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A General Solution for the Errors in Variables (EIV) Model with Equality and Inequality Constraints

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Featured Application: It is a classical method for people to address the likewise engineering problems and to observe the deformation of buildings many times and calculate the most probable value of observation by using the least square method. If the least square model is constrained by equality or inequality based on some prior information in engineering, a new calculation model is constructed, which will be feasible to improve the accuracy of the model parameter values. In this paper, a general calculation method based on the penalty function and weighted observation value is proposed to understand and calculate the value of observation for this new model, which is easier to understand and calculate than the previous model.

Abstract: Targeting the adjustment of the errors-in-variables (EIV) model with equality and inequality constraints, a general solution that is similar to the classical least square adjustment is proposed based on the penalty function and the weight in measurement. Firstly, we take the equality constraints as inequality constraints that do not satisfy the constraint conditions and construct the penalty functions of equality and inequality constraints, respectively. Thus, the inequality constrained optimization problem is transformed into an unconstrained optimization problem. Then the detailed calculation formula and approximate accuracy evaluation formula of the general solution are deduced. The iteration formula of the general solution is easy regarding comprehension and applicable in implementation. It can not only solve the EIV model with equality and inequality constraints respectively, but also address the EIV model with equality and inequality constraints simultaneously. In addition, it can promote the Gauss–Markov (G-M) model with equality and inequality constraints. Finally, three examples (i.e., equality constraints, inequality constraints and those with equality and inequality constraints) are validated, indicating that the general solution is effective and feasible. The results show that the general solution is effective and feasible.

Keywords: total least square (TLS); errors in variables (EIV) model; equality and inequality constraints; penalty function

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1. Introduction

The total least square (TLS) is a method of parameter estimation that was first proposed by Golub and Loan [1]. The error of the coefficient matrix needs to be considered in the TLS method that differs from the least squares (LS) method. In general, we need to construct the Gauss–Markov (G-M) model and use the LS method for adjustment when the error of the coefficient matrix is excluded. However, the errors-in-variables (EIV) model needs to be built if we adopt the TLS method for adjustment [2]. In recent years, the research on TLS/EIV has received more attention in the field of surveying data processing [3,4].

Unlike the G-M model, the solution of the EIV model usually needs to be calculated by iteration due to the non-linearity of the EIV model. The common iterative algorithms were derived by using the Lagrange method [5,6]. To consider that some elements of the coefficient matrix contain errors, the PEIV model is proposed based on the EIV model [7,8]. In addition, other weighted total least squares methods are extended [9,10]. However, there are a number of studies involving the extended form of the EIV model and the extended algorithm of TLS, such as structured TLS [11], outliers processing in TLS [12], multivariate EIV model [2], variance component estimation algorithm for EIV model [13,14], TLS prediction [15], equality and inequality constrained TLS [11,16–21] and so on.

In the field of surveying data processing, the adjustment models with equality or inequality constraints need to be constructed when some prior information is available. The algorithm of the EIV model with equality constraints is generally elucidated based on the principle of Lagrange [17,19]. Zeng, Liu and Yao [20], Zhang, Tong and Zhang [21] and Xie, Lin and Long [18] investigated the EIV model with inequality constraints. Although their methods are available to solve this inequality constraints problem, the optimization theory involved in the derivation of these algorithms is far from the traditional measurement adjustment theory. In addition, EIV models with equality and inequality constraints are discussed separately in most studies, while their combination is less investigated [19].

In fact, both equality and inequality constraints may exist in surveying data processing. Thus, it is necessary to figure out simpler and more easily implemented algorithms. Three EIV models with constraints, namely the EIV model with equality constraints (EC-EIV), EIV model with inequality constraints (IC-EIV) and EIV model with equality and inequality constraints (EIC-EIV), shall be included when equality and inequality constraints are available. These three EIV models with constraints are degenerated into G-M models with constraints if the coefficient matrix error is excluded. Thus, a general solution for the EIV model with equality and inequality constraints is in high demand.

In this paper, a general solution that is similar to classical least square adjustment is proposed based on the penalty function and the weight in measurement. In Section 2, the EIV model with equality and inequality constraints is introduced. In Section 3, the detailed calculation formula and approximate accuracy evaluation formula of the general solution are presented. In Section 4, three examples, including the equality constraint, inequality constraint, and equality and inequality, are presented to illustrate the validity of the proposed general solution. The results show that the general solution is effective, stated in Section 5.

