The minimum 1-day, 7-days and 30-days data are presented in table S1.

Table S1. The observed data from Prigor hydrometric station

minimum 1-day low flows

		1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
Flow	[m³/s]	0.257	0.335	0.250	0.249	0.170	0.310	0.235	0.796	0.400	0.500	0.258
		2001	**2002**	**2003**	**2004**	**2005**	**2006**	**2007**	**2008**	**2009**	**2010**	**2011**
Flow	[m³/s]	0.268	0.330	0.294	0.300	0.450	0.610	0.450	0.241	0.350	0.520	0.193
		2012	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**	**2019**	**2020**		
Flow	[m³/s]	0.216	0.173	0.687	0.180	0.388	0.207	0.182	0.185	0.390		

minimum 7-days low flows

		1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
Flow	[m³/s]	0.270	0.442	0.250	0.295	0.180	0.310	0.292	0.811	0.501	0.589	0.300
		2001	**2002**	**2003**	**2004**	**2005**	**2006**	**2007**	**2008**	**2009**	**2010**	**2011**
Flow	[m³/s]	0.362	0.369	0.295	0.363	0.684	0.738	0.450	0.241	0.350	0.609	0.224
		2012	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**	**2019**	**2020**		
Flow	[m³/s]	0.257	0.192	0.713	0.193	0.428	0.214	0.199	0.201	0.390		

minimum 30-days low flows

		1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
Flow	[m³/s]	0.280	0.577	0.326	0.357	0.268	0.477	0.380	1.085	0.733	0.690	0.344
		2001	**2002**	**2003**	**2004**	**2005**	**2006**	**2007**	**2008**	**2009**	**2010**	**2011**
Flow	[m³/s]	0.499	0.470	0.320	0.500	0.797	0.771	0.696	0.255	0.364	1.014	0.264
		2012	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**	**2019**	**2020**		
Flow	[m³/s]	0.273	0.254	0.847	0.270	0.445	0.261	0.233	0.241	0.408		

The values of the main statistical indicators presented in table S2.

Table S2. The statistical indicators of the observed values

Low flows	Statistical indicators											
	μ	σ	C_v	C_s	C_k	L_1	L_2	L_3	L_4	τ_2	τ_3	τ_4
	[m³/s]	[m³/s]	[-]	[-]	[-]	[m³/s]	[m³/s]	[m³/s]	[m³/s]	[-]	[-]	[-]
minimum 1-day	0.335	0.157	0.469	1.359	4.593	0.335	0.084	0.026	0.013	0.251	0.307	0.152
minimum 7-day	0.378	0.179	0.475	1.043	3.10	0.378	0.098	0.028	0.01	0.260	0.282	0.102
minimum 30-day	0.474	0.240	0.507	1.086	3.266	0.474	0.131	0.041	0.009	0.276	0.310	0.07

The deficit volumes calculated for the thresholds of 95%, 90%, 80% are presented in table S3.

Table S3. The annual volumes deficit

95% threshold

		1990	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
W	[1000 m³]	28	0.0	49.6	0.4	164.4	0.0	4.4	0.0	0.0	0.0	1.4
		2001	**2002**	**2003**	**2004**	**2005**	**2006**	**2007**	**2008**	**2009**	**2010**	**2011**
W	[1000 m³]	1.5	0.0	0.0	0.0	0.0	0.0	0.0	131.7	0.0	0.0	100.8

		2012	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**	**2019**	**2020**
W	[1000 m³]	82	133.8	0.0	179.7	0.0	233	141.1	122.8	0.0

90% threshold

		1990	**1991**	**1992**	**1993**	**1994**	**1995**	**1996**	**1997**	**1998**	**1999**	**2000**
W	[1000 m³]	344.5	0.0	228.5	106.9	506.8	77.1	51.9	0.0	0.0	0.0	116.7
		2001	**2002**	**2003**	**2004**	**2005**	**2006**	**2007**	**2008**	**2009**	**2010**	**2011**
W	[1000 m³]	13	6.3	146.6	12.8	0.0	0.0	0.0	596.1	7.0	0.0	409.7
		2012	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**	**2019**	**2020**		
W	[1000 m³]	897.2	553.4	0.0	437	0.0	739.2	450.3	326.8	0.0		

80% threshold

		1990	**1991**	**1992**	**1993**	**1994**	**1995**	**1996**	**1997**	**1998**	**1999**	**2000**
W	[1000 m³]	1741.2	80.9	1099.5	1448.6	1742	644.1	411.6	0.0	4.6	0.0	1117.5
		2001	**2002**	**2003**	**2004**	**2005**	**2006**	**2007**	**2008**	**2009**	**2010**	**2011**
W	[1000 m³]	428.4	350.3	1057.5	577.6	3.2	0.0	39.6	2023	757.6	0.0	1592.8
		2012	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**	**2019**	**2020**		
W	[1000 m³]	3511.8	1907.7	0.0	1276.2	359.1	2402.3	1642.2	1338.4	366.9		

The drought durations determined for the thresholds of 95%, 90%, 80% are presented in table S4.

