_{\mathbf{e}_2}^2 = \frac{q_{\Delta l + \mathbf{v}_{\Delta l}}}{f_{\mathbf{e}_2}} = \frac{(\Delta\mathbf{l} + \mathbf{v}_{\Delta l})^{\mathrm{T}} \mathbf{Q}_{\Delta l}^{-1} (\Delta\mathbf{l} + \mathbf{v}_{\Delta l})}{n} = \frac{(\mathbf{H}_{\Delta l} \cdot \mathbf{e}_2)^{\mathrm{T}} \mathbf{Q}_{\Delta l}^{-1} (\mathbf{H}_{\Delta l} \cdot \mathbf{e}_2)}{n}, \quad (51)$$

which is different than the equation

$$q_{\mathbf{v}_{\Delta l}} = \mathbf{v}_{\Delta l}^{\mathrm{T}} \mathbf{Q}_{\Delta l}^{+} \mathbf{v}_{\Delta l} = \mathbf{v}_{\Delta l}^{\mathrm{T}} \mathbf{P}_{\Delta l} \mathbf{v}_{\Delta l} = q_{\mathbf{v_u}}, \quad (52)$$

written by Welsch [12] in Equation (32), and Welsch and Zhang [13] in Equations (3)–(13), where

$$\Delta\mathbf{l} = \begin{bmatrix} \Delta\mathbf{l}_{\Delta D} \\ \Delta\mathbf{l}_{\Delta\alpha} \end{bmatrix} = \begin{bmatrix} D_{ij_2} - D_{ij_1} \\ \alpha_{ijk_2} - \alpha_{ijk_1} \end{bmatrix} = \begin{bmatrix} \text{dist. change} \\ \text{angle change} \end{bmatrix}, \quad (53)$$

is the vector of changes in measurement values in Equation (16),

$$\mathbf{v}_{\Delta l} = \mathbf{H}_{\Delta l} \mathbf{e}_2 - \Delta\mathbf{l}, \quad (54)$$

is the residual vector of changes in measurement values,

$$\mathbf{e}_2 = \left(\mathbf{H}_{\Delta\mathbf{l}}^{\mathrm{T}}\mathbf{P}_{\Delta\mathbf{l}}\mathbf{H}_{\Delta\mathbf{l}}\right)^{-1}\mathbf{H}_{\Delta\mathbf{l}}^{\mathrm{T}}\mathbf{P}_{\Delta\mathbf{l}}\Delta\mathbf{l}, \tag{55}$$

is the auxiliary vector and $n = 3$ is the number of degrees of freedom.

Test statistic is distributed according to the Fischer distribution $F_{n,f,1-\alpha}$:

- if $T_{22}^2 \leq F_{n,f,1-\alpha}$ the null hypothesis cannot be rejected and with the significance level α we cannot claim that deformations have occurred in the triangle,
- if $T_{22}^2 > F_{n,f,1-\alpha}$ the null hypothesis is rejected and with the significance level of less than α we can claim that deformations have occurred in the triangle.

2.3. Computation of Additional Deformation Parameters

Other deformation parameters can be computed based on the basic parameters using the following equations [12,15]:

$$\Delta = e_{xx} + e_{yy} \text{ is the change of area,} \tag{56}$$

$$e_1 = \frac{1}{2}\left(e_{xx} + e_{yy} + ee\right) \text{ is the principal (the largest) normal deformation or strain,} \tag{57}$$

$$e_2 = \frac{1}{2}\left(e_{xx} + e_{yy} - ee\right) \text{ is the second principal (the smallest) normal deformation or strain,} \tag{58}$$

$$ee^2 = \left(e_{xx} - e_{yy}\right)^2 + 4e_{xy}^2, \tag{59}$$

$$e_{\mathrm{I}} = \frac{e_1 - e_2}{2} \text{ is the principal shear deformation or strain,} \tag{60}$$

$$\gamma = 2e_{xy} \text{ is the engineering shear strain and represents the change of the right angle between } x \text{ and } y \text{ directions,} \tag{61}$$

$$\tan 2\vartheta = \frac{2e_{xy}}{e_{xx} - e_{yy}} \text{ is the bearing of the principal normal strains,} \tag{62}$$

$$\Psi = \vartheta + 45^\circ \text{ is the bearing angle of the principal shear strain.} \tag{63}$$

