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Study on Stress and Displacement of Axisymmetric Circular Loess Tunnel Surrounding Rock Based on Joint Strength

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Abstract: The development of an effective evaluation method suitable for loess-tunnel excavation is necessary to avoid the collapse accidents caused by tunnel excavation and any secondary disasters. Although the Fenner formulas and the modified Fenner formulas are widely used in tunnel engineering, a defect still exists in these formulas because the Mohr–Coulomb (M–C) criterion exaggerates the tensile strength of the surrounding rock of the loess tunnel. A newly modified Fenner formula was derived based on joint strength to overcome this deficiency. First, the expressions of stress and the radius of the plastic zone of the surrounding rock of the loess tunnel and the expressions of radial displacement were derived based on the stress-equilibrium equation of the axisymmetric plane and the joint strength. Then, the difference in the modified Fenner formulas based on the two kinds of strength criteria for the loess tunnel were compared. The results showed that the radius of the plastic zone and the radial displacement of the loess tunnel determined by the modified Fenner formula based on joint strength were larger than those determined by the modified Fenner formula based on M–C strength. However, the plastic stress of the plastic zone determined by the modified Fenner formula based on joint strength was smaller. The comparative analysis reveals that the modified Fenner formula based on joint strength can evaluate the stress and plastic-displacement field of the surrounding rock of a loess tunnel more reasonably.

Keywords: loess tunnel; modified Fenner formula; joint strength; stress; radius of plastic zone

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1. Introduction

Loess is a multiphase porous medium with a special structural behavior. With the rapid development of the transportation infrastructure in China over the last few decades, many new tunnels have been constructed or are being constructed through regions with challenging geological conditions. Hundreds of tunnels with large cross-sections have been constructed in the ground, such as in the loess area in China [1,2]. Several engineering facilities have been produced, such as highways, dams, bridges, and tunnels, to expand the living space of mankind. Jefferson et al. updated a series of research topics proposed by G.A. Mavlyanov concerning engineering geology and discussed key concepts [3]. A collapse accident caused by tunnel excavation can cause ground collapse, which may harm the ground-water systems and ecological environments. Based on a probabilistic background, Oggeri et al. proposed a global retrospective analysis method which can define the unknown parameters of the rock and soil in the construction stage, so as to obtain more accurate results [4]. The stress field in the original stratum can change during tunnel excavation, which can lead to stress redistribution and the formation of a plastic zone or even a fracture zone in the rock surrounding the tunnel [5]. The stress and displacement field in the surrounding rock of the tunnel can be used to evaluate the stability of the surrounding rock and the design of the lining. Thus, accurately determining the radius of the plastic zone based on different strength criteria has great engineering importance in optimising the design of underground support structures and ensuring the construction safety of tunnels.

The Mohr–Coulomb (M–C) strength criterion is commonly used in geotechnical materials and still plays an important role at present. The damaged zone of rock masses in some deep circular tunnels was analysed in view of the strain-softening-based plastic zone using the M–C strength criterion [6]. However, the Drucker–Prager (D–P) strength criterion was put forward to overcome the corner defect in M–C strength in π plane; it is widely used in numerical analysis of geotechnical engineering. Multiphase-field modelling of crack propagation in geological materials and porous media was analysed based on D–P strength criterion [7]. The king of theoretical solutions in tunnel-excavation processes was deduced through the cavity-expansion theory based on the D–P strength criterion [8]. The Hoek–Brown (H–B) strength criterion was developed in 1980 to reasonably consider the tensile and compressive strengths of rock mass [9], and has been widely used due to its adequacy and convenience in rock-behaviour predictions and applications in rock engineering [10]. Over the past few decades, the analysis of tunnel mechanical properties has been mainly based on the M–C, D–P and H–B strength criteria to determine the plastic zone and its elastic–plastic stress for the evaluation of the stress environment and displacement field in the surrounding rock during tunnel excavation.

The Fenner formula was proposed on the basis of the M–C strength criterion [11], and the expressions of stress and the radius of the plastic zone in a circular tunnel under the condition of an infinite homogeneous stratum were derived. However, the radial stress σ_R at the interface between the elastic and plastic zones in the surrounding rock of the tunnel was determined. Fenner adopted a simplified formula by neglecting the cohesion c in the place. On the basis of the M–C strength criterion, Kastner's formula of plastic stress and the radius of the plastic zone in a circular tunnel were derived for a circular tunnel under the condition of infinite homogeneous stratum by using the unconfined compressive strength σ_c and the internal friction angle φ as parameters [12]. The cohesion c at the interface between the elastic and plastic zones in the surrounding rock of a tunnel could be expressed by the unconfined compressive strength σ_c and internal-friction angle φ. Therefore, the cohesion c at the interface between the elastic and plastic zones was appropriately considered in the Kastner formula. By comparing the difference between the Fenner and Kastner formulas, a modified Fenner formula of elastic–plastic stress and the plastic radius in a circular tunnel, which considered the formula at the interface between the elastic and plastic zones, was derived on the basis of M–C strength criterion [13]. As proven, the modified Fenner formula based on the M–C strength criterion was equivalent to the Kastner formula. Fenner, Kastner and modified Fenner formulas are widely used in tunnel engineering and are mainly used to determine the plastic zone of a tunnel and to analyse the radial displacement of a tunnel.

The stress and displacement of surrounding rock of the tunnel were solved by Park, and on the basis of M–C criterion and the large strain similarity solution, the effect of small and large strain solutions on displacement was studied [14]. Ren et al. studied the effect of the hydraulic pressure on the direction of the principal stress in plastic zone of surrounding rock of the circular rock tunnel; then, they analysed the relationship between the hydraulic pressure of tunnel wall and the plastic zone of surrounding rock of the tunnel based on Fenner formula [15]. On the basis of the M–C strength criterion, Li et al. deduced the analytical solutions of stress displacement and the range of the plastic zone in the elastic, elastoplastic and rheological stages [16]. Liu et al. presented a semi-analytical solution for the tunnel excavation problem based on an extended D–P model [17]. Some elastic–plastic studies were conducted on the basis of the H–B strength criterion to better analyse the elastic–plastic mechanical properties of a rock tunnel. For the rock tunnel, Pan et al. considered the structural plane of the surrounding rock, performed an elastic–plastic analysis of the surrounding rock of a circular tunnel and obtained the Fenner formula based on the H–B strength criterion [18]. Zou et al. studied the theoretical solutions of stress, displacement and the plastic radius of a circular tunnel based on the H–B strength criterion and the non-associated flow law of elastic–brittle–plastic rock [19]. Wang et al. believed that the intermediate principal stress was an important factor to determine the plastic zone; thus, the H–B strength criterion was modified by considering the intermediate principal stress, and the theoretical formula of the plastic zone radius of a rock tunnel was proposed [20]. Ranjbarnia et al. developed a new analytical-numerical procedure based on the H–B strength criterion to provide the stresses and strains around a circular tunnel in rock masses exhibiting different stress–strain behaviours; a simple relationship was also presented to determine the closure of a circular tunnel in terms of the rock mass strength and tunnel depth [21]. Sharan compared the applicability of the M–C and H–B strength criteria in a rock tunnel, analysed the elastic–brittle–plastic behaviour of tunnel and evaluated the stress difference of the surrounding rock in a tunnel between the expansive and non-expansive rocks using a numerical method [22]. On the basis of the difference in the analytical solutions of displacement of the surrounding rock between the M–C and H–B strength criteria, Park et al. used the non-associated flow rule to compare the displacement difference of the surrounding rock in a tunnel between soft and hard rocks with the effect of the angle of dilatancy [23]. Serrano et al. deduced a mathematical expression to determine the radius of the plastic zone based on M–C and H–B strength criteria; then, they compared the convergence of circular tunnels in elastoplastic rock masses [24].