2. EIV Model with Equality and Inequality Constraints

The errors in variables model (EIV) can be defined as the following [6]:

$$\boldsymbol{L} + \boldsymbol{V_L} = (\boldsymbol{A} + \boldsymbol{E_A})\boldsymbol{X} \tag{1}$$

where $\boldsymbol{L}$ and $\boldsymbol{V_L}$ are the $\boldsymbol{m} \times \boldsymbol{1}$ observation vector and its random error vector, respectively. $\boldsymbol{A}$ and $\boldsymbol{E_A}$ are the $\boldsymbol{m} \times \boldsymbol{n}$ coefficient matrix and the corresponding error matrix, respectively. $\boldsymbol{X}$ is the $\boldsymbol{n} \times 1$ unknown parameter matrix.

The corresponding error vector and the stochastic model are expressed as [19]

$$\boldsymbol{V} = \begin{bmatrix} \boldsymbol{V_L} \\ \boldsymbol{V_A} \end{bmatrix} = \begin{bmatrix} \boldsymbol{V_L} \\ \boldsymbol{vec}(\boldsymbol{E_A}) \end{bmatrix} \sim \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \sigma_0^2 \boldsymbol{Q} = \sigma_0^2 \begin{bmatrix} \boldsymbol{Q_L} & \boldsymbol{Q_{LA}} \\ \boldsymbol{Q_{AL}} & \boldsymbol{Q_A} \end{bmatrix} \right) \tag{2}$$

where $\boldsymbol{V_A} = \boldsymbol{vec}(\boldsymbol{E_A})$ is the $\boldsymbol{mn} \times 1$ order error vector and $\boldsymbol{vec}$ denotes the operator of the vectorization resulting a column vector by stacking the columns of a matrix on top of one another. σ_0^2 is the variance factor, $\boldsymbol{Q_L}$ and $\boldsymbol{Q_A}$ are the cofactor matrices for $\boldsymbol{V_L}$ and $\boldsymbol{V_A}$, and $\boldsymbol{Q_{LA}}$ is the cofactor matrix, which refers to the correlations of $\boldsymbol{V_L}$ and $\boldsymbol{V_A}(\boldsymbol{Q_{LA}} = \boldsymbol{Q_{AL}^T})$.

The total least squares objective function can be defined as follows [3]:

$$\boldsymbol{V^T Q^{-1} V} = \mathbf{min} \tag{3}$$

The functional and stochastic models of Equations (1) and (2) are equivalent to

$$\hat{\boldsymbol{V}} = \hat{\boldsymbol{A}}\boldsymbol{X} - \hat{\boldsymbol{L}} \qquad \hat{\boldsymbol{V}} \sim (0, \sigma_0^2 \hat{\boldsymbol{Q}}) \tag{4}$$

where $\hat{Q} = (I_m - X^T \otimes I_m)Q(I_m - X \otimes I_m)$, $\hat{V} = V_L - E_A X$, $\hat{A} = A + E_A$, $\hat{L} = L + E_A X$ and $\otimes$ stands for the Kronecker–Zehfuss product. The corresponding objective function can be expressed as follows:

$$\hat{V}^T \hat{Q}^{-1} \hat{V} = \mathbf{min} \tag{5}$$

When equality and inequality constraints are available simultaneously, the corresponding EIV model (EIC-EIV model) can be defined as the following:

$$\left.\begin{array}{l} L + V_L = (A + E_A)X \\ G_1 X = W_1 \\ G_2 X \leq W_2 \end{array}\right\} \tag{6}$$

where G_1 is the $s_1 \times n$ coefficient matrix of equality constraints, G_2 is the $s_2 \times n$ coefficient matrix of inequality constraints, W_1 is the $s_1 \times 1$ constant vector, and W_2 is the $s_2 \times 1$ constant vector.