2.4. Analysis of Distance between Two Points

In the previous subsection we analyzed the deformation of a single triangle, we did not address the movements of a single point or pair of points. By testing point by point with respect to the other $n - 1$ points of the geodetic network, we can find out which points have moved in a statistically significant way. The test is done by examining the changes in all $n - 1$ (datum-independent) distances connecting a single point to other points. We formulate the null and alternative hypothesis [12]:

H_0: the distance between two points has not changed between two epochs;
H_a: the distance between two points has changed between two epochs.

We use the test statistic [12]:

$$T_{23}^2 = \frac{\Delta\mathbf{l}^{\mathrm{T}}\mathbf{Q}_{\Delta\mathbf{l}}^{-}\Delta\mathbf{l}}{n_D\, s^2}, \tag{64}$$

where $\Delta\mathbf{l}$ is the vector of distance differences between the selected point and the other $n - 1$ points of the geodetic network—Equation (13), $\mathbf{Q}_{\Delta\mathbf{l}} = \mathbf{L}\mathbf{Q}_{\mathbf{u}}\mathbf{L}^{\mathrm{T}}$ is the corresponding cofactor matrix, $\mathbf{L} = \left[\frac{\partial \mathbf{l}}{\partial \hat{\mathbf{x}}}\right]$ is the matrix of partial derivatives of measurements with respect to the coordinate unknowns with $n-1$ lines—from Equation (12) and $n_D = 1$ is the number of degrees of freedom.

Test statistic is distributed according to the Fischer distribution $F_{n_D,f,1-\alpha}$:

- if $T_{23}^2 \leq F_{n_D,f,1-\alpha}$ the null hypothesis cannot be rejected and with the significance level α we cannot claim that the considered distance has changed,
- if $T_{23}^2 > F_{n_D,f,1-\alpha}$ the null hypothesis is rejected and with the significance level of less than α we can claim that the considered distance has changed.

3. Results

We will demonstrate the usefulness of the described method (with the corrections of the original Munich approach) with a simulated example from the literature [16]—Figure 1. For all tests, we choose the significance level $\alpha = 0.05$. Three different geometries of geodetic networks were tried in which different characteristics of triangles were implemented. The Montsalvens geodetic network includes 12 points, of which 7 are reference points (these are points 1, 2, 3, 4, 6, 7 and 9) and 5 points on the crest of the dam (these are points from 10 to 14). At 5 points, 49 directions and 6 lengths are simulated in each epoch. In LS adjustment we have 24 coordinate unknowns and 5 orientation unknowns. Thus, the number of redundant measurements is $f_1 = f_2 = n - u + d = (49+6) - (24+5) + 3 = 29$. In the LS adjustments, we compute the a posteriori reference variance $s_1^2 = 1.044$ and $s_2^2 = 0.996$. Other results are not given here, as they can be found in the literature [16].

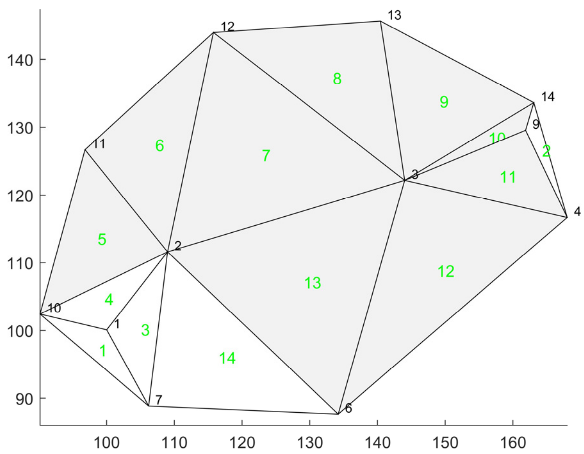

Figure 1. First geometry of triangles in geodetic network.