The abovementioned considerable literature has promoted research on the elastic–plastic mechanical properties of tunnels. However, two problems still need to be solved: one is that the defect still exists in the Fenner formula and in the modified Fenner formula derived based on M–C strength criterion because this criterion exaggerates the tensile strength of surrounding rock of the tunnel; the other is that the H–B strength criterion is unsuitable for the analysis of a tunnel in earth, especially a loess tunnel, given that it is mainly based on the empirical criterion of rock failure, which is put forward by the curve-failure envelope of a large number of rock tests. Xie et al. studied the effects of matrix suction and net confining pressure on the deformation and strength characteristics of compacted loess by carrying out the triaxial consolidation shear test of unsaturated loess, and the results showed that under certain matrix suction conditions, the shear strength of compacted loess increased with the increase of net confining pressure, and under constant net confining pressure, the shear strength increased with the increase of matrix suction [25]. When the strength of structural loess is higher, it not only has higher shear strength but also has relatively higher tensile strength, and tensile strength considerably affects the mechanical properties of structural loess [26]. Therefore, the influence of tensile strength of structural loess should be considered when the stress and displacement of the surrounding rock of structural loess tunnel need to be studied. Li et al. put forward the joint strength formula of structural loess to reasonably consider the characteristics of tensile strength of structural loess; it overcame the shortcoming that M–C strength exaggerated the tensile strength of loess [27]. Through the formula and calculation of loess earth pressures based on joint strength, it can be verified that the joint strength formula can not only can reflect the characteristics of tensile strength of loess but also can study the characteristics of the shear strength of the loess [28].

Therefore, accurately determining the radius of plastic zone of loess tunnel based on joint strength criteria is an issue to be solved in loess tunnel. A newly modified Fenner formula for the loess tunnel will be derived on the basis of the joint strength theory that can consider the tensile and shear properties of loess. Then, the change in stress environment in the surrounding rock of a loess tunnel under the action of the average stratum stress should be studied. Lastly, the differences in the stress distribution, the radius of plastic zone and the radial displacement will be analysed and compared on the basis of the joint strength and M–C strength.

2. The Basis of Derivation

2.1. Basic Hypothesis for Derivation

The deep axisymmetric tunnel may be described as the following plane-strain problem in polar coordinates (r, θ): a tunnel of circular cross-section with radius a is excavated in a homogeneous isotropic rock mass of infinite extent, which is subjected to a uniform in situ stress σ_s in all directions, and a radial support pressure σ_a is applied on the tunnel wall (Figure 1).

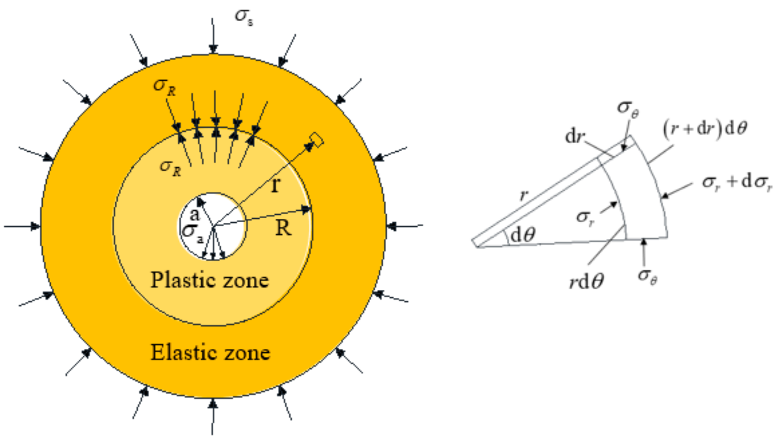

Figure 1. Stress component of surrounding rock in plane-strain analysis.

In Figure 1, a uniform in situ stress $\sigma_s = \gamma h$ (γ is weight per unit volume and h is the central depth of tunnel), r is vector diameter in the surrounding rock, R is radius of plastic zone, σ_r is radial stress, σ_R is radial stress at the interface between the elastic and plastic zones and σ_θ is tangential stress.

Therefore, the stress equilibrium equation of the axisymmetric plane problem in polar coordinates can be expressed as [5].

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{1}$$

2.2. Joint Strength of Loess

The loess area is generally a semi-arid area in the middle temperate zone. The groundwater level is deep, the water content of the loess is low, and the liquid index of the natural loess is generally 0.19 [29]. Natural loess is in a hard plastic state, with high hardness and high strength. In the process of tunnel excavation, the surrounding rock of the loess shows a certain degree of self-stability. After tunnel excavation, the stress of the surrounding rock is redistributed. In the tensile shear-stress zone, the tensile stress and shear stress of a loess with high strength do not exceed the limit.

The M–C strength line of loess can be determined according to conventional triaxial shear strength test and the three failure stress circles, as shown in Figure 2, where σ is normal stress and τ is shear stress. By determining the point of intersection of M–C strength line and horizontal axis, the value of tensile strength corresponding to M–C strength can be obtained.

In order to unite the tensile strength and shear strength of loess, the strength that can take the tensile characteristics and shear characteristics into consideration is called the joint strength, and the corresponding failure strength curve is expressed in the form of hyperbola. Here "joint" means a kind of united consideration.

