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Self-Organization When Pedestrians Move in Opposite Directions. Multi-Lane Circular Track Model

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Abstract: When pedestrians walk along a corridor in both directions, a frequently observed phenomenon is the segregation of the whole group into lanes of individuals moving in the same direction. While this formation of lanes facilitates the flow and benefits the whole group, it is believed that results from the actions of the individuals acting on their behalf, without considering others. This phenomenon is an example of self-organization and has attracted the attention of a number of researchers in diverse fields. We introduce and analyze a simple model. We assume that individuals move around a multi-lane circular track. All of them move at the same speed. Half of them in one direction and the rest in the opposite direction. Each time two individuals collide, one of them moves to a neighboring lane. The individual changing lanes is selected randomly. We prove that the system self-organizes. Eventually, each lane is occupied with individuals moving in only one direction. Our analysis supports the belief that global self-organization is possible even if each member of the group acts without considering the rest.

Keywords: mathematical modeling; dynamics of crowds; self-organization; probabilistic models

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1. Introduction

Spontaneous organization in systems composed of several units is known as self-organization. Self-organization is pervasive and is observed even when the units act responding to local stimuli, without considering the rest of the group. Researchers are interested in understanding how local interactions among units and with the environment leads to self-organization. Some books and review articles on self-organization in biological systems include [1–4].

Self-organization is observed when humans are walking in crowded environments. As a first example, consider two adjacent rooms connected by a door that is initially closed. Assume a large number of individuals is in each room and they all want to go to the other room. Suddenly, the door is opened. Under these circumstances, the spontaneous formation of an alternating flow is frequently observed. By alternating flow, we mean that the individuals take turns. Individuals from only one of the rooms cross the door for a period of time and then there is a sudden switch, individuals from the other room are the only ones crossing the door after the switch. This switching, or turn taking, persists while the environment remains crowded [5].

A second example of self-organization due to walking individuals interacting among themselves and with the environment is the formation of trails. More precisely, assume individuals need to cross, possibly in different directions, the same mildly dense grass field on a regular basis. Individuals will try to walk along the areas where the grass is shorter and less dense, even if the path they take as a result is not the shortest. Stepping on the grass is how the individuals affect the environment. Grass has a harder time growing in the areas heavily transited. As a result, dirt path trails form [6].

A third example of self-organization is the different pattern formations observed when the crowd is a collection of several smaller groups. For example, groups of friends or families. We refer the reader to [7] for more details. In this and the previous two examples, the coordinated behaviors observed are not planned by the crowd as a whole or by any member or group of members within the crowd. Instead, they emerge spontaneously, as a consequence of the individuals acting in response to local stimuli and motivated by their own goals.

The study of the dynamics of crowds is of interest for several reasons. It is a source of examples of self-organization. It may lead to strategies to increase the safety in crowded areas such as bridges [8] or stadiums. It can provide guidelines in the design of movie theaters, shopping malls, or other similar types of heavily transited buildings, where optimal crowd flow is desired because of economical and safety reasons. Accordingly, the study of dynamics of crowds, both theoretically (early works include [9,10], more recent work includes [11–16], see also [1,17] for reviews) and experimentally [18–20], is a very active area of research.

Microscopic mathematical models to study dynamics of crowds are those that keep track of each individual. Cellular automata models [18,21–23], lattice gas automata models [24,25], algorithms [26], and large systems of odes, are all examples of microscopic models. When large systems of odes are used, the mass times acceleration of each individual is set equal to the sum of generalized or social forces the individual feels [27–31]. These social forces are not real forces. They model the responses of the individuals to the environment and the presence and actions of the other individuals [32]. Some of these models are known as self-propelled particles models [33] and others, as individual based models [32,34].

Mesoscopic, kinetic or Boltzman-type models to study the dynamics of crowds, are integro-partial differential equations that describe the evolution of probability densities of the position and velocities of the individuals [35,36]. Macroscopic or continuum models are partial differential equations (conservation equations), where the dependent variables are the density and local average velocity of individuals [1,37–40]. Some works connect microscopic to macroscopic models [41–43]. Also, network models have been introduced [44] and optimal control theory has been used [45] to study the dynamics of crowds.

Assume a corridor or street is crowded with persons walking. Some of the pedestrians are walking in one direction, while the others, in the opposite direction. A self-organizing phenomenon frequently observed is that the individuals segregate into lanes of individuals moving only in one direction [46–48]. Needless to say, this formation of lanes benefits the whole group, as it results in an easier flow in both directions [49]. Motivated by this phenomenon, we introduce and analyze a new mathematical model. Our work supports the claim that the simple behavior by the individuals of moving out of the way to avoid imminent collisions leads to the self-organization of the system.

We describe the model in Section 2. Briefly, individuals move around a multi-lane circular track with the same angular speed. Half of them walk clockwise and the rest, counterclockwise. Each individual remains in its lane unless it collides with an other individual walking in the same lane but in the opposite direction. When such a collision occurs, one of the colliding individuals moves to a neighboring lane. The individual changing lanes is selected randomly. In Section 3, we present numerical simulations.

This is not the first article of the author studying this problem. The two-lane version of this model was introduced and studied in [50,51]. However, in the model introduced and studied here, the number of lanes is not restricted to two, it can have any number of lanes. Going from two to any number of lanes is not a simple extension. As a result, the analysis and methods used here are completely different. In short, this is a novel article and not a mere extension of previous work of the author.