We continue the computation by checking the congruence of the geodetic network. The results are shown in Table 1.

Table 1. Testing the congruence of geodetic network (critical value $F_{21,58} = 1.74$).

Method		Test Statistic	Equation
X-method	coordinates	$T_{11}^2 = 11.15$	(4)
L-method	distances	$T_{12}^2 = 11.15$	(6), (12) and (13)
	angles	$T_{12}^2 = \text{NaN}\ *$	(6), (14) and (15)
	dist. + angles	$T_{12}^2 = 11.15$	(6) and (16)

* If only angles of the triangle are known, the triangle cannot be analyzed.

Since in all cases the test statistic T_{11}^2 or T_{12}^2 is greater than the critical value $F_{21,58,0.95} = 1.74$, we reject the null hypothesis and claim with a significance level of less than $\alpha = 0.05$ that deformations have occurred in the network.

The next step of the procedure is the testing of the deformation of the geodetic network, which we will use to determine the resulting deformations. We connect the points of the geodetic network to form a triangular grid of the first geometry shown in Figure 1, and the results are given in Table 2 where *p*-value is the statistical measure used to decide if the null-hypothesis is to be rejected.

Table 2. Testing the deformation of the first geometry of geodetic network (critical value $F_{3,58,0.95} = 2.76$).

Par. / Trian.	e_{xx} [ppm]	e_{xy} [ppm]	e_{yy} [ppm]	ω [″]	t_x [m]	t_y [m]	T^2_{21} or T^2_{22}	H_0 Rejected?	*p*-Value [%]
1	−6.44	−16.08	−4.88	−0.9	0.002	0.003	2.35	no	8.20
2	24.35	47.82	41.74	−7.4	−0.017	−0.008	0.76	no	52.10
3	0.00	−8.81	0.29	−1.8	0.000	0.002	0.37	no	77.74
4	6.70	−11.61	−3.49	−0.6	0.000	0.002	2.48	no	7.04
5	15.97	−28.67	10.89	−3.2	−0.001	0.004	7.41	yes	0.03
6	42.26	9.18	64.29	−2.1	−0.007	−0.007	14.97	yes	0.00
7	51.37	−3.22	−25.33	4.3	−0.003	0.001	28.59	yes	0.00
8	66.13	14.39	−6.93	5.6	−0.007	−0.005	7.19	yes	0.03
9	66.03	15.74	15.08	6.0	−0.007	−0.008	18.93	yes	0.00
10	50.48	5.36	33.12	1.9	−0.006	−0.007	6.13	yes	0.11
11	−6.28	14.28	35.43	−1.1	−0.003	−0.007	7.63	yes	0.02
12	−19.16	−3.02	28.19	−4.0	−0.001	−0.002	11.51	yes	0.00
13	−13.18	−7.11	−17.09	−0.5	0.002	0.003	5.80	yes	0.15
14	−0.87	−3.88	−22.07	−2.3	−0.001	0.004	2.73	no	5.22

Based on the statistical testing of the deformation of the first geometry of the geodetic network, the null hypothesis cannot be rejected because the test statistic T^2_{21} or T^2_{22} is smaller than the critical value $F_{3,58,0.95} = 2.76$ and with a significance level smaller than $\alpha = 0.05$ it cannot be claimed that the deformations between two dimensions occurred in triangles 1, 2, 3, 4 and 14.

Connecting the points of the geodetic network to a triangular network of second geometry shown in Figure 2, we obtain the results given in Table 3.