Table S4. The annual drought durations

95% threshold

		1990	**1991**	**1992**	**1993**	**1994**	**1995**	**1996**	**1997**	**1998**	**1999**	**2000**
D	[days/year]	23	0.0	23	2.0	54	0.0	1.0	0.0	0.0	0.0	2.0
		2001	**2002**	**2003**	**2004**	**2005**	**2006**	**2007**	**2008**	**2009**	**2010**	**2011**
D	[days/year]	1.0	0.0	0.0	0.0	0.0	0.0	0.0	59	0.0	0.0	34
		2012	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**	**2019**	**2020**		
D	[days/year]	59	39	0.0	35	0.0	67	33	27	0.0		

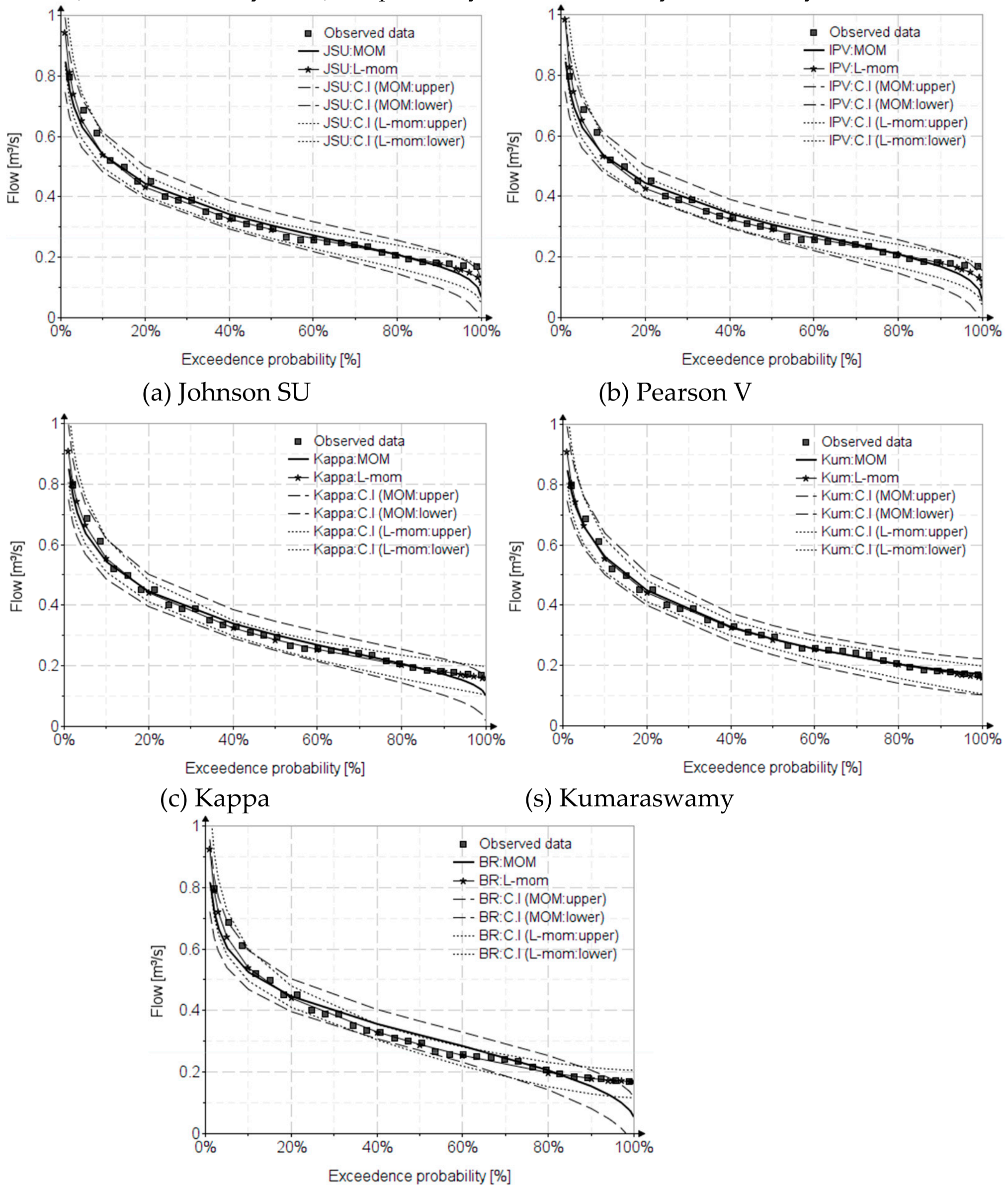
90% threshold

		1990	**1991**	**1992**	**1993**	**1994**	**1995**	**1996**	**1997**	**1998**	**1999**	**2000**
D	[days/year]	99	0.0	40	21	68	21	10	0.0	0.0	0.0	29
		2001	**2002**	**2003**	**2004**	**2005**	**2006**	**2007**	**2008**	**2009**	**2010**	**2011**
D	[days/year]	2.0	5.0	42	11	0.0	0.0	0.0	94	16	0.0	69
		2012	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**	**2019**	**2020**		
D	[days/year]	179	86	0.0	48	0.0	99	64	54	0.0		

80% threshold

		1990	**1991**	**1992**	**1993**	**1994**	**1995**	**1996**	**1997**	**1998**	**1999**	**2000**
D	[days/year]	122	15	105	147	133	57	42	0.0	1.0	0.0	142
		2001	**2002**	**2003**	**2004**	**2005**	**2006**	**2007**	**2008**	**2009**	**2010**	**2011**
D	[days/year]	62	41	92	73	1.0	0.0	11	131	89	0.0	118
		2012	**2013**	**2014**	**2015**	**2016**	**2017**	**2018**	**2019**	**2020**		
D	[days/year]	218	117	0.0	84	81	163	126	95	57		

Figure S1 shows the frequency curves using the annual minimum flow (minimum 1-day flow), respectively minimum 7-days and 30-days.



