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Dimensional Synthesis of Watt II and Stephenson III Six-Bar Slider-Crank Function Generators for Nine Prescribed Positions

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Abstract: This paper proposes an efficient exact dimensional synthesis method for finding all the link lengths of the Watt II and Stephenson III six-bar slider-crank function generators, satisfying nine prescribed precision points using the homotopy continuation method. The synthesis equations of each mechanism are initially constructed as a system of 56 quadratic polynomials whose Bézout number, which represents the maximum number of solutions, is $2^{56} \cong 7.21 \times 10^{16}$. In order to reduce the size of the system, multi-homogeneous formulation is applied to transform the system into 12 equations in 12 unknowns, and the multi-homogeneous Bézout number of the system is 286,720. The Bertini solver, based on the homotopy continuation method, is used to solve the synthesis equations to obtain the dimensions of the two mechanisms. For the arbitrarily given nine precision points, the proposed method yields 37 and 31 defect-free solutions of Watt II and Stephenson III six-bar slider-crank mechanisms, respectively, and it is confirmed that they pass through the prescribed positions.

Keywords: function generator; dimensional synthesis; exact synthesis; homotopy continuation; six-bar slider-crank mechanism

1. Introduction

The dimensional synthesis of a mechanism determines the dimensions of the individual links that generate a desired motion. Since the development of an analytical approach to the design of four-bar linkages introduced by Freudenstein [1], numerous studies have been conducted on the dimensional synthesis, which can be classified into two categories. One is the exact synthesis method, known as the precision point approach, in which a finite number of exact positions are to be satisfied. The other is the approximate synthesis, which seeks the dimensions that approximately satisfy prescribed positions.

In general, dimensional synthesis problems are formulated as a system of nonlinear equations and require heavy computation. For this reason, various approximate synthesis approaches using numerical methods [2,3], optimization methods such as least square techniques [4–6], genetic algorithms [7–10], and evolutionary algorithms [11–15] have been used widely in recent times. However, the approximate synthesis methods require appropriate initial values that are close to the actual solutions of the synthesis equations. When unsuitable initial values are assumed, the solutions may not be found. In addition, the methods may not find all the solutions of the synthesis equations.

In contrast, since the exact synthesis method can find all the solutions without estimating an initial value, it yields diverse mechanism dimensions that satisfy a desired motion. Among the studies on the precision point approach, Huang et al. [16] and Al-mandeel et al. [17] used elimination methods to perform the dimensional synthesis of four-bar path generators for five precision points and slider-crank function generators for four precision points, respectively. Subbian and Flugrad [18] and Wampler et al. [19] carried out the dimensional synthesis of four-bar path generators for five precision points and nine precision points, respectively. For the dimensional synthesis of six-bar linkages with revolute joints only, Dhingra et al. [20] and Plecnik and McCarthy [21–23] applied the

homotopy continuation method for the synthesis of Watt II, Stephenson II, and Stephenson III six-link function generators, proposing design methods for eight, nine, and eleven precision points, respectively. It is worth noting that the continuation method is developed from the parameter perturbation procedure introduced by Freudenstein and Roth [24] for solving a system of nonlinear equations starting from a known solution of a similar system.

Most studies on the precision point approach deal with four-bar linkages, four-link slider-crank mechanisms, and six-bar linkages with revolute joints only, but the exact synthesis of six-link slider-crank mechanisms with six revolute joints and one prismatic joint are extremely rare. Since the six-bar slider-crank linkage has more design parameters, it can satisfy more precision points than its four-link counterpart. Different types of six-bar slider-crank linkages are widely used in various applications such as ore crushers, mechanical presses [25], punching mechanisms [26], and quick return mechanisms [27] in machinery. These precision mechanisms are required to satisfy precise motion between input and output links. Furthermore, by adding one link and one joint, the six-link slider-crank can be modified to a two-degree-of-freedom seven-link adjustable mechanism that can be used in controllable presses for punching and metal forming [28,29], variable pumps [30], and variable compression ratio engines [31–34]. For the design of this machinery, the dimensional synthesis of the six-link slider-crank is essential.

This paper proposes a systematic exact dimensional synthesis method to find all Watt II and Stephenson III six-bar slider-crank function generators that satisfy nine precision points of the output slider with respect to prescribed input crank angles. The synthesis equations of each mechanism for nine precision points are derived as a system of 56 quadratic polynomial equations with 58 unknowns, respectively. By assuming two unknowns as free choices, the system can be solved. However, since the system size is very large, it is necessary to decrease its size so as to reduce the calculation time for pursuing an efficient dimensional synthesis. To this end, some of the unknowns are eliminated from the synthesis equations to simplify the system to eight equations in eight unknowns. Then, in order to apply the multi-homogeneous Bézout theorem [35,36], four auxiliary equations that render the system a set of homogeneous polynomials are defined, and the unknowns are divided into appropriate two-homogeneous groups. Through this process, the synthesis equations for Watt II and Stephenson III six-bar slider-crank linkages are formulated into eight quartic polynomials and four auxiliary quadratic polynomials in 12 unknowns, and the multi-homogeneous Bézout number of the system that represents the maximum number of solutions is 286,720. The synthesis equations for the two mechanisms are solved by using the Bertini software program [37] that is based on the homotopy continuation method. The computation results provided 25,630 and 36,061 nonsingular solutions, respectively. Among the nonsingular solutions, mechanisms that allow full rotation of the input crank and satisfy nine precision points on a single stroke of the slider are finally selected as feasible solutions. As a result, 37 Watt II and 31 Stephenson III six-bar slider-crank linkages without kinematic defects are determined.

The paper is arranged as follows. In Section 2, the dimensional synthesis equations for Watt II and Stephenson III six-bar slider-crank function generators are derived and a process for simplifying the synthesis equations is presented. Section 3 describes the procedure for calculating the multi-homogeneous Bézout number by applying the multi-homogeneous Bézout theorem. Section 4 then explains how to select feasible solutions that enable prescribed precision points to lie on a single stroke as well as the input link to rotate continuously. Section 5 presents examples of applying the dimensional synthesis process proposed in this paper. Finally, Section 6 concludes with a summary and discussion of the research.

2. Dimensional Synthesis Equations for Six-Bar Slider-Crank Mechanisms

Each of the dimensional synthesis equations for six-bar slider-crank function generators can be formulated by a system of polynomial equations. As the number of prescribed positions increases, the system size becomes very large and the calculation time to obtain

the solution could increase tremendously. Hence, in order to carry out the dimensional synthesis efficiently, it is necessary to reduce the size of the equation system. In this section, the synthesis equations for the two mechanisms are derived and simplified by eliminating some unknowns.

2.1. Synthesis Equations of Watt II Six-Bar Slider-Crank Function Generator

A Watt II six-bar slider-crank linkage is shown in Figure 1a, in which the input link r_1 rotates about the fixed pivot O, and the output slider translates vertically. A fixed frame is positioned so that its origin is at O and oriented so that its x-axis is perpendicular to the direction of slide. The horizontal distance from O to the line of path of the moving pivot E is the offset denoted by h. In the fixed frame, the input angle θ is measured from the positive x-axis and the output displacement of the slider p is the vertical distance from the x-axis to E, the center of the revolute joint attached to the slider. The angle of link r_0 is η, and the angle between links r_3 and r_4 is β. The angular displacements of links r_2, r_3, r_4, and r_5 are denoted by ϕ, α, $\alpha + \beta$, and δ, respectively.
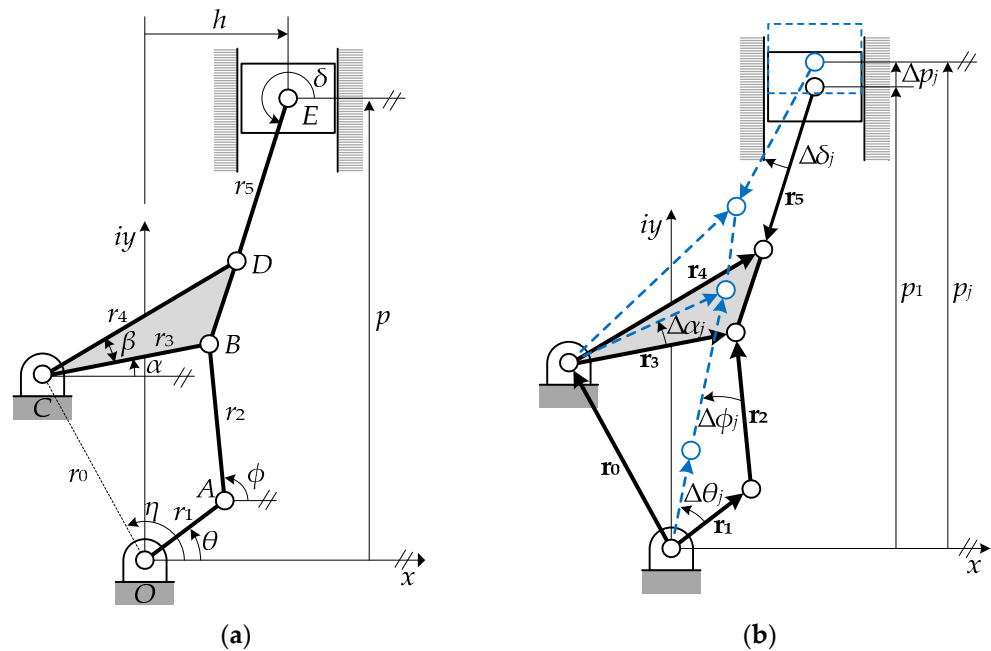
Figure 1. Watt II six-bar slider-crank mechanism: (**a**) Design parameters; (**b**) The link vectors drawn in its first and jth positions.