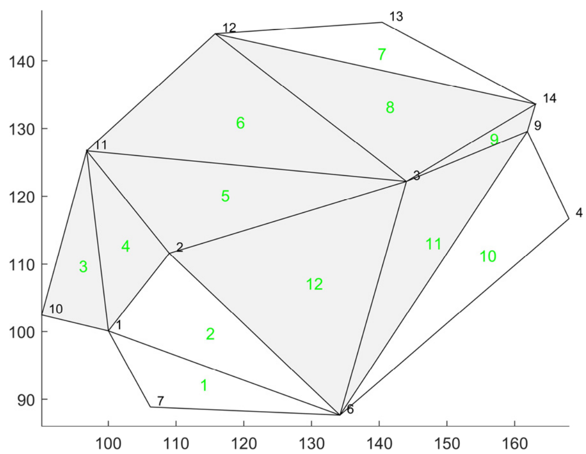

Figure 2. Second geometry of triangles in geodetic network.

Table 3. Testing the deformation of the second geometry of geodetic network (critical value $F_{3,58,0.95} = 2.76$).

Trian. \ Par.	e_{xx} [ppm]	e_{xy} [ppm]	e_{yy} [ppm]	ω [″]	t_x [m]	t_y [m]	T^2_{21} or T^2_{22}	H_0 Rejected?	*p*-Value [%]
1	4.03	−11.50	−22.73	−3.9	−0.001	0.006	2.54	no	6.54
2	−3.56	−0.93	−14.04	−1.1	0.000	0.002	1.51	no	22.11
3	14.24	−22.75	−9.23	−3.3	−0.001	0.005	8.60	yes	0.01
4	13.05	−25.81	22.43	−1.9	0.000	0.001	3.93	yes	1.28
5	15.76	−34.48	−3.18	−4.4	−0.001	0.007	11.84	yes	0.00
6	72.29	26.38	7.90	7.1	−0.008	−0.009	14.44	yes	0.00
7	67.14	−2.42	−4.61	2.1	−0.007	0.000	0.89	no	45.41
8	65.61	22.95	6.59	7.4	−0.007	−0.009	18.59	yes	0.00
9	50.48	5.36	33.12	1.9	−0.006	−0.007	6.13	yes	0.11
10	−11.03	3.48	11.02	−1.2	0.000	−0.001	1.87	no	14.55
11	−21.82	−1.61	51.19	−5.7	−0.002	−0.005	8.23	yes	0.01
12	−13.18	−7.11	−17.09	−0.5	0.002	0.003	5.80	yes	0.15

Based on the testing of the deformation of the second geometry of the geodetic network, the null hypothesis cannot be rejected because the test statistic T^2_{21} or T^2_{22} is smaller than the critical value $F_{3,58,0.95} = 2.76$ and with a significance level smaller than $\alpha = 0.05$ it cannot be claimed that the deformations between two epochs occurred in triangles 1, 2, 7 and 10.

Connecting the points of the geodetic network into the triangular network of the third geometry shown in Figure 3, we obtain the results given in Table 4.

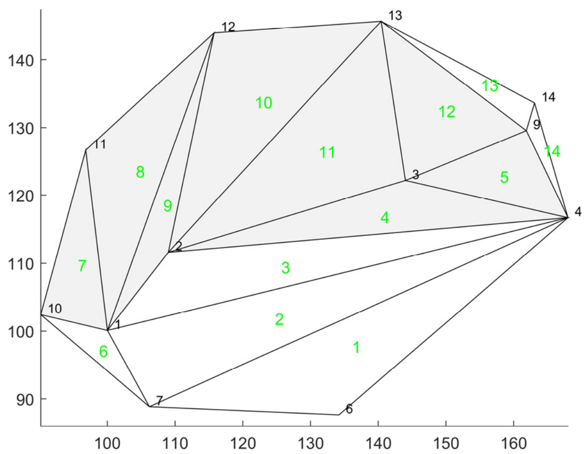

Figure 3. Third geometry of triangles in geodetic network.