Sections 4–8 consist of analysis of the system. In Section 8, we prove that, with probability 1, the system will self-organize, i.e., eventually, all the individuals within each lane move in the same direction. We finish the article with a short discussion in Section 9.

2. The Model

We consider the following scenario: $2N$ individuals or pedestrians walk around a multi-lane circular track. Half of the individuals move in a clockwise direction and the other half, in a counterclockwise direction. Each time two individuals moving in opposite directions and in the same lane meet, we say they collide. When two individuals collide, exactly one of them, randomly chosen, with each having the same probability of $1/2$ of being chosen, moves to a neighboring lane. If the collision occurs in the most inner or most outer lane, there is only one neighboring lane. If the collision takes place in another lane, the neighboring lane to which the individual changing lanes moves is also chosen randomly.

Next, we enumerate a list of statements. The ith statement will be referred as statement i from Section 2. These statements are either rules that help precisely define the dynamics of our system, or observations that are consequences of those rules and will be needed in the analysis in subsequent sections.

1. $2N$ individuals move around a circular track with L lanes. Each lane is labeled with a number. The inner lane is lane 1, the lane next to lane 1 is lane 2, and so on. Note that the outer lane is lane L.
2. N individuals move in the counterclockwise direction and the other N in the clockwise direction.
3. All individuals move with the same constant angular speed ω and thus, it takes each individual a time of $2\pi/\omega$ to complete a loop.
4. Each individual moving counterclockwise is labeled with an integer i, where $1 \leq i \leq N$. The position of the individual i moving counterclockwise at time t is described by an angle $\theta_i^{(+)}(t)$ and by $\ell_i^{(+)}(t)$, the number of the lane the individual is at time t. For simplicity, we require $\theta_i^{(+)}(t)$ to be a continuous function and thus, since the angular speed ω is constant, we have $\theta_i^{(+)}(t) = \theta_i^{(+)}(0) + \omega t$. For convenience, we assume the initial angles to satisfy $0 \leq \theta_i^{(+)}(0) < 2\pi$. Note that the function $\ell_i^{(+)}(t)$ will be discontinuous at the times t when the counterclockwise moving individual i changes lanes. Strictly speaking, $\ell_i^{(+)}(t)$ is not defined at the times it is discontinuous. This will not cause any problems.
5. Each individual moving clockwise is also labeled with an integer j, with $1 \leq j \leq N$. The position of the individual j moving clockwise at time t is described by the angle $\theta_j^{(-)}(t) = \theta_j^{(-)}(0) - \omega t$, where $0 \leq \theta_j^{(+)}(0) < 2\pi$, and by $\ell_i^{(-)}(t)$, the number of the lane the individual is at time t.
6. We assume that initially, i.e., at time $t = 0$, all the $2N$ angles defined above are different. This means not only that individuals start at different positions, but also that, at $t = 0$, any two individuals do not have positions that correspond to the same angle. If an individual at $t = 0$ looks to its sides, it will not see any other individual in the other lanes with the same initial angle.
7. Given the last statement, two individuals moving in the same direction will never have the same angle.
8. Two individuals collide at time t if they reach the same location at that time. Given the last statement, if two individuals collide, they move in opposite directions. In mathematical terms, the individual i moving counterclockwise and the individual j moving clockwise collide at time t if $\theta_i^{(+)}(t) = \theta_j^{(-)}(t) + 2\pi k$ for some integer k and $\lim_{s\to t^-} \ell_i^{(+)}(s) = \lim_{s\to t^-} \ell_j^{(-)}(s)$. In the last equation, $\lim_{s\to t^-}$ denotes the right limit as s tends to t, that is, s approaches t but under the restriction $s < t$. This limit is taken because one of the individuals will change lanes at time t and thus, either $\theta_i^{(+)}$ or $\theta_j^{(-)}$ will not be defined at exactly that time. Note also that the angles of the colliding individuals do not have to be equal. It is enough that they differ by an integer multiple of 2π, which includes the case of them being equal. Adding or subtracting 2π to an angle does not change the position.

9. For all pairs i, j such that $1 \leq i, j \leq N$, we define

$$\tau_{ij} = \begin{cases} \frac{\theta_j^{(-)}(0) - \theta_i^{(+)}(0)}{2\omega} & \text{if } \theta_j^{(-)}(0) > \theta_i^{(+)}(0) \\ \frac{\theta_j^{(-)}(0) - \theta_i^{(+)}(0)}{2\omega} + \frac{\pi}{\omega} & \text{if } \theta_j^{(-)}(0) < \theta_i^{(+)}(0). \end{cases} \tag{1}$$

   We assume that $\tau_{i_1 j_1} \neq \tau_{i_2 j_2}$ if $i_1 \neq i_2$ or $j_1 \neq j_2$. Equation (1) will be further discussed in the next section. The discussion in that section should make the reader clear that the assumption $\tau_{i_1 j_1} \neq \tau_{i_2 j_2}$ if $i_1 \neq i_2$ or $j_1 \neq j_2$ implies that two different collisions never occur at the same time.
10. If two individuals, the individual i moving counterclockwise and the individual j moving clockwise, collide at time t, exactly one of them changes lanes at that time. The probability that the individual i changes lanes is $1/2$ and thus, $1/2$ is also the probability that j changes lanes. Assume the collision occurs in lane ℓ. If $1 < \ell < L$, the individual changing lanes moves to lane $\ell - 1$ with probability $1/2$ or to lane $\ell + 1$, also with probability $1/2$. If $\ell = 1$, the individual changing lanes moves to lane 2, and if $\ell = L$, it moves to lane $L - 1$.
11. An individual can only change lanes when it collides with an other individual.