Combining Equations (5) and (6), the corresponding objective function of EIC-EIV model is stated as follows:

$$\left.\begin{array}{l} \hat{V}^T \hat{Q}^{-1} \hat{V} = \mathbf{min} \\ G_1 X = W_1 \\ G_2 X \leq W_2 \end{array}\right\} \tag{7}$$

3. A General Solution for EIC-EIV Model

There are many research studies about the solution of the EIV model. For example, the Lagrange method, Newton method or Gauss–Newton method can be adopted to solve the problem [3]. Although the solution of EIV model requires iterative processing, the solution should be numerically efficient as well as identical to the classical least square adjustment. According to previous literature research [3,5], the formula of the parameter estimation for the EIV model similar to classical least square adjustment can be derived from Equation (4):

$$X = \left(\hat{A}^T \hat{Q}^{-1} \hat{A}\right)^{-1} \cdot \hat{A}^T \hat{Q}^{-1} \hat{L} \tag{8}$$

It is worth noting that Equation (4) needs to be iterated successively. The coefficient matrix $\hat{A}$, observation vector $\hat{L}$ and cofactor matrix $\hat{Q}$ need to be updated after each iteration, and the iteration is stopped when the difference between the two results is less than the given threshold. Similarly, the approximate accuracy evaluation formula of the parameter estimates can be obtained directly (see Jazaeri, Amiri-Simkooei and Sharifi [5]).

Targeting at the situation that most algorithms based on inequality-constrained least squares adjustment are complex, a simple iterative algorithm is proposed based on penalty functions and zero or infinite weight. It is similar to the classical least square adjustment and easy to implement. In this paper, the method is applied to the EIC-EIV model and extended to equality constraints. First, we take equality constraints as inequality constraints that do not satisfy the constraint conditions, and the penalty function of equality and inequality constraints are constructed respectively. Thus, the corresponding objective function of Equation (7) can be expressed as follows:

$$\hat{V}^T \hat{Q}^{-1} \hat{V} + P_1(X) + P_2(X) = \mathbf{min} \tag{9}$$

where $P_1(X)$ and $P_2(X)$ are the penalty functions of equality constrained and inequality constrained, respectively. $P_1(X)$ is a positive large value, meaning that equality constraints need to be punished. $P_2(X)$ takes 0 or a large value according to the inequality constraint. To construct a penalty function, Equation (6) can be expressed as follows:

$$\left.\begin{array}{l} \hat{V} = \hat{A}X - \hat{L} \\ V_1 = G_1 X - W_1 \\ V_2 = G_2 X - W_2 \end{array}\right\} \tag{10}$$

The penalty function of equality constraint and inequality constraint is constructed as

$$\left.\begin{array}{l} \boldsymbol{P}_1(\boldsymbol{X}) = \boldsymbol{V}_1{}^T \boldsymbol{P}_1 \boldsymbol{V}_1 \\ \boldsymbol{P}_2(\boldsymbol{X}) = \boldsymbol{V}_2{}^T \boldsymbol{P}_2 \boldsymbol{V}_2 \end{array}\right\} \tag{11}$$

The weight of the penalty function in Equation (6) can be explained as the following:

$$\boldsymbol{P}_1(\boldsymbol{s}_1) = \mu \quad \boldsymbol{P}_2(\boldsymbol{i}) = \begin{cases} \mu & \boldsymbol{V}_2(\boldsymbol{s}_2) > 0 \\ 0 & \boldsymbol{V}_2(\boldsymbol{s}_2) \le 0 \end{cases} \tag{12}$$

where μ is a positive large value, and we set $\mu = 10^6$ in this paper. Equation (12) indicates that the weight of the penalty function for the equality constraint takes a large value, while the weight of the penalty function for the inequality constraint takes 0 or a large value according to the inequality constraint. Thus, the corresponding objective function of Equation (9) can be expressed as follows:

$$\hat{\boldsymbol{V}}_L{}^T \hat{\boldsymbol{Q}}^{-1} \hat{\boldsymbol{V}}_L + \boldsymbol{V}_1{}^T \boldsymbol{P}_1 \boldsymbol{V}_1 + \boldsymbol{V}_2{}^T \boldsymbol{P}_2 \boldsymbol{V}_2 = \mathbf{min} \tag{13}$$

When the error of the coefficient matrix is excluded, the EIV model degenerates into a traditional G-M model. According to Equations (10) and (13), the parameter estimation formula of the G-M model with equations and inequality constraints can be demonstrated as follows:

$$\boldsymbol{X} = (\boldsymbol{A}^T \boldsymbol{Q}_L^{-1} \boldsymbol{A} + \boldsymbol{G}_1^T \boldsymbol{P}_1 \boldsymbol{G}_1 + \boldsymbol{G}_2^T \boldsymbol{P}_2 \boldsymbol{G}_2)^{-1} \cdot (\boldsymbol{A}^T \boldsymbol{Q}_L^{-1} \boldsymbol{L} + \boldsymbol{G}_1^T \boldsymbol{P}_1 \boldsymbol{W}_1 + \boldsymbol{G}_2^T \boldsymbol{P}_2 \boldsymbol{W}_2) \tag{14}$$