Figure 1b shows the mechanism in its first and jth positions. In the Figure, $\Delta\theta_j$, $\Delta\phi_j$, $\Delta\alpha_j$, and $\Delta\delta_j$ denote the rotation angles of links r_1 to r_5, respectively, and Δp_j represents the displacement of the slider, all measured from the initial position to jth position of the mechanism. Let θ_1 and p_1 denote the angle of input link r_1 and the position of output slider in the initial position, respectively. Then, the prescribed positions of the function generator to be synthesized are given pairwise as

$$\Delta\theta_j = \theta_j - \theta_1, \Delta p_j = p_j - p_1, j = 2, \ldots, n, \tag{1}$$

where θ_j and p_j are the input angle and the position of the slider in jth position, respectively, and n is the number of precision points to be satisfied.

The dimensional synthesis equations of the Watt II six-bar slider-crank function generator can be constructed by the closure equations of the two loops $OABC$ and $OCDE$ shown in Figure 1 as follows. Loop $OABC$:

For the initial and *j*th positions, the loop closure equations can be written as

$$\mathbf{r}_1 + \mathbf{r}_2 = \mathbf{r}_0 + \mathbf{r}_3, \tag{2}$$

and

$$\mathbf{r}_1 e^{i\Delta\theta_j} + \mathbf{r}_2 e^{i\Delta\phi_j} = \mathbf{r}_0 + \mathbf{r}_3 e^{i\Delta\alpha_j}, j = 2, \ldots, n, \tag{3}$$

respectively. Subtracting Equation (2) from Equation (3) gives

$$\mathbf{r}_1(e^{i\Delta\theta_j} - 1) + \mathbf{r}_2(e^{i\Delta\phi_j} - 1) = \mathbf{r}_3(e^{i\Delta\alpha_j} - 1), j = 2, \ldots, n. \tag{4}$$

Loop *OCDE*:
For the initial and *j*th positions, the loop closure equations can be written as

$$\mathbf{r}_0 + \mathbf{r}_4 = h + ip_1 + \mathbf{r}_5, \tag{5}$$

and

$$\mathbf{r}_0 + \mathbf{r}_4 e^{i\Delta\alpha_j} = h + ip_j + \mathbf{r}_5 e^{i\Delta\delta_j}, j = 2, \ldots, n, \tag{6}$$

respectively. Subtracting Equation (5) from Equation (6) gives

$$\mathbf{r}_4(e^{i\Delta\alpha_j} - 1) = i\Delta p_j + \mathbf{r}_5(e^{i\Delta\delta_j} - 1), j = 2, \ldots, n. \tag{7}$$

Separating the real and imaginary parts of Equations (4) and (7) leads to

$$r_1\{\cos(\theta_1 + \Delta\theta_j) - \cos\theta_1\} + r_2\{\cos(\phi_1 + \Delta\phi_j) - \cos\phi_1\} = r_3\{\cos(\alpha_1 + \Delta\alpha_j) - \cos\alpha_1\}, j = 2, \ldots, n, \tag{8}$$

$$r_1\{\sin(\theta_1 + \Delta\theta_j) - \sin\theta_1\} + r_2\{\sin(\phi_1 + \Delta\phi_j) - \sin\phi_1\} = r_3\{\sin(\alpha_1 + \Delta\alpha_j) - \sin\alpha_1\}, j = 2, \ldots, n, \tag{9}$$

and

$$r_4\{\cos(\alpha_1 + \beta + \Delta\alpha_j) - \cos(\alpha_1 + \beta)\} = r_5\{\cos(\delta_1 + \Delta\delta_j) - \cos\delta_1\}, j = 2, \ldots, n, \tag{10}$$

$$r_4\{\sin(\alpha_1 + \beta + \Delta\alpha_j) - \sin(\alpha_1 + \beta)\} = r_5\{\sin(\delta_1 + \Delta\delta_j) - \sin\delta_1\} + \Delta p_j, j = 2, \ldots, n, \tag{11}$$

respectively. Note that θ_1, ϕ_1, α_1, and δ_1 denote the angles of the corresponding links shown in Figure 1a in the initial position. Among the unknowns in Equations (8) through (11), $\Delta\phi_j$, $\Delta\alpha_j$, and $\Delta\delta_j$ should satisfy trigonometric identities

$$\begin{aligned} \cos^2\Delta\phi_j + \sin^2\Delta\phi_j &= 1, \\ \cos^2\Delta\alpha_j + \sin^2\Delta\alpha_j &= 1, \\ \cos^2\Delta\delta_j + \sin^2\Delta\delta_j &= 1, \end{aligned} \tag{12}$$

where $j = 2, \ldots, n$.

The unknowns in the synthesis equations, Equations (8)–(12), are r_1, r_2, r_3, r_4, r_5, θ_1, ϕ_1, α_1, β, δ_1, $\cos\Delta\phi_j$, $\sin\Delta\phi_j$, $\cos\Delta\alpha_j$, $\sin\Delta\alpha_j$, $\cos\Delta\delta_j$, and $\sin\Delta\delta_j$, where $j = 2, \ldots, n$. Hence, the problem of the dimensional synthesis for the function generator that satisfies n prescribed positions is formulated as $7(n-1)$ equations in $10 + 6(n-1)$ unknowns. For 9 prescribed positions, n is equal to 9 and the synthesis equations consist of 56 quadratic polynomials with respect to 58 unknowns. Therefore, to match the number of unknowns and the number of equations, two free choices can be made. The total degree of a system of m polynomial equations, which is referred to as the Bézout number, is defined as $\prod_{i=1}^{m} d_i$, where d_i is the degree of the ith polynomial [35]. Hence, the total degree of the synthesis equations for 9 precision points is $2^{56} \cong 7.21 \times 10^{16}$. Since the system size is too large to solve, it is necessary to reduce the size of the synthesis equations as explained in the following sections.

2.2. Procedure for Simplifying the Synthesis Equations of Watt II Six-Bar Slider-Crank

The system of $7(n-1)$ synthesis equations, Equation (8) through (12), can be reduced to a system of $n-1$ polynomial equations by eliminating the unknowns $\Delta\phi_j$, $\Delta\delta_j$, and $\Delta\alpha_j$, which corresponds to the elimination of $6(n-1)$ unknowns $\cos\Delta\phi_j$, $\sin\Delta\phi_j$, $\cos\Delta\delta_j$, $\sin\Delta\delta_j$, $\cos\Delta\alpha_j$, and $\sin\Delta\alpha_j$, where $j = 2, \ldots, n$. For the elimination of $\Delta\phi_j$ from Equations (8) and (9), isolate the term $r_2\cos(\phi_1 + \Delta\phi_j)$ in Equation (8) and $r_2\sin(\phi_1 + \Delta\phi_j)$ in Equation (9) to one side. Then, square both sides of the equations, add, and simplify the resulting equation using the trigonometric identity $\cos^2(\phi_1 + \Delta\phi_j) + \sin^2(\phi_1 + \Delta\phi_j) = 1$ to obtain

$$
\begin{aligned}
r_2^2 &= [r_3\{\cos(\alpha_1 + \Delta\alpha_j) - \cos\alpha_1\} - r_1\{\cos(\theta_1 + \Delta\theta_j) - \cos\theta_1\} + r_2\cos\phi_1]^2 \\
&+ [r_3\{\sin(\alpha_1 + \Delta\alpha_j) - \sin\theta_3\} - r_1\{\sin(\theta_1 + \Delta\theta_j) - \sin\theta_1\} + r_2\sin\phi_1]^2,
\end{aligned} \quad j = 2, \ldots, n. \tag{13}
$$

Similarly, the unknown $\Delta\delta_j$ can be eliminated from Equations (10) and (11) as

$$
\begin{aligned}
r_5^2 &= [r_4\{\cos(\alpha_1 + \beta + \Delta\alpha_j) - \cos(\alpha_1 + \beta)\} + r_5\cos\delta_1]^2 \\
&+ [r_4\{\sin(\alpha_1 + \beta + \Delta\alpha_j) - \sin(\alpha_1 + \beta)\} + r_5\sin\delta_1 - \Delta p_j]^2,
\end{aligned} \quad j = 2, \ldots, n. \tag{14}
$$

Let the x- and y-components of the initial positions of the links r_1, r_2, r_3, r_4, and r_5 be

$$
\begin{aligned}
&r_1\cos\theta_1 = r_{1x}, r_2\cos\phi_1 = r_{2x}, r_3\cos\alpha_1 = r_{3x}, r_4\cos(\alpha_1 + \beta) = r_{4x}, r_5\cos\delta_1 = r_{5x}, \\
&r_1\sin\theta_1 = r_{1y}, r_2\sin\phi_1 = r_{2y}, r_3\sin\alpha_1 = r_{3y}, r_4\sin(\alpha_1 + \beta) = r_{4y}, r_5\sin\delta_1 = r_{5y}.
\end{aligned} \tag{15}
$$