We note that the assumptions in points 6 and 9 above are not restrictive at all. For example, if the initial angles were to be selected randomly with a uniform probability distribution around the track, assumptions in points 6 and 9 would be satisfied with probability 1. Those assumptions are made to simplify the analysis but are not fundamental.

Our model is very simple and is based on local responses only. The selection of the individual that changes lanes, and the lane it moves to, are independent of the location of all the other individuals. The individuals do not make smart decisions attempting to prevent future collisions.

3. Numerical Simulations

Figure 1 shows the results of a numerical simulation with 120 individuals in a circular track with 4 lanes. The individuals move at one revolution per unit time. Figure 1 shows the positions of the individuals at three different times: $t = 0$, $t = 4$ and $t = 5$. The number of collisions that occurred by those times are also indicated in Figure 1. The initial angles of the individuals were randomly chosen with uniform probability distribution around the track. The lane where each individual started was also randomly selected, with each lane having $1/4$ as the probability of being chosen.

Note that, in the realization of Figure 1, at $t = 5$ (in fact slightly before that), after 2192 collisions, all the pedestrians moving in the counterclockwise direction are in the inner lane, which is lane 1, and in lane 3. All those moving in the clockwise direction are in lanes 2 and 4, the outer lane. No more collisions occur after this time. We say that the system has self-organized.

We have run several simulations with different randomly generated initial conditions (not shown here) and in all those simulations, the system self-organized. Our numerical simulations suggest that the system always self-organizes. We will prove that this is the case with probability 1.

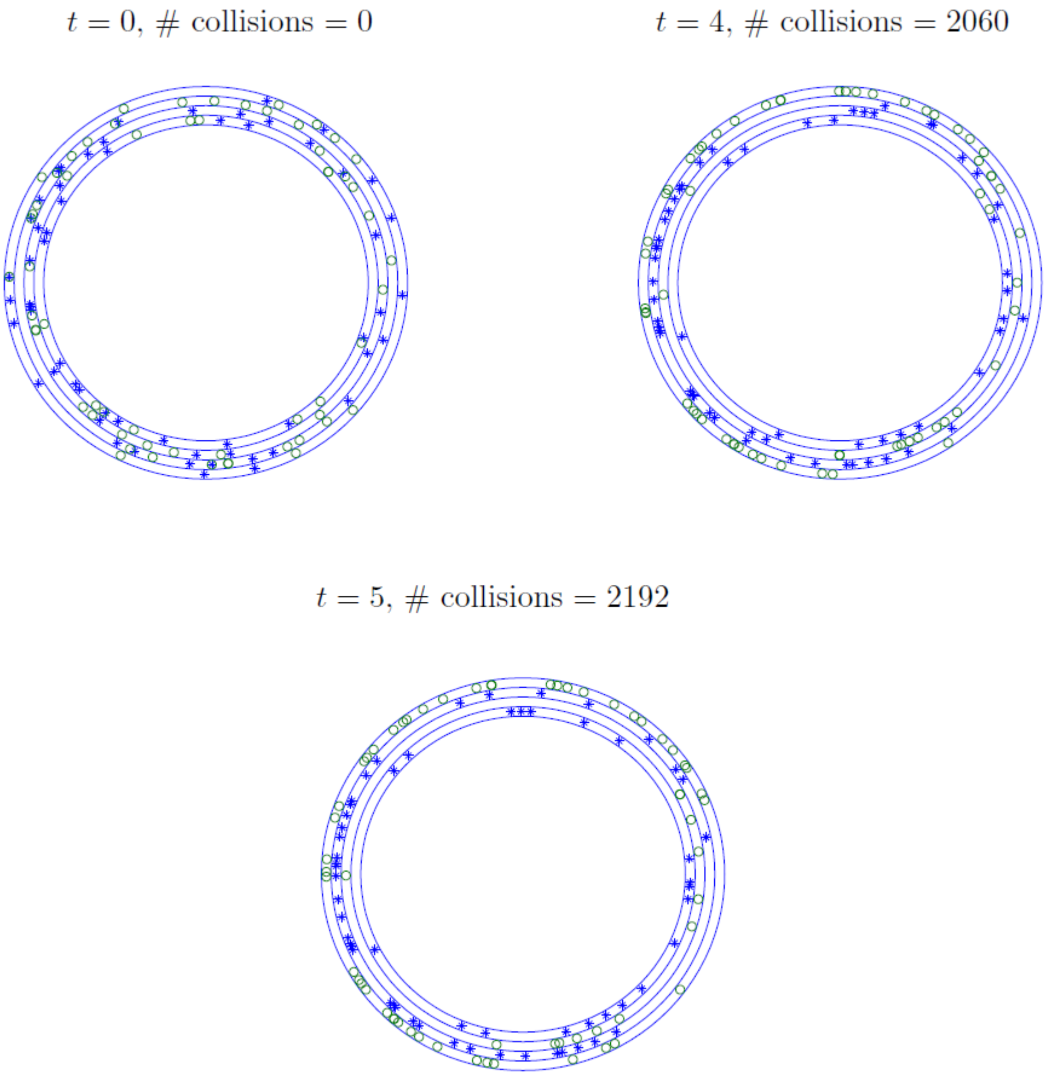

Figure 1. Positions of the pedestrians at different times. The circles are the pedestrians moving clockwise. The exes are the pedestrians moving counterclockwise.