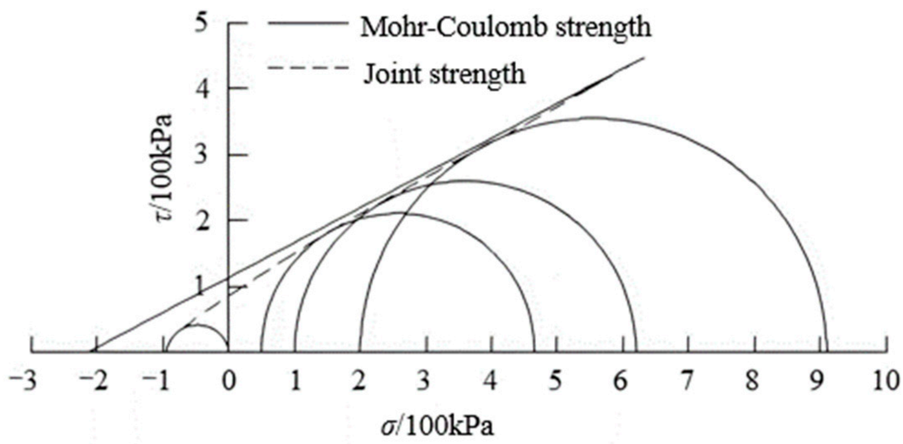

Figure 2. M–C strength and the joint strength.

To fit a hyperbolic curve of the joint strength line of structural loess on the σ-τ plane, Li et al. assumed that the horizontal axis intercept of the curve represents the tensile strength σ_t, and that the M–C strength envelope was regarded as the asymptote [26]. Then, the joint strength formula corresponding to the dotted line, which can describe the tensile and shear strengths of structural loess comprehensively and reasonably, was derived on the basis of the tensile and conventional triaxial shear strength tests.

$$\tau_f{}^2 = (c + \sigma \tan \varphi)^2 - (c + \sigma_t \tan \varphi)^2 \quad (2)$$

Figure 2 shows that the tensile strength determined by the M–C strength is more than twice that of the tensile test. The results show that the M–C strength criterion obviously overestimates the tensile strength of the loess, whilst the joint strength overcomes the shortcoming of the M–C strength exaggerating the tensile strength of the loess. Therefore, the joint strength can reasonably evaluate the stress and displacement of the surrounding rock of the loess tunnel.

2.3. Equivalent Equation in the Form of Principal Stress of the Joint Strength

The ultimate equilibrium state of the joint strength on the σ-τ plane is shown in Figure 3, where σ_1 is major principal stress and σ_3 is minor principal stress.

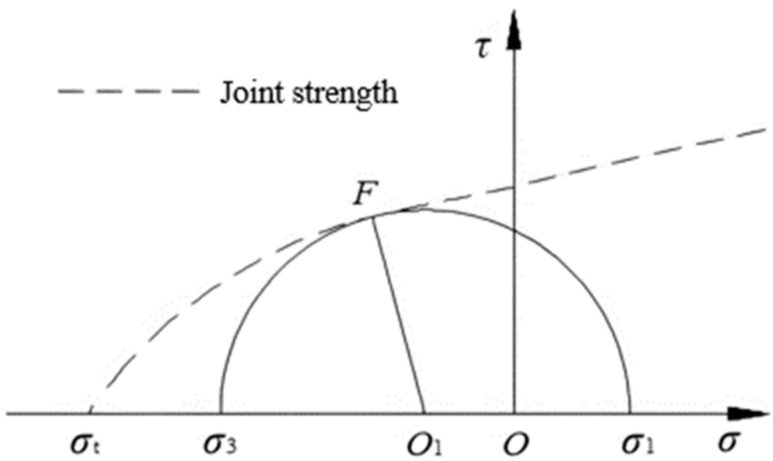

Figure 3. Ultimate equilibrium state in the joint strength.

The stress Circle O_1 is tangent to the joint strength line, with the assumption that the coordinates of tangent Point F are (σ_F, τ_F), because tangent Point F is on the joint strength line. Thus,

$$\tau_F^2 = (c + \sigma_F \tan\varphi)^2 - \beta^2 \tag{3}$$

Or

$$\tau_F = \sqrt{(c + \sigma_F \tan\varphi)^2 - \beta^2} \tag{4}$$

In Formulas (3) and (4),

$$\beta = c + \sigma_t \tan\varphi \tag{5}$$

The tangent slope k_F at the differentiable Point F of Formula (4) is

$$k_F = \frac{d\tau_F}{d\sigma_F} = \frac{(c + \sigma_F \tan\varphi)\tan\varphi}{\sqrt{(c + \sigma_F \tan\varphi)^2 - \beta^2}} \tag{6}$$

Then, the slope of the normal Line O_1F at Point F is k_N:

$$k_N = -\frac{1}{k_F} = -\frac{\sqrt{(c + \sigma_F \tan\varphi)^2 - \beta^2}}{(c + \sigma_F \tan\varphi)\tan\varphi} \tag{7}$$

According to the coordinates of Point F and Formula (7), the equation of straight Line O_1F is

$$\tau - \tau_F = k_N(\sigma - \sigma_F) \tag{8}$$

The intersection of Formula (8) and the σ coordinate axis is the central coordinate of the Circle O_1 $(\sigma_F + (c + \sigma_F \tan\varphi)\tan\varphi,\ 0)$.

The distance between Points O_1 and F is the radius of Circle O_1. Thus, the equation of Circle O_1 is

$$\sqrt{[(c + \sigma_F \tan\varphi)\tan\varphi]^2 + \tau_F^2} = \frac{\sigma_1 - \sigma_3}{2} \tag{9}$$

According to the half relation between the abscissa of centre of Circle O_1 and the value of (σ_1 + σ_3), we have

$$\sigma_F + (c + \sigma_F \tan\varphi)\tan\varphi = \frac{\sigma_1 + \sigma_3}{2} \tag{10}$$

Introducing Formula (4) into Formula (9) yields

$$\sqrt{\frac{(c + \sigma_F \tan\varphi)^2}{\cos^2\varphi} - \beta^2} = \frac{\sigma_1 - \sigma_3}{2} \tag{11}$$

With the simultaneous solution of Formulas (10) and (11) and the elimination of σ_F, the equivalent equation in the form of principal stress of the joint strength can be obtained as

$$\frac{\sigma_1 - \sigma_3}{2} = \sqrt{\left[\left(\frac{\sigma_1 + \sigma_3}{2}\right)\sin\varphi + c\cos\varphi\right]^2 - \beta^2} \tag{12}$$

3. Derivation of the New Modified Fenner Formula from Joint Strength

The influence parameters are shown in Figure 1. The surrounding rock of the loess adopts peak strength, which is expressed by the shear strength index.