Substituting Equation (15) into Equation (13) and gathering the coefficients of $\cos\Delta\alpha_j$ and $\sin\Delta\alpha_j$ gives

$$
L_{1j} + L_{2j}\cos\Delta\alpha_j + L_{3j}\sin\Delta\alpha_j = 0, \, j = 2, \ldots, n, \tag{16}
$$

where

$$
\begin{aligned}
L_{1j} &= -2\cos\Delta\theta_j\left(r_{1x}^2 + r_{1y}^2 + r_{1x}r_{2x} - r_{1x}r_{3x} + r_{1y}r_{2y} - r_{1y}r_{3y}\right) + 2\sin\Delta\theta_j\left(-r_{1x}r_{2y} + r_{1y}r_{2x} + r_{1x}r_{3y} - r_{1y}r_{3x}\right) \\
&+ 2\left(r_{3x}^2 + r_{3y}^2 - r_{2x}r_{3x} - r_{2y}r_{3y}\right) + 2\left(r_{1x}^2 + r_{1y}^2 + r_{1x}r_{2x} - r_{1x}r_{3x} + r_{1y}r_{2y} - r_{1y}r_{3y}\right),
\end{aligned} \tag{16a}
$$

$$
L_{2j} = -2\cos\Delta\theta_j\left(r_{1x}r_{3x} + r_{1y}r_{3y}\right) - 2\sin\Delta\theta_j\left(r_{1x}r_{3y} - r_{1y}r_{3x}\right) + 2\left(r_{1x}r_{3x} + r_{1y}r_{3y}\right) - 2\left(r_{3x}^2 + r_{3y}^2 - r_{2x}r_{3x} - r_{2y}r_{3y}\right), \tag{16b}
$$

$$
L_{3j} = 2\cos\Delta\theta_j\left(r_{1x}r_{3y} - r_{1y}r_{3x}\right) - 2\sin\Delta\theta_j\left(r_{1x}r_{3x} + r_{1y}r_{3y}\right) - 2\left(r_{1x}r_{3y} - r_{1y}r_{3x}\right) - 2\left(r_{2x}r_{3y} - r_{2y}r_{3x}\right). \tag{16c}
$$

Similarly, substituting Equation (15) into Equation (14) and combining the coefficients of $\cos\Delta\alpha_j$ and $\sin\Delta\alpha_j$ yields

$$
Q_{1j} + Q_{2j}\cos\Delta\alpha_j + Q_{3j}\sin\Delta\alpha_j = 0, \, j = 2, \ldots, n, \tag{17}
$$

where

$$
Q_{1j} = 2\left(r_{4x}^2 + r_{4y}^2 - r_{4x}r_{5x} - r_{4y}r_{5y}\right) + \Delta p_j^2 + 2\Delta p_j\left(r_{4y} - r_{5y}\right), \tag{17a}
$$

$$
Q_{2j} = -2\left(r_{4x}^2 + r_{4y}^2 - r_{4x}r_{5x} - r_{4y}r_{5y}\right) - 2r_{4y}\Delta p_j, \tag{17b}
$$

$$
Q_{3j} = 2\left(r_{4x}r_{5y} - r_{4y}r_{5x}\right) - 2r_{4x}\Delta p_j. \tag{17c}
$$

In order to eliminate $\Delta\alpha_j$ from Equations (16) and (17), solve the two equations for $\cos\Delta\alpha_j$ and $\sin\Delta\alpha_j$ to obtain

$$
\cos\Delta\alpha_j = \frac{-L_{1j}Q_{3j} + L_{3j}Q_{1j}}{L_{2j}Q_{3j} - L_{3j}Q_{2j}}, \sin\Delta\alpha_j = \frac{L_{1j}Q_{2j} - L_{2j}Q_{1j}}{L_{2j}Q_{3j} - L_{3j}Q_{2j}}, j = 2, \ldots, n. \tag{18}
$$

Squaring both sides of Equation (18), adding the result, and simplifying using the trigonometric identity $\cos^2(\Delta\alpha_j) + \sin^2(\Delta\alpha_j) = 1$ gives

$$
\left(-L_{1j}Q_{3j} + L_{3j}Q_{1j}\right)^2 + \left(L_{1j}Q_{2j} - L_{2j}Q_{1j}\right)^2 - \left(L_{2j}Q_{3j} - L_{3j}Q_{2j}\right)^2 = 0, j = 2, \ldots, n. \tag{19}
$$

Equation (19), the system of synthesis equations for n prescribed positions, has $n - 1$ eighth-degree polynomials in 10 unknowns, r_{ix} and r_{iy}, $i = 1, \ldots, 5$, given in Equation (15). Hence, for 9 position synthesis, n is equal to 9 and there are 8 equations in 10 unknowns, and two unknowns can be selected as free choices. Among the 10 unknowns, r_{1x} and r_{1y} are chosen as the free choices in this study to reduce the calculation time as explained in Section 3.1. Now, the total degree of the 9 precision point synthesis equations is $8^8 = 16,777,216$, which is much smaller than that of the synthesis equations derived in Section 2.1. However, the system size of the simplified synthesis equations can be further reduced by using auxiliary equations and applying the multi-homogeneous Bézout theorem. This process is described in Section 3.

2.3. Synthesis Equations of Stephenson III Six-Bar Slider-Crank Function Generator

A Stephenson III slider-crank linkage is shown in Figure 2a. The input link r_1 rotates about the fixed pivot O, and the output slider translates along a line parallel to y-axis. The angle between links r_2 and r_4 is β. The angular displacements of links r_2, r_3, r_4, and r_5 are denoted by ϕ, α, $\phi + \beta$, and δ, respectively. As shown in Figure 2b, $\Delta\theta_j$, $\Delta\phi_j$, $\Delta\alpha_j$, and $\Delta\delta_j$ represent the rotation angles of links r_1 to r_5, measured from their initial positions to jth positions, respectively. The prescribed precision points are given as Equation (1).
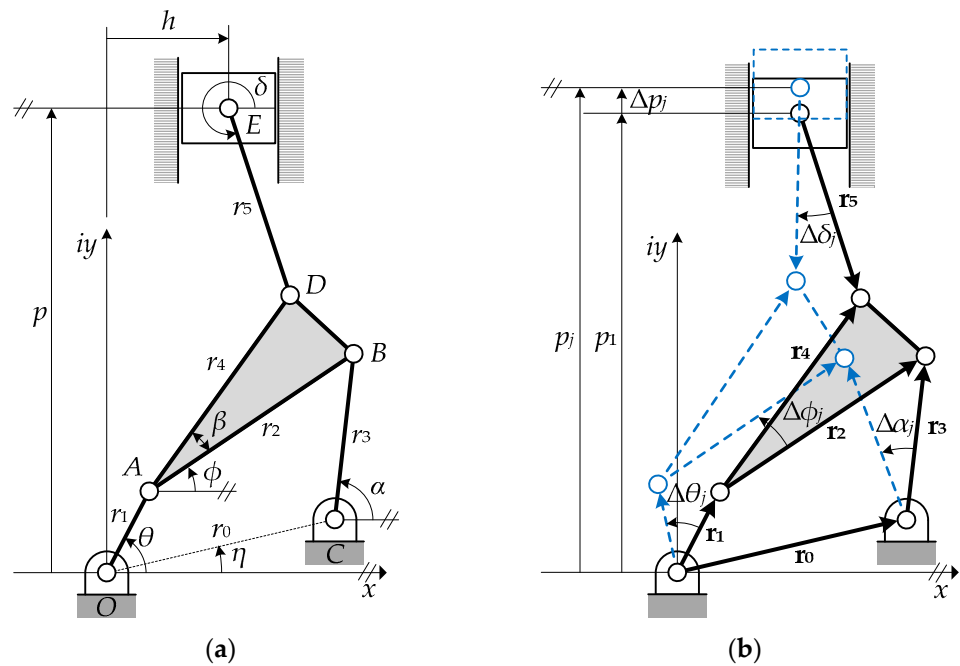
Figure 2. Stephenson III six-bar slider-crank mechanism: (**a**) Design parameters; (**b**) The link vectors drawn in its first and jth positions.

The dimensional synthesis equations of the Stephenson III six-bar slider-crank function generator can be derived using the closure equations of the two loops, *OABC* and *OADE*, shown in Figure 2 as follows.

