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Augmented Reality as a Didactic Resource for Teaching Mathematics

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Abstract: This paper is an example of how to use new technologies to produce didactic innovative resources that ease the teaching and learning process of mathematics. This paper is focused on augmented reality technology with the aim of achieving the creation of didactic resources related to the polyhedra taught in a course of compulsory secondary education in Spain. First, we introduce the basis of this technology and present the theoretical framework in which we make an exhaustive analysis that justifies its usage with educative purposes. Secondly, we explain how to build the polyhedra in augmented reality using the Unity game engine and the Vuforia software development kit (SDK), which enables the use of augmented reality. Using both tools, we create an augmented reality application and some augmented reality notes with the purpose of helping in the process of visualization and comprehension of the three-dimensional geometry related to polyhedra. Finally, we design an innovative, didactic proposal for teaching the polyhedra in the third course of compulsory Secondary Education in Spain, using the resources created with the augmented reality technology.

Keywords: augmented reality; didactic resources; educative innovation; polyhedra

1. Introduction

In our society, new technologies are increasingly important in our daily lives: they facilitate tasks and allow us instant access to information in many ways. In addition, in recent years the number of jobs that depend on them has been gradually increasing, so it is necessary that children and adolescents get in touch and become familiar with them from an early age.

This importance is reflected in the Spanish Education Law. It considers new technologies as a fundamental piece of the methodological change that achieved the objective of improving educational quality. Likewise, information and communication technologies will also be key tools in teacher training and in the learning of citizens throughout life.

Our intention in this study is the creation of a didactic proposal for the teaching of mathematics, especially in the geometry content block, for secondary mathematics teachers, and also for future teachers in their master training courses. Although it is true that educational innovation does not necessarily have to be linked to the use of new technologies, the problem of visualization in geometry cannot be solved simply with manipulative teaching material, which has traditionally been used in the classroom, such as polyhedra made of paper and wood or with vertex and edge assemblies.

The use of augmented reality (AR) would allow us to develop mandatory aspects of the curriculum, such as the use of new technologies, and improve visualization in geometry so that this block does not remain as a mere calculation of areas and volumes by applying formulas.

One of the problems of new technologies is not only their economic cost, but also the fact that the teacher must be familiar with them, so that it is less expensive to implement them in the classroom.

The idea of this project is to facilitate this point, explaining in detail how to make an app with accessible hardware and free software so that any teacher can use it in their math classes.

In previous work [1,2] we discussed the importance of the use of accessible hardware and free software so that the educational use of augmented reality (AR from now on) is feasible and can be used in classrooms without problems. In addition, augmented reality educational experiences have been successfully designed to teach different educational contents of science and mathematics, although the interface has not always been the screen of a mobile device.

The motivation of the students and their involvement with learning are key to achieve effective learning. In this case, AR-related technology plays a fundamental role as several investigations have shown [3–7], in which the increase in student motivation with the use of AR technologies has been measured. The advantage of AR is its similarity to games to which students are accustomed in their leisure time. Hence, its implementation in the classroom is very natural for them.

AR has become an innovative, useful, and effective teaching tool in different educational fields [8–10]. Furthermore, some authors consider that AR systems help students to develop skills which can be developed in other environments, but they are developed more effectively through the use of AR [11].

New technologies are a fundamental tool to facilitate the learning of curricular contents of different subjects. In this paper we focus on mathematics, and specifically on the geometry content block. In usual geometry classes everything is focused on figure recognition and the application of formulae to calculate areas and volumes. However, when the content is more complex and involves visualization, the acquisition of spatial skills, and the application of advanced strategies, traditional resources do not work. AR helps in these aspects, as seen in similar works in other educational levels [12–14]. Moreover, understanding geometric concepts and relationships, students have an increase in creativity with AR technology [15]. This paper proposes the creation of an application and textbook using AR, for the teaching of polyhedra in the third year of secondary compulsory education, as well as the design of an adapted didactic proposal making use of the resources created. This idea is inspired by the work of Kaufmann and Schamalstieg [16], who carried out a huge project for the construction of a complex augmented reality application (Construct 3D) that implements knowledge of geometry with the aim of being used in a classroom. For this, two tools are used: the powerful Unity video game design engine and the software Vuforia. Although this paper is focused on a design for a specific subject, such as mathematics, and for a single teacher to implement it in their classroom, in the case of the design of AR experiences, due to its technical complexity and versatility, it would be positive and advisable to organize a collaborative work between teachers of different subjects [17]. For this, teachers should be assessed on which educational fields and levels AR technology would be the most effective, and what would be the most suitable materials [18]. Following this idea, we propose highly accessible hardware and software that are detailed with sufficient information for AR implementation.

AR consists essentially in the superposition of new information to the existing reality using computer devices such as a computer, tablet, or smartphone. When the device detects a certain image, pattern, position on the Earth's surface, etc., it generates the appearance of the new information that is added to the actual existing information.

Azuma [19] was one of the pioneers in studying this technology. He conceived AR as a combination of elements of reality with virtual elements, interactive in real time and recorded in 3D. In his work, he presents the potentialities of applying this technology in many different fields. These proposals were successful, since currently AR is present in many different areas: commercial and tourism to advertise products and places; education sector; in the design of videogames; architecture and industrial design; military training; for airplane pilots, doctors, astronauts, etc.

The types of AR have evolved over the years. A previous work [20] established essentially two types of augmented reality depending on the elements that act as "activators" of the information:

1. Augmented reality based on markers: markers are nowadays the most common way to activate information in augmented reality applications (when the device's camera recognizes any of these markers, digital information emerges). There are two types of markers: flat markers which

adopt flat geometric shapes (usually black and white images) that allow a device's software to recognize them (e.g., barcodes and QR (Quick Response), codes) and markerless NFT (natural feature tracking) which uses real-world objects or images to activate information

2. AR geolocated is when the new information is activated by elements of the device that indicate its positioning, such as geolocation (given by GPS (Global Positioning System), GLONASS, Global Navigation Satellite System, etc.), orientation, device angle, speed, and acceleration. All these data are given by sensors of the device such as the compass, the gyroscope, or the accelerometer.

Depending on the degree of complexity that applications use to implement it, there are four types of AR [21]:

Level 0 (creation of hyperlinks): Applications make use of 2D codes (for example, barcodes or QR codes), which serve as links to other content, so that a marker tracking is not necessary. For example, reading a QR code with our mobile device can take us directly to a web page.

Level 1 (augmented reality with markers): In this type, applications use 2D and 3D images as information triggers. When a certain predefined marker is displayed by the camera of the device, the digital information emerges.

Level 2 (augmented reality without marks): The markers that activate the information are replaced by the geolocation and user orientation given by the device's GPS or compass gyroscope. Thus, when the user is in a specific place on the Earth's surface, the digital information appears.

Level 3 (augmented vision): Examples of this level would be Google Glass or Microsoft HoloLens glasses. This type of AR allows a totally personalized, immersive, and contextualized experience. In this case the information may arise due to the geolocation of the user or due to existing markers.

Although it is not a new technology and despite being a great promise as a learning facilitator, its implementation in Spanish classrooms is rather discreet. The 2016 Horizon report [22] considered AR as an emerging technology that should be present in many classrooms in the academic year 2018–2019.

