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The Extended Galerkin Method for Approximate Solutions of Nonlinear Vibration Equations

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Featured Application: This is a novel procedure for solving nonlinear equations of vibrations with asymptotic solutions. It is an extension to the popular Galerkin method by adding an integration of time over one period of vibrations. The method is applicable to a broad class of nonlinear equations as a systematic procedure for approximate solutions.

Abstract: An extension has been made to the popular Galerkin method by integrating the weighted equation of motion over the time of one period of vibrations to eliminate the harmonics from thee deformation function. A set of successive equations of coupled higher-order vibration amplitudes is resulted, and a nonlinear eigenvalue problem is obtained for the frequency-amplitude dependence of nonlinear vibrations with successive displacements. The subsequent solutions of vibration frequencies and deformation are consistent with other successive approximate methods, such as the harmonics balance method. This is an extension of the Galerkin method which has broad applications for asymptotic solutions, particularly for problems in solid mechanics. This extended Galerkin method can also be utilized for the analysis of free and forced nonlinear vibrations of structures as a new technique with significant advantages in calculations.

Keywords: vibration; nonlinear; frequency; approximation; Galerkin

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1. Introduction

The Galerkin method has been a popular choice for the approximate calculation of natural frequencies of elastic components and structures, particularly if there are no analytical solutions or the equation of vibrations is hard to modify and solve [1–4]. Applications of the Galerkin method can be found in the literature with details of implementation and some novel techniques [5–7]. Engineers and students can use the method conveniently because it usually involves calculations of weighted integrations of functions from equations of equilibrium or motion over the physical domain of problem. Of course, it is also the basis of the finite element method with omnipresent application nowadays. There are plenty of resources on the Galerkin method, including popular textbooks [8–10], but the essence is the minimization of the error from an approximate solution through the diminishing of the weighted integration of the error over the solution domain.

It is evident that most discussions and applications of the Galerkin method are on the linear analysis of problems of structural vibrations. If the vibrations are related to material nonlinearity or larger deformation, there are not many discussions on the utilization of the Galerkin method, at least in a systematic manner, because the method is known primarily for linear problems as demonstrated. In a recent effort to study nonlinear vibrations of elastic solids, it was found that such problems can be solved effectively with an extension of the Galerkin method through adding integration over time with the full expression of displacement. In fact, this is an extension to the standard procedure for effective and accurate solutions of nonlinear vibration problems with the resulting nonlinear eigenvalue equations. Furthermore, it is inspiring that such an extension of the Galerkin method for nonlinear vibration problems can be useful for problems with periodic properties. Through this extension, the Galerkin method is now capable of analyzing both linear and nonlinear vibrations as a unified technique and also adds another powerful tool for nonlinear vibration problems as demonstrated in this paper and forced nonlinear vibrations of a multiple-degree-of-freedom system [11]. The objective of this paper is to demonstrate the effectiveness of the extended Galerkin method in solving nonlinear vibrations and promote its adoption in the study and development of tools and techniques for more broad nonlinear problems.

2. The Galerkin Formulation

2.1. State of the Art

The utilization of the Galerkin method in this study starts from a general nonlinear differential equation which is frequently encountered in vibrations and solid mechanics in the form of [8–10]

$$N\left(u, \dot{u}, \ddot{u}, u^k, \varepsilon,\ t\right) = 0, \tag{1}$$

where $u,\ \dot{u}\left(\ddot{u}\right), u^k, \varepsilon,$ and t are the displacement, derivatives with respect to time t, nonlinear term with power of k, small parameter, and time, respectively. It should be emphasized that the displacement function u is defined in the physical domain V which is not shown but implied. Then the standard Galerkin method requires

$$\int_V N\left(u, \dot{u}, \ddot{u}, u^k, \varepsilon,\ t\right)\delta u \mathrm{d}V = 0, \tag{2}$$

with δu as arbitrary variation of displacement. Equation (2) is usually referred to as the weak form of Equation (1). It should be pointed out that in the standard Galerkin method, there is no integration over time because the temporal variable is not considered in the formulation and solution process. The effectiveness and validity of the standard Galerkin method is based on the principle of the least square method, which implies an accurate solution is achieved with the displacement solution in Equation (2).

Without losing generality, the approximate solution is assumed as

$$u = \sum A_n \cos n\omega t, \tag{3}$$

where A_n, n, and ω are amplitudes, integers, and the angular frequency, respectively.