Based on the testing of the deformation of the third geometry of the geodetic network, the null hypothesis cannot be rejected because the test statistic T^2_{21} or T^2_{22} is smaller than the critical value $F_{3,58,0.95} = 2.76$ and with a significance level smaller than $\alpha = 0.05$ it cannot be claimed that the deformations between two epochs occurred in triangles 1, 2, 3, 4, 5, 12 and 14.

Table 4. Testing the deformation of the third geometry of geodetic network (critical value $F_{3,58,0.95} = 2.76$).

Par. / Trian.	e_{xx} [ppm]	e_{xy} [ppm]	e_{yy} [ppm]	ω [″]	t_x [m]	t_y [m]	T^2_{21} Ali T^2_{22}	H_0 Rejected?	p-Value [%]
1	−7.93	20.22	−20.00	2.8	0.000	0.000	2.76	no	5.02
2	1.43	−6.90	2.53	−2.0	−0.001	0.002	0.99	no	40.40
3	−2.84	−8.22	3.43	−2.4	0.000	0.002	0.98	no	40.73
4	−55.38	−46.14	10.36	−11.2	0.005	0.010	14.76	yes	0.00
5	−6.28	14.28	35.43	−1.1	−0.003	−0.007	7.63	yes	0.02
6	−6.44	−16.08	−4.88	−0.9	0.002	0.003	2.35	no	8.20
7	14.24	−22.75	−9.23	−3.3	−0.001	0.005	8.60	yes	0.01
8	19.85	6.43	87.98	−6.9	−0.006	−0.006	15.40	yes	0.00
9	63.24	−28.82	−51.92	10.7	0.001	0.003	10.81	yes	0.00
10	48.73	0.95	−4.98	2.6	−0.004	−0.001	11.87	yes	0.00
11	63.82	4.95	−31.40	6.8	−0.004	−0.001	29.59	yes	0.00
12	66.45	17.51	20.36	5.8	−0.007	−0.009	19.13	yes	0.00
13	55.02	2.15	3.72	4.4	−0.004	−0.004	1.41	no	25.01
14	24.35	47.82	41.74	−7.4	−0.017	−0.008	0.76	no	52.10

The final step of the procedure/approach is to analyze the stability of a single point or a pair of points. The results are shown in Table 5 and graphically presented in Figure 4.

Table 5. Analysis of each or pair of points. The test statistic is computed according to Equation (64) (critical value $F_{1,58} = 4.01$).

pair	1	2	3	4	6	7	9	10	11	12	13	14
1	/	0.63	16.89	0.03	3.45	0.23	0.41	0.01	2.50	20.38	9.66	0.00
2		/	15.80	0.23	1.75	0.05	0.00	0.45	6.89	25.58	14.75	0.37
3			/	11.69	8.94	16.96	14.74	6.55	0.31	0.62	20.49	16.46
4				/	0.82	0.49	0.48	0.04	7.01	1.04	3.90	0.00
6					/	7.03	0.04	1.13	2.11	8.04	9.26	0.27
7						/	1.18	0.55	3.38	15.99	8.10	0.12
9							/	0.28	5.73	0.05	3.54	1.35
10								/	0.01	12.52	4.28	0.01
11									/	38.65	22.36	6.92
12										/	0.21	0.00
13											/	1.60
14												/

After analyzing individual points or pairs of points marked in green in Table 5, the null hypothesis cannot be rejected because the test statistic T^2_{23} is smaller than the critical value $F_{1,58,0.95} = 4.01$, and with the significance level of less than $\alpha = 0.05$, it cannot be claimed that the discussed points or pairs have changed their position.