Loop *OABC*:

For the initial and jth positions, the loop closure equations can be written as

$$\mathbf{r}_1 + \mathbf{r}_2 = \mathbf{r}_0 + \mathbf{r}_3, \tag{20}$$

and

$$\mathbf{r}_1 e^{i\Delta\theta_j} + \mathbf{r}_2 e^{i\Delta\phi_j} = \mathbf{r}_0 + \mathbf{r}_3 e^{i\Delta\alpha_j}, \ j = 2, \ldots, n, \tag{21}$$

respectively. Subtracting Equation (20) from Equation (21) gives

$$\mathbf{r}_1(e^{i\Delta\theta_j} - 1) + \mathbf{r}_2(e^{i\Delta\phi_j} - 1) = \mathbf{r}_3(e^{i\Delta\alpha_j} - 1), j = 2, \ldots, n. \tag{22}$$

Loop *OADE*:
For the initial and *j*th positions, the loop closure equations can be written as

$$\mathbf{r}_1 + \mathbf{r}_4 = h + ip_1 + \mathbf{r}_5, \tag{23}$$

and

$$\mathbf{r}_1 e^{i\Delta\theta_j} + \mathbf{r}_4 e^{i\Delta\alpha_j} = h + ip_j + \mathbf{r}_5 e^{i\Delta\delta_j}, j = 2, \ldots, n, \tag{24}$$

respectively. Subtracting Equation (23) from Equation (24) gives

$$\mathbf{r}_1(e^{i\Delta\theta_j} - 1) + \mathbf{r}_4(e^{i\Delta\alpha_j} - 1) = i\Delta p_j + \mathbf{r}_5(e^{i\Delta\delta_j} - 1), j = 2, \ldots, n. \tag{25}$$

Separating the real and imaginary parts of Equations (22) and (25) yields

$$r_1\{\cos(\theta_1 + \Delta\theta_j) - \cos\theta_1\} + r_2\{\cos(\phi_1 + \Delta\phi_j) - \cos\phi_1\} = r_3\{\cos(\alpha_1 + \Delta\alpha_j) - \cos\alpha_1\}, j = 2, \ldots, n, \tag{26}$$

$$r_1\{\sin(\theta_1 + \Delta\theta_j) - \sin\theta_1\} + r_2\{\sin(\phi_1 + \Delta\phi_j) - \sin\phi_1\} = r_3\{\sin(\alpha_1 + \Delta\alpha_j) - \sin\alpha_1\}, j = 2, \ldots, n, \tag{27}$$

and

$$r_1\{\cos(\theta_1 + \Delta\theta_j) - \cos\theta_1\} + r_4\{\cos(\phi_1 + \beta + \Delta\alpha_j) - \cos(\phi_1 + \beta)\} = r_5\{\cos(\delta_1 + \Delta\delta_j) - \cos\delta_1\}, j = 2, \ldots, n, \tag{28}$$

$$r_1\{\sin(\theta_1 + \Delta\theta_j) - \sin\theta_1\} + r_4\{\sin(\phi_1 + \beta + \Delta\alpha_j) - \sin(\phi_1 + \beta)\} = r_5\{\sin(\delta_1 + \Delta\delta_j) - \sin\delta_1\} + \Delta p_j, j = 2, \ldots, n, \tag{29}$$

respectively. Among the unknowns in Equations (26) through (29), $\Delta\phi_j$, $\Delta\alpha_j$, and $\Delta\delta_j$ should satisfy trigonometric identities

$$\begin{aligned} \cos^2\Delta\phi_j + \sin^2\Delta\phi_j &= 1, \\ \cos^2\Delta\alpha_j + \sin^2\Delta\alpha_j &= 1, \\ \cos^2\Delta\delta_j + \sin^2\Delta\delta_j &= 1. \end{aligned} \tag{30}$$

where $j = 2, \ldots, n$.

As in case of the Watt II slider-crank mechanism, the dimensional synthesis equations, Equations (26)–(30), for the Stephenson III function generator that satisfies *n* prescribed positions are formulated as a system of $7(n-1)$ equations in $10 + 6(n-1)$ unknowns which are $r_1, r_2, r_3, r_4, r_5, \theta_1, \phi_1, \alpha_1, \beta, \delta_1, \cos\Delta\phi_j, \sin\Delta\phi_j, \cos\Delta\alpha_j, \sin\Delta\alpha_j, \cos\Delta\delta_j$, and $\sin\Delta\delta_j$, with $j = 2, \ldots, n$. For 9 prescribed positions, *n* is equal to 9 and there are 56 equations in 58 unknowns. Hence, two unknowns need to be assumed as free choices in order to obtain the solutions. The total degree of the synthesis equations is $2^{56} \cong 7.21 \times 10^{16}$. In the following section, the synthesis equations for *n* precision points are simplified by eliminating some of the unknowns.

2.4. Procedure for Simplifying the Synthesis Equations of Stephenson III Six-Bar Slider-Crank

The synthesis equations, Equations (26)–(30), can be reduced from $7(n-1)$ to $n-1$ by eliminating $\Delta\alpha_j$, $\Delta\delta_j$, and $\Delta\phi_j$, $j = 2, \ldots, n$. Applying the similar procedure used in Section 2.2, the unknown $\Delta\alpha_j$ can be eliminated from Equations (26) and (27) as

$$\begin{aligned} r_3^2 &= [r_1\{\cos(\theta_1 + \Delta\theta_j) - \cos\theta_1\} + r_2\{\cos(\phi_1 + \Delta\phi_j) - \cos\phi_1\} + r_3\cos\alpha_1]^2 \\ &+ [r_1\{\sin(\theta_1 + \Delta\theta_j) - \sin\theta_1\} + r_2\{\sin(\phi_1 + \Delta\phi_j) - \sin\phi_1\} + r_3\sin\alpha_1]^2, \end{aligned} \quad j = 2, \ldots, n, \tag{31}$$

and the unknown $\Delta\delta_j$ can be eliminated from Equations (28) and (29) as

$$\begin{aligned} r_5^2 &= [r_1\{\cos(\theta_1 + \Delta\theta_j) - \cos\theta_1\} + r_4\{\cos(\phi_1 + \beta + \Delta\phi_j) - \cos(\phi_1 + \beta)\} + r_5\cos\delta_1]^2 \\ &+ [r_1\{\sin(\theta_1 + \Delta\theta_j) - \sin\theta_1\} + r_4\{\sin(\phi_1 + \beta + \Delta\phi_j) - \sin(\phi_1 + \beta)\} + r_5\sin\delta_1 - \Delta p_j]^2, \end{aligned} \quad j = 2, \ldots, n. \tag{32}$$

Let the x- and y-components of the initial positions of the links r_1, r_2, r_3, r_4, and r_5 be

$$r_1 \cos \theta_1 = r_{1x}, r_2 \cos \phi_1 = r_{2x}, r_3 \cos \alpha_1 = r_{3x}, r_4 \cos(\phi_1 + \beta) = r_{4x}, r_5 \cos \delta_1 = r_{5x},$$
$$r_1 \sin \theta_1 = r_{1y}, r_2 \sin \phi_1 = r_{2y}, r_3 \sin \alpha_1 = r_{3y}, r_4 \sin(\phi_1 + \beta) = r_{4y}, r_5 \sin \delta_1 = r_{5y}. \tag{33}$$

Substituting Equation (33) into Equations (31) and (32), and gathering the coefficients of $\cos\Delta\phi_j$ and $\sin\Delta\phi_j$ gives

$$L_{1j} + L_{2j} \cos \Delta\phi_j + L_{3j} \sin \Delta\phi_j = 0, \; j = 2, \ldots, n, \tag{34}$$

where

$$
\begin{aligned}
L_{1j} = \; & -2\cos\Delta\theta_j\left(r_{1x}^2 + r_{1y}^2 + r_{1x}r_{2x} - r_{1x}r_{3x} + r_{1y}r_{2y} - r_{1y}r_{3y}\right) + 2\sin\Delta\theta_j\left(-r_{1x}r_{2y} + r_{1y}r_{2x} + r_{1x}r_{3y} - r_{1y}r_{3x}\right) \\
& +2\left(r_{2x}^2 + r_{2y}^2 - r_{2x}r_{3x} - r_{2y}r_{3y}\right) + 2\left(r_{1x}^2 + r_{1y}^2 + r_{1x}r_{2x} - r_{1x}r_{3x} + r_{1y}r_{2y} - r_{1y}r_{3y}\right),
\end{aligned}
\tag{34a}
$$

$$L_{2j} = 2\cos\Delta\theta_j(r_{1x}r_{2x} + r_{1y}r_{2y}) + 2\sin\Delta\theta_j(r_{1x}r_{2y} - r_{1y}r_{2x}) - 2(r_{1x}r_{2x} + r_{1y}r_{2y}) - 2\left(r_{2x}^2 + r_{2y}^2 - r_{2x}r_{3x} - r_{2y}r_{3y}\right), \tag{34b}$$

$$L_{3j} = 2\cos\Delta\theta_j(-r_{1x}r_{2y} + r_{1y}r_{2x}) + 2\sin\Delta\theta_j(r_{1x}r_{2x} + r_{1y}r_{2y}) + 2(r_{1x}r_{2y} - r_{1y}r_{2x}) + 2(r_{2x}r_{3y} - r_{2y}r_{3x}), \tag{34c}$$

and

$$Q_{1j} + Q_{2j} \cos \Delta\phi_j + Q_{3j} \sin \Delta\phi_j = 0, \; j = 2, \ldots, n, \tag{35}$$

where

$$
\begin{aligned}
Q_{1j} = \; & -2\cos\Delta\theta_j\left(r_{1x}^2 + r_{1y}^2 + r_{1x}r_{4x} - r_{1x}r_{5x} + r_{1y}r_{4y} - r_{1y}r_{5y} + r_{1y}\Delta p_j\right) \\
& +2\sin\Delta\theta_j\left(-r_{1x}r_{4y} + r_{1y}r_{4x} + r_{1x}r_{5y} - r_{1y}r_{5x} - r_{1x}\Delta p_j\right) + 2\left(r_{1x}^2 + r_{1y}^2 + r_{1x}r_{4x} - r_{1x}r_{5x} + r_{1y}r_{4y} - r_{1y}r_{5y}\right) \\
& +2\left(r_{4x}^2 + r_{4y}^2 - r_{4x}r_{5x} - r_{4y}r_{5y}\right) + 2\Delta p_j\left(r_{1y} + r_{4y} - r_{5y}\right) + \Delta p_j^2,
\end{aligned}
\tag{35a}
$$

$$
\begin{aligned}
Q_{2j} = \; & 2\cos\Delta\theta_j\left(r_{1x}r_{4x} + r_{1y}r_{4y}\right) + 2\sin\Delta\theta_j\left(r_{1x}r_{4y} - r_{1y}r_{4x}\right) - 2\left(r_{1x}r_{4x} + r_{1y}r_{4y}\right) \\
& -2\left(r_{4x}^2 + r_{4y}^2 - r_{4x}r_{5x} - r_{4y}r_{5y}\right) - 2r_{4y}\Delta p_j,
\end{aligned}
\tag{35b}
$$

$$
\begin{aligned}
Q_{3j} = \; & -2\cos\Delta\theta_j\left(r_{1x}r_{4y} - r_{1y}r_{4x}\right) + 2\sin\Delta\theta_j\left(r_{1x}r_{4x} + r_{1y}r_{4y}\right) + 2\left(r_{1x}r_{4y} - r_{1y}r_{4x}\right) \\
& +2\left(r_{4x}r_{5y} - r_{4y}r_{5x}\right) - 2r_{4x}\Delta p_j,
\end{aligned}
\tag{35c}
$$

respectively.