According to Dunleavy and Dede [23], the fact that AR is considered as a technology that improves learning, is based on two completely differentiated and independent theoretical frameworks:

- The constructivist learning theories proposed by Bruner and Vygotsky affirm that people build their new knowledge based on what they already know and believe (this will depend on very different aspects such as their social and cultural backgrounds, the prevailing social context or previous personal experiences). According to these theories, AR has enormous potential to improve the construction of knowledge by students.
- Situated learning theory states that meaningful learning takes place in a specific context, the quality of which depends directly on the interactions that occur between people, objects, places, processes, and culture (Brown, Collins and Duguid, 1989, cited by [23]). The use of AR produces a new mode of interaction for the student with the curricular contents, teachers, and other students, as well as with processes, places or culture, which would be beneficial for learning.

The advantages of the implementation of this technology in classrooms are discussed in the literature [24,25] and include the following:

1. It would allow students to familiarize with new technologies and they would learn to handle them.
2. Education would be more interactive and didactic.
3. It would facilitate the development of cognitive, spatial, and motor skills.
4. It would increase the motivation and curiosity of students to learn, in addition to improving their creativity and imagination.
5. It would stimulate attention, concentration, short-term and long-term memory, and reasoning.
6. It would allow students to form critical attitudes of reflection to explain an observed phenomenon or the solution to a given problem.
7. It would facilitate understanding of the subject to be learned and encourage the autonomous learning of the student. For this, materials such as a textbook with AR tools could be developed.

Disadvantages and problems of AR are also discussed in the literature [26] and include the following:

1. It could be difficult to train teachers to be able to implement activities using this technology, although AR tools designed by other professionals could be used.
2. AR is difficult to implement in the classroom due to the high economic cost of computer equipment (mobile phones, tablets, or computers with quite powerful processors that allow the fluid use of this technology).
3. The general use of new technologies could deteriorate communication and relationships between people, making these relationships much more superfluous (humans communicate less face to face because they do so more and more through the new technologies). In education, this fact must be considered because it could cause students to interact less with teachers and other students, dehumanizing the teaching-learning process.
4. It could widen the digital divide between different population groups (especially among socially disadvantaged groups).

Various empirical studies [3,27,28] demonstrate an improvement in the results obtained by students who have been taught mathematics using AR technology. For example, it showed great success in teaching a course of functions of several variables to university engineering students in a Colombian university. It was found that students taught through AR obtained significantly better grades than the control group that was taught in a traditional way [27].

In addition, there are many other authors who talk about the potential of augmented reality in education [22,25,29–32]. For all these reasons, we can consider this technology as one that has great potential to produce a beneficial methodological change for the teaching of mathematics and justifies the creation of math teaching materials in augmented reality that facilitate this change for teachers.

This paper originates from the training of teachers for secondary education. In Spain, to be a secondary school teacher, in addition to obtaining a university degree, you must take a training master's degree, with theoretical classes, practices in high schools, and a final proposal for the master thesis that must include different aspects related to teaching math in a classroom. Our idea was to develop some geometry classes focused on an innovative methodology such as AR. This is a line of research in which we have been working for the last years at our university.

Our research question is: is it possible to design geometry classes in a high school course so that any teacher with a basic knowledge and, most importantly, with simple hardware and accessible software, could develop the content with this resource?

The accessibility of AR resources is one of the biggest problems when designing classes, and their lack of implementation in the classroom is conditioned by this fact. In the Spanish public education system, after many years of economic recession and given the current uncertainty, one cannot assume that the implementation of new technologies in the classroom depends on the acquisition of very expensive hardware or software that needs to be paid for. If this were so, we would increase the digital gap and divide our students into first and second classes, depending on the devices they manage. Most of the technological innovations that are made fall on teachers, on their technological skills, their own resources, and their ability to learn.

To avoid this, the design of our project is based on simple hardware and open software. The process to follow is detailed in Sections 2 and 3 of this paper, so that any teacher, both senior and in training, without excessive programming knowledge can implement it in their classroom.

2. Materials and Methods

One of the most widely used technology resources in math classrooms is GeoGebra. It has many advantages: it is free, its use is very intuitive, and there is a vast amount of resources developed by teachers and available to anyone. However, we decided not to use GeoGebra in our project for technical reasons.

Geogebra uses Google's augmented reality library (ARCore, it is a software development kit developed by Google (Google LLC, California, USA) to built AR apps) to implement augmented reality. To use ARCore you need to have a fairly new and expensive mobile phone. Mobile phones and tablets before certain dates do not support ARCore and therefore you cannot use augmented reality with Geogebra (it does not work even if you update to new versions of Android). In addition, not all new mobile phones use Android, so in principle you will not be able to use ARCore.

In this project, the development of AR teaching resources will be carried out using Unity software. The reason for this choice is mainly because it is a very powerful and versatile tool that allows the creation of different types of applications.

Using Unity we can create applications for any platform without problems. Today, videogame design engines are the best to design AR experiences. Moreover, something very important for us, being coherent and realistic with the Spanish public educational system, is to design resources in AR which can be applied without having very expensive hardware, but rather something in a mid-range, available to students and teachers.

Unity is the most popular game development software among developers around the world (about half of the videogames that are designed are made using this tool). It creates applications and designs AR experiences for all types of platforms such as mobile phones, consoles, or personal computers. The company that provides this software is Unity Technologies, which is based in San Francisco. It was founded in 2004 by David Helgason, Nicholas Francis, and Joachim Ante in Copenhagen (originally under the name Over the Edge Entertainment and changed its name to Unity Technologies in 2007).

The huge community of Unity users is very active in forums, where there is a solution for many of the questions and doubts that may arise during its use. In addition, on the official Unity website [33], questions about programming, examples, and mini tutorials can be consulted.

Vuforia is a software development kit (SDK) to create AR applications. It uses a series of computational algorithms that allows applications to recognize different types of markers (also called "ImageTarget") in two (for example, a QR code) and three dimensions (real-world objects) in real time. When the marker is recognized by the application thanks to Vuforia, accurate digital information is shown to the user. This ability to recognize images allows positioning and orientation of predesigned virtual objects (3D models) and other information in relation to real images that are captured by the device's camera. In addition, if the information shown is a digital model in two or three dimensions, the digital object will "detect" the position and orientation of the ImageTarget in real time, so that if we move the marker, the virtual object will move synchronized with it. This effect makes the user have the impression that the virtual model is an object present in the real world, even if it is not really.

Vuforia allows the implementation of AR on Android and iOS (by Apple) and the creation of AR applications on Unity (easily exportable to any platform). Therefore, with Unity and Vuforia augmented reality applications can be created for almost any system. That is why both have been chosen to develop this project.

2.1. GameObjects Unity Components

A GameObject is any element that is within our game scene. It constitutes the basic entity of the game and its development and it is based on the interactions that exist between them. Each GameObject in Unity contains what is known as components. A component is basically a piece of code that provides some functionality or property to the GameObject. In fact, it is Unity's way of performing modular programming, reusing components that will be added to game objects when necessary. The basic example of a Unity component is the Transform component, which stores the position, rotation and scale of the object. Every GameObject in Unity will always at least have the Transform component.