3.1. Stress and Plastic Radius of Plastic Zone in Surrounding Rock of the Tunnel Based on Joint Strength

3.1.1. Formula of Stress in Plastic Zone of Surrounding Rock of the Tunnel Based on Joint Strength

The surrounding rock of the plastic zone ($a \le r \le R$) is in the state of ultimate equilibrium. If the compressive stress is positive, then the major principal stress σ_1 is tangential stress $\sigma_{\theta\text{p}}$ and the minor principal stress σ_3 is the radial stress σ_{rp} in the plastic zone. Introducing them into Formula (12) yields

$$\frac{\sigma_{\theta\text{p}} - \sigma_{\text{rp}}}{2} = \sqrt{\left[\left(\frac{\sigma_{\theta\text{p}} + \sigma_{\text{rp}}}{2}\right)\sin\varphi + c\cos\varphi\right]^2 - \beta^2} \tag{13}$$

where σ_{rp} is radial stress in the plastic zone and $\sigma_{\theta\text{p}}$ is tangential stress in the plastic zone.

To simplify the calculation, we let

$$\eta = \left(\frac{\sigma_{\theta\text{p}} + \sigma_{\text{rp}}}{2}\right)\sin\varphi + c\cos\varphi \tag{14}$$

Introducing Formula (14) into Formula (13) yields

$$\frac{\sigma_{\theta\text{p}} - \sigma_{\text{rp}}}{2} = \sqrt{\eta^2 - \beta^2} \tag{15}$$

With simultaneous solution of Formulas (14) and (15), the formulas of σ_{rp} and $\sigma_{\theta\text{p}}$ based on joint strength can be obtained with η as a parameter.

$$\left.\begin{array}{l} \sigma_{\text{rp}} = \frac{\eta - c\cos\varphi}{\sin\varphi} - \sqrt{\eta^2 - \beta^2} \\ \sigma_{\theta\text{p}} = \frac{\eta - c\cos\varphi}{\sin\varphi} + \sqrt{\eta^2 - \beta^2} \end{array}\right\} \tag{16}$$

The derivative can be obtained by Formula (16).

$$\frac{\text{d}\sigma_{\text{rp}}}{\text{d}r} = \left(\frac{1}{\sin\varphi} - \frac{\eta}{\sqrt{\eta^2 - \beta^2}}\right)\frac{\text{d}\eta}{\text{d}r} \tag{17}$$

Introducing Formulas (16) and (17) into Formula (1) yields

$$\left(\frac{1}{\sin\varphi} - \frac{\eta}{\sqrt{\eta^2 - \beta^2}}\right)\frac{\text{d}\eta}{\text{d}r} - \frac{2\sqrt{\eta^2 - \beta^2}}{r} = 0 \tag{18}$$

After separating the variables in Formula (18), Formula (19) can be obtained by the integral.

$$\ln\frac{\left(\eta + \sqrt{\eta^2 - \beta^2}\right)^{\frac{1}{\sin\varphi}}}{\sqrt{\eta^2 - \beta^2}} - \ln r^2 = C \tag{19}$$

where C is constant of integration.

When r equals a, $\sigma_{\text{rp}} = \sigma_{\text{a}}$, $\eta = \eta_0$. Introducing them into Formulas (16) and (19) yields

$$\eta_0 = \frac{c + \sigma_{\text{a}}\tan]\varphi}{\cos\varphi} + \sqrt{(c + \sigma_{\text{a}}\tan\varphi)^2 - \beta^2}\tan\varphi \tag{20}$$

$$C = \ln\frac{\left(\eta_0 + \sqrt{\eta_0^2 - \beta^2}\right)^{\frac{1}{\sin\varphi}}}{\sqrt{\eta_0^2 - \beta^2}} - \ln a^2 \tag{21}$$

By introducing Formula (21) into Formula (19), the equation between η and r can be obtained.

$$\left(\frac{\eta + \sqrt{\eta^2 - \beta^2}}{\eta_0 + \sqrt{\eta_0^2 - \beta^2}}\right)^{\frac{1}{\sin\varphi}}\frac{\sqrt{\eta_0^2 - \beta^2}}{\sqrt{\eta^2 - \beta^2}} = \frac{r^2}{a^2} \tag{22}$$

Formulas (16) and (22) show that the expressions of the tangential stress $\sigma_{\theta p}$, the radial stress σ_{rp} in the plastic zone and the vector diameter r of the calculated point regard η as a variable parameter based on the joint strength. By introducing the vector diameter r into Formula (22), the value of η the calculated point can be obtained. Introducing η into Formula (16) can determine the corresponding tangential stress $\sigma_{\theta p}$ and radial stress σ_{rp} in the plastic zone.

3.1.2. Formula of Radius of Plastic Zone in Surrounding Rock of the Tunnel Based on Joint Strength

According to Lomé's solution and boundary condition, the elastic stress [5] in the surrounding rock of the tunnel can be obtained from Formula (23).

$$\left.\begin{array}{l}\sigma_{re} = \sigma_s\left(1 - \frac{R^2}{r^2}\right) + \sigma_R \frac{R^2}{r^2} \\ \sigma_{\theta e} = \sigma_s\left(1 + \frac{R^2}{r^2}\right) - \sigma_R \frac{R^2}{r^2}\end{array}\right\} \tag{23}$$

where σ_R is radial stress at the interface between the elastic and plastic zones and R is radius of plastic zone.

The condition of stress continuity is satisfied at the interface between the elastic and plastic zones. When r equals R, Formula (24) holds.

$$\left.\begin{array}{l}\sigma_{re} = \sigma_{rp} \\ \sigma_{\theta e} = \sigma_{\theta p}\end{array}\right\} \tag{24}$$

where σ_{re} is radial stress in the elastic zone and $\sigma_{\theta e}$ is tangential stress in the elastic zone.

At the interface between the elastic and plastic zones, r equals R. Thus, Formula (23) becomes Formula (25).

$$\left.\begin{array}{l}\sigma_{re} = \sigma_R \\ \sigma_{\theta e} = 2\sigma_s - \sigma_R\end{array}\right\} \tag{25}$$

At the interface between the elastic and plastic zones, r equals R and η equals η_R. Accordingly, Formula (16) becomes Formula (26).

$$\left.\begin{array}{l}\sigma_{rp} = \frac{\eta_R - c\cos\varphi}{\sin\varphi} - \sqrt{\eta_R^2 - \beta^2} \\ \sigma_{\theta p} = \frac{\eta_R - c\cos\varphi}{\sin\varphi} + \sqrt{\eta_R^2 - \beta^2}\end{array}\right\} \tag{26}$$

Introducing Formulas (24) and (25) into Formula (26) yields

$$\sigma_s = \frac{\eta_R - c\cos\varphi}{\sin\varphi} \tag{27}$$

Or

$$\eta_R = \sigma_s \sin\varphi + c\cos\varphi \tag{28}$$

At the interface between the elastic and plastic zones, r equals R and η equals η_R. Thus, plastic radius R based on joint strength can be derived from Formulas (22) and (28), as shown in Formula (29).