With the solution in Equation (3), it is obvious that

$$\begin{array}{cc} u = \sum\ A_n \cos n\omega t, & u^k = \left(\sum\ A_n \cos n\omega t\right)^k, \\ \dot{u} = -\omega \sum\ nA_n \sin n\omega t, & \ddot{u} = -\omega^2 \sum\ n^2 A_n \cos n\omega t. \end{array} \tag{4}$$

Now by substituting Equation (4) into Equation (1), it yields

$$N\left(A_n, A_n^k, \omega^m, \cos^m n\omega t\ , \sin^n n\omega t, \varepsilon\right) = 0, \tag{5}$$

where m is a combination of integers n and k.

As it is known, generally, there is no analytical solution to such a nonlinear equation, and approximate methods such as the harmonic balance method (HBM) [12], the Krylov–Bogoliubov–Mitropolsky method (KBM) [13], the Lindstedt–Poincaré method (L-P) [14], the homotopy analysis method (HAM) [15,16], and variations have been utilized to find solutions with different degrees of accuracy at conditions [8–10].

2.2. The Extended Galerkin Formulation

As an alternative approach, treating time as an independent variable, the Galerkin method in one period of vibrations is used to let [5–7]

$$\int_0^T N\left(u, \dot{u}, \ddot{u}, u^k, \varepsilon, t\right)\delta u \mathrm{d}t = 0, T = \frac{2\pi}{\omega}, \tag{6}$$

Implying the best approximation of displacement amplitudes over the period of vibrations. The full advantage of the Galerkin method is taken with a reasonable choice of the displacement function, but the addition of integration over time should be noted because the periodic property has been taken into consideration with the generalization of the weighted integration. It has to be pointed out that Equation (6) is the Galerkin method with the addition of temporal variable in one period of vibrations. This procedure has not been used before for either linear or nonlinear vibrations. For the vibration analysis of a continuous system, it is the extended Galerkin method because of the additional integration over time domain of one period of vibrations.

Since the variation of solution in Equation (3) is

$$\delta u = \sum \delta A_n \cos n\omega t, \tag{7}$$

the arbitrary δA_n will enforce the vanish of the weighted integration as the optimal approximation to the equation through the known technique of the Galerkin method.

As a result of this operation, Equation (6) will be

$$\int_0^T N\left(A_n, A_n^k, \omega^m, \cos^m n\omega t\ , \sin^n n\omega t,\ \varepsilon\right) \cos n\omega t \mathrm{d}t = 0, \tag{8}$$

for amplitudes A_n and vibration frequency ω as a system of coupled nonlinear algebraic equations.

Clearly, this approach is based on the known fact that the Galerkin method and the weighted integration over a proper interval of time will provide a reasonable approximation to the equation, which cannot be solved exactly. The Galerkin method has been widely used with linear equations and spatial domains as the basis of many approximation procedures of analytical and numerical nature but applying to the time domain for nonlinear vibrations is not considered until recently with the Rayleigh–Ritz method by the first author [11,17,18]. Due to the interchangeability or equivalence of the Galerkin and Rayleigh-Ritz methods, it is intuitive that the Galerkin method should also yield solutions to nonlinear vibration equations similar to other methods like the harmonic balance method [12], as it has been stated with the same formulation in Equation (8) by Liu and Chen [19]. The Galerkin approach has also been implied by Nayfeh and Mook but not formulated in the same manner [20]. A similar procedure with integration over a quarter of vibration period is demonstrated by He [21] and Anderson et al. [22]. Since the procedure presented in this study has not been used before, an extra exploration of these two methods for asymptotic solutions to typical nonlinear vibration equations is recommended based on current approaches. Naturally, this is an extension to the standard Galerkin method with the interval of time as one period of the fundamental mode of vibrations with the underline principle of the least square method.

3. Application Examples

3.1. The Duffing Equation

A similar approach in solving Duffing equation with good results has been reported as the extended Rayleigh–Ritz method (ERRM) by Wang [17,18,23]. The current Galerkin method is more flexible if the Lagrangian functional is not readily available, as it is known in many cases of nonlinear vibration problems [24].

Now, the popular Duffing equation is considered as [8–10]

$$\ddot{u} + u + \varepsilon u^3 = 0, \tag{9}$$

where ε is a parameter which is usually assumed small as also in this study. Solution techniques for larger ε are available for numerical analysis, but it is not the topic of this paper. This is one of the standard nonlinear vibration equations and also is used as a test problem for different solution techniques. Naturally, Duffing equation is taken as a test problem for the simplicity and accuracy of the currently proposed extended Galerkin method (EGM) with integration over time.