(a) Johnson SU (b) Pearson V

(c) Kappa (s) Kumaraswamy

(e) Burr

Figure S1. The probability distributions curves for minimum 1-day low flows

For the transparency of the verification of the graphically presented results, in the following section all the equations necessary for the application of these distributions are presented, for both parameter estimation methods.

Johnson SU (JSU)

The family of Johnson distribution includes four statistical distributions, namely the Johnson bounded distribution (SB), the Johnson unbounded distribution (SU), the Johnson semi-bounded distribution (SL) and the

Johnson normal distribution (SN). In this article, only the Johnson SU distribution will be presented, further denoted by JSU [1,2].

$$f(x)=\frac{\lambda}{\sqrt{(x-\gamma)^2+\beta^2}}\cdot\frac{1}{\sqrt{2\cdot\pi}}\cdot\exp\left(-\frac{1}{2}\cdot\left(\lambda\cdot a\sinh\left(\frac{x-\lambda}{\beta}\right)+\alpha\right)^2\right) \tag{1}$$

$$F(x)=1-\text{pnorm}\left(a\sinh\left(\frac{x-\gamma}{\beta}\right),-\frac{\alpha}{\lambda},\frac{1}{\lambda}\right) \tag{2}$$

$$\begin{aligned} x(p)=\gamma+\beta\cdot\frac{\exp\left(\text{qnorm}\left(1-p,-\frac{\alpha}{\lambda},\frac{1}{\lambda}\right)\right)+\exp\left(-\text{qnorm}\left(1-p,-\frac{\alpha}{\lambda},\frac{1}{\lambda}\right)\right)}{2}= \\ \lambda+\beta\cdot\sinh\left(\text{qnorm}\left(1-p,-\frac{\alpha}{\lambda},\frac{1}{\lambda}\right)\right) \end{aligned} \tag{3}$$

where α,λ are the shape parameters and β,λ are the scale and the position parameters; x can take any values of range $\gamma<x<\infty$.

The equations needed to estimate the parameters with MOM have the following expressions:

$$\mu=\gamma-\beta\cdot\exp\left(\frac{1}{2\cdot\lambda^2}\right)\cdot\sinh\left(\frac{\alpha}{\lambda}\right) \tag{4}$$

$$\sigma^2=\beta^2\cdot\frac{\exp\left(\frac{1}{\lambda^2}\right)-1}{2}\cdot\left(\exp\left(\frac{1}{\lambda^2}\right)\cdot\cosh\left(2\cdot\frac{\alpha}{\lambda}\right)+1\right) \tag{5}$$

$$C_s=\frac{\exp\left(\frac{1}{2\cdot\lambda^2}\right)\cdot\sqrt{\exp\left(\frac{1}{\lambda^2}\right)-1}\cdot\left(3\cdot\sinh\left(\frac{\alpha}{\lambda}\right)+\exp\left(\frac{1}{\lambda^2}\right)\cdot\left(\exp\left(\frac{1}{\lambda^2}\right)+2\right)\cdot\sinh\left(\frac{3\cdot\alpha}{\lambda}\right)\right)}{\sqrt{2}\cdot\left(1+\exp\left(\frac{1}{\lambda^2}\right)\cdot\cosh\left(\frac{2\cdot\alpha}{\lambda}\right)\right)^{1.5}} \tag{6}$$

$$C_k=\frac{\begin{gathered}3+6\cdot\exp\left(\frac{1}{\lambda^2}\right)+4\cdot\exp\left(\frac{2}{\lambda^2}\right)\cdot\left(\exp\left(\frac{1}{\lambda^2}\right)+2\right)\cdot\cosh\left(\frac{2\cdot\alpha}{\lambda}\right)+ \\ \exp\left(\frac{2}{\lambda^2}\right)\cdot\left(3\cdot\exp\left(\frac{2}{\lambda^2}\right)+2\cdot\exp\left(\frac{3}{\lambda^2}\right)+\exp\left(\frac{4}{\lambda^2}\right)-3\right)\cdot\cosh\left(\frac{4\cdot\alpha}{\lambda}\right)\end{gathered}}{2\cdot\left(1+\exp\left(\frac{1}{\lambda^2}\right)\cdot\cosh\left(\frac{2\cdot\alpha}{\lambda}\right)\right)^2} \tag{7}$$

The parameter estimation with the L-moment method is done numerically (definite integrals) based on the equations using the quantile of the function.

$$L_1=\int_0^1\gamma+\beta\cdot\sinh\left(\text{qnorm}\left(1-p,-\frac{\alpha}{\lambda},\frac{1}{\lambda}\right)\right)\cdot dp \tag{8}$$

$$L_2=\int_0^1\left(\gamma+\beta\cdot\sinh\left(\text{qnorm}\left(1-p,-\frac{\alpha}{\lambda},\frac{1}{\lambda}\right)\right)\right)\cdot(1-2\cdot p)\cdot dp \tag{9}$$

$$L_3=\int_0^1\left(\gamma+\beta\cdot\sinh\left(\text{qnorm}\left(1-p,-\frac{\alpha}{\lambda},\frac{1}{\lambda}\right)\right)\right)\cdot\left(6\cdot p^2-6\cdot p+1\right)\cdot dp \tag{10}$$

$$L_4=\int_0^1\left(\gamma+\beta\cdot\sinh\left(\text{qnorm}\left(1-p,-\frac{\alpha}{\lambda},\frac{1}{\lambda}\right)\right)\right)\cdot\left(1-12\cdot p+30\cdot p^2-20\cdot p^3\right)\cdot dp \tag{11}$$