$$R = a\left(\frac{\eta_R + \sqrt{\eta_R^2 - \beta^2}}{\eta_0 + \sqrt{\eta_0^2 - \beta^2}}\right)^{\frac{1}{2\sin\varphi}} \left(\frac{\eta_0^2 - \beta^2}{\eta_R^2 - \beta^2}\right)^{\frac{1}{4}} \tag{29}$$

3.2. Formula of Elastic Stress and Displacement in Elastic Zone of Surrounding Rock of the Tunnel Based on Joint Strength

3.2.1. Formula of Elastic Stress in Surrounding Rock of the Tunnel Based on Joint Strength

Introducing Formula (27) into Formula (26) yields

$$\left.\begin{aligned}\sigma_{\mathrm{rp}} &= \sigma_{\mathrm{s}} - \sqrt{\eta_{\mathrm{R}}^{2} - \beta^{2}} \\ \sigma_{\theta\mathrm{p}} &= \sigma_{\mathrm{s}} + \sqrt{\eta_{\mathrm{R}}^{2} - \beta^{2}}\end{aligned}\right\} \tag{30}$$

Introducing Formulas (24) and (25) into Formula (30) yields

$$\sigma_{\mathrm{R}} = \sigma_{\mathrm{s}} - \sqrt{\eta_{\mathrm{R}}^{2} - \beta^{2}} \tag{31}$$

Introducing Formulas (29) and (31) into Formula (23) can obtain the elastic stresses of surrounding rock of the tunnel based on joint strength, as shown in Formula (32).

$$\left.\begin{aligned}\sigma_{\mathrm{re}} &= \sigma_{\mathrm{s}} - \tfrac{a^{2}}{r^{2}}\left(\tfrac{\eta_{\mathrm{R}}+\sqrt{\eta_{\mathrm{R}}^{2}-\beta^{2}}}{\eta_{0}+\sqrt{\eta_{0}^{2}-\beta^{2}}}\right)^{\frac{1}{\sin\varphi}}\sqrt{\eta_{0}^{2}-\beta^{2}} \\ \sigma_{\theta\mathrm{e}} &= \sigma_{\mathrm{s}} + \tfrac{a^{2}}{r^{2}}\left(\tfrac{\eta_{\mathrm{R}}+\sqrt{\eta_{\mathrm{R}}^{2}-\beta^{2}}}{\eta_{0}+\sqrt{\eta_{0}^{2}-\beta^{2}}}\right)^{\frac{1}{\sin\varphi}}\sqrt{\eta_{0}^{2}-\beta^{2}}\end{aligned}\right\} \tag{32}$$

3.2.2. Formula of Displacement in Surrounding Rock of the Tunnel Based on Joint Strength

At the interface between the elastic and plastic zones, the displacement of the surrounding rock of the tunnel can be determined from Formula (33) when r equals R and η equals η_{R}.

$$u_{\mathrm{R}} = \frac{1+\mu}{E}R(\sigma_{\mathrm{s}} - \sigma_{\mathrm{R}}) \tag{33}$$

Introducing Formulas (29) and (31) into Formula (33) can derive the displacement at the interface between the elastic and plastic zones based on joint strength.

$$u_{\mathrm{R}} = \frac{1+\mu}{E}a\left(\frac{\eta_{\mathrm{R}}+\sqrt{\eta_{\mathrm{R}}^{2}-\beta^{2}}}{\eta_{0}+\sqrt{\eta_{0}^{2}-\beta^{2}}}\right)^{\frac{1}{2\sin\varphi}}\left(\eta_{0}^{2}-\beta^{2}\right)^{\frac{1}{4}}\left(\eta_{\mathrm{R}}^{2}-\beta^{2}\right)^{\frac{1}{4}} \tag{34}$$

With the assumption that the plastic zone conforms to the volume-compatibility condition in the deformation process, Formula (35) must hold.

$$\pi\left(R^{2} - a^{2}\right) = \pi\left[(R - u_{\mathrm{R}})^{2} - (a - u_{\mathrm{a}})^{2}\right] \tag{35}$$

After expanding Formula (35) and omitting higher-order terms of u_{R}^{2} and u_{a}^{2}, radial displacement of tunnel wall based on joint strength can be approximatively derived, as shown in Formula (36).

$$u_{\mathrm{a}} = \frac{R}{a}u_{\mathrm{R}} \tag{36}$$

Introducing Formulas (29) and (34) into Formula (36) can obtain the radial displacement of the tunnel wall based on joint strength, as shown in Formula (37).

$$u_{\mathrm{a}} = \frac{1+\mu}{E}a\left(\frac{\eta_{\mathrm{R}}+\sqrt{\eta_{\mathrm{R}}^{2}-\beta^{2}}}{\eta_{0}+\sqrt{\eta_{0}^{2}-\beta^{2}}}\right)^{\frac{1}{\sin\varphi}}\left(\eta_{0}^{2}-\beta^{2}\right)^{\frac{1}{2}} \tag{37}$$

4. Results and Discussion

According to the abovementioned results, which include Formulas (16), (29), (32), (34) and (37), a series of analytical solutions comprises the modified Fenner formula based on joint strength. In this section, the difference between the modified Fenner formula based on joint strength and the modified Fenner formula based on M–C strength [7] for the unlined loess tunnel will be analysed and discussed.

For the unlined loess tunnel, the radius of the loess tunnel is 2.0 m, weight per unit volume γ is 15 kN/m^3, cohesion c is 60 kPa, internal friction angle φ is 25°, tensile strength σ_t is 28 kPa, elastic modular E is 72 MPa and Poisson's ratio μ is 0.35.

4.1. Comparison of the Stresses in Surrounding Rock of Loess Tunnel

When the buried depth of unlined loess tunnel is 50 m, the uniform in situ stress σ_s is 750 kPa. Thus, the distribution of stresses in the surrounding rock of the unlined loess tunnel of the modified Fenner formula based on M–C strength and the newly modified Fenner formula based on joint strength can be calculated and is shown in Figures 4 and 5.