The asymptotic solution now is formulated as

$$u = \sum_{j=0}^{M} A_{2j+1} \cos(2j+1)\omega t, \tag{10}$$

where M and $A_{2j+1}(\varepsilon)$ are an integer and amplitudes, and a substitution of Equation (10) into Equation (9) will give

$$\begin{aligned} N = {} & A_1\left(1-\omega^2\right)\cos\omega t + A_3\left(1-9\omega^2\right)\cos 3\omega t + \varepsilon A_1^3 \cos^3\omega t \\ & + 3\varepsilon A_1^2 A_3 \cos^2\omega t \cos 3\omega t + 3\varepsilon A_1 A_3^2 \cos\omega t \cos^2 3\omega t + \varepsilon A_3^3 \cos^3 3\omega t. \end{aligned} \tag{11}$$

From Equation (6), the weighted integrations will be

$$\int_0^T N \cos\omega t \mathrm{d}t = 0, \quad \int_0^T N \cos 3\omega t \mathrm{d}t = 0, \tag{12}$$

or

$$\begin{aligned} & \tfrac{A_1\left(1-\omega^2\right)}{2} + \tfrac{3\varepsilon A_1^3}{8} + \tfrac{3\varepsilon A_1^2 A_3}{8} + \tfrac{3\varepsilon A_1 A_3^2}{4} = 0, \\ & \tfrac{A_3\left(1-9\omega^2\right)}{2} + \tfrac{\varepsilon A_1^3}{8} + \tfrac{3\varepsilon A_1^2 A_3}{4} + \tfrac{3\varepsilon A_3^3}{8} = 0. \end{aligned} \tag{13}$$

The typical initial conditions are

$$u(t=0) = A, \quad \dot{u}(t=0) = 0, \tag{14}$$

which are related to the solutions through

$$A_1 + A_3 = A, \quad A_1 = A - A_3. \tag{15}$$

Substituting Equation (15) into the first part of Equation (13), it gives

$$\omega^2 = 1 + \frac{3\varepsilon A^2}{4} - \frac{3\varepsilon A A_3}{4} + \frac{3\varepsilon A_3^2}{2}. \tag{16}$$

Substituting Equation (15) into the second part of Equation (13) and making necessary simplifications by dropping higher-order terms of ε, it yields

$$A_3 = \frac{\varepsilon A^3}{4}\frac{1}{8+6\varepsilon A^2} = \frac{\varepsilon A^3}{32}\left(1 - \frac{3\varepsilon A^2}{4}\right). \tag{17}$$

Substituting Equation (17) into Equation (16), the approximate frequency solution is

$$\omega = 1 + \frac{3\varepsilon A^2}{8} - \frac{3\varepsilon^2 A^4}{256}, \tag{18}$$

while other approximate solutions from the harmonic balance method (HBM) [12], the Krylov–Bogoliubov–Mitropolsky method (KBM) [13], the Lindstedt–Poincaré method (L-P) [14], and the homotopy analysis method (HAM) [15,16] are

$$\begin{aligned} \omega &= 1 + \tfrac{3\varepsilon A^2}{8} - \tfrac{15\varepsilon^2 A^4}{256}, \quad (\mathrm{KBM}), \\ \omega &= 1 + \tfrac{3\varepsilon A^2}{8} - \tfrac{21\varepsilon^2 A^4}{256}, \quad (\mathrm{L-P}), \\ \omega &= 1 + \tfrac{3\varepsilon A^2}{8} + \tfrac{9\varepsilon^2 A^4}{192}, \quad (\mathrm{HAM}). \end{aligned} \tag{19}$$

It is clear that through a simple and elegant procedure, the proposed extended Galerkin method presented an approximate solution for Duffing equation with the small parameter ε. Although there are differences in the solution in Equation (18) in comparison with Equation (19) with the coefficient for the $\varepsilon^2 A^4$ from other approximate techniques and variations [25–27], they are in the same degree of accuracy, nonetheless. To illustrate the accuracy of the extended Galerkin method shown in this study, a numerical analysis has been completed to a multi-degree-of-freedom nonlinear vibration problem under excitation with comparisons to results from other methods [11]. The same results exactly from the harmonic balance method are obtained with the new method. Another demonstration of the accuracy for the nonlinear surface wave analysis in comparison with the finite element analysis is also presented recently [24]. The efficiency of the proposed solution procedure can be seen through operations of integration, which is generally a one step process in comparison with the split and combination of harmonic terms. Since the main objective of this study is to validate the extended Galerkin method which can be used to solve nonlinear vibration equations with approximate solutions, the numerical comparisons of specific solutions and parameters can be done with details by following the procedure shown here in separated studies in the future. The close results in Equation (18) show that a new and convenient technique for solving such a nonlinear vibration equation has been presented and validated with the fundament vibration frequency.