Kappa (KAP)

This distribution was introduced in 1994 by Hosking [34], representing a generalized form of the three-parameter Mielke-Johnson Kappa distribution. As particular forms of it, we find the following distributions: Generalized Pareto ($\lambda=1$), GEV ($\lambda=0$), Generalized Logistic ($\lambda=-1$), Exponential ($\lambda=1,\alpha=0$), Gumbel ($\lambda=0,\alpha=0$), Logistic ($\lambda=-1,\alpha=0$), Uniform ($\lambda=1,\alpha=1$), Burr type III ($\lambda,\alpha<0$), Inverse Burr type XII ($\lambda<0,\alpha>0$) [3,4].

$$f(x)=\frac{1}{\beta}\cdot\left(1-\frac{\alpha}{\beta}\cdot(x-\gamma)\right)^{\frac{1}{\alpha}-1}\cdot\left(1-\lambda\cdot\left(1-\frac{1}{\beta}\cdot(x-\gamma)\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{\lambda}-1} \tag{12}$$

$$F(x)=1-\left(1-\lambda\cdot\left(1-\frac{\alpha}{\beta}\cdot(x-\gamma)\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{\lambda}} \tag{13}$$

$$x(p)=\gamma+\frac{\beta}{\alpha}\cdot\left(1-\left(\frac{1-(1-p)^{\lambda}}{\lambda}\right)^{\alpha}\right) \tag{14}$$

where α,λ are the shape parameters; β,γ the scale and the position parameters; $\alpha,\beta,\lambda>0$; the following conditions are imposed: $\gamma+\frac{\beta}{\alpha}\cdot\left(1-\lambda^{-\alpha}\right)\le x\le\gamma+\frac{\beta}{\alpha}$ if $\lambda,\alpha>0$; $\gamma+\frac{\beta}{\alpha}\cdot\left(1-\lambda^{-\alpha}\right)\le x<\infty$ if $\lambda>0,\alpha<0$; $-\infty<x<\gamma+\frac{\beta}{\alpha}$ if $\lambda\le 0,\alpha>0$; $\gamma+\frac{\beta}{\alpha}\le x<\infty$ if $\lambda\le 0,\alpha<0$.

The equations needed to estimate the parameters with MOM have the following expressions [3,4,5]:

if $\lambda>0$:

$$\mu=\gamma+\frac{\beta}{\alpha}\cdot\left(1+\frac{\Gamma(1+\alpha)\cdot\Gamma\left(\frac{1}{\lambda}\right)}{\lambda^{1+\alpha}\cdot\Gamma\left(1+\alpha+\frac{1}{\lambda}\right)}\right) \tag{15}$$

$$\sigma^2=\frac{\beta^2}{\alpha^2}\cdot\left(\frac{2\cdot\alpha\cdot\Gamma(2\cdot\alpha)\cdot\Gamma\left(\frac{1}{\lambda}\right)}{\lambda^{1+2\cdot\alpha}\cdot\Gamma\left(1+2\cdot\alpha+\frac{1}{\lambda}\right)}-\left(\frac{\Gamma(1+\alpha)\cdot\Gamma\left(\frac{1}{\lambda}\right)}{\lambda^{1+\alpha}\cdot\Gamma\left(1+\alpha+\frac{1}{\lambda}\right)}\right)^2\right) \tag{16}$$

$$C_s=\frac{\frac{3\cdot\Gamma(1+2\cdot\alpha)\cdot\Gamma\left(\frac{1}{\lambda}\right)}{\lambda^{1+2\cdot\alpha}\cdot\Gamma\left(1+2\cdot\alpha+\frac{1}{\lambda}\right)}\cdot\frac{\Gamma(1+\alpha)\cdot\Gamma\left(\frac{1}{\lambda}\right)}{\lambda^{1+\alpha}\cdot\Gamma\left(1+\alpha+\frac{1}{\lambda}\right)}-\frac{\Gamma(1+3\cdot\alpha)\cdot\Gamma\left(\frac{1}{\lambda}\right)}{\lambda^{1+3\cdot\alpha}\cdot\Gamma\left(1+3\cdot\alpha+\frac{1}{\lambda}\right)}+2\cdot\left(\frac{\Gamma(1+\alpha)\cdot\Gamma\left(\frac{1}{\lambda}\right)}{\lambda^{1+\alpha}\cdot\Gamma\left(1+\alpha+\frac{1}{\lambda}\right)}\right)^3}{\left(\frac{\Gamma(1+2\cdot\alpha)\cdot\Gamma\left(\frac{1}{\lambda}\right)}{\lambda^{1+2\cdot\alpha}\cdot\Gamma\left(1+2\cdot\alpha+\frac{1}{\lambda}\right)}-\left(\frac{\Gamma(1+\alpha)\cdot\Gamma\left(\frac{1}{\lambda}\right)}{\lambda^{1+\alpha}\cdot\Gamma\left(1+\alpha+\frac{1}{\lambda}\right)}\right)^2\right)^{3/2}} \tag{17}$$