4. Possible Collision Times

We recall the reader that two individuals collide at time t if they are in the same lane at that time and their angles differ by an integer multiple of 2π. When we say they are in the same lane at time t, we mean they are in the same lane just before one of them changes lanes. Note that, since 0 is an integer, the angles differing by an integer multiple of 2π includes the case when the angles are equal.

We define τ_{ij} to be the first time that the angle of the individual i moving counterclockwise and the angle of the individual j moving clockwise differ by an integer multiple of 2π. These two individuals do not collide for any t satisfying $0 < t < \tau_{ij}$ and they collide at $t = \tau_{ij}$ if and only if they are in the same lane at that time. Thus, we refer to τ_{ij} as the first or smallest possible collision time between the individual i moving counterclockwise and the individual j moving clockwise.

Recall that the angle of the individual i moving counterclockwise and the angle of the individual j moving clockwise are given by $\theta_i^{(+)}(t) = \theta_i^{(+)}(0) + \omega t$ and $\theta_j^{(-)}(t) = \theta_j^{(-)}(0) - \omega t$, respectively. Recall also that all the initial angles are greater than or equal to 0 and less than 2π. Given these two facts, little thought is necessary to convince the reader about the validity of the Equation (1). Note that the time τ_{ij} always satisfies $0 < \tau_{ij} < \pi/\omega$.

We define $\mathcal{T}_{ij}$ to be the set of all positive times when the difference between the angles of the individual i moving counterclockwise and the individual j moving clockwise differ by an integer multiple of 2π. Straight forward algebra shows that $\mathcal{T}_{ij}$ is the set of all times of the form $\tau_{ij} + n\pi/\omega$, where n is any non-negative integer,

$$\mathcal{T}_{ij} = \left\{ t = \tau_{ij} + n\frac{\pi}{\omega} \text{ for some non-negative integer } n \right\}. \tag{2}$$

If the individual i moving counterclockwise and the individual j moving clockwise collide, they so do at some time t in $\mathcal{T}_{ij}$. In fact, these two individuals collide at time t if and only if $t \in \mathcal{T}_{ij}$ and they are both in the same lane at time t. Given these facts, we refer to the set $\mathcal{T}_{ij}$ as the set of possible collision times between the individual i moving counterclockwise and the individual j moving clockwise.

We define $\mathcal{T}$ to be the union of the sets $\mathcal{T}_{ij}$, where the union is taken over all individuals i moving counterclockwise and individuals j moving clockwise

$$\mathcal{T} = \bigcup_{1 \le i,j \le N} \mathcal{T}_{ij}. \tag{3}$$

If there is a collision at time t, then t is in the set $\mathcal{T}$. Thus, we refer to $\mathcal{T}$ as the set of possible collision times.

We sort the elements in $\mathcal{T}$ and label them as t_n with $n \ge 1$. In other words, the sequence t_n, with $n \ge 1$, is defined by the facts that $t_n < t_{n+1}$ for all positive integers n, and

$$\mathcal{T} = \{t_n : n \ge 1'\}. \tag{4}$$

For convenience, we define $t_0 = 0$, but we do keep t_0 out of the set $\mathcal{T}$.

Note that the assumption in Point 9 in Section 2 implies that the sets $\mathcal{T}_{ij}$ are pairwise disjoint. Thus, t_n is a first possible collision time between two individuals, i.e., $t_n = \tau_{ij}$ for some i and j, if and only if $1 \le n \le N^2$. Thus, we have that $t_{N^2} < \pi/\omega < t_{N^2+1}$. Note also that $t_{N^2+n} = t_n + \pi/\omega$ for all $n \ge 1$. Also note that, for every pair i, j, with $1 \le i, j \le N$, there exists n such that $1 \le n \le N^2$ and $t_n = \tau_{ij}$.

5. Evolution Equations

Given any positive integer n, the time t_n is a possible collision time between two individuals. We denote by $I(n)$, the integer corresponding to the individual moving counterclockwise and by $J(n)$, the integer corresponding to the individual moving clockwise involved in the possible collision at time t_n. In other words, the functions $I = I(n)$ and $J = J(n)$ are defined by the following statement: t_n is a possible collision time between the individual $i = I(n)$ moving counterclockwise and the individual $j = J(n)$ moving clockwise. Following the discussion of Section 4, it can be easily shown that $I(n + N^2) = I(n)$ and $J(n + N^2) = J(n)$ for all n.

A simple but important observation is that each individual remains in the same lane in between consecutive possible collisions times. i.e., $\ell_i^{(+)}$ and $\ell_j^{(-)}$ remain constant in time intervals of the form $t_n < t < t_{n+1}$ for all n, i and j. Thus, for any $n \ge 0$ and $1 \le i, j \le N$, we define

$$\ell_i^{(+,n)} = \ell_i^{(+)}(t) \text{ and } \ell_j^{(-,n)} = \ell_j^{(-)}(t) \text{ where } t_n < t < t_{n+1}. \tag{5}$$

For each integer n, we regard $\ell_i^{(+,n)}$ as the ith component of a $2N$-vector that we call $\ell^{(n)}$ and we regard $\ell_j^{(-,n)}$ as the $(N + j)$th component of the same vector,

$$\ell^{(n)} = \left(\ell_1^{(+,n)}, \ldots, \ell_N^{(+,n)}, \ell_1^{(-,n)}, \ldots, \ell_N^{(-,n)} \right). \tag{6}$$

We also denote the kth component of $\ell^{(n)}$ by $\ell_k^{(n)}$. Thus, we have $\ell_k^{(n)} = \ell_k^{(+,n)}$ if $1 \le k \le N$, and $\ell_k^{(n)} = \ell_{k-N}^{(-,n)}$ if $N < k \le 2N$.