Scripting is essential in all games to ensure that game events run at the right time. Scripts can be used for many different tasks such as controlling the physical behavior of objects, creating graphic effects, implementing artificial intelligence for the game's characters, etc. Unity uses the

C# programming language to perform scripting. The main utility of Unity scripts is that it allows the creation of components defined by the programmer according to the needs that arise during the development of their applications.

In our project, to create the application it has been necessary to design several components, mainly for the creation of polyhedra. To start a new script in the Unity window, within the Assets tab there is the Create tab in which the C# Script option can be selected. Immediately afterwards we will have to name the new script (see Figure 1).

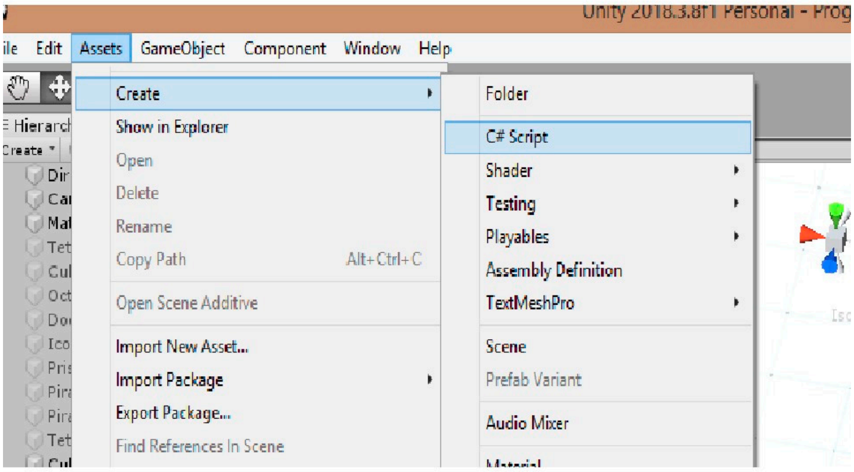


Figure 1. Creating a script in Unity.

This script will be saved in the project files folder for when we need to add it to a GameObject. If we open a script for the first time after its creation, we will find a series of Unity and C# libraries and a class that has the name of the script and also inherits from the MonoBehavior Class, which indicates that the script is a component (MonoBehavior is the basic class that all components inherit). In addition, there are two functions called Start and Update, which are functions that can be used within the component and that have been inherited from the MonoBehavior class. The "functions" of inherited MonoBehavior will be very useful to customize our components.

2.2. Hierarchy and Parenting in Unity

In Unity there is a concept known as "parenting" to establish a relationship between GameObjects, which gives rise to what is known as a hierarchy of objects (parents and children): A GameObject B is a "child" of a GameObject A when B inherits the Transform component of A. This relationship is essential to move or change the scale of a series of objects, in our case, for the creation of polyhedra. This behavior is very useful: each polyhedron is made up of different objects (faces, edges, and vertices) that will be the children of an object of any type given polyhedron. In this way, if we move, rotate, or change the scale of a polyhedron type object, then its faces, edges, and vertices will do so in an appropriate way so that the shape of the polyhedron is preserved.

2.3. The Unity Work Window

Opening Unity for the first time, you can see the work window essentially composed of the following elements:

1. The scene of the game (Scene): It allows us to visualize the distribution in the scene of all the objects of the game. In addition, we can modify the position, rotation, and scale of our objects to see what their appearance would be.
2. Hierarchy: It shows all the objects (GameObjects) existing in our game, in addition to the relationship between them.

3. The Unity Asset Store: Allows us to access assets or resources programmed by other users.
4. The project folder (Project): It stores the useful elements to edit the game, for example scripts, materials, prefabs, assets, music, etc.
5. The game screen: It allows us to visualize an emulation of our game or application without compiling it for any platform.
6. The inspector (Inspector): It shows us the information of the different elements of the game (components of objects, materials, scripts, etc.), in addition to allowing new components to be added to a GameObject using the Add Component option once that the GameObject has been selected in the hierarchy. It also allows us to modify the public attributes of object components, as well as to enable and disable GameObjects and GameObjects components.
7. The console: This is an essential element during programming because it detects the syntax errors we made during the creation of our scripts in C#.

2.4. Using Vuforia inside Unity

To use Vuforia we must register on the Vuforia page, after that on the website we will request a Vuforia license. Then, we must upload on the Vuforia page the images (in jpg format) that we want to use as markers within our application. The web will rate each image from zero to five stars indicating the quality it has to be used as an augmented reality marker (0 stars indicates that the image does not serve as a marker and five that the image is perfect as a marker). The option chosen in this project to obtain markers is to use QR codes.

In order to use AR inside Unity, we must remove or deactivate the main camera and include an AR camera in the GameObject menu and select the AR camera from the Vuforia Engine drop-down table. We will add to the scene the markers that we want to use as information triggers, so in the drop-down tab of Vuforia Engine we select Image, by doing so an object called ImageTarget will appear on the scene and in the hierarchy. To configure it with the image we want, we only need to select the ImageTarget object and within its component called ImageTargetBehavior choose the image database that we have downloaded from Vuforia and select the image we want to act as a marker.

Finally, we must associate the information that we want to arise contained in a GameObject, when the device's camera detects the image associated with the ImageTarget object. In the case of a 3D model, it is enough to make the GameObject that represents it the child of the ImageTarget object (it can be done directly in the hierarchy or using the SetParent method of the Transform component of the GameObject) and then adjust the position, rotation, and scale of the model.

2.5. Creating Polyhedra in Unity

Unity includes a series of predefined very limited geometric figures, known as PrimitiveType, but they are not enough for the application we wanted to create. For the construction of the vertices and edges of the polyhedra the predefined spheres and cylinders can be used. Therefore:

1. To construct a vertex of a polyhedron, it will be enough to instantiate a sphere of a suitable radius, positioned on the coordinates of the vertex.
2. To build an edge of a polyhedron it will be enough to instantiate a cylinder of a suitable radius, modify its height so that it measures the same as the length of the edge and make a series of translations and turns to it so that it is positioned joining the vertices of the edge.

The construction of faces is the most complicated part. Each face of a polyhedron will be an empty object (it only has one Transform component that gives the object's rotation and scale position) to which two new predefined components in Unity have been added:

1. The MeshFilter component: there is a mesh type variable that will allow the creation of faces. To create the face of a polyhedron, the different triangles in which the face can be decomposed must be constructed.

2. The MeshRenderer component: this takes the network (points and triangles) stored in the MeshFilter component and considering the position, rotation, scale, and kinship of the object, renders it properly on the screen.

2.6. Example: The Creation of a Tetrahedron. The FTetrahedron Script

To create the tetrahedron we need to calculate the coordinates of the vertices and from them build faces, vertices, and edges. All this information and construction will be done in the FTetrahedron Script. In it, a method called Create is designed that uses the methods CreateSpheres (from the Spheres script), CreateEdgesFaces (from the Edges script), and BuildFaces (from the CreateFace script) to allow you to create a Tetrahedron. The Create method receives as parameters the colors of the faces, vertices, and edges.