$$C_k = \frac{\frac{\Gamma(1+4\cdot\alpha)\cdot\Gamma\left(\frac{1}{\lambda}\right)}{\lambda^{1+4\cdot\alpha}\cdot\Gamma\left(1+4\cdot\alpha+\frac{1}{\lambda}\right)} - 4\cdot\frac{\Gamma(1+3\cdot\alpha)\cdot\Gamma\left(\frac{1}{\lambda}\right)}{\lambda^{1+3\cdot\alpha}\cdot\Gamma\left(1+3\cdot\alpha+\frac{1}{\lambda}\right)}\cdot\frac{\Gamma(1+\alpha)\cdot\Gamma\left(\frac{1}{\lambda}\right)}{\lambda^{1+\alpha}\cdot\Gamma\left(1+\alpha+\frac{1}{\lambda}\right)} + 6\cdot\frac{\Gamma(1+2\cdot\alpha)\cdot\Gamma\left(\frac{1}{\lambda}\right)}{\lambda^{1+2\cdot\alpha}\cdot\Gamma\left(1+2\cdot\alpha+\frac{1}{\lambda}\right)}\cdot\left(\frac{\Gamma(1+\alpha)\cdot\Gamma\left(\frac{1}{\lambda}\right)}{\lambda^{1+\alpha}\cdot\Gamma\left(1+\alpha+\frac{1}{\lambda}\right)}\right)^2 - 3\cdot\left(\frac{\Gamma(1+\alpha)\cdot\Gamma\left(\frac{1}{\lambda}\right)}{\lambda^{1+\alpha}\cdot\Gamma\left(1+\alpha+\frac{1}{\lambda}\right)}\right)^4}{\left(\frac{\Gamma(1+2\cdot\alpha)\cdot\Gamma\left(\frac{1}{\lambda}\right)}{\lambda^{1+2\cdot\alpha}\cdot\Gamma\left(1+2\cdot\alpha+\frac{1}{\lambda}\right)} - \left(\frac{\Gamma(1+\alpha)\cdot\Gamma\left(\frac{1}{\lambda}\right)}{\lambda^{1+\alpha}\cdot\Gamma\left(1+\alpha+\frac{1}{\lambda}\right)}\right)^2\right)^2} \tag{18}$$

if $\lambda < 0$:

$$\mu = \gamma + \frac{\beta}{\alpha}\cdot\left(1 + \frac{\Gamma(1+\alpha)\cdot\Gamma\left(-\alpha-\frac{1}{\lambda}\right)}{(-\lambda)^{1+\alpha}\cdot\Gamma\left(1-\frac{1}{\lambda}\right)}\right) \tag{19}$$

$$\sigma^2 = \frac{\beta^2}{\alpha^2}\cdot\left(\frac{\Gamma(1+2\cdot\alpha)\cdot\Gamma\left(-2\cdot\alpha-\frac{1}{\lambda}\right)}{(-\lambda)^{1+2\cdot\alpha}\cdot\Gamma\left(1-\frac{1}{\lambda}\right)} - \left(\frac{\Gamma(1+\alpha)\cdot\Gamma\left(-\alpha-\frac{1}{\lambda}\right)}{(-\lambda)^{1+\alpha}\cdot\Gamma\left(1-\frac{1}{\lambda}\right)}\right)^2\right) \tag{20}$$

$$C_s = \frac{\frac{3\cdot\Gamma(1+2\cdot\alpha)\cdot\Gamma\left(-2\cdot\alpha-\frac{1}{\lambda}\right)}{(-\lambda)^{1+2\cdot\alpha}\cdot\Gamma\left(1-\frac{1}{\lambda}\right)}\cdot\frac{\Gamma(1+\alpha)\cdot\Gamma\left(-\alpha-\frac{1}{\lambda}\right)}{(-\lambda)^{1+\alpha}\cdot\Gamma\left(1-\frac{1}{\lambda}\right)} - \frac{\Gamma(1+3\cdot\alpha)\cdot\Gamma\left(-3\cdot\alpha-\frac{1}{\lambda}\right)}{(-\lambda)^{1+3\cdot\alpha}\cdot\Gamma\left(1-\frac{1}{\lambda}\right)} + 2\cdot\left(\frac{\Gamma(1+\alpha)\cdot\Gamma\left(-\alpha-\frac{1}{\lambda}\right)}{(-\lambda)^{1+\alpha}\cdot\Gamma\left(1-\frac{1}{\lambda}\right)}\right)^3}{\left(\frac{\Gamma(1+2\cdot\alpha)\cdot\Gamma\left(-2\cdot\alpha-\frac{1}{\lambda}\right)}{(-\lambda)^{1+2\cdot\alpha}\cdot\Gamma\left(1-\frac{1}{\lambda}\right)} - \left(\frac{\Gamma(1+\alpha)\cdot\Gamma\left(-\alpha-\frac{1}{\lambda}\right)}{(-\lambda)^{1+\alpha}\cdot\Gamma\left(1-\frac{1}{\lambda}\right)}\right)^2\right)^{3/2}} \tag{21}$$