We define Ω to be the set of $2N$-vectors, whose components are positive integers no greater than L

$$\Omega = \{\mathbf{v} = (v_1, v_2, \ldots, v_{2N}) \text{ such that } v_i \text{ is an integer and } 1 \le v_i \le L \text{ for all } i\}. \tag{7}$$

Note that $\ell^{(n)}$ is in Ω for all n.

Let $\mathbf{u}$ and $\mathbf{v}$ be two elements in Ω. Next, we describe the probability that $\ell^{(n)} = \mathbf{v}$ given that $\ell^{(n-1)} = \mathbf{u}$. The standard notation for this probability, known as conditional probability, is $P\left(\ell^{(n)} = \mathbf{v} \,\middle|\, \ell^{(n-1)} = \mathbf{u}\right)$. Before we proceed, note that $\ell^{(n-1)} = \mathbf{u}$ implies that $\ell_{I(n)}^{(+,n-1)} = \ell_{I(n)}^{(n-1)} = u_{I(n)}$ and $\ell_{J(n)}^{(-,n-1)} = \ell_{N+J(n)}^{(n-1)} = u_{N+J(n)}$.

1. If $u_{I(n)} \neq u_{N+J(n)}$, there is no collision at time t_n. Thus,

$$P\left(\ell^{(n)} = \mathbf{v} \,\middle|\, \ell^{(n-1)} = \mathbf{u}\right) = \begin{cases} 1 & \text{if } \mathbf{v} = \mathbf{u} \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

2. If $u_{I(n)} = u_{N+J(n)} = 1$, at time t_n, the individual $I(n)$ moving counterclockwise and the individual $J(n)$ moving clockwise collide in lane 1. Thus,

$$P\left(\ell^{(n)} = \mathbf{v} \,\middle|\, \ell^{(n-1)} = \mathbf{u}\right) = \begin{cases} 1/2 & \text{if } v_{I(n)} = 2 \text{ and } v_k = u_k \text{ for all } k \neq I(n) \\ 1/2 & \text{if } v_{N+J(n)} = 2 \text{ and } v_k = u_k \text{ for all } k \neq N + J(n) \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

The first condition in the above equation corresponds to the individual $I(n)$ moving counterclockwise, changing from lane 1 to lane 2. The second condition corresponds to the individual $J(n)$ moving clockwise, changing from lane 1 to lane 2.

3. If $u_{I(n)} = u_{N+J(n)} = L$, at time t_n, the individual $I(n)$ moving counterclockwise and the individual $J(n)$ moving clockwise collide in lane L. Thus,

$$P\left(\ell^{(n)} = \mathbf{v} \,\middle|\, \ell^{(n-1)} = \mathbf{u}\right) = \begin{cases} 1/2 & \text{if } v_{I(n)} = L - 1 \text{ and } v_k = u_k \text{ for all } k \neq I(n) \\ 1/2 & \text{if } v_{N+J(n)} = L - 1 \text{ and } v_k = u_k \text{ for all } k \neq N + J(n) \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

The first condition in the above equation corresponds to the individual $I(n)$ moving counterclockwise, changing from lane L to lane $L-1$. The second condition corresponds to the individual $J(n)$ moving clockwise, changing from lane L to lane $L-1$.

4. If $u_{I(n)} = u_{N+J(n)} = \ell$, with $1 < \ell < L$, at time t_n, the individual $I(n)$ moving counterclockwise and the individual $J(n)$ moving clockwise collide in a lane other than lanes 1 and L. Thus,

$$P\left(\ell^{(n)} = \mathbf{v} \,\middle|\, \ell^{(n-1)} = \mathbf{u}\right) = \begin{cases} 1/4 & \text{if } v_{I(n)} = \ell - 1 \text{ and } v_k = u_k \text{ for all } k \neq I(n) \\ 1/4 & \text{if } v_{I(n)} = \ell + 1 \text{ and } v_k = u_k \text{ for all } k \neq I(n) \\ 1/4 & \text{if } v_{N+J(n)} = \ell - 1 \text{ and } v_k = u_k \text{ for all } k \neq N + J(n) \\ 1/4 & \text{if } v_{N+J(n)} = \ell + 1 \text{ and } v_k = u_k \text{ for all } k \neq N + J(n) \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

The first condition in the above equation corresponds to the individual $I(n)$ moving counterclockwise, changing from lane ℓ to lane $\ell - 1$. The second condition corresponds to the individual $I(n)$ moving counterclockwise, changing from lane ℓ to lane $\ell + 1$. The third condition corresponds to the individual $J(n)$ moving clockwise, changing from lane ℓ to lane

$\ell - 1$. The fourth condition corresponds to the individual $J(n)$ moving clockwise, changing from lane ℓ to lane $\ell + 1$.