In order that the FTetrahedron script has access to these scripts, first an empty object is created (we call it Tetrahedron) and all scripts are added except the Materials script. In addition, the FTetrahedron script contains the public variable sidePolyhedron to which a public object named Face is assigned in the inspector, which allows instantiating faces and assigning internal networks corresponding within the Create function to obtain the desired result. Finally, when you want to instantiate a tetrahedron we only need to use the Create function of the FTetrahedron script within the Start or Awake functions of some other script (or in the Ftetrahedron script itself) that will perform the task of instantiating the polyhedron (we can see the working window in Figure 2).

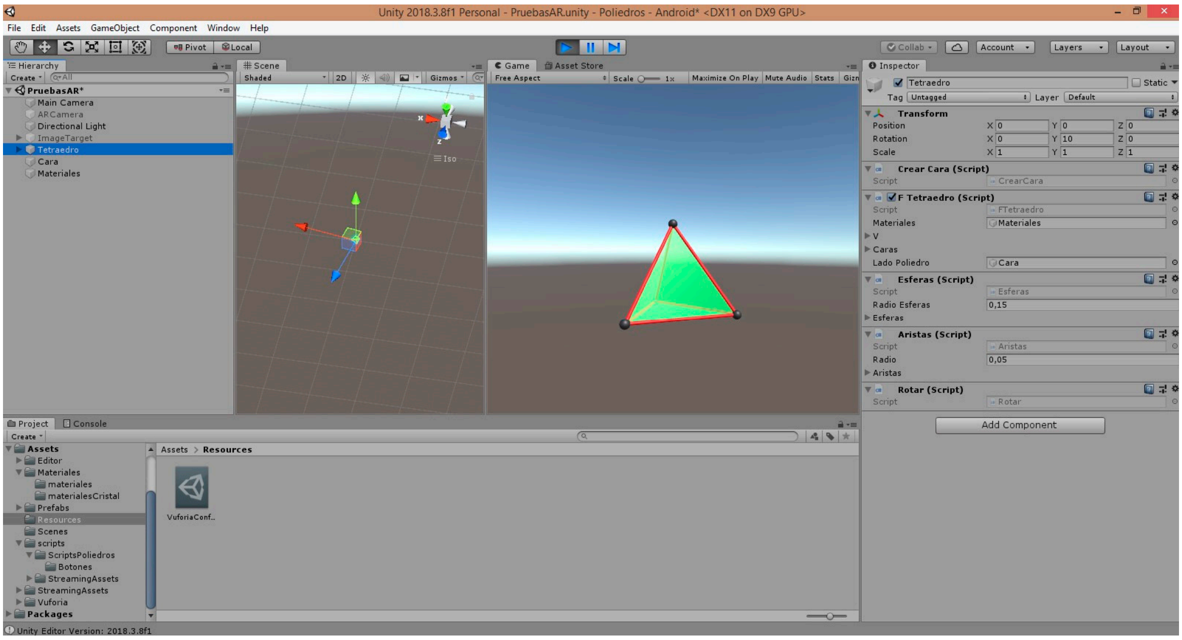


Figure 2. Tetrahedron created in Unity.

Once we have all the polyhedra designed, to visualize them in augmented reality we simply have to assign the GameObject (which contains the corresponding scripts of the polyhedron and that instantiate it) as a child of the ImageTarget (it contains the image that we want to act as a marker of the polyhedron) that acts as a marker thanks to Vuforia.

3. Results

3.1. A Textbook with AR Resources to Teach Polyhedra

In Spain there are 17 regions with legislative powers in education. Taking into account the place where this work was carried out, we will use as our legislative framework the ORDER EDU/362/2015, of May 4, which establishes the curriculum and regulates the implementation, evaluation, and development of compulsory secondary education in the community of Castilla y León, Spain.

In that document, article 19 deals with general considerations related to secondary education, and materials and resources for curriculum development are discussed. This article says that the materials and resources should be both traditional and more innovative, integrating different supports, to promote the ability to learn for yourself. An interesting aspect included in this article is the invitation for teachers to prepare their own resources. This is a fundamental aspect of our work, so some technical aspects are detailed above.

The curricular contextualization is made for the third year of compulsory secondary education. In Spain, there are two types of mathematics in this year and students can choose between both of them, depending on their future academic career: mathematics for academic teaching, which will facilitate a more orthodox training designed for those who choose A-Levels and then university studies; and applied mathematics, more oriented for those who choose a professional training that enables them for a profession.

We have chosen the first of those courses, which is the majority choice of students. In the specific contents of this course, we find the justification for our work in two sections. One is related to the content common to the entire course. In it, the use of technological means in the learning process is emphasized to facilitate the understanding of geometric properties and the design of simulations.

In the case of the content block specifically focused on geometry, our proposal focuses on the following content:

- Use of technological tools to study and build geometric shapes, configurations, and relationships.
- Geometry of space, polyhedra, plans of symmetry in the polyhedra, and Euler's formula for simple polyhedra.
- Regular polyhedra and dual polyhedra.
- Calculation of areas and volumes of geometric bodies.

All these aspects determine, define, and concretize our proposal, as it could not be otherwise because it is a mandatory legal framework for compliance. With a regular textbook, drawings on the board, or simple manipulative material, as is very common in the math classroom, these contents are very difficult, if not impossible, to develop.

To develop all these aspects, a textbook has been written with mathematical content about polyhedra. The strong point of this work is the creation of this AR textbook to teach mathematics, which is innovative in itself because almost nobody currently does it.

When we want the student to visualize a polyhedron or some aspect of it, we will include in the text an image that will act as a marker (a QR code) of the content we want to show students (this content will have been previously programmed and can be used through the created AR application). When students use the application and point their device's camera at the markers, the new information we want to show them will appear on their device.

The application has two parts:

1. The first one is related to AR notes on polyhedra. When we want to visualize some of the models of the notes, we have only to install and open this application and point to the QR codes of the notes to visualize the AR model.
2. The other allows us to visualize AR models of a wide range of polyhedra. It basically consists of a menu in which the polyhedron that we wish to visualize in AR is selected with a single marker to each polyhedra.

Our project target is to use these two applications, together with the AR textbook, to facilitate and enrich the teaching-learning process of polyhedra. This will be detailed in the didactic proposal that we present in the next section.

3.2. Classroom Design Using AR

Now we will explain the didactic proposal to teach the didactic unit of polyhedra. For its teaching, AR is used as a novel teaching resource that aims to facilitate learning, and to improve the visualization of geometric concepts and spatial capacity. In addition, using AR to teach will serve as a catalyst to improve the motivation of students to mathematics.

The mathematical contents correspond, mostly, to the official curriculum in force in Spain for the third year of compulsory secondary education. These mathematical contents are:

Common contents to all math blocks:

1. Planning of the problem-solving process: analysis of the situation, selection, and relationship between the data, selection, and application of the appropriate resolution strategies, analysis of the solutions and, where appropriate, extension of the initial problem.
2. Choice of strategies and procedures implemented: Use of appropriate language (graphic and geometric) and good notation; construction of a figure, scheme, or diagram. Experimentation using the error test method. Search for analogies and similar problems. Reformulation of the problem, subproblem resolution dividing the problem into parts; exhaustive counting, start by simple individual cases, search for regularities and laws; introduction of auxiliary and complementary elements; working back assuming the problem is solved.
3. Reflection on the results: review of the operations used, assignment of units to the results, verification and interpretation of the solutions in the context of the situation, search for other forms of resolutions, etc.
4. Verbal and written expression in mathematics.
5. Approach of mathematical school research in geometric contexts, appropriate to the educational level and the difficulty of the situation.
6. Practice of the processes of mathematization and modeling, in contexts of reality and in mathematical contexts.
7. Confidence in one's own abilities to develop appropriate attitudes and face the difficulties of scientific work.