For the elimination of $\Delta\phi_j$ from Equations (34) and (35), solve the two equations for $\cos\Delta\phi_j$ and $\sin\Delta\phi_j$, and simplify using the trigonometric identity to obtain

$$\left(-L_{1j}Q_{3j} + L_{3j}Q_{1j}\right)^2 + \left(L_{1j}Q_{2j} - L_{2j}Q_{1j}\right)^2 - \left(L_{2j}Q_{3j} - L_{3j}Q_{2j}\right)^2 = 0, \; j = 2, \ldots, n. \tag{36}$$

As in the case of the Watt II six-link slider-crank mechanism, the synthesis equations of the Stephenson III six-link slider-crank function generator for n prescribed positions derived in Equation (36) are formulated by $n - 1$ eighth-degree polynomials in 10 unknowns which are given in Equation (33). For 9 position synthesis, n is equal to 9 and there are 8 equations in 10 unknowns. Hence, 2 unknowns should be assumed as free choices. In this research, the x- and y-components of link r_1 in the initial position are selected as free choices. The total degree of the synthesis equations for 9 precision points is $8^8 = 16,777,216$.

3. Additional Degree Reduction with the Multi-Homogeneous Theorem

The Bézout number of a polynomial system represents the largest number of solutions that the system can have, and is also the number of paths to be tracked to find all isolated solutions in the continuation method. According to the multi-homogeneous Bézout theorem, the number of solution paths and thus the computing time can be significantly reduced by dividing the unknowns into appropriate homogeneous groups by the multi-homogeneous formulation [35,36].

In order to apply the theorem, the 9 position synthesis equations, Equations (19) and (36), need to be converted into homogeneous structures first. For this purpose, quadratic auxiliary equations that replace particular terms in Equations (16a)–(16c), (17a)–(17c), (34a)–(34c), and (35a)–(35c) are defined as new unknowns as follows.

For the Watt II six-bar slider-crank mechanism:

$$
\begin{aligned}
M_1 &= r_{3x}^2 + r_{3y}^2 - r_{2x}r_{3x} - r_{2y}r_{3y}, \\
M_2 &= r_{2x}r_{3y} - r_{2y}r_{3x}, \\
M_3 &= r_{4x}^2 + r_{4y}^2 - r_{4x}r_{5x} - r_{4y}r_{5y}, \\
M_4 &= r_{4x}r_{5y} - r_{4y}r_{5x}.
\end{aligned}
\tag{37}
$$

For the Stephenson III six-bar slider-crank mechanism:

$$
\begin{aligned}
N_1 &= r_{2x}^2 + r_{2y}^2 - r_{2x}r_{3x} - r_{2y}r_{3y}, \\
N_2 &= r_{2x}r_{3y} - r_{2y}r_{3x}, \\
N_3 &= r_{4x}^2 + r_{4y}^2 - r_{4x}r_{5x} - r_{4y}r_{5y}, \\
N_4 &= r_{4x}r_{5y} - r_{4y}r_{5x}.
\end{aligned}
\tag{38}
$$

By substituting M_k in Equation (37) into L_{ij} and Q_{ij} of Equation (19), and substituting N_k in Equation (38) into L_{ij} and Q_{ij} of Equation (36), where $i = 1, \ldots, 3$, $j = 2, \ldots, 9$, and $k = 1, \ldots, 4$, Equations (19) and (36) become 8 homogenous quartic polynomials, respectively. Hence, now the 9 position synthesis equations are formulated into a system of 12 polynomial equations —8 quartic polynomials, Equations (19) or (36), and 4 quadratic polynomials, Equations (37) or (38)—in 12 unknowns, r_{ix} and r_{iy}, where $i = 2, \ldots, 5$, and M_k, or N_k, with $k = 1, \ldots, 4$. The total degree of the synthesis equations for each mechanism is $4^8 2^4 = 1{,}048{,}576$, which is $1/16$ of $8^8 = 16{,}777{,}216$, the total degree of the simplified synthesis equations described in Section 2.

As the last step for reducing the system size, the multi-homogeneous Bézout theorem is applied to the synthesis equations. Then, the synthesis equations can be solved by the tracking of only 286,720 paths as explained in Sections 3.1 and 3.2.

3.1. 2-Homogeneous Formulation for Watt II Six-Bar Slider-Crank Mechanism

For the multi-homogeneous formulation of the synthesis equations for the Watt II slider-crank mechanism, the 12 unknowns are arranged into two groups λ_1 and λ_2 that constitute the loop *OABC* and loop *OCDE* shown in Figure 1 as follows.

$$
\lambda_1 : \left(r_{2x}, r_{2y}, r_{3x}, r_{3y}, M_1, M_2 \right), \quad \lambda_2 : \left(r_{4x}, r_{4y}, r_{5x}, r_{5y}, M_3, M_4 \right).
\tag{39}
$$

In order to determine the multi-homogenous Bézout number for a polynomial system whose unknowns are arranged into m homogeneous groups λ_1 to λ_m, let l be the number of polynomial equations to solve, $d_{i,j}$ be the degree of the ith equation with respect to the unknowns in group λ_j, and k_j be the number of unknowns in group λ_j. Then, the multi-homogenous Bézout number of the system is defined as the coefficient of the term $\prod_{j=1}^m \lambda_j^{k_j}$ of the following equation [35].

$$
\prod_{i=1}^l \left(\sum_{j=1}^m d_{i,j}\lambda_j \right) = (d_{1,1}\lambda_1 + \cdots + d_{1,m}\lambda_m)(d_{2,1}\lambda_1 + \cdots + d_{2,m}\lambda_m) \cdots (d_{l,1}\lambda_1 + \cdots + d_{l,m}\lambda_m),
\tag{40}
$$

Note that λ_j in Equation (40) is used as a null value for calculating the multi-homogenous Bézout number.

Since the system of 12 synthesis equations for the Watt II slider-crank function generator, Equations (19) and (37) with Equation (39), is a 2-homogeneous system, substituting $l = 12$, $m = 2$, $k_1 = k_2 = 6$, and $d_{i,j}$ shown in Table 1 into Equation (40) gives

$$
\prod_{i=1}^{12} \left(\sum_{j=1}^2 d_{i,j}\lambda_j \right) = (2\lambda_1 + 2\lambda_2)^8 (2\lambda_1)^2 (2\lambda_2)^2.
\tag{41}
$$

Table 1. Degrees of the 2-homogenous system for Watt II six-bar slider-crank synthesis equations.

Equations ($i = 1, \ldots, 12$)	Equation (19) ($i = 1, \ldots, 8$)	M_1 ($i = 9$)	M_2 ($i = 10$)	M_3 ($i = 11$)	M_4 ($i = 12$)
$d_{i,1}$ (Group λ_1)	2	2	2	0	0
$d_{i,2}$ (Group λ_2)	2	0	0	2	2

Hence, the multi-homogenous Bézout number of the system, the coefficient of the term $\lambda_1^6 \lambda_2^6$ in Equation (41), can be obtained by using the binomial theorem as $(2^8 8!/4!4!)2^2 2^2 = 286{,}720$, which agrees with the results reported by Dhingra et al. [20] for the synthesis of the Watt II six-link function generator with revolute joints only for 9 precision positions.

If other unknowns than r_{1x} and r_{1y} are selected as the free choices, the number of auxiliary equations to convert the system into a homogenous structure increases and 2-homogenous Bézout number becomes higher. For example, if r_{2x} and r_{2y} are assumed as the free choices, it is necessary to define 6 auxiliary equations and the 2-homogeneous Bézout number is $(2^8 8!/4!4!)2^4 2^2 = 1{,}146{,}880$. Therefore, it is important to select the unknown that minimize the number of auxiliary equations as the free choices. In this case, it is efficient to assume r_{1x} and r_{1y} as the free choices to reduce the 2-homogeneous Bézout number in consideration of the calculation time.

3.2. 2-Homogeneous Formulation for Stephenson III Six-Bar Slider-Crank

In order to make the Stephenson III six-bar slider-crank synthesis equations a 2-homogeneous system, the 12 unknowns are divided into two groups γ_1 and γ_2 for the loop *OABC* and loop *OADE* shown in Figure 2 as follows.

$$\gamma_1 : \left(r_{2x}, r_{2y}, r_{3x}, r_{3y}, N_1, N_2\right), \gamma_2 : \left(r_{4x}, r_{4y}, r_{5x}, r_{5y}, N_3, N_4\right). \tag{42}$$

As shown in Table 2, the degrees of the individual equations with respect to the unknowns in each group are the same as those of the Watt II six-bar slider-crank function generator. Therefore, the 2-homogeneous Bézout number of the system is also 286,720, which agrees with the results of Dhingra et al. [20] for the synthesis of the Stephenson III six-bar function generator with revolute joints only for 9 precision points. For the same reason as described in Section 3.1, r_{1x} and r_{1y} are assumed to be the free choices in this study to reduce the 2-homogeneous Bézout number.