6. Self-Organized Configurations

We say that the system is self-organized if there are no more collisions. This can only occur if each lane contains only individuals moving in the same direction. Note that once the system is self-organized, it remains self-organized for all later times since there are no more collisions and thus, no more lane changes by any individual.

Assume the system is self-organized after the nth possible collision time, i.e., for $t > t_n$. Thus, for any pair i, j, where $1 \le i, j \le N$, the individual i moving counterclockwise is in a different lane than the individual j moving clockwise, i.e., $\ell_i^{(+,n)} \neq \ell_j^{(-,n)}$. Equivalently, $\ell_i^{(n)} \neq \ell_{j+N}^{(n)}$. A more mathematical description is possible. We define the set

$$A = \left\{ \mathbf{v} \in \Omega \text{ such that } v_i \neq v_{j+N} \text{ for all } 1 \le i, j \le N \right\}. \tag{12}$$

The system is self-organized for $t > t_n$ if and only if $\ell^{(n)}$ is in A.

7. Probabilities to Reach Self-Organization

For each non-negative integer n and each $\mathbf{v}$ in Ω, we define $f_n(\mathbf{v})$ to be the probability that the system eventually self-organizes given that $\ell^{(n)} = \mathbf{v}$. According to the discussion of the last section,

$$f_n(\mathbf{v}) = P\left(\ell^{(k)} \text{ is in } A \text{ for some } k \ge n \,\middle|\, \ell^{(n)} = \mathbf{v} \right). \tag{13}$$

Given the periodicity of the pairs of individuals involved in the possible collisions, i.e., $I(n + N^2) = I(n)$ and $J(n + N^2) = J(n)$, we have that $f_{n+N^2}(\mathbf{v}) = f_n(\mathbf{v})$ for all non-negative integers n and all $\mathbf{v}$ in Ω. Note also that $f_n(\mathbf{v}) = 1$ for all $\mathbf{v}$ in A.

Following standard arguments in the analysis of Markov Chains, we can obtain the following equation

$$f_n(\mathbf{u}) = \sum_{\mathbf{v} \in \Omega} P\left(\ell^{(n+1)} = \mathbf{v} \,\middle|\, \ell^{(n)} = \mathbf{u} \right) f_{n+1}(\mathbf{v}). \tag{14}$$

8. Self-Organization Occurs with Probability 1

Since Ω has only a finite number of elements, L^{2N} to be precise, and $f_{n+N^2}(\mathbf{v}) = f_n(\mathbf{v})$ for all non-negative integers n and all $\mathbf{v}$ in Ω, the set $\{f_n(\mathbf{v})$ such that n is a non-negative integer and $\mathbf{v}$ is in $\Omega\}$ contains, at most, $N^2 L^{2N}$ different numbers. Thus, since it is finite, this set has a minimum. For future reference, we summarize this statement in the next observation.

Observation 1. *There exists a non-negative integer n, that can be taken to be no greater than N^2, and $\mathbf{v}$ in Ω such that $f_n(\mathbf{v}) \le f_k(\mathbf{u})$ for any non-negative integer k and $\mathbf{u}$ in Ω. For future reference, we call this minimum value λ, i.e.,*

$$\lambda = \min_{k,\mathbf{u}} f_k(\mathbf{u}), \tag{15}$$

where the minimum is taken over all non-negative integers k and all $\mathbf{u}$ in Ω.

Let $x, x_1, \ldots, x_r$ be all real numbers. We say that x is a convex combination of $x_1, \ldots, x_r$ if there exits non-negative numbers $a_1, \ldots, a_r$ such that $a_1 + \ldots + a_r = 1$ and $x = a_1 x_1 + \ldots + a_r x_r$. There is a very simple and well known fact that will be useful to us. Namely, if x is a convex combination of $x_1, \ldots, x_r$ with positive coefficients (i.e., $a_i > 0$ for all i) and $x \le x_i$ for all i, then $x_i = x$ for all $1 \le i \le r$.

Note that Equation (14) implies that $f_n(\mathbf{u})$ is a convex combination of $f_{n+1}(\mathbf{v})$ with $\mathbf{v}$ in Ω. By restricting the sum on the right hand side of Equation (14) to only those $\mathbf{v}$ such that $P\left(\ell^{(n+1)} = \mathbf{v} \,\middle|\, \ell^{(n)} = \mathbf{u}\right) > 0$, we can apply the fact stated in the previous paragraph to prove the validity of the following observation.

Observation 2. *Let n be a non-negative integer and $\mathbf{u}$ in Ω. Assume $f_n(\mathbf{u}) = \lambda$ (as defined in Equation (15)). Let $\mathbf{v}$ be also in Ω. If $P\left(\ell^{(n+1)} = \mathbf{v} \,\middle|\, \ell^{(n)} = \mathbf{u}\right) > 0$, then $f_{n+1}(\mathbf{v}) = \lambda$*

Repeated applications of Observation 2 leads to the next Observation.