Related to the objectives to be achieved with this project using new technologies in the learning process, our targets are:

- To elaborate and create geometric representations.
- To facilitate the understanding of geometric properties.
- To communicate and share, in appropriate environments, information and mathematical ideas.

Specifically about the geometry content:

- Use of technological tools to study and build shapes, configurations, and geometric relationships
- Space geometry: polyhedra
- Symmetry planes in polyhedra
- Euler's formula for simple polyhedra
- Regular polyhedra
- Dual polyhedra
- Calculation of areas and volumes of polyhedra
- Contextualization
- Truncation of polyhedra
- Archimedes solids
- Kepler–Poinsot solids

3.3. Classroom Sessions

Next, we explain how we would teach polyhedra using the created materials as support, with which we intend to facilitate their study and visualization, as well as to improve their motivation and to encourage the use of new technologies.

Whenever it is mentioned during the activities that the polyhedra are displayed, we will remind that they are visualized using the AR, through markers and a computer or mobile device with camera and the applications installed. It is also assumed that students have previously been taught the geometry of the plane, because without doing so it would not make sense to explain the most advanced geometric concepts related to polyhedra. Finally, work groups of 3–4 students will be formed to carry out the activities. Groups will be determined based on students that complement each other and are able to work well together. This is intended for some students to help others to understand the concepts and to work as a team.

To structure the contents and the implementation of the application of AR in the classroom, different sessions are planned:

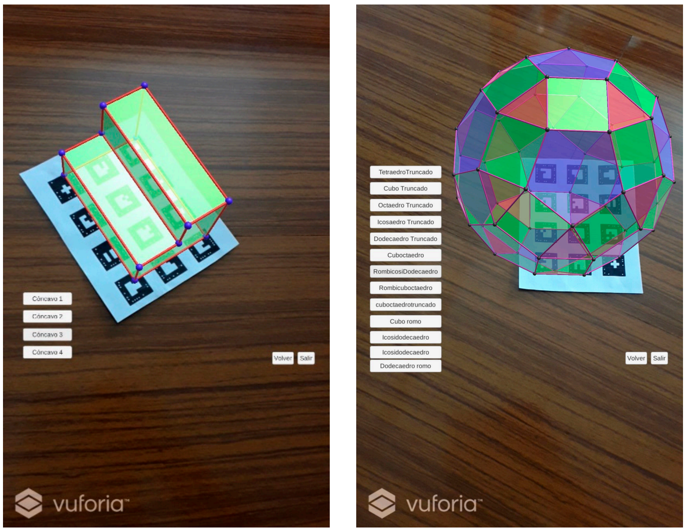
Class session 1

This session will be dedicated to teaching students how to use augmented reality technology with markers, to visualize and describe polyhedra and the elements that define them (faces, vertices, edges, diagonal).

We start teaching how to use the application we have created. In groups, we distribute a marker so that students can visualize various types of polyhedra using the AR application. The AR models will facilitate visualization (they can move, rotate the marker and the polyhedron will also move with the marker) and understand concepts like faces, vertices, and edges, that will be in different colors.

Class session 2

The notions of concave and convex polyhedra are introduced. Eight different concave and convex polyhedra will be shown with AR (see an example in Figure 3). Groups will be asked to identify what differentiates concave from convex polyhedra. They are asked to count the number of edges, vertices, and faces of convex polyhedra and try to find a relationship between these numbers. Euler's theorem for convex polyhedra is explained. They are asked to verify with other convex polyhedra that Euler's theorem is fulfilled and verify that the theorem is not fulfilled in the case of concave polyhedra.

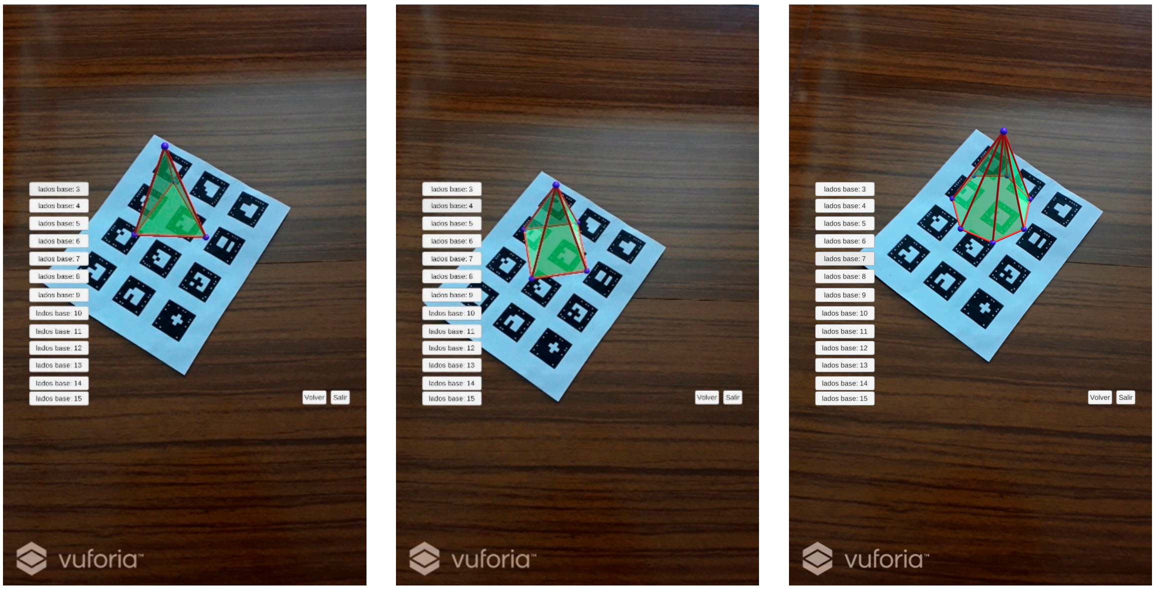


(a) (b)

Figure 3. Examples of concave and convex polyhedra with augmented reality (AR). (**a**) Concave polyhedron; (**b**) Convex polyhedron.