$$C_k = \frac{\frac{\Gamma(1+4\cdot\alpha)\cdot\Gamma\left(-4\cdot\alpha-\frac{1}{\lambda}\right)}{(-\lambda)^{1+4\cdot\alpha}\cdot\Gamma\left(1-\frac{1}{\lambda}\right)} - 4\cdot\frac{\Gamma(1+3\cdot\alpha)\cdot\Gamma\left(-3\cdot\alpha-\frac{1}{\lambda}\right)}{(-\lambda)^{1+3\cdot\alpha}\cdot\Gamma\left(1-\frac{1}{\lambda}\right)}\cdot\frac{\Gamma(1+\alpha)\cdot\Gamma\left(-\alpha-\frac{1}{\lambda}\right)}{(-\lambda)^{1+\alpha}\cdot\Gamma\left(1-\frac{1}{\lambda}\right)} + 6\cdot\frac{\Gamma(1+2\cdot\alpha)\cdot\Gamma\left(-2\cdot\alpha-\frac{1}{\lambda}\right)}{(-\lambda)^{1+2\cdot\alpha}\cdot\Gamma\left(1-\frac{1}{\lambda}\right)}\cdot\left(\frac{\Gamma(1+\alpha)\cdot\Gamma\left(-\alpha-\frac{1}{\lambda}\right)}{(-\lambda)^{1+\alpha}\cdot\Gamma\left(1-\frac{1}{\lambda}\right)}\right)^2 - 3\cdot\left(\frac{\Gamma(1+\alpha)\cdot\Gamma\left(-\alpha-\frac{1}{\lambda}\right)}{(-\lambda)^{1+\alpha}\cdot\Gamma\left(1-\frac{1}{\lambda}\right)}\right)^4}{\left(\frac{\Gamma(1+2\cdot\alpha)\cdot\Gamma\left(-2\cdot\alpha-\frac{1}{\lambda}\right)}{(-\lambda)^{1+2\cdot\alpha}\cdot\Gamma\left(1-\frac{1}{\lambda}\right)} - \left(\frac{\Gamma(1+\alpha)\cdot\Gamma\left(-\alpha-\frac{1}{\lambda}\right)}{(-\lambda)^{1+\alpha}\cdot\Gamma\left(1-\frac{1}{\lambda}\right)}\right)^2\right)^2} \tag{22}$$

The equations needed to estimate the parameters with L-moments have the following expressions [3,4,5]:

if $\lambda > 0$:

$$L_1 = \gamma + \frac{\beta}{\alpha} \cdot \left(1 + \frac{\Gamma(1+\alpha) \cdot \Gamma\left(\frac{1}{\lambda}\right)}{\lambda^{1+\alpha} \cdot \Gamma\left(1+\alpha+\frac{1}{\lambda}\right)}\right) \tag{23}$$

$$L_2 = \frac{\beta \cdot \Gamma(\alpha)}{\lambda^{1+\alpha}}\left(\frac{\Gamma\left(\frac{1}{\lambda}\right)}{\Gamma\left(1+\alpha+\frac{1}{\lambda}\right)} - \frac{2 \cdot \Gamma\left(\frac{2}{\lambda}\right)}{\Gamma\left(1+\alpha+\frac{2}{\lambda}\right)}\right) \tag{24}$$

$$L_3 = \frac{\beta \cdot \Gamma(\alpha)}{\lambda^{1+\alpha}} \cdot \left(\frac{6 \cdot \Gamma\left(\frac{2}{\lambda}\right)}{\Gamma\left(1+\alpha+\frac{2}{\lambda}\right)} - \frac{6 \cdot \Gamma\left(\frac{3}{\lambda}\right)}{\Gamma\left(1+\alpha+\frac{3}{\lambda}\right)} - \frac{\Gamma\left(\frac{1}{\lambda}\right)}{\Gamma\left(1+\alpha+\frac{1}{\lambda}\right)}\right) \tag{25}$$

$$L_4 = \frac{\beta \cdot \Gamma(\alpha)}{\lambda^{1+\alpha}} \cdot \left(\frac{\Gamma\left(\frac{1}{\lambda}\right)}{\Gamma\left(1+\alpha+\frac{1}{\lambda}\right)} - \frac{12 \cdot \Gamma\left(\frac{2}{\lambda}\right)}{\Gamma\left(1+\alpha+\frac{2}{\lambda}\right)} + \frac{30 \cdot \Gamma\left(\frac{3}{\lambda}\right)}{\Gamma\left(1+\alpha+\frac{3}{\lambda}\right)} - \frac{20 \cdot \Gamma\left(\frac{4}{\lambda}\right)}{\Gamma\left(1+\alpha+\frac{4}{\lambda}\right)}\right) \tag{26}$$

if $\lambda < 0$:

$$L_1 = \gamma + \frac{\beta}{\alpha} \cdot \left(1 + \frac{\Gamma(1+\alpha) \cdot \Gamma\left(-\alpha-\frac{1}{\lambda}\right)}{(-\lambda)^{1+\alpha} \cdot \Gamma\left(1-\frac{1}{\lambda}\right)}\right) \tag{27}$$

$$L_2 = \frac{\beta \cdot \Gamma(\alpha)}{(-\lambda)^{1+\alpha}} \cdot \left(\frac{\Gamma\left(-\alpha-\frac{1}{\lambda}\right)}{\Gamma\left(1-\frac{1}{\lambda}\right)} - \frac{\Gamma\left(-\alpha-\frac{2}{\lambda}\right)}{\Gamma\left(1-\frac{2}{\lambda}\right)}\right) \tag{28}$$

$$L_3 = \frac{\beta \cdot \Gamma(\alpha)}{(-\lambda)^{1+\alpha}} \cdot \left(\frac{6 \cdot \Gamma\left(-\alpha-\frac{2}{\lambda}\right)}{\Gamma\left(1-\frac{2}{\lambda}\right)} - \frac{6 \bullet \Gamma\left(-\alpha-\frac{3}{\lambda}\right)}{\Gamma\left(1-\frac{3}{\lambda}\right)} - \frac{\Gamma\left(-\alpha-\frac{1}{\lambda}\right)}{\Gamma\left(1-\frac{1}{\lambda}\right)}\right) \tag{29}$$