Table 2. Degrees of the 2-homogenous system for the Stephenson III six-bar slider-crank synthesis equations.

Equations ($i = 1, \ldots, 12$)	Equation (36) ($i = 1, \ldots, 8$)	N_1 ($i = 9$)	N_2 ($i = 10$)	N_3 ($i = 11$)	N_4 ($i = 12$)
$d_{i,1}$ (Group γ_1)	2	2	2	0	0
$d_{i,2}$ (Group γ_2)	2	0	0	2	2

Consequently, the 2-homogeneous Bézout number of the synthesis equations for each mechanisms is about 1/58 of the total degree of the polynomial systems that are derived by eliminating some unknowns in Sections 2.2 and 2.4. Hence, the process proposed in this section can significantly reduce the time to solve the synthesis equations.

In this research, the calculation of the synthesis equations is carried out by using the Bertini software program [37], which is based on the homotopy continuation method, for obtaining all the solutions to the Watt II and Stephenson III six-bar slider-crank synthesis equations that satisfy 9 prescribed precision points. Among the real solutions, however, some may have kinematic defects. The following section describes the process of sorting out feasible solutions.

4. Sorting Out Feasible Solutions

In order for the synthesized mechanisms to function as six-link slider-crank linkages, they should satisfy the following two conditions. The first is that the input link needs to be capable of complete rotation, and the second is the prescribed positions should lie on a single stroke of the slider. Hence, the solutions that do not satisfy these two conditions should be excluded. This section describes the process for screening the feasible mechanisms by means of the full rotation condition of the input link and the displacement analysis of synthesized mechanisms to check whether prescribed positions are present on a single stroke of the slider.

4.1. Full Rotatability of Input Link

The input link r_1 of each mechanism is in the four-bar loop $OABC$ shown in Figures 1 and 2. Hence, it is required for the four-bar to have a full rotatable crank as the input link. Among the four-bar linkages, those that allow complete rotation of the input link are the crank-rocker and double-crank, and each mechanism satisfies the following condition, respectively [38].

$$\text{Crank-rocker mechanism}: H_1 > 0, H_2 > 0, H_3 > 0,$$
$$\text{Double-crank mechanism}: H_1 < 0, H_2 < 0, H_3 > 0, \tag{43}$$

where

$$\begin{aligned} H_1 &= r_0 - r_1 + r_2 - r_3, \\ H_2 &= r_0 - r_1 - r_2 + r_3, \\ H_3 &= r_2 + r_3 - r_0 - r_1. \end{aligned} \tag{44}$$

The value of r_0 in Equation (44) can be determined by the loop closure equation for the four-bar in each mechanism. From Figure 1 and Equation (2) for the Watt II slider-crank and Figure 2 and Equation (20) for the Stephenson III slider-crank, the x- and y-components of $\mathbf{r}_0$ can be written as $r_0\cos\eta = r_{1x} + r_{2x} - r_{3x}$ and $r_0\sin\eta = r_{1y} + r_{2y} - r_{3y}$, respectively, where r_{1x} and r_{1y} are the prescribed values as free choices, and $r_{2x}, r_{2y}, r_{3x},$ and r_{3y} are obtained from the solution of the synthesis equations. Then, $r_0 = (r_0{}^2\cos^2\eta + r_0{}^2\sin^2\eta)^{1/2}$. Similarly, since r_{ix} and r_{iy} ($i = 1, 2, 3$) represent the x- and y-components of $\mathbf{r}_i$ as defined in Equations (15) and (33), r_i in Equation (44) can be determined by $r_i = (r_{ix}{}^2 + r_{iy}{}^2)^{1/2}$.

Based on the condition provided in Equation (43), the solutions that allow the full rotation of the input link can be found from among all the solutions of the synthesis equations.

The next section describes the displacement analysis for further discerning the mechanisms in which all the prescribed precision points are present on a single stroke of the output slider.

4.2. Displacement Analysis

If all link lengths r_i, $i = 0, \ldots, 5$, and the offset h are given, for a given value of the input crank angle, the Watt II and Stephenson III six-bar slider-crank mechanisms can each be assembled into four different configurations called assembly modes or branches, and some of the prescribed positions of a synthesized mechanism may lie on different assembly modes. If all the prescribed positions do not lie on a single stroke of the slider on one branch, the solution is unacceptable. Therefore, it is necessary to select the solution mechanisms that satisfy this condition by analyzing the displacement of each mechanism.

For the analysis of the six-bar slider-crank mechanism, the four-bar linkages commonly used in the two mechanisms need to be analyzed first. The angular displacements α of link r_3 and ϕ of link r_2 shown in Figures 1 and 2 can be determined as [39]

$$\alpha = 2\tan^{-1}\left(\frac{-B \pm \sqrt{B^2 - C^2 + A^2}}{C - A}\right), \quad -\pi \leq \alpha \leq \pi, \tag{45}$$

where

$$
\begin{aligned}
A &= 2r_0 r_3 \cos \eta - 2r_1 r_3 \cos \theta, \\
B &= 2r_0 r_3 \sin \eta - 2r_1 r_3 \sin \theta, \\
C &= r_0^2 + r_1^3 + r_3^2 - r_2^2 - 2r_0 r_1 (\cos \eta \cos \theta + \sin \eta \sin \theta),
\end{aligned}
\tag{46}
$$

and

$$
\phi = \tan^{-1} \left(\frac{r_0 \sin \eta + r_3 \sin \alpha - r_1 \sin \theta}{r_0 \cos \eta + r_3 \cos \alpha - r_1 \cos \theta} \right),
\tag{47}
$$

respectively.

Notice that when only the link lengths of the four-bar mechanism and the angle of the ground link η shown in Figures 1 and 2 are given, there are two sets of α and ϕ for a given crank angle θ due to the positive (α^+) or negative (α^-) sign in Equation (45), which correspond to the two assembly modes. However, since the solution of the synthesis equations in this study yields the x- and y-components of each link in its initial position, the sign to be used in Equation (45) and the assembly mode of the four-bar can be readily identified.

4.2.1. Watt II Six-Bar Slider-Crank Mechanism

In order to analyze the slider displacement of the Watt II slider-crank mechanism, the position of the joint E in Figure 1 needs to be determined. To find the coordinates of E, the position of joint D can be expressed as

$$
\mathbf{D} = \begin{bmatrix} D_x \\ D_y \end{bmatrix} = \begin{bmatrix} r_0 \cos \eta \\ r_0 \sin \eta \end{bmatrix} + \begin{bmatrix} r_4 \cos(\alpha + \beta) \\ r_4 \sin(\alpha + \beta) \end{bmatrix},
\tag{48}
$$

where α is calculated from Equation (45), and β is the angle between link r_3 and link r_4 that can be determined using the solution of the synthesis equations as

$$
\beta = \tan^{-1} \left(\frac{r_{4y}}{r_{4x}} \right) - \tan^{-1} \left(\frac{r_{3y}}{r_{3x}} \right).
\tag{49}
$$

Now, the position of the moving pivot E is

$$
\mathbf{E} = \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} h \\ D_y \pm \sqrt{r_5^2 - (h - D_x)^2} \end{bmatrix},
\tag{50}
$$

where

$$
h = r_0 \cos \eta + r_{4x} - r_{5x}.
\tag{51}
$$

Equation (50) indicates that there are two solutions for the position of the joint E. That is, the slider and link r_5 can be assembled above (with the positive sign: E_y^+) or below (with the negative sign: E_y^-) the moving pivot D. As mentioned in Section 4.1, however, since the result of this research gives the orientation of each link in its initial position, the sign to be used in Equation (50) and the assembly mode of a synthesized mechanism can be identified.

4.2.2. Stephenson III Six-Bar Slider-Crank Mechanism

The position of the moving pivot D in Figure 2 can be determined by

$$
\mathbf{D} = \begin{bmatrix} D_x \\ D_y \end{bmatrix} = \begin{bmatrix} r_1 \cos \theta \\ r_1 \sin \theta \end{bmatrix} + \begin{bmatrix} r_4 \cos(\phi + \beta) \\ r_4 \sin(\phi + \beta) \end{bmatrix},
\tag{52}
$$

where β is the angle between link r_2 and link r_4 that can be determined using the solution of the synthesis equations as

$$
\beta = \tan^{-1} \left(\frac{r_{4y}}{r_{4x}} \right) - \tan^{-1} \left(\frac{r_{2y}}{r_{2x}} \right).
\tag{53}
$$

Then, the coordinates of the moving pivot E are

$$\mathbf{E} = \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} h \\ D_y \pm \sqrt{r_5^2 - (h - D_x)^2} \end{bmatrix}, \tag{54}$$

where

$$h = r_{1x} + r_{4x} - r_{5x}. \tag{55}$$

As in the case with the Watt II slider-crank mechanism, the $\pm$ sign to be used in Equation (54) can be determined by considering the x- and y-components of link r_5 of the synthesized function generator.