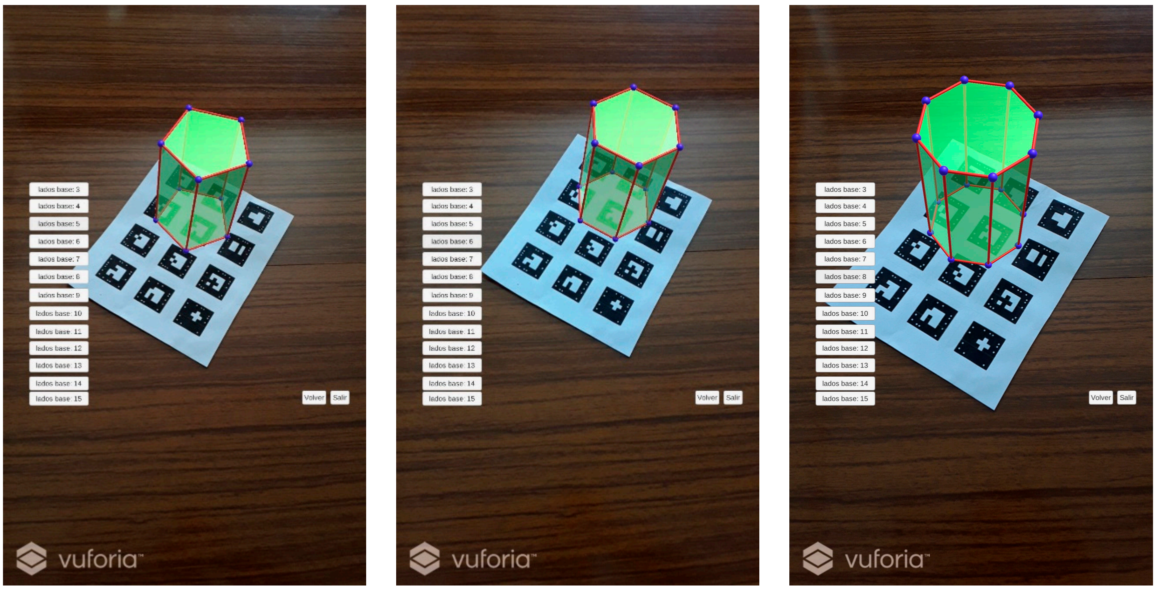
Class session 3

Different types of polyhedra are presented: orthohedron, parallelepipeds, prisms, and pyramids (straight and oblique) (see Figures 4 and 5). We teach the calculation of the diagonal of an orthohedron and the apothem of a pyramid as an application of the Pythagoras' theorem with the help of AR. The flat developments of polyhedra are visualized and explained and the formulas of the area deduced.



(**a**) (**b**) (**c**)

Figure 4. Images of regular pyramids with AR application. (**a**) Pyramid with a triangle base; (**b**) Pyramid with a square base; (**c**) Pyramid with an octagonal base.



(**a**) (**b**) (**c**)

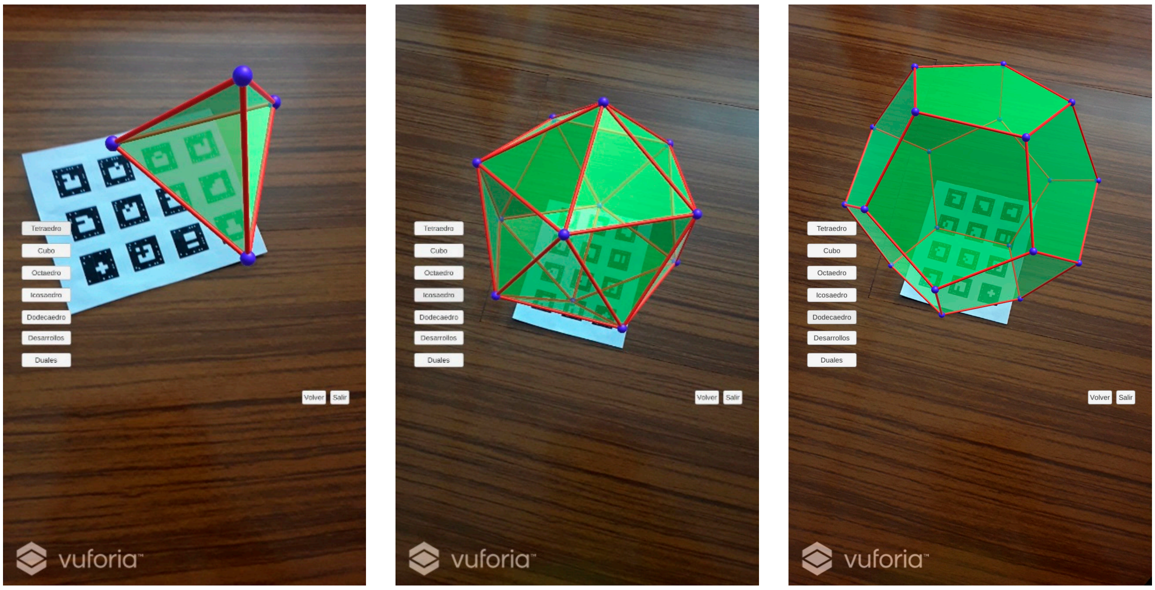
Figure 5. Images of regular prisms with AR application. (**a**) Prism with a pentagonal base; (**b**) Prism with a hexagonal base; (**c**) Prism with an octagonal base.

Class session 4

The Cavalieri principle is explained, and the formulas of the area and volume of the polyhedra are deduced, helping us with the AR application.

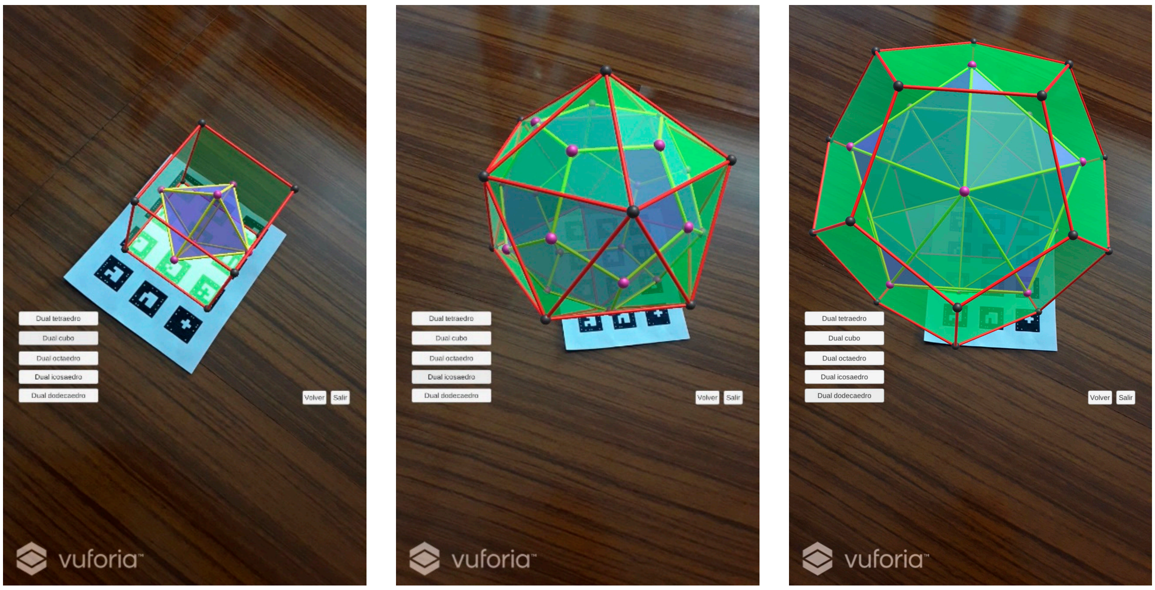
Class session 5

The concept of a regular polyhedron is explained and the fact that there are only five regular convex polyhedra displayed within the AR application: tetrahedron, cube, octahedron, icosahedron, dodecahedron (see Figure 6). In addition, the notion of dual or conjugate polyhedron for the previous five, is introduced (see some of these conjugated polyhedra in Figure 7).