$$L_4 = \frac{\beta \cdot \Gamma(\alpha)}{(-\lambda)^{1+\alpha}} \cdot \left(\frac{\Gamma\left(-\alpha-\frac{1}{\lambda}\right)}{\Gamma\left(1-\frac{1}{\lambda}\right)} - \frac{12 \cdot \Gamma\left(-\alpha-\frac{2}{\lambda}\right)}{\Gamma\left(1-\frac{2}{\lambda}\right)} + \frac{30 \cdot \Gamma\left(-\alpha-\frac{3}{\lambda}\right)}{\Gamma\left(1-\frac{3}{\lambda}\right)} - \frac{20 \cdot \Gamma\left(-\alpha-\frac{4}{\lambda}\right)}{\Gamma\left(1-\frac{4}{\lambda}\right)}\right) \tag{30}$$

Burr (BR)

This distribution represents a generalized case of the Burr Type XII distribution by adding a position parameter. When $\gamma = 0$ we have the Burr Type XII distribution (Dagum, Beta_k, inverse Burr III); when $\gamma = 0$ and $\beta = 1$ we have the 2 parameter Burr Type XII distribution; when $\gamma = 0$ and $\beta = -1$

we have the Lomax distribution; when $\alpha=1$ we have the Log-logistic distrinution [6,7].

$$f(x)=\frac{\frac{\beta\cdot\alpha}{x-\gamma}\cdot\left(\frac{\lambda}{x-\gamma}\right)^{\beta}}{\left(1+\left(\frac{\lambda}{x-\gamma}\right)^{\beta}\right)^{\alpha+1}} \tag{31}$$

$$F(x)=1-\frac{1}{\left(1+\left(\frac{\lambda}{x-\gamma}\right)^{\beta}\right)^{\alpha}} \tag{32}$$

$$x(p)=\gamma+\lambda\cdot\left(\frac{1}{\left(\frac{1}{1-p}\right)^{\frac{1}{\alpha}}-1}\right)^{\frac{1}{\beta}} \tag{33}$$

where α,β are the shape parameters; λ,γ the scale and the position parameters; $\alpha,\beta,\lambda>0$; x can take any values in the range $\gamma<x<\infty$.

The equations needed to estimate the parameters with MOM have the following expressions:

$$\mu=\gamma+\lambda\cdot\frac{\Gamma\left(1-\frac{1}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{1}{\beta}\right)}{\Gamma(\alpha)} \tag{34}$$

$$\sigma^2=\frac{\lambda^2}{\Gamma(\alpha)}\cdot\left(\Gamma\left(1-\frac{2}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{2}{\beta}\right)-\frac{\Gamma\left(1-\frac{1}{\beta}\right)^2\cdot\Gamma\left(\alpha+\frac{1}{\beta}\right)^2}{\Gamma(\alpha)}\right) \tag{35}$$

$$C_s=\frac{\frac{\Gamma\left(1-\frac{3}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{3}{\beta}\right)}{\Gamma(\alpha)}-3\cdot\frac{\Gamma\left(1-\frac{2}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{2}{\beta}\right)}{\Gamma(\alpha)}\cdot\frac{\Gamma\left(1-\frac{1}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{1}{\beta}\right)}{\Gamma(\alpha)}+2\cdot\left(\frac{\Gamma\left(1-\frac{1}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{1}{\beta}\right)}{\Gamma(\alpha)}\right)^3}{\left(\frac{\Gamma\left(1-\frac{2}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{2}{\beta}\right)}{\Gamma(\alpha)}-\left(\frac{\Gamma\left(1-\frac{1}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{1}{\beta}\right)}{\Gamma(\alpha)}\right)^2\right)^{3/2}} \tag{36}$$

$$C_k=\frac{\frac{\Gamma\left(1-\frac{4}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{4}{\beta}\right)}{\Gamma(\alpha)}-4\cdot\frac{\Gamma\left(1-\frac{3}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{3}{\beta}\right)}{\Gamma(\alpha)}\cdot\frac{\Gamma\left(1-\frac{1}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{1}{\beta}\right)}{\Gamma(\alpha)}+6\cdot\frac{\Gamma\left(1-\frac{2}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{2}{\beta}\right)}{\Gamma(\alpha)}\cdot\left(\frac{\Gamma\left(1-\frac{1}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{1}{\beta}\right)}{\Gamma(\alpha)}\right)^2-3\cdot\left(\frac{\Gamma\left(1-\frac{1}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{1}{\beta}\right)}{\Gamma(\alpha)}\right)^4}{\left(\frac{\Gamma\left(1-\frac{2}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{2}{\beta}\right)}{\Gamma(\alpha)}-\left(\frac{\Gamma\left(1-\frac{1}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{1}{\beta}\right)}{\Gamma(\alpha)}\right)^2\right)^2} \quad (37)$$