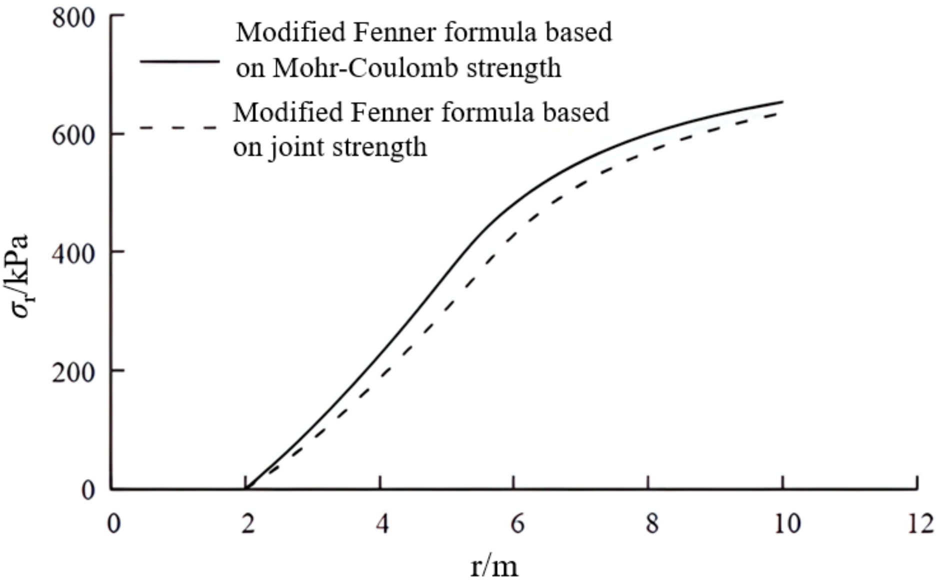

Figure 4. Comparison of radial stresses in surrounding rock of loess tunnel.

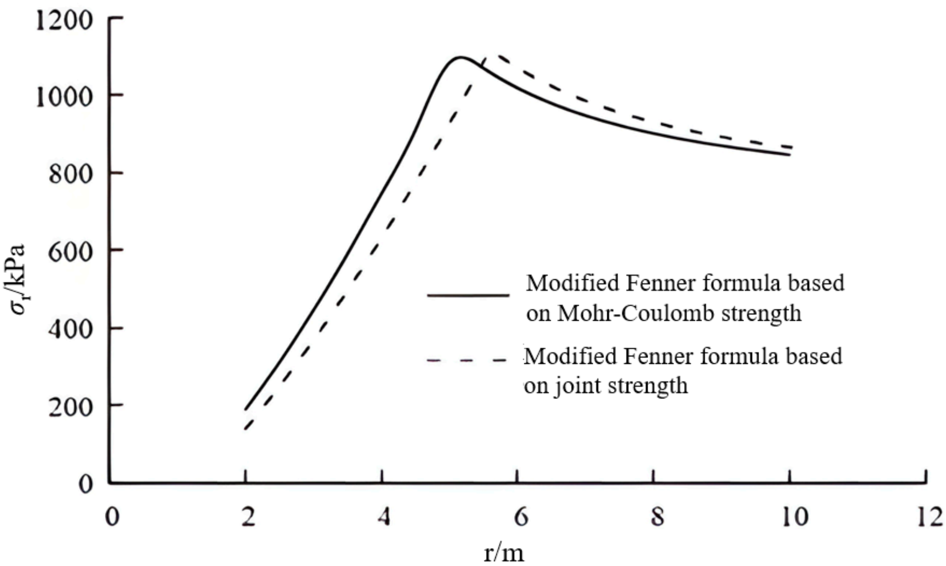

Figure 5. Comparison of tangential stresses in surrounding rock of loess tunnel.

The radial stresses of the modified Fenner formula based on M–C strength and the newly modified Fenner formula based on joint strength increase when the distance is far from the axis of the loess tunnel. However, the radial stress of surrounding rock of loess tunnel determined by modified Fenner formula based on joint strength is less than that determined by M–C strength. This is consistent with the conclusion in the literature [28] that the M–C strength criterion overestimates the tensile strength of loess, resulting in a deviation in the calculation results.

The tangential stresses of the modified Fenner formula based on M–C strength and the newly modified Fenner formula based on joint strength increase firstly and then decrease when the distance is far from the axis of the loess tunnel. In the meantime, the maximum tangential stresses occur at the interface of the elastic–plastic zone.

The tangential stress in the plastic zone of the surrounding rock determined by the modified Fenner formula based on joint strength is obviously smaller than that determined by the M–C strength, but the phenomenon is reversed in the elastic zone. The largest difference in tangential stress in the two kinds of modified Fenner formulas is observed at the elastic–plastic interface.

4.2. Comparison of the Radius of Plastic Zone and Radial Displacement

The radii of the plastic zone in the surrounding rock of the unlined loess tunnel of the modified Fenner formula based on M–C strength and the newly modified Fenner formula based on joint strength can be calculated is shown in Figure 6.

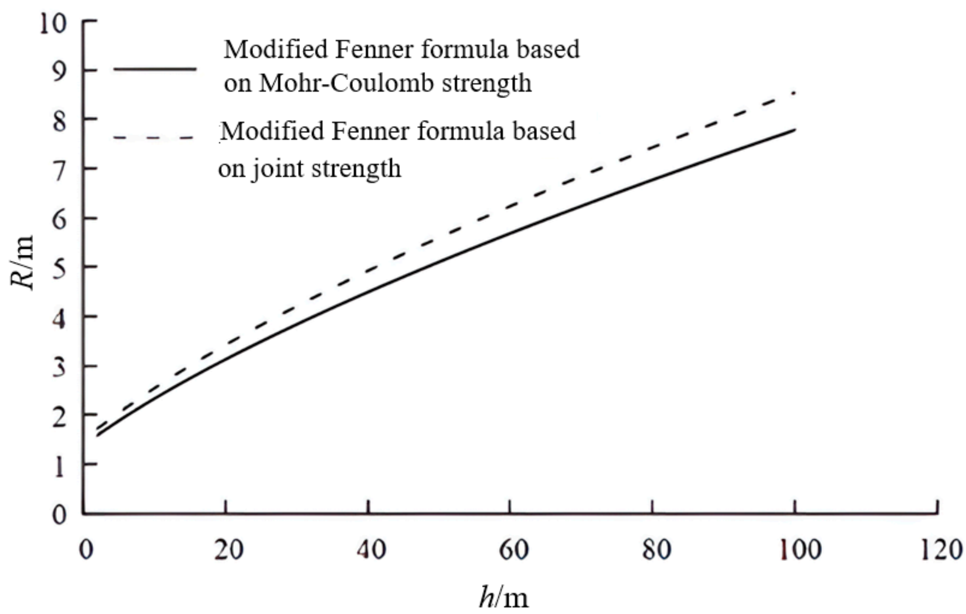

Figure 6. Relationship between radius of plastic zone and depth of loess tunnel.

The radii of the plastic zone calculated by the modified Fenner formula based on joint strength and the modified Fenner formula based on M–C strength increased with the increase in tunnel depth. However, the radius of the plastic zone determined by the modified Fenner formula based on joint strength was larger than that based on the latter under the same depth.