Observation 3. *Let m be a non-negative integer and $\mathbf{u}^{(0)}$ in Ω. Assume $f_m\left(\mathbf{u}^{(0)}\right) = \lambda$. Let $\mathbf{u}^{(1)}, \dots, \mathbf{u}^{(s)}$ be elements in Ω. Assume that $P\left(\ell^{(m+k+1)} = \mathbf{u}^{(k+1)} \,\middle|\, \ell^{(m+k)} = \mathbf{u}^{(k)}\right) > 0$ for all $0 \le k < s$, then $f_{m+s}\left(\mathbf{u}^{(s)}\right) = \lambda$.*

For each positive integer n, we define a set of functions $\mathbf{F}_n$ defined on Ω that also take values on Ω. Setting $\mathbf{v} = \mathbf{F}_n(\mathbf{u})$, these functions are determined by the following rules:

1. If $u_{I(n)} \neq u_{N+J(n)}$ then $\mathbf{v} = \mathbf{u}$.
2. If $u_{I(n)} = u_{N+J(n)} = 1$ then $v_{N+J(n)} = 2$ and $v_i = u_i$ for all $i \neq N + J(n)$.
3. If $u_{I(n)} = u_{N+J(n)} \neq 1$ then $v_{I(n)} = u_{I(n)} - 1$ and $v_i = u_i$ for all $i \neq I(n)$.

Note that, if $\ell^{(n-1)} = \mathbf{u}$ and $\ell^{(n)} = \mathbf{F}_n(\mathbf{u})$, each individual stayed in its lane if there was no collision at time t_n; the $J(n)$ individual moving clockwise moved to lane 2 if there was a collision at time t_n and that collision occurred in lane 1; the $I(n)$ individual moving counterclockwise moved to lane $\ell - 1$ if there was a collision at time t_n and that collision occurred in lane ℓ with $\ell > 1$.

Note that, when we think of the individuals moving around the track, the effect of applying the functions $\mathbf{F}_n$ successively, i.e., for $n = 1, 2, \dots$ is to get any clockwise moving individual out of lane 1 and into lane 2, and to move any counterclockwise moving individual toward lane 1. All these changes of lanes are possible within our model with positive probability.

Observation 4. $P\left(\ell^{(n)} = \mathbf{F}_n(\mathbf{u}) \,\middle|\, \ell^{(n-1)} = \mathbf{u}\right) > 0$ *for all positive integers n and all $\mathbf{u}$ in Ω.*

The validity of this last Observation can be easily verified using the discussion of Section 5 and the definition of the functions $\mathbf{F}_n$. In fact, we have

$$P\left(\ell^{(n)} = \mathbf{F}_n(\mathbf{u}) \,\middle|\, \ell^{(n-1)} = \mathbf{u}\right) = \begin{cases} 1 & \text{if } u_{I(n)} \neq u_{N+J(n)} \\ 1/2 & \text{if } u_{I(n)} = u_{N+J(n)} = 1 \\ 1/2 & \text{if } u_{I(n)} = u_{N+J(n)} = L \\ 1/4 & \text{otherwise.} \end{cases} \tag{16}$$

Observation 5. *Let $\mathbf{v} = \mathbf{F}_n(\mathbf{u})$. Then, $v_i \le u_i$ and $v_{j+N} \ge u_{j+N}$ for all $1 \le i, j \le N$, with exactly one of these inequalities being a strict inequality unless $u_{I(n)} \neq u_{N+J(n)}$.*

The validity of this last Observation is immediate from the definition of the functions $\mathbf{F}_n$.

Observation 6. *Let r be a non-negative integer. Let $\mathbf{v}^{(r)}$ be in Ω and let $\mathbf{v}^{(n)}$ be defined recursively by $\mathbf{v}^{(n)} = \mathbf{F}_n(\mathbf{v}^{(n-1)})$, where $n > r$. Assume there exist $k \ge r$ such that $\mathbf{v}^{(k+1)} \neq \mathbf{v}^{(k)}$. Then, $\mathbf{v}^{(n)} \neq \mathbf{v}^{(k)}$ for all $n > k$.*

This Observation results from Observation 5. For any $1 \le i \le 2N$, the ith component of $\mathbf{v}^{(n)}$ with $n \ge k$ forms a monotone sequence. Thus, unless it remains constant, it can not return to its initial value (with $n = k$). Since $\mathbf{v}^{(k+1)} \neq \mathbf{v}^{(k)}$, there exists i_0 such that the i_0th component of $\mathbf{v}^{(k+1)}$ is different than the i_0th component of $\mathbf{v}^{(k)}$. Thus, the i_0th component of $\mathbf{v}^{(n)}$ is different to the i_0th component of $\mathbf{v}^{(k)}$ for all $n > k$.

We recall that the set A was defined in Equation (12). The system is self-organized for $t > t_n$ if and only if $\ell^{(n)}$ is in A.

Observation 7. *Let r be a non-negative integer. Let $\mathbf{v}^{(r)}$ be in Ω and let $\mathbf{v}^{(n)}$ be defined recursively by $\mathbf{v}^{(n)} = \mathbf{F}_n(\mathbf{v}^{(n-1)})$, where $n \geq r$. Assume there exist $k \geq r$ and $m \geq N^2$ such that $\mathbf{v}^{(k+m)} = \mathbf{v}^{(k)}$. Then, $\mathbf{v}^{(k)}$ belongs to A.*

From Observation 6, $\mathbf{v}^{(k+m)} = \mathbf{v}^{(k)}$ implies that $\mathbf{v}^{(n)} = \mathbf{v}^{(k)}$ for all n satisfying $k \leq n \leq k+m$. This can only happen if $v^{(k)}_{I(n)} \neq v^{(k)}_{N+J(n)}$ for all $k < n \leq k+m$. Thus, since $I(s+N^2) = I(s)$ and $J(s+N^2) = J(s)$ for all s, and $m \geq N^2$, we have that in fact $v^{(k)}_{I(n)} \neq v^{(k)}_{N+J(n)}$ for all $k < n$. However, this implies that there will no be any more collisions for $t > t_k$ if $\ell^{(k)} = \mathbf{v}^{(k)}$ and thus, $\mathbf{v}^{(k)}$ belongs to A.