From Equation (14), we also include the equality constraint as a special inequality constraint and obtain a general solution that has not been discussed and implemented. For the EIV model with equality and inequality constraints, the general solution can be expressed:

Step 1: Obtain an initial value for $\boldsymbol{X}^0$ from LS solution with $\boldsymbol{Y}, \boldsymbol{A}, \boldsymbol{Q}_Y, \boldsymbol{Q}_A$.

$$\boldsymbol{X}^0 = (\boldsymbol{A}^T \hat{\boldsymbol{Q}}_L^{-1} \boldsymbol{A})^{-1} \boldsymbol{A}^T \hat{\boldsymbol{Q}}_L^{-1} \boldsymbol{L}$$

Step 2: Start the following iterative process with the initial value of:

$$\begin{array}{c} \boldsymbol{F}^{i+1} = \left[\boldsymbol{I}_m - (\boldsymbol{X}^i)^T \otimes \boldsymbol{I}_m\right],\ \hat{\boldsymbol{Q}}^{i+1} = \boldsymbol{F}^{i+1} \boldsymbol{Q} (\boldsymbol{F}^{i+1})^T, \\ \boldsymbol{V}_A^{i+1} = \left[\boldsymbol{Q}_{AL} \quad \boldsymbol{Q}_A\right] (\boldsymbol{F}^{i+1})^T (\hat{\boldsymbol{Q}}^{i+1})^{-1} (\boldsymbol{L} - \boldsymbol{A}\boldsymbol{X}^i),\ \boldsymbol{E}_A^{i+1} = \boldsymbol{vec}^{-1}(\boldsymbol{V}_A^{i+1}), \\ \hat{\boldsymbol{A}}^{i+1} = \boldsymbol{A} + \boldsymbol{E}_A^{i+1},\ \hat{\boldsymbol{L}}^{i+1} = \boldsymbol{L} + \boldsymbol{E}_A^{i+1} \boldsymbol{X}^i \end{array}$$

where $\boldsymbol{vec}^{-1}$ is the opposite of the $\boldsymbol{vec}$ operator which reshapes the vector into the original matrix.

Step 3: To determine the weight of the penalty function,

$$\boldsymbol{P}_1{}^{i+1}(\boldsymbol{s}_1) = \mu \quad \boldsymbol{P}_2{}^{i+1}(\boldsymbol{s}_2) = \begin{cases} \mu & \boldsymbol{V}_2{}^{i+1}(\boldsymbol{s}_2) > 0 \\ 0 & \boldsymbol{V}_2{}^{i+1}(\boldsymbol{s}_2) \le 0 \end{cases}$$

Step 4: To obtain the new value for $\boldsymbol{X}$,

$$\boldsymbol{X}^{i+1} = ((\hat{\boldsymbol{A}}^{i+1})^T \hat{\boldsymbol{Q}}^{-1} \hat{\boldsymbol{A}}^{i+1} + \boldsymbol{G}_1^T \boldsymbol{P}_1{}^{i+1} \boldsymbol{G}_1 + \boldsymbol{G}_2^T \boldsymbol{P}_2{}^{i+1} \boldsymbol{G}_2)^{-1} \cdot ((\hat{\boldsymbol{A}}^{i+1})^T \hat{\boldsymbol{Q}}^{-1} \hat{\boldsymbol{L}}^{i+1} + \boldsymbol{G}_1^T \boldsymbol{P}_1{}^{i+1} \boldsymbol{W}_1 + \boldsymbol{G}_2^T \boldsymbol{P}_2{}^{i+1} \boldsymbol{W}_2)$$

Step 5: Repeat Steps 2 and 4 until $\left\|\boldsymbol{X}^{i+1} - \boldsymbol{X}^i\right\| < \varepsilon$, and we can obtain the final value $\boldsymbol{X}_{EIC-EIC} = \boldsymbol{X}^{i+1}$.