By applying the solution screening process explained in this section, the Watt II and Stephenson III six-bar slider-cranks without kinematic defects can be finally selected from among all the solutions of the synthesis equations. The next section provides examples of the dimensional synthesis method proposed in this research.

5. Numerical Examples

In this section, examples are given for the dimensional synthesis of the Watt II and Stephenson III six-bar slider-crank function generators that satisfy 9 precision points. The synthesis equations of the two mechanisms derived in this study were calculated by using the homotopy continuation method, through which all the solutions were obtained. Then, the feasible solutions were selected based on the two criteria described in Section 4.

5.1. Dimensional Synthesis of Watt II Six-Bar Slider-Crank Function Generator

The prescribed positions that the Watt II six-bar slider-crank function generator should satisfy are given in Table 3, which shows the relative displacement of the output slider, $\Delta p_j = p_j - p_1$, with respect to the relative input crank angle, $\Delta \theta_j = \theta_j - \theta_1$, for $i = 2, \dots, 9$, each measured from the initial positions of the slider and input crank, respectively. The free choices were assumed as $r_{1x} = 0.12268$ and $r_{1y} = 0.87294$.

Table 3. Prescribed positions for the Watt II slider-crank function generator.

Precision Points (j)	1	2	3	4	5	6	7	8	9
$\Delta \theta_j$ (deg)	0	21	70	100	124	164	193	224	298
Δp_j	0	−0.49087	−1.45837	−1.69238	−1.77397	−1.77643	−1.67172	−1.42028	−0.13685

For the given precision points, Equations (19) and (37) are solved by using the Bertini software package. The computation time was 3.42 h on a single node of Intel® Xeon® W-2245 CPU@3.90 GHz. Among 25,630 nonsingular solutions obtained, those that have less than 0.01% structural error are selected as feasible solutions. As a result, there were 37 mechanisms that have completely rotatable cranks and pass all the prescribed positions. Table 4 shows the number of the feasible mechanisms for each assembly configuration. The kinematic diagrams in their initial positions and slider displacements of two synthesized mechanisms are shown in Figure 3 and the corresponding solutions are listed in Table 5.

Table 4. The number of feasible mechanisms for each configuration.

Configuration	Number of Feasible Mechanisms
1 (Assembled by α^-, $E_y{}^+$)	0
2 (Assembled by α^-, $E_y{}^-$)	1
3 (Assembled by α^+, $E_y{}^+$)	25
4 (Assembled by α^+, $E_y{}^-$)	11

A (0.12268, 0.87294)
B (1.95988, 2.80321)
C (-0.33185, 2.9918)
D (2.6244, 3.6179)
E (0.11854, 5.6438)

(**a**)

A (0.12268, 0.87294)
B (-0.36957, 0.43326)
C (-0.20214, 0.00545)
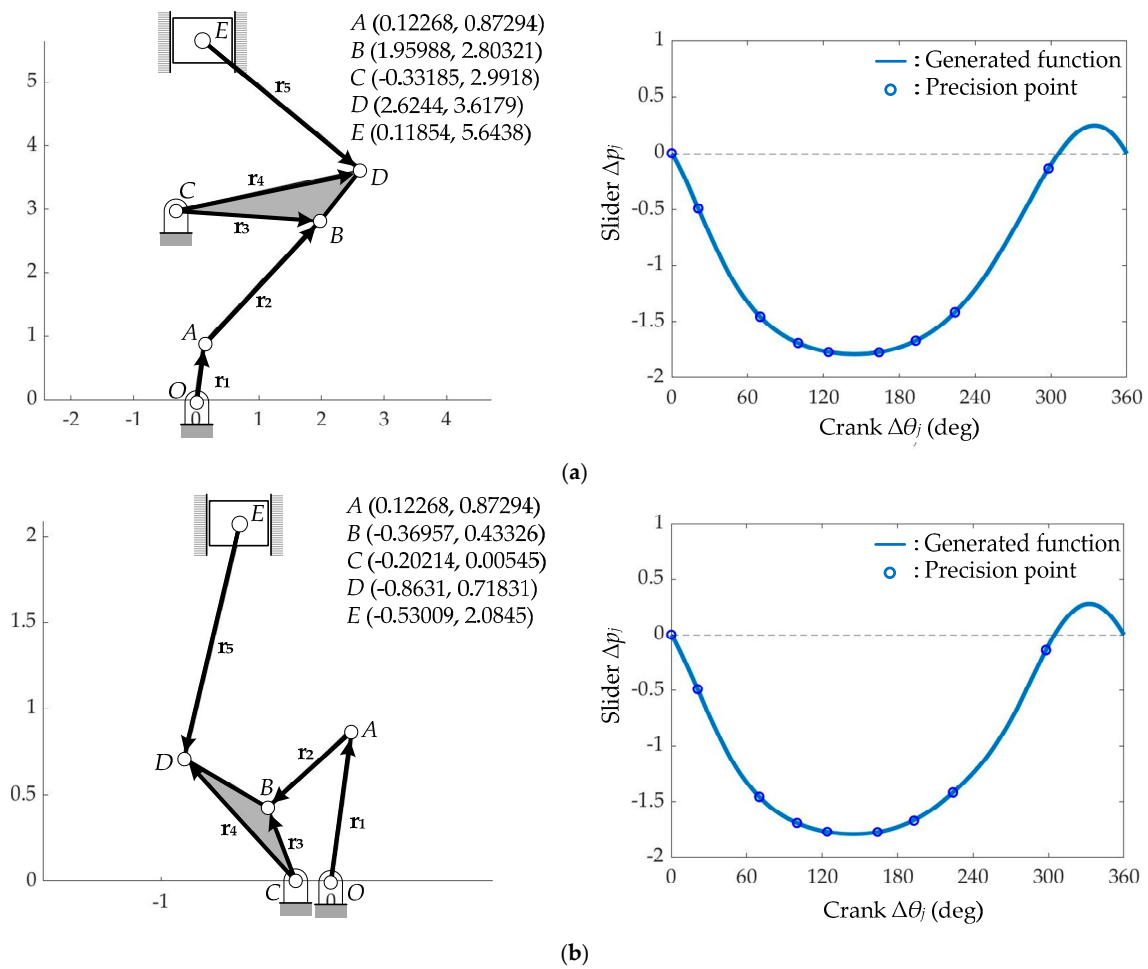D (-0.8631, 0.71831)
E (-0.53009, 2.0845)

(**b**)

Figure 3. Watt II six-bar slider-crank function generator: (**a**) (α^+, E_y^+) with a crank and the displacement of the slider; (**b**) (α^+, E_y^+) with double cranks and the displacement of the slider.

Table 5. The solutions of the Watt II six-bar slider-crank mechanism shown in Figure 3.

Configuration	Type of Four-Bar Mechanism	Solutions		Link Lengths		Orientations (Deg)	
3	Crank-rocker	r_{2x}	1.83719848448098	r_2	2.66481499912287	ϕ	46.4151406996215
		r_{2y}	1.93026959468645				
		r_{3x}	2.29173698831327	r_3	2.29948451051778	α	−4.70463367830402
		r_{3y}	−0.188601671540832				
		r_{4x}	2.95628516701656	r_4	3.02185450092804	$\alpha + \beta$	11.9574803117514
		r_{4y}	0.626085166776018				
		r_{5x}	2.50589035665077	r_5	3.22236485855068	δ	−38.9535693977896
		r_{5y}	−2.02586988774379				
3	Double-crank	r_{2x}	−0.492252951916725	r_2	0.660020417350853	ϕ	−138.229126201616
		r_{2y}	−0.439674860151526				
		r_{3x}	−0.167431742713909	r_3	0.459407273662191	α	111.373840719186
		r_{3y}	0.427810302149809				
		r_{4x}	−0.66096098576403	r_4	0.97212798670023	$\alpha + \beta$	132.83672677491
		r_{4y}	0.712855804369779				
		r_{5x}	−0.333010353208065	r_5	1.40620185002117	δ	−103.698662763378
		r_{5y}	−1.36620194248844				

5.2. Dimensional Synthesis of Stephenson III Six-Bar Slider-Crank Function Generator

The prescribed positions for the Stephenson III six-bar slider-crank function generator are given in Table 6 and the free choices were assumed as r_{1x} = 0.12859 and r_{1y} = 1.0473.

Table 6. Prescribed positions for the Stephenson III slider-crank function generator.

Precision Points (*j*)	1	2	3	4	5	6	7	8	9
$\Delta\theta_j$ (deg)	0	39	88	140	182	225	253	287	333
Δp_j	0	−0.16691	−1.08488	−2.29326	−2.83569	−2.59666	−1.93088	−0.95797	−0.18975

Using the Bertini software package, the computation time to solve the synthesis equations given in Equations (36) and (38) was 4.38 h on the same work station referred to in Section 5.1, and a total of 36,061 nonsingular solutions were obtained. Among them, 31 Stephenson III six-bar slider-cranks that have less than 0.01% structural error were found, and the number of the feasible mechanisms for each assembly configuration is tabulated in Table 7. The kinematic diagrams in the initial positions and slider displacements of two synthesized mechanisms are shown in Figure 4 and the corresponding solutions are listed in Table 8.

Table 7. The number of feasible mechanisms for each configuration.

Configuration	Number of Feasible Mechanisms
1 (Assembled by α^-, $E_y{}^+$)	11
2 (Assembled by α^-, $E_y{}^-$)	0
3 (Assembled by α^+, $E_y{}^+$)	12
4 (Assembled by α^+, $E_y{}^-$)	8

Table 8. The solutions of the Stephenson III six-bar slider-crank mechanism shown in Figure 4.