Observation 8. *Let r be a non-negative integer. Let $\mathbf{v}^{(r)}$ be in Ω and let $\mathbf{v}^{(n)}$ be defined recursively by $\mathbf{v}^{(n)} = \mathbf{F}_n(\mathbf{v}^{(n-1)})$ for $n > r$. Then, there exists $k \geq r$ such that $\mathbf{v}^{(k)}$ belongs to A.*

The proof of this Observation is as follows. Ω is a finite set. Thus, there exists $k \geq r$ integer and $m > N^2$ such that $\mathbf{v}^{(k+m)} = \mathbf{v}^{(k)}$. Then, from Observation 7, $\mathbf{v}^{(k)}$ belongs to A.

Observation 9. *Let λ be as defined in Equation (15). Then $\lambda = 1$.*

Let $\mathbf{w}$ be in Ω and r an integer satisfying $0 \leq r < N^2$ such that $\lambda = f_r(\mathbf{w})$. Let $\mathbf{v}^{(r)} = \mathbf{w}$ and let $\mathbf{v}^{(n)}$ be defined recursively by $\mathbf{v}^{(n)} = \mathbf{F}_n(\mathbf{v}^{(n-1)})$ for all $n > r$.

Apply Observation 4 with $\mathbf{u} = \mathbf{v}^{(n-1)}$ to get $P\left(\ell^{(n)} = \mathbf{F}_n(\mathbf{v}^{(n)}) \,\middle|\, \ell^{(n-1)} = \mathbf{v}^{(n-1)}\right) > 0$ for all $n > r$.

Now apply Observation 3, with the role of $\mathbf{u}^{(s)}$ in that observation being played by $\mathbf{v}^{(r+s)}$ for all $s \geq 0$ and the role of m being played by r to get that $f_{r+s}(\mathbf{v}^{(r+s)}) = \lambda$ for all $s \geq 0$. Note that we are under the hypothesis of that Observation since $f_r(\mathbf{v}^{(r)}) = f_r(\mathbf{w}) = \lambda$.

From Observation 8 there exists $k \geq r$ such that $\mathbf{v}^{(k)}$ belongs to A. Thus, $f_m(\mathbf{v}^{(k)}) = 1$ for all integers m. However, we have obtained that $f_{r+s}(\mathbf{v}^{(r+s)}) = \lambda$ for all $s \geq 0$. Setting $s = k - r$, we get $f_k(\mathbf{v}^{(k)}) = \lambda$ and thus, the statements of this paragraph imply that $\lambda = 1$.

Theorem 1. *The system self-organizes with probability 1 no matter the initial conditions. In other words, let $\mathbf{v}$ in Ω. Assume $\ell^{(0)} = \mathbf{v}$. Then, $P\left(\ell^{(k)} \text{ is in } A \text{ for some } k \geq n \,\middle|\, \ell^{(0)} = \mathbf{v}\right) = 1$.*

The proof is immediate. By definition, $f_0(\mathbf{v}) = P\left(\ell^{(k)} \text{ is in } A \text{ for some } k \geq n \,\middle|\, \ell^{(0)} = \mathbf{v}\right)$. However, $\lambda \leq f_0(\mathbf{v})$. Since we proved that $\lambda = 1$ and probabilities can not exceed 1, the theorem is proved.

9. Discussion

We have introduced a simple model to study the formation of lanes in crowds of individuals moving in opposite directions. Our model has proved simple enough to be amenable to analytical analysis.

The model studied in this article is different in nature from the existing models that can be found in the literature. This model is the simplest model we could think of that isolates the effect of pedestrians avoiding collisions in a very simple way. This model, that could be considered a toy model, has the usual advantage of simple models: (1) The results obtained are clear, concrete and easy to interpret, (2) Its simplicity makes it amenable to analytical analysis.

Our model belongs to the class of microscopic models, i.e., the model follows the trajectory of each individual. Many microscopic models that can be found in the literature consist of a set of coupled odes. Our model is different. Our model is a Markov Chain. Cellular automata models are somewhat closer in flavor to our model, but our set up and the rules governing the dynamics of our system are different to what can be found in the literature.

Our model, as presented and analyzed here, cannot be used to make any deep comparison with experimental data. The only comparison that can be made is that our model predicts that self-organization occurs.

There is a trade off, and with simplicity we sometimes lose generality. Most models in the literature do not lend themselves to the simple analytical analysis presented in this article. Instead, they have to be solved numerically. However, they are more general in the sense that different phenomena can be studied by simply changing boundary or initial conditions or parameters. We believe that both strategies, the study of simple toy models, as in this article, and the study of comprehensive more complex models, as most of the studies found in the literature, are very valuable and complement each other.

While several extensions of the work presented here are possible, understanding the expected time that it takes the system to self-organize is, in the view of the author, both challenging and interesting. The author plans to pursue this research direction.

We believe this article is a step towards the understanding of self-organization in biological systems, and we hope the modeling style of this article will be adopted by other researchers to study this and other self-organization phenomena in biological systems.

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