The radial displacement at the interface of elastic–plastic zone in the surrounding rock of the unlined loess tunnel for the modified Fenner formula based on M–C strength and the newly modified Fenner formula based on joint strength can be calculated and is shown in Figure 7. In the meantime, the radial displacement of the tunnel wall determined by the two strength theories is shown in Figure 8.

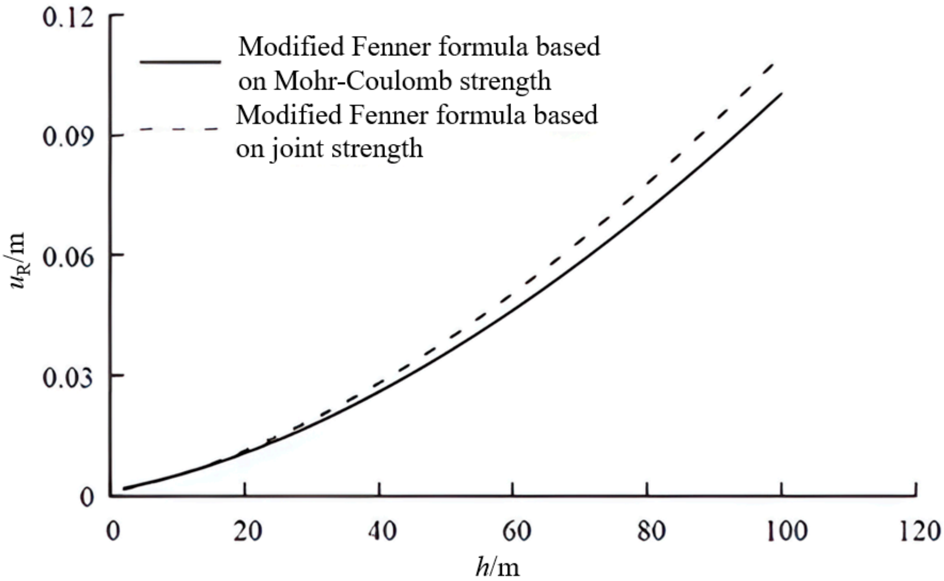

Figure 7. Relationship between radial displacement at elastic–plastic interface and depth of tunnel.

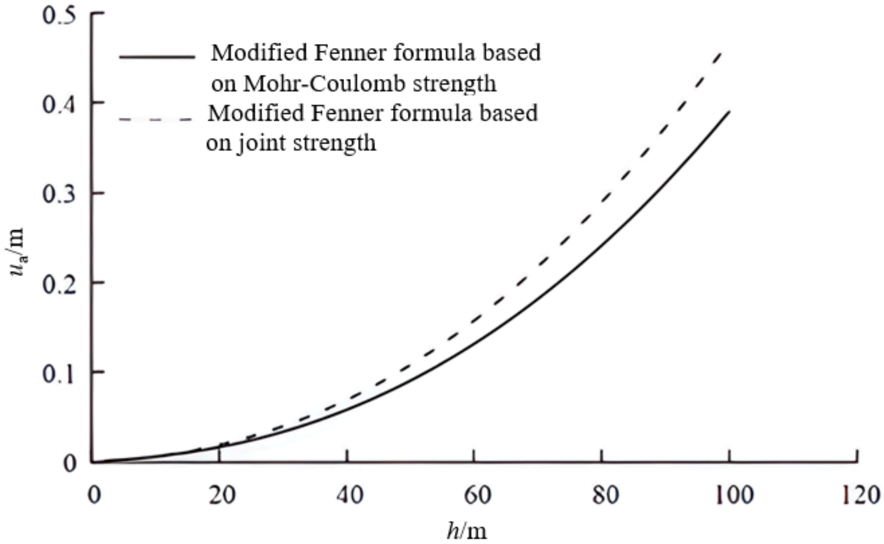

Figure 8. Relationship between radial displacement of the tunnel wall and depth of tunnel.

The radial displacements determined by the two strength theories increase with the increase in buried depth. However, the radial displacement determined by the joint strength theory is slightly larger than that determined by M–C strength. In a word, the M–C strength overestimates the tensile strength of structural loess, which leads to the smaller radius of the plastic zone and the smaller displacement of the surrounding rock of the unlined loess tunnel calculated by the modified Fenner formula based on M–C strength.

The joint strength can be used to reasonably evaluate the tensile and shear strengths of structural loess. Thus, the modified Fenner formula based on joint strength is more practical in the analysis of the stress environment and displacement field of sthe urrounding rock of a loess tunnel. Therefore, a reasonable evaluation of the tunnel excavation impact not only can eliminate potential collapse accidents in advance but also can avoid damage caused by ground subsidence to groundwater systems and ecological environments.

5. Conclusions

A newly modified Fenner formula for a loess tunnel was derived on the basis of the joint-strength theory that can consider the tensile and shear properties of loess. Then, the change in the stress environment in the surrounding rock of a loess tunnel under the action of the average stratum stress was studied. Lastly, the differences in the stress distribution, the radius of the plastic zone and the radial displacement were analysed and compared on the basis of the joint strength and M–C strength. The main conclusions are as follows:

(1) With respect to the axisymmetric plane problem of the loess tunnel, a new modified Fenner formula based on joint strength was derived by means of the ultimate equilibrium equation on the basis of the principal stress expression of joint strength. The stress expression of the elastic zone, the expression of the radial displacement at the elastic–plastic interface and the expression of the radial displacement of the loess tunnel were determined on the basis of joint strength by conforming to the conditions of stress continuity and volume compatibility at the elastic–plastic interface;

(2) The radius of the plastic zone and the radial displacement of the unlined loess tunnel determined by the modified Fenner formula based on joint strength were larger than those determined by the modified Fenner formula based on M–C strength. However, the radial stress in the plastic zone and the tangential stress in the plastic zone determined by the modified Fenner formula based on joint strength were smaller than those based on M–C strength. In addition, the radial stress in the elastic zone determined by the modified Fenner formula based on joint strength was smaller than that based on M–C strength, while the tangential stress in the elastic zone was the opposite;

(3) The joint strength reasonably evaluates the tensile strength of the structural loess. Thus, the modified Fenner formula based on joint strength is a new method for reasonably evaluating the stress and displacement field of the surrounding rock of a loess tunnel.