3.2. The Van Der Pol Equation

As a popular nonlinear vibration equation different from the Duffing equation, van der Pol equation is also a typical test problem for solution techniques and accuracy [27]. The standard form of van der Pol equation is [9,28]

$$\ddot{u} + u = \varepsilon\left(1 - u^2\right)\dot{u}, \tag{20}$$

where ε is a small parameter in this study. The approximate displacement of the first two frequencies is assumed to be

$$u = A_1^s \sin\omega t + A_1^c \cos\omega t + A_3^s \sin 3\omega t + A_3^c \cos 3\omega t\,, \tag{21}$$

where $A\left(A_1^s, A_1^c, A_3^s, A_3^c\right)$ and ω are amplitudes and frequency to be determined.

Now, the general form of the nonlinear vibration equation with the approximate solution in Equation (21) is written as

$$\begin{aligned} N = {} & A_1^s\left(1-\omega^2\right)\sin\omega t + A_1^c\left(1-\omega^2\right)\cos\omega t + A_3^s\left(1-9\omega^2\right)\sin 3\omega t \\ & + A_3^c\left(1-9\omega^2\right)\cos 3\omega t - \varepsilon\left[1 - \left(A_1^s\sin\omega t + A_1^c\cos\omega t + A_3^s\sin 3\omega t\right.\right. \\ & \left.\left. + A_3^c\cos 3\omega t\,\right)^2\right]\omega\left(A_1^s\cos\omega t - A_1^c\sin\omega t + 3A_3^s\cos 3\omega t - 3A_3^c\sin 3\omega t\,\right). \end{aligned} \tag{22}$$

Applying the general procedure of the extended Galerkin method, it requires

$$\begin{aligned} & \textstyle\int_0^T N(A,\ \omega,\varepsilon,t)\sin\omega t\mathrm{d}t = 0, \qquad \textstyle\int_0^T N(A,\ \omega,\varepsilon,t)\cos\omega t\mathrm{d}t = 0, \\ & \textstyle\int_0^T N(A,\ \omega,\varepsilon,t)\sin 3\omega t\mathrm{d}t = 0, \quad \textstyle\int_0^T N(A,\ \omega,\varepsilon,t)\cos 3\omega t\mathrm{d}t = 0. \end{aligned} \tag{23}$$

Substituting Equation (22) for Equation (23), it yields

$$\begin{aligned}
&(A_1^s)^2(A_1^c - A_3^c)\varepsilon\omega + A_1^c\Big\{(A_1^c)^2 + A_1^c A_3^c + 2\Big[-2 + (A_3^c)^2 + (A_3^s)^2\Big]\Big\}\varepsilon\omega \\
&+2A_1^s(-2 + A_1^c A_3^s \varepsilon\omega + 2\omega^2) = 0, \\
&(A_1^c)^2(A_1^s + A_3^s)\varepsilon\omega + A_1^s\Big\{(A_1^s)^2 - A_1^s A_3^s + 2\Big[-2 + (A_3^c)^2 + (A_3^s)^2\Big]\Big\}\varepsilon\omega \\
&-2A_1^c(-2 + A_1^s A_3^c \varepsilon\omega + 2\omega^2) = 0, \\
&4A_3^s - \Big\{(A_1^c)^3 - 3A_1^c(A_1^s)^2 + 6A_3^c(A_1^c)^2 + 3A_3^c\Big[-4 + 2(A_1^s)^2 + (A_3^c)^2 + (A_3^s)^2\Big]\Big\}\varepsilon\omega \\
&-36A_3^s\omega^2 = 0, \\
&4A_3^c + \Big\{-(A_1^s)^3 + 6A_3^s(A_1^s)^2 + 3(A_1^c)^2[A_1^s + 2A_3^s] + 3A_3^s\Big[-4 + (A_3^c)^2 + (A_3^s)^2\Big]\Big\}\varepsilon\omega \\
&-36A_3^c\omega^2 = 0.
\end{aligned} \tag{24}$$

Since van der Pol equation is dominantly even by nature, the focus is on the even function solution of time through the elimination of coefficients accordingly.