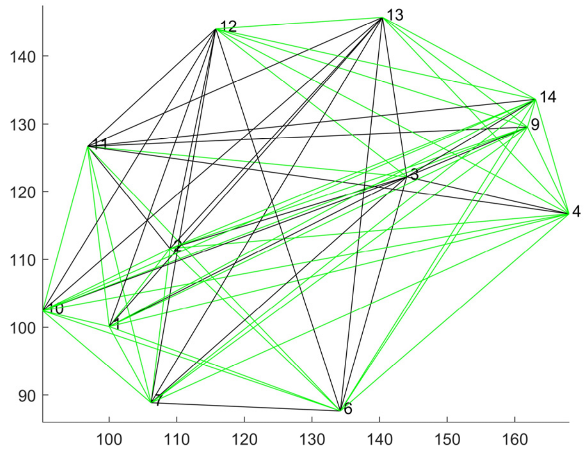

Figure 4. Analysis of each or pair of points—graphical representation.

4. Analysis and Discussion

After the LS adjustment of geodetic network as a free network for each epoch separately, we tested the congruence and proved that the coordinates of points in the network have changed between two repeated epochs.

To identify stable points, we divided the geodetic network into triangles and tested the transformation of each triangle. The subdivision into triangles was done with three different geometries of the geodetic network.

- Through the analysis in the first geometry of the geodetic network, we found that no statistically significant deformations occurred between two epochs in triangles 1, 2, 3, 4, and 14. It can be concluded that the vertices of these triangles have not moved, i.e., the reference points 1, 2, 4, 6, 7, 9 and 10 and 14 (the last two are points on the dam).
- Through the analysis in the second geometry of the geodetic network, we found that no statistically significant deformations occurred between two epochs in triangles 1, 2, 7 and 10. It can be concluded that the vertices of these triangles have not moved, i.e., the reference points 1, 2, 4, 6, 7, 9, and 12, 13 and 14 (the last three are points on the dam).
- Through the analysis in the third geometry of the geodetic network, we found that no statistically significant deformations occurred between two epochs in triangles 1, 2, 3, 6, 13 and 14. It can be concluded that the vertices of these triangles have not moved, i.e., the reference points 1, 2, 4, 6, 7, 9 and 10, 13 and 14 (the last three are points on the dam).

Using different geometries of geodetic networks lead to a situation where an observed point forms triangles with different points. If this point is a part of several different triangles which do not exhibit statistically significant deformations we conclude that the point doesn't move significantly. Based on these procedure and results, we can say that the displacements of the reference points 1, 2, 4, 6, 7 and 9 were not statistically significant. In all analyzed geodetic network geometries, reference point 3 has statistically significant displacement. The analysis also shows that point 14, the point on the dam, has not moved significantly either.

We also performed an analysis of the stability of each pair of points:

- The distances between reference points 1–2, 1–4, 1–6, 1–7, 1–9, 2–4, 2–6, 2–7, 2–9, 4–6, 4–7, 4–9, 6–9, and 7–9 have not changed in a statistically significant way. However, all distances between reference point 3 and the other reference points 3–1, 3–2, 3–4, 3–6, 3–7, and 3–9 have changed in a statistically significant manner. This analysis confirms the results of the individual triangle transformation testing that reference points 1, 2, 4, 6, 7, and 9 have not moved significantly, and that point 3 has. Only the distance between reference points 6–7 has changed in a statistically significant way (since the movements of these two points point to each other).

- Considering the pairs between the reference points (1, 2, 4, 6, 7, and 9) and the points on the dam (10 and 14) that have not moved as a result of the congruence test, we find that all distances between pairs 1–10, 1–11, 1–14, 2–10, 2–14, 4–10, 4–14, 6–10, 6–11, 6–14, 7–10, 7–11, 7–14, 9–10, 9–14 have not statistically significantly changed. The distances between the reference points and the points on the dam 1–12, 1–13, 2–12, 2–13, 3–13, 6–12, 6–13, 7–12 and 7–13 have changed statistically significantly. All of the above changes confirm the findings from testing the transformation of the individual triangle that reference points 1, 2, 4, 6, 7, and 9 have not moved significantly, and that point 3 has. Distances between 2–11, 3–10, 3–14, 4–11, and 9–11 show a statistically significant change as the points of the pair move toward each other; the distances between pairs 3–11, 3–12, 4–12, 4–13, 9–12, 9–13 do not show a statistically significant change as the two points of the pair have moved in a similar direction relative to each other.
- The distances between the points on the dam 10–11, 10–13, 10–14, 12–13, 12–14, 13–14 have not changed in a statistically significant way. The reason for this assertion is that at least one point of the listed pairs moved a little or the directions of their movement were similar. The distances between the points on dam 10–12, 11–12, 11–13, and 11–14 changed in a statistically significant way.