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References

1. Li, J.; Shao, S.J.; Shao, S. Collapsible characteristics of loess tunnel site and their effects on tunnel structure. *Tunn. Undergr. Space Technol.* **2018**, *83*, 509–519. [CrossRef]
2. Shao, S.; Shao, S.J.; Li, J.; Qiu, B. An Analysis of Loess Tunnel Failure and Its Mechanism. *Adv. Civ. Eng.* **2021**, *2021*, 6671666. [CrossRef]
3. Jefferson, I.F.; Mavlyanova, N.; Hara-Dhand, K.O.; Smalley, I.J. Engineering geology of loess ground: 15 tasks for investigators—The Mavlyanov programme of loess research. *Eng. Geol.* **2004**, *74*, 33–37. [CrossRef]
4. Oggeri, C.; Oreste, P. Tunnel static behavior assessed by a probabilistic approach to the back-analysis. *Am. J. Appl. Sci.* **2012**, *9*, 1137–1144.
5. Mambou, L.L.N.; Ndop, J.; Ndjaka, J.-M.B. Numerical Investigations of Stresses and Strains Redistribution around the Tunnel: Influence of Transverse Isotropic Behavior of Granitic Rock, In Situ Stress and Shape of Tunnel. *J. Min. Sci.* **2015**, *51*, 497–505. [CrossRef]
6. Zareifard, M.R. Ground Reaction Curve for Deep Circular Tunnels in Strain-Softening Mohr-Coulomb Rock Masses Considering the Damaged Zone. *Int. J. Geomech.* **2020**, *20*, 04020190. [CrossRef]
7. Späth, M.; Herrmann, C.; Prajapati, C.; Prajapati, N.; Schneider, D.; Schwab, F.; Selzer, M.; Nestler, B. Multiphase-field Modelling of Crack Propagation in Geological Materials and Porous Media with Drucker-Prager Plasticity. *Comput. Geosci.* **2021**, *25*, 325–343. [CrossRef]
8. Yang, X.F.; Yuan, H.; Wu, J.Y.; Li, S.Q. Elastoplastic Analysis of Circular Tunnel based on Drucker-Prager Criterion. *Adv. Civ. Eng.* **2018**, *2018*, 5149789. [CrossRef]
9. Hoek, E.; Brown, E.T. Empirical Strength Criterion for Rock Masses. *J. Geotech. Geoenviron. Eng.* **1980**, *106*, 1013–1035. [CrossRef]

10. Zhu, H.H.; Zhang, Q.; Huang, B.Q.; Zhang, L.Y. A Constitutive Model based on the Modified Generalized Three-Dimensional Hoek-Brown Strength Criterion. *Int. J. Rock Mech. Min. Sci.* **2017**, *98*, 78–87. [CrossRef]
11. Fenner, R. Untersuchungen zur Erkenntnis des Gebirgsdruckes. *Glückauf* **1938**, *74*, 681.
12. Kastner, H. *Über den Echten Gebirgsdruck beim Bau Tiefliegender Tunnel*; Springer: Berlin/Heidelberg, Germany, 1949.
13. Zheng, Y.R. Discussion on Surrounding Rock Pressure Theory of Circular Cavern. *Undergr. Eng.* **1979**, *3*, 1–17.
14. Park, K.H. Large Strain Similarity Solution for a Spherical or Circular Opening Excavated in Elastic-perfectly Plastic Media. *Int. J. Numer. Anal. Methods Geomech.* **2015**, *39*, 724. [CrossRef]
15. Ren, Q.W.; Zhang, H.C. A Modification of Fenner Formula. *J. Hohai Univ.* **2001**, *29*, 109.
16. Li, M.; Mao, X.B.; Yu, Y.L.; Li, K.; Ma, C.; Peng, Y. Stress and Deformation Analysis on Deep Surrounding Rock at Different Time Stages and Its Application. *Int. J. Min. Sci. Technol.* **2012**, *22*, 301–306. [CrossRef]
17. Liu, K.; Chen, S.L.; Gu, X.Q. Analytical and Numerical Analyses of Tunnel Excavation Problem using an Extended Drucker-Prager Model. *Rock Mech. Rock Eng.* **2020**, *53*, 1777–1790. [CrossRef]
18. Pan, Y.; Zhao, G.M.; Meng, X.R. Elastro-plastrc Research of Surrounding Rock based on Hoek-Brown Strength Criterion. *J. Eng. Geol.* **2011**, *19*, 637.
19. Zou, J.F.; Su, Y. Theoretical Solutions of a Circular Tunnel with the Influence of the Out-of-plane Stress based on the Generalized Hoek–Brown Failure Criterion. *Int. J. Geomech.* **2016**, *16*, 06015006.
20. Wang, R.; Deng, X.H.; Meng, Y.Y.; Yuan, D.Y.; Xia, D.H. Case Study of Modified H-B Strength Criterion in Discrimination of Surrounding Rock Loose Circle. *Korean Soc. J. Civ. Eng.* **2019**, *23*, 1395. [CrossRef]
21. Ranjbarnia, M.; Rahimpour, N.; Oreste, P. A New Analytical-Numerical Solution to Analyze a Circular Tunnel Using 3D Hoek-Brown Failure Criterion. *Geomech. Eng.* **2020**, *22*, 11–23.
22. Sharan, S.K. Exact and Approximate Solutions for Displacements around Circular Openings in Elastic-brittle-plastic Hoek-Brown Rock. *Int. J. Rock Mech. Min. Sci.* **2005**, *42*, 542. [CrossRef]
23. Park, K.H.; Kim, Y.J. Analytical Solution for a Circular Opening in an Elastic-brittle-plastic Rock. *Int. J. Rock Mech. Min. Sci.* **2006**, *43*, 616. [CrossRef]
24. Serrano, A.; Olalla, C.; Reig, I. Convergence of Circular Tunnels in Elastoplastic Rock Masses with Non-linear Failure Criteria and Non-associated Flow Laws. *Int. J. Rock Mech. Min. Sci.* **2011**, *48*, 878–887. [CrossRef]
25. Xie, X.; Qi, L.; Li, X.M. Deformation, strength and water variation characteristics of unsaturated compacted loess. *Case Stud. Constr. Mater.* **2022**, *16*, e01129. [CrossRef]
26. Li, R.J.; Liu, J.D.; Yan, R.; Zheng, W.; Shao, S.J. Evaluation of Loess Landslide Disaster Based on the Developed Hyperbola Strength of Structural Loess. *Disaster Adv.* **2013**, *6*, 316.
27. Li, R.J.; Liu, J.D.; Yan, R.; Zheng, W.; Shao, S.J. Characteristics of Structural Loess Strength and Preliminary Framework for Joint Strength Formula. *Water Sci. Eng.* **2014**, *7*, 319.
28. Sun, P.; Li, R.J.; Igwe, O.; Luo, H. A new formula of loess earth pressures based on the joint strength theory. *Bulg. Chem. Commun.* **2017**, *49*, 78–82.
29. Liang, Q.G.; Li, J.; Wu, X.Y.; Zhou, A.N. Anisotropy of Q(2) loess in the Baijiapo Tunnel on the Lanyu Railway, China. *Bull. Eng. Geol. Environ.* **2016**, *75*, 109–124. [CrossRef]