Configuration	Type of Four-Bar Mechanism	Solutions		Link Lengths		Orientations (Deg)	
1	Crank-rocker	r_{2x}	2.54452041055486	r_2	2.99784510000088	ϕ	31.9204361635379
		r_{2y}	1.58508394852419				
		r_{3x}	−0.439101961903118	r_3	3.00115483999674	α	98.4132035930371
		r_{3y}	2.96885833961284				
		r_{4x}	0.265943554102272	r_4	2.55948745154112	$\phi + \beta$	84.0359171635551
		r_{4y}	2.54563352441547				
		r_{5x}	0.272684581643968	r_5	3.33598841545769	δ	−85.3113966753684
		r_{5y}	−3.32482508216621				
3	Double-crank	r_{2x}	−0.448150890683856	r_2	0.943859211257232	ϕ	−118.346779745901
		r_{2y}	−0.830681039782654				
		r_{3x}	0.003969289812873	r_3	0.854831121637962	α	89.7339540343908
		r_{3y}	0.854821906164786				
		r_{4x}	−0.725026834630641	r_4	0.729835893584505	$\phi + \beta$	−173.418970576143
		r_{4y}	−0.083645206854701				
		r_{5x}	0.650197621387553	r_5	2.41279005227063	δ	−74.3666995186773
		r_{5y}	-2.32353155551589				

A (0.12859, 1.0473)
B (2.6731, 2.6324)
C (3.1122, -0.33648)
D (0.39453, 3.5929)
E (0.12185, 6.9177)

(**a**)

A (0.12859, 1.0473)
B (-0.31956, 0.21661)
C (-0.32353, -0.63821)
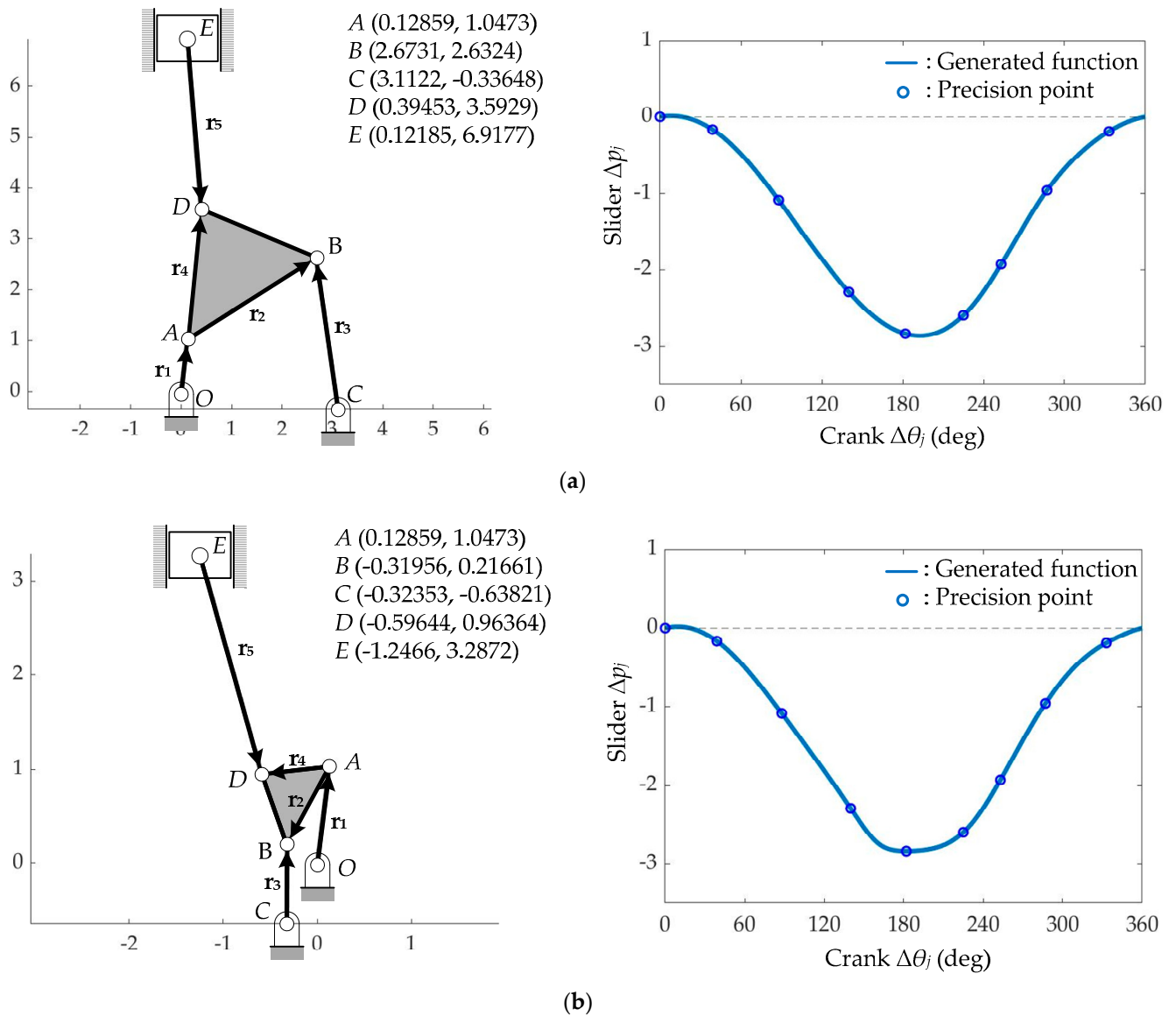D (-0.59644, 0.96364)
E (-1.2466, 3.2872)

(**b**)

Figure 4. Stephenson III six-bar slider-crank function generator: (**a**) (α^-, E_y^+) with a crank and the displacement of the slider; (**b**) (α^+, E_y^+) with double cranks and the displacement of the slider.

6. Conclusions

This paper presents the dimensional synthesis of the Watt II and Stephenson III six-bar slider-crank function generators that satisfy nine precision points of the output slider for the prescribed rotational angles of the input link. In this study, the system of synthesis equations for each mechanism initially derived by 56 quadratic polynomials is simplified to eight eighth-degree polynomial equations by eliminating some unknowns. Then, by defining four auxiliary quadratic polynomials as new unknowns, the synthesis equations are converted into eight quartic and four quadratic homogeneous polynomials in 12 unknowns. By this process, the Bézout number of the system that represents the maximum number of isolated solutions is reduced from $2^{56} \cong 7.21 \times 10^{16}$ to 1,048,576. In order to decrease the computation time of the synthesis equations further, 2-homogeneous formulation is applied by arranging the unknowns of the system into two groups and the multi-homogeneous Bézout number of the system is 286,720. In this research, the synthesis equations are solved by using the Bertini solver, which uses the homotopy continuation method.

As is known, all the solutions obtained by solving the synthesis equations do not yield feasible six-link slider-crank mechanisms. For the solution mechanism to function properly as desired without any defects, it must have a full rotatable crank and the prescribed precision points should lie on a single stroke of the slider. The screening process to select suitable linkages among the solutions is explained in detail.

The proposed method is verified by carrying out synthesis examples. For arbitrarily given nine precision points, the methods determined 37 feasible Watt II slider-crank mechanisms among 25,630 nonsingular solutions and 31 Stephenson III six-bdar slider-crank linkages out of 36,061 solutions. The structural errors of the synthesized mechanisms determined in this research are ranged from $5.42 \times 10^{-3}\%$ and $7.95 \times 10^{-12}\%$.

In order to synthesize six-link slider-crank function generators for other numbers of precision points than nine, the n-1 polynomials in 10 unknowns given in Equation (19) for the Watt II and Equation (36) for the Stephenson III slider-crank mechanism would be the starting point for the synthesis. Once the number of precision points to be synthesized for is chosen, the number of free choices is determined. When selecting free choices among the 10 unknowns in Equations (15) or (33), it is efficient to choose the unknowns that can minimize the number of auxiliary equations to obtain the lower value of the multi-homogeneous Bézout number of the system. Then, define auxiliary equations that can convert the system into a homogeneous structure and apply multi-homogeneous formulation by arranging the variables into homogeneous groups. By solving the resulting system of equations and sorting out feasible solutions, the dimensions of the six-link slider-crank function generators can be determined.

In order to reduce the computation time to solve the system of synthesis equations, the use of newly developed homotopy continuation methods [40–43] can be considered. Among them, the Diagonal homotopy method [40] seeks only the nonsingular solutions for a system of nonlinear equations. Since there exist many degenerate solutions in a system of non-linear equations, using this method to solve the system may reduce the calculation time significantly.

Author Contributions: Conceptualization, J.K.S. and M.S.J.; Methodology, M.S.J., H.S.P. and J.K.S.; Software, M.S.J., H.S.P. and W.R.K.; Validation, M.S.J., J.K.S., H.S.P. and W.R.K.; Investigation, M.S.J. and W.R.K.; Data curation, M.S.J. and W.R.K.; Writing—Original Draft Preparation, M.S.J.; Writing—Review and Editing, J.K.S.; Visualization, M.S.J.; Supervision, J.K.S.; project administration, M.S.J. and W.R.K. All authors have read and agreed to the published version of the manuscript.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The data presented in this study are available on request from the corresponding author.

Conflicts of Interest: The authors declare no conflict of interest.

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