After all the analyses performed, we can calculate deformation parameters Equations (56)–(63). The results are shown for the first geometry of the geodetic network (Figure 1). The results are shown in Table 6.

Table 6. Deformation parameters of the first geometry of geodetic network.

Triangle \ Par.	Δ [ppm]	e_1 [ppm]	e_2 [ppm]	e_I [ppm]	γ [ppm]	ϑ [°]	Ψ [°]
1	−11.32	10.44	−21.76	16.10	−32.16	43	88
2	66.09	81.65	−15.57	48.61	95.65	140	5
3	0.29	8.96	−8.67	8.81	−17.62	44	89
4	3.20	14.28	−11.08	12.68	−23.23	147	12
5	26.86	42.21	−15.35	28.78	−57.33	138	3
6	106.55	67.62	38.94	14.34	18.37	160	25
7	26.04	51.50	−25.46	38.48	−6.44	178	43
8	59.20	68.86	−9.67	39.27	28.79	10	55
9	81.11	70.50	10.61	29.95	31.48	16	61
10	83.60	52.00	31.60	10.20	10.71	16	61
11	29.15	39.85	−10.70	25.27	28.55	163	28
12	9.03	28.38	−19.35	23.87	−6.05	3	48
13	−30.27	−7.76	−22.50	7.37	−14.21	143	8
14	−22.94	−0.18	−22.76	11.29	−7.77	170	35

The meaning of the parameters given in this table is explained in Section 2.3.

5. Conclusions

In this article we first introduced the theory of deformation analysis, the Munich approach, presented first by W.M. Welsch [9,12], which includes X- and L- method. We performed a congruence test of the geodetic network, and in the next step a strain test. We have always considered all the possibilities that can occur in a single triangle: in the X-method we can consider a different number of equations in the system (22) and in the L-method a different number of directions and distances can occur. Finally, the method is supplemented by the analysis of the distance between two points.

We demonstrated the usefulness of Munich approach using a simulated example from the literature [16]. Three different geometries of triangles were constructed using the points of the geodetic network. Each geometry was characterized by different triangles characteristics, in the first geometry, the triangles were as regular as possible, in the second and third we formed also very narrow triangles which are numerically less suitable. For a selected point that occurs in nondeformed triangle in all three network geometries, we conclude that the point has not moved statistically significantly. If at least one of the triangles has deformed in a statistically significant way and the selected point is located in the vertex of this triangle, we can conclude that the point has moved. From the results of our computations and analysis we can state that at the chosen significance level $\alpha = 0.05$, points 1, 2, 4, 6, 7, and 9 have not moved significantly between the two measuring epochs, while the reference point 3 has. Moreover, we can say that points 10 and 14, located on the dam, have not moved significantly either. These results were obtained by both L- and X-method.

Similar results about points that move or points that remain their positions were also obtained by other researchers using other methods of deformation analysis, described in the introduction of this article, such as Caspary [16], Nowel [17,18], and Batilović et al. [19].

In our opinion the main contribution of this article is that we proposed some corrections to the original Munich approach presented by W. Welsch in the papers of Welsch (1983) [12] and Welsch and Zhang (1983) [13] which in our opinion are crucial for appropriate numerical consideration of the presented deformation problem.

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