From the first one of Equation (24), it is clear that if higher-order coefficients vanish, simple solutions are $A_1^c = 2$ and $A_1^s = 0$, which are known from approximate solutions of other methods [9]. With these known coefficients, the frequency solution from the above equations is

$$\omega^2 = 1 + \frac{1}{2}A_3^s\varepsilon\omega = 1 + \frac{A_3^s}{2}\varepsilon, \;\; \omega \approx 1. \tag{25}$$

A substitution of Equation (25) into Equation (24) will give

$$A_3^s = \frac{2\varepsilon\omega}{1 - 9\omega^2} = \frac{2\varepsilon}{1 - 9\omega^2} = -\frac{\varepsilon}{4}, \;\; A_3^c = -\frac{3A_3^s\varepsilon}{1 - 9\omega^2} = -\frac{3\varepsilon^2}{32}. \tag{26}$$

In summary, approximate solutions of amplitudes and frequency from the above equations are

$$A_1^c = 2, \; A_1^s = 0, \; A_3^s = -\frac{\varepsilon}{4}, \; A_3^c = -\frac{3\varepsilon^2}{32}, \; \omega^2 = 1 - \frac{\varepsilon^2}{8}. \tag{27}$$

The solution from the KBM method is [9]

$$A_1^c = 2, \; A_1^s = 0, \; A_3^s = -\frac{\varepsilon}{4}, \; A_3^c = 0, \; \omega^2 = 1 - \frac{\varepsilon^2}{8}. \tag{28}$$

Clearly, using the solution technique and procedure of the extended Galerkin method, a reasonably accurate solution to the van der Pol equation is also obtained. With the limited assumption of approximation and a simple procedure, the results are quite reasonable and accurate. Again, the approximate solutions present a good validation of the extended Galerkin method for lower-order asymptotic solutions. Further examinations of accuracy can be made with higher-order solutions and parameters by following the procedure in a systematic manner.

3.3. Results and Discussion

From the two examples analyzed and solved by applying the extended Galerkin method to popular nonlinear vibration equations, it is clear that the equivalent approximate solutions are obtained with a new procedure. The fundamental solutions for both Duffing and van der Pol equations shown in Equations (18) and (27) are close to other methods. By making comparisons with existing solution methods, it shows that the extended Galerkin method is consistent with the standard Galerkin method and more systematic in generating the equations for undetermined amplitudes for approximate solutions. The extended Galerkin method can be used for higher-order solutions of such nonlinear vibrations in a systematic manner and iterative procedure.

4. Conclusions

A procedure based on the popular Galerkin method has been proposed for solutions of nonlinear vibrations with a good approximation of the natural frequency and asymptotic deformation. It is done by representing the deformation with a harmonic series, and the nonlinear equation is solved approximately in the fundamental period of vibrations through adding integration with the weighting function and harmonic terms over time. Then a nonlinear eigenvalue problem is obtained just like in the standard Galerkin method. By solving the nonlinear eigenvalue problem, the natural frequency and mode shapes are obtained approximately and asymptotically. The procedure is actually the extension of the Galerkin method by adding an integration of the weighting function with time over one period of the fundamental vibration mode. The same procedure can also be applied to linear vibration equations to obtain identical results from the Galerkin method. The effectiveness and accuracy are stemmed from the Galerkin method itself and the periodicity of vibrations to guarantee excellent approximation with the weak form of the original problem in differential equations, as demonstrated in approximate solutions in this study and earlier numerical results [11,24]. This is a unique and systematic procedure for the analysis of both linear and nonlinear vibrations of elastic structures and solids by the Galerkin method with possible applications in other fields involving solutions of linear and nonlinear differential equations, as was proposed and suggested in an earlier study [11]. The simplicity, elegance, effectiveness, and accuracy with advantages of operations of integrations, particularly with symbolic mathematical tools, offer a favorable and powerful method for nonlinear vibration analysis and approximate solutions for more general nonlinear differential equations. Further applications and rigorous formulation of this technique for the analysis of nonlinear vibrations of solids and structures will provide a significant addition to available methods for some complications like the presence of external excitations and bias fields in addition to free vibrations shown here.

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