(**a**) (**b**) (**c**)

Figure 6. Images of (**a**) tetrahedron; (**b**) icosahedron, and (**c**) dodecahedron with AR.



(**a**) (**b**) (**c**)

Figure 7. Images of dual polyhedra with AR. (**a**) Octahedron dual polyhedron; (**b**) and (**c**) Dodecahedron dual polyhedron.

Class session 6

In this session, we visualize the flat developments of the platonic solids (see some examples in Figure 8) and how to calculate their area is explained. The formulas of the volume of the tetrahedron, cube, and octahedron are also deduced and the formulas that allow the calculation of the volume of the icosahedron and the dodecahedron are explained.

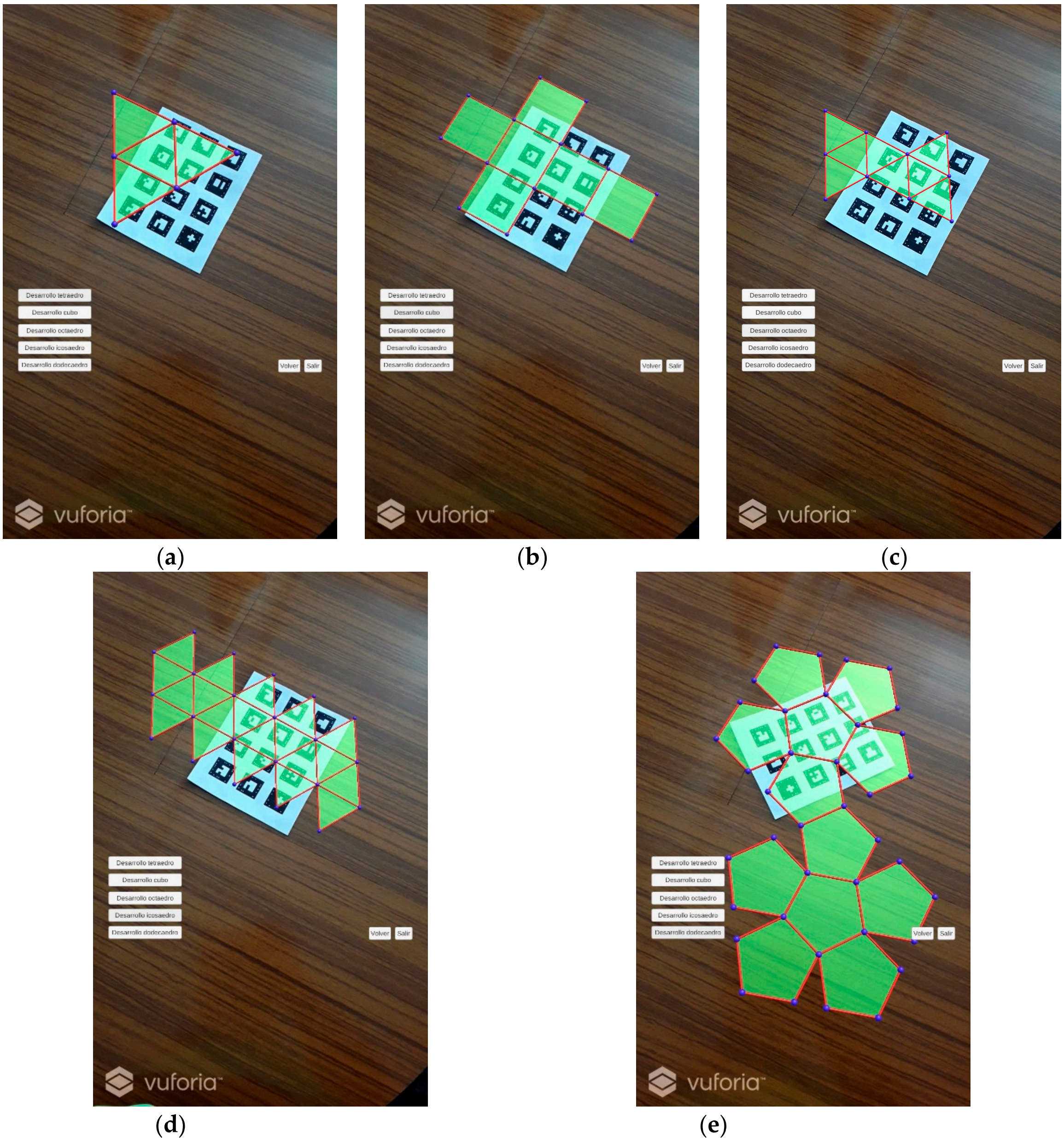


Figure 8. Images of flat developments of the platonic solids. (**a**) Tetrahedron; (**b**) Cube; (**c**) Octahedron; (**d**) Icosahedron; (**e**) Dodecahedron.

Class session 7

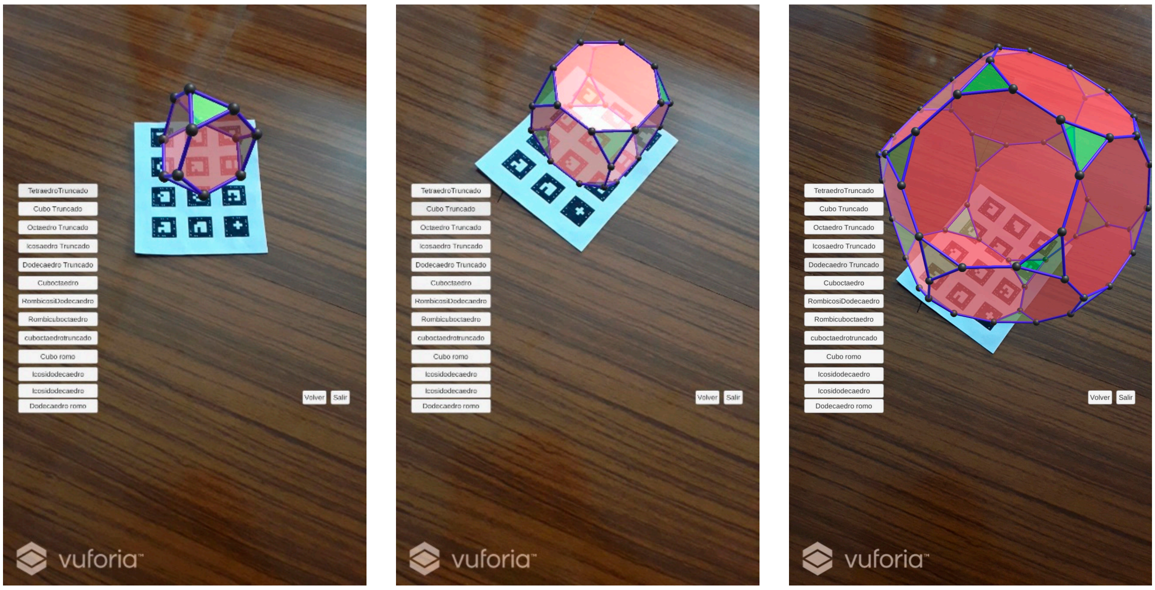
In this session, we explain the symmetry plane of a polyhedron. In addition, we visualize them for different polyhedra as regular prisms, regular pyramids, platonic solids, and parallelepipeds.