Then we can obtain a biased variance component estimator of the unit weight [19]

$$\sigma_0^2 = \frac{\hat{V}^T \hat{Q}^{-1} \hat{V}}{m - n + s} \tag{15}$$

where s denotes the number of constraint equations, according to Formula (14). The approximate accuracy evaluation formula of parameter estimation can be derived:

$$D(X) = \sigma_0^2 \cdot \left(\left(\hat{A}^{i+1}\right)^T \hat{Q}^{-1} \hat{A}^{i+1} + G_1^T {P_1}^{i+1} G_1 + G_2^T {P_2}^{i+1} G_2 \right)^{-1} \tag{16}$$

In fact, there are two main approaches for accuracy evaluation of IC-EIV estimates, including active constraint method and aggregate constraint method [20]. The first approach is adopted in our proposed Formula (16), while the other method is adopted in Fang and Wu [19]. The results of the accuracy evaluation obtained by using the two methods are different.

Note that the general solution can not only solve the EIV model with equality and inequality constraints respectively, but also solve the EIV model with equality and inequality constraints. Moreover, it can also solve the Gauss–Markov (G-M) model with equality and inequality constraints. It was not discussed by Fang and Wu [19].

4. Experiment Analysis

To verify the effectiveness of this method, three examples are gathered to validate the proposed general solution. First, an example with equality constraints is conducted to verify. Then, an example with inequality constraints is adopted for solving. Finally, an example with both equality and inequality constraints is applied for further analysis.

4.1. Experiment with Equality Constraints

In this example, we use the data with equality constraints presented by Fang and Wu [19]. The coefficient matrix A, observation vector L and cofactor matrix Q are represented as follows:

$$A = \begin{bmatrix} -0.5 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix},\ L = \begin{bmatrix} 6 \\ 3 \\ 4 \\ 10 \end{bmatrix},\ Q = I_{16}$$

Equality constraints: $\begin{bmatrix} -2 & 0 & 3 \end{bmatrix} X = 16$.

The general solution for EIC-EIV model presented in this paper and the algorithm presented by Fang and Wu [19] are adopted to solve the EIV model with equality constraints. The corresponding adjustment results are listed in Table 1.

Table 1. Adjustment results of equality constraints.

	Fang	EIC-EIV
X_1	2.36823	2.36823
X_2	5.69850	5.69850
X_3	6.91215	6.91215
$TSSR$	0.21284	0.21284
$D(X_1)$	2.91866	2.91866
$D(X_2)$	5.09582	5.09582
$D(X_3)$	1.29718	1.29718

The results of Table 1 indicate that the estimated parameters obtained by the two approaches are all identical. This shows that the processing method adopted in this paper is feasible. The equality constraint is an inequality constraint that does not satisfy the constraint conditions. Our general solution for EIV model with equality is simpler than other methods in the literature.

4.2. Experiment with Inequality Constraints

In the second example, a linear regression data with inequality constraints is implemented, which was adopted by Zeng, Liu and Yao [20]. The coefficient matrix $\boldsymbol{A}$, observation vector $\boldsymbol{L}$, constraints matrix $\boldsymbol{G}_2$ and constraints constant vector W_2 are located in Table 2. Since the coefficient matrix of the linear regression model contains constant columns, the cofactor matrix $\boldsymbol{Q} = \boldsymbol{blkdiag}(\boldsymbol{I}_{10} \quad \boldsymbol{O}_{10} \quad \boldsymbol{I}_{10})$.

Table 2. Data set and inequality constraints of the regression model.

$\boldsymbol{A}$		$\boldsymbol{L}$	$\boldsymbol{G}_2$		$\boldsymbol{W}_2$
1	−39.7312	−18.6749	−1	0	−1.9500
1	−29.0831	−11.4825	1	0	2.0500
1	−21.1294	−7.6373	0	−1	−0.4500
1	−9.5689	−3.0315	0	1	0.5500
1	0.1594	2.3574	−2	1	−3.4500
1	9.3462	6.8975	2	−1	3.5500
1	19.7832	11.9379			
1	30.1713	17.7448			
1	41.7892	22.7045			
1	51.3847	27.7086			

The general solution for EIC-EIV model presented in this paper and the Fang algorithm presented by Zeng, Liu and Yao [20] are adopted to solve the EIV model with inequality constraints, and the corresponding adjustment results are listed in Table 3.

Table 3. Adjustment results of inequality constrained.