Class session 8

The concept of truncation of a polyhedron is explained. Several cases are displayed. The pyramid logs are defined, and their formulas of areas and volumes are deduced.

Class session 9

We teach the Archimedes solids obtained from the platonic solids. They are polyhedra whose faces are regular polygons of two or three different types. We intend that students do the truncation from platonic solids using the AR application (see examples in Figure 9).



(**a**) (**b**) (**c**)

Figure 9. Images of several truncated polyhedra. (**a**) Truncated tetrahedron; (**b**) Truncated octahedron; (**c**) Truncated dodecahedron.

Class session 10

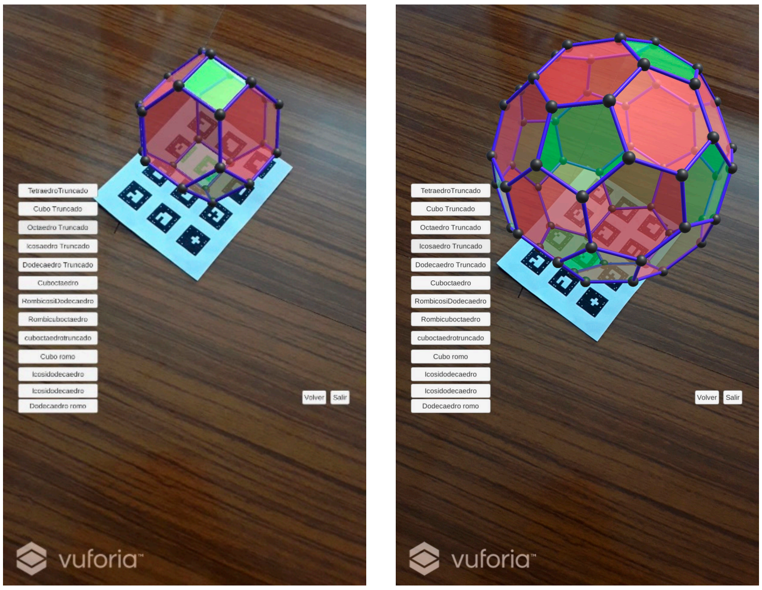
The truncated icosahedron is presented to the students as a soccer ball. They are also asked to calculate the number of faces, edges, and vertices of the truncated icosahedron as well as the area and volume of a soccer ball with a specific edge value. They can find help in the notes and with the AR application to visualize the icosahedron. We can see the images of these polyhedra in Figure 10.

Class session 11

In this session, the Kepler–Poinsot solids will be treated as a curricular extension, and the history of their discovery and the elements that define them are discussed. We will also talk about the great presence of polyhedral forms in the famous Sagrada Familia temple, where Kepler–Poinsot polyhedra can be found [34].

3.4. Teaching Methodology

The usual organization of the teaching will be: first, theory explanation about polyhedra and then, practical application exercises that allow students to connect theory with practical aspects, trying to make it easier for them to understand the theoretical concepts. It is intended that students be able to build new concepts on well-established previous concepts (constructivist view of mathematics). Therefore, as far as possible, we will adapt the teaching pace to the learning pace of the class, teaching more advanced concepts when the easiest ones have been well assimilated and building on them.



(**a**) (**b**)

Figure 10. Images of (**a**) truncated octahedron and (**b**) truncated icosahedron with AR.

As a visual support tool that helps us to facilitate the explanation and understanding of the students of the geometric concepts related to polyhedra, we use augmented reality with markers (markers will be used to visualize the polyhedra when they are pointed by the device camera). For this we will need computers with a camera and a projector or tablets. In addition, if the student has a mobile device, the software and bookmarks will be provided and allowed for use, so that they can visualize the polyhedra.

The class sessions will be dynamic and both individual and group activities will be carried out, which will facilitate student learning and inclusion, as well as train them to work as a team. We intend to generate a climate of confidence in the classroom and get students to participate actively in the classes, asking and answering questions.

The didactical proposal activities contribute positively to the achievement of the seven key competences established in the Spanish Education Law (LOMCE, Ley Orgánica para la Mejora de la Calidad Educativa, Organic law for the improvement of educational quality). For the evaluation of the success or failure of the proposal, it will be considered if the activities have been successfully carried out and if the stated objectives have been achieved. This would allow, if necessary, to make changes that can improve them.

4. Discussion

The reflections on our proposal are based on the experience during its preparation and the objectives set for it, namely the creation of an application and an augmented reality textbook, as well as an adapted teaching proposal making use of materials created in augmented reality.

Regarding the experience that the elaboration of these resources has involved, the conclusions we draw are as follows:

1. Resources are created and ready to be used. Augmented reality applications can be used on any device: computer, mobile phone, tablet. If low-end phones and tablets are used, the applications take a little longer to load, but they work in a fluid way after a few seconds (this proves that it does not take very expensive equipment to use the application, which makes it viable to use in the classroom).
2. The construction of similar resources is possible with the appropriate advanced computer skills.

3. The cost of creating similar teaching materials is relatively low, since only one computer is required to produce them, and the software is completely free.
4. The creation of these materials is a real learning process of different fields of computer science and technology such as 3D modeling, videogame programming, AR, application creation, etc.
5. Unity in combination with Vuforia, are powerful and versatile tools that can allow the creation of innovative teaching resources that greatly facilitate the teaching-learning process.

Regarding the didactic proposal elaborated, we can say that the use of AR technology constitutes an educational innovation, which can contribute positively to the improvement of the understanding of geometric concepts, the development of spatial visualization, as well as the improvement of the motivation of the students (all this is thought with a view to facilitate the explanations of the teacher and ultimately to improve the teaching-learning process of the students). It also helps to acquire basic competency in science and technology, in addition to digital competence. The proposed activities are a good example of how the classic mathematical concepts could be taught using new technologies that facilitate different aspects of the teaching-learning process. In addition, it can be noted that the methodologies proposed during the activities are designed to encourage students in different aspects such as:

- Cooperation and teamwork with other colleagues
- Development of critical thinking to make logical decisions
- Progressive gain of autonomy

As possible new lines of work and as proposals for improvement for the future, the following measures are proposed. Since Unity also allows applications to be implemented in virtual reality and mixed reality technologies, it would be convenient to investigate and experiment with them to create new didactic materials, as well as new innovative instructive proposals making use of these technologies.

The use of AR technologies also implies different learning environments, since part of what is exposed in this work, and what a teacher should do if he/she wanted to develop this tool, involves learning its use in forums and open spaces where it is shared knowledge. These new learning environments could be a subject of research itself as some researchers have concluded [35].

We are aware of the limitations of this study. It could not be implemented directly in the classroom due to a problem unrelated to our research, over which we had no control. However, its implementation is something that we contemplate in the short term, and we would like to effectively measure the results and motivation of the students. Technical development takes a lot of time, resources, and effort, and for implementation in the classroom, the entire technical and curricular part must be perfectly organized.

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