	Zeng	**EIC-EIV**
$\boldsymbol{X}_1$	2.02504	2.02504
$\boldsymbol{X}_2$	0.50007	0.50007
$\boldsymbol{TSSR}$	2.56497	2.56497
$\boldsymbol{D}(\boldsymbol{X}_1)$	0.000009	0.000009
$\boldsymbol{D}(\boldsymbol{X}_2)$	0.000036	0.000036

The results of Table 3 indicate that the adjustment result of our general solution is exactly the same as those of Zeng, Liu and Yao [20]. In other words, our general solution can effectively deal with the EIV model with inequality constraints. The active constraint method is used in Table 3.

4.3. Experiment with Both Equality and Inequality Constraints

Data of the third example using equality and inequality constraints come from Fang and Wu [19]. The coefficient matrix $\boldsymbol{A}$, observation vector $\boldsymbol{L}$, constraints matrix $\boldsymbol{G}_2$ and constraints constant vector W_2 are located in Table 4.

Table 4. Data from Zhang et al. (2013) [21].

		$\boldsymbol{A}$		$\boldsymbol{L}$
0.9501	0.7620	0.6153	0.4057	0.0578
0.2311	0.4564	0.7919	0.9354	0.3528
0.6068	0.0185	0.9218	0.9169	0.8131
0.4859	0.8214	0.7382	0.4102	0.0098
0.8912	0.4447	0.1762	0.8936	0.1388
		$\boldsymbol{G}_2$		W_2
0.2027	0.2721	0.7467	0.4659	0.5251
0.1987	0.1988	0.4450	0.4186	0.2026
0.6037	0.0152	0.9318	0.8462	0.6721
$-0.1 \leq x_i \leq 2.0, \boldsymbol{i} = 1,2,3,4$				

The inequality constrined equation can be expressed as

$$\begin{bmatrix} G_2^T & I_4 \otimes [1\ -1] \end{bmatrix}^T X \leq \begin{bmatrix} W_2^T & I_4 \otimes [0.1\ 2] \end{bmatrix}^T$$

Equality constraints: $\begin{bmatrix} -2 & 1 & 3 & 0 \end{bmatrix} X = 2$.

Table 5 presents the adjustment results in three different methods: (1) only inequality constraints are considered by using the Fang algorithm presented by Fang and Wu (2016), IC-Fang; (2) only inequality constraints are considered by using our general solution, IC-EIV; (3) equations and inequality constraints are considered simultaneously by using our general solution, EIC-EIV; and (4) the Fang algorithm presented byFang and Wu [19], EIC-Fang.

Table 5. Adjustment results of equality and inequality constraints.

	IC-Fang	IC-EIV	EIC-Fang	EIC-EIV
X_1	−0.10000	−0.10000	−0.10000	−0.10000
X_2	−0.10000	−0.10000	−0.10000	−0.10000
X_3	0.16870	0.16870	0.63333	0.63333
X_4	0.39961	0.39961	−0.09432	−0.09432
$TSSR$	0.13974	0.13974	0.21074	0.21074
$D(X_1)$	0.09069	0.04×10^{-6}	0.06845	0.04×10^{-6}
$D(X_2)$	0.10011	0.04×10^{-6}	0.07127	0.04×10^{-6}
$D(X_3)$	0.08969	0.056	0.06866	0.03×10^{-6}
$D(X_4)$	0.09747	0.063	0.09309	0.3×10^{-6}

From Table 5, we can estimate that the parameter estimates of considering inequality constraints only is different from that of considering both equality and inequality constraints. The results of our general solution are identical to those presented in Fang and Wu [19]. It discloses the feasibility of our general solution. It is obvious that the TSSR of EIC-EIV and EIC-Fang are larger than that of IC-EIV and IC-Fang, which is due to the newly added equality constraint. However, the precision of the parameter estimates is improved [19]. It is worth noting that different results of precision for the parameter estimates are shown in two appraisal approaches.

5. Conclusions

In this paper, a general solution for the errors-in-variables (EIV) model with equality and inequality constraints is proposed based on the penalty function and the weight in measurement. The iteration formula of the general solution is similar to that of the least squares. It can maintain the EIV model with equality and inequality constraints on the one hand, and it can support the EIV model with equality and inequality constraints simultaneously on the other hand. Furthermore, it can also solve Gauss–Markov (G-M) model with equality and inequality constraints. Finally, three examples are testified, showing that the proposed general solution is effective and feasible.

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