The equations needed to estimate the parameters with L-moments have the following expressions:

$$L_1=\gamma+\frac{\lambda\cdot\Gamma\left(1-\frac{1}{\beta}\right)\cdot\Gamma\left(\alpha+\frac{1}{\beta}\right)}{\Gamma(\alpha)} \quad (38)$$

$$L_2=\lambda\cdot\Gamma\left(1-\frac{1}{\beta}\right)\cdot\left(\frac{\Gamma\left(2\cdot\alpha+\frac{1}{\beta}\right)}{\Gamma(2\cdot\alpha)}-\frac{\Gamma\left(\alpha+\frac{1}{\beta}\right)}{\Gamma(\alpha)}\right) \quad (39)$$

$$L_3=\lambda\cdot\Gamma\left(1-\frac{1}{\beta}\right)\cdot\left(\frac{2\cdot\Gamma\left(3\cdot\alpha+\frac{1}{\beta}\right)}{\Gamma(3\cdot\alpha)}-\frac{3\cdot\Gamma\left(2\cdot\alpha+\frac{1}{\beta}\right)}{\Gamma(2\cdot\alpha)}+\frac{\Gamma\left(\alpha+\frac{1}{\beta}\right)}{\Gamma(\alpha)}\right) \quad (40)$$

$$L_4=\lambda\cdot\Gamma\left(1-\frac{1}{\beta}\right)\cdot\left(\frac{5\cdot\Gamma\left(4\cdot\alpha+\frac{1}{\beta}\right)}{\Gamma(4\cdot\alpha)}-\frac{10\cdot\Gamma\left(3\cdot\alpha+\frac{1}{\beta}\right)}{\Gamma(3\cdot\alpha)}+\frac{6\cdot\Gamma\left(2\cdot\alpha+\frac{1}{\beta}\right)}{\Gamma(2\cdot\alpha)}-\frac{\Gamma\left(\alpha+\frac{1}{\beta}\right)}{\Gamma(\alpha)}\right) \quad (41)$$

Kumaraswamy (KUM)

The distribution represents a generalized form of the 2-parameter Kumaraswamy distribution, presented in 1980 by P.Kumaraswamy [8,9,10]. When $\lambda=1,\gamma=0$ we have two parameters Kumaraswamy distribution. It is closely related to the Gamma and the Beta family of distributions.

$$f(x)=\frac{\alpha\cdot\beta}{\lambda}\cdot\left(\frac{x-\gamma}{\lambda}\right)^{\alpha-1}\cdot\left(1-\left(\frac{x-\gamma}{\lambda}\right)^{\alpha}\right)^{\beta-1} \quad (42)$$

$$F(x)=1-\left(1-\left(\frac{x-\gamma}{\lambda}\right)^{\alpha}\right)^{\beta} \quad (43)$$

$$x(p)=\gamma+\lambda\cdot\left(1-p^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}} \quad (44)$$

where α,β are the shape parameters; γ is the position parameters; λ is the scale parameters; x can take any values in the range $\gamma<x<\infty$.

The equations needed to estimate the parameters with MOM have the following expressions [8,9]:

$$\mu = \gamma + \lambda \cdot \frac{\Gamma\left(1+\frac{1}{\alpha}\right)\cdot\Gamma(\beta+1)}{\Gamma\left(1+\beta+\frac{1}{\alpha}\right)} \tag{45}$$

$$\sigma^2 = \lambda^2 \cdot \left(\frac{\Gamma\left(1+\frac{2}{\alpha}\right)\cdot\Gamma(\beta+1)}{\Gamma\left(1+\beta+\frac{2}{\alpha}\right)} - \frac{\Gamma\left(1+\frac{1}{\alpha}\right)^2\cdot\Gamma(\beta+1)^2}{\Gamma\left(1+\beta+\frac{1}{\alpha}\right)^2}\right) \tag{46}$$

$$C_s = \frac{\frac{\Gamma\left(1+\frac{3}{\alpha}\right)\cdot\Gamma(\beta+1)}{\Gamma\left(1+\beta+\frac{3}{\alpha}\right)} - 3\cdot\frac{\Gamma\left(1+\frac{2}{\alpha}\right)\cdot\Gamma\left(1+\frac{1}{\alpha}\right)\cdot\Gamma(\beta+1)^2}{\Gamma\left(1+\beta+\frac{2}{\alpha}\right)\cdot\Gamma\left(1+\beta+\frac{1}{\alpha}\right)} + 2\cdot\frac{\Gamma\left(1+\frac{1}{\alpha}\right)^2\cdot\Gamma(\beta+1)^2}{\Gamma\left(1+\beta+\frac{1}{\alpha}\right)^2}}{\left(\frac{\Gamma\left(1+\frac{2}{\alpha}\right)\cdot\Gamma(\beta+1)}{\Gamma\left(1+\beta+\frac{2}{\alpha}\right)} - \frac{\Gamma\left(1+\frac{1}{\alpha}\right)^2\cdot\Gamma(\beta+1)^2}{\Gamma\left(1+\beta+\frac{1}{\alpha}\right)^2}\right)^{3/2}} \tag{47}$$

$$C_k = \frac{\frac{\Gamma\left(1+\frac{4}{\alpha}\right)\cdot\Gamma(\beta+1)}{\Gamma\left(1+\beta+\frac{4}{\alpha}\right)} - 4\cdot\frac{\Gamma\left(1+\frac{1}{\alpha}\right)\cdot\Gamma\left(1+\frac{3}{\alpha}\right)\cdot\Gamma(\beta+1)^2}{\Gamma\left(1+\beta+\frac{3}{\alpha}\right)\cdot\Gamma\left(1+\beta+\frac{1}{\alpha}\right)} + 6\cdot\frac{\Gamma\left(1+\frac{2}{\alpha}\right)\cdot\Gamma\left(1+\frac{1}{\alpha}\right)^2\cdot\Gamma(\beta+1)^3}{\Gamma\left(1+\beta+\frac{2}{\alpha}\right)\cdot\Gamma\left(1+\beta+\frac{1}{\alpha}\right)^2} - 3\cdot\frac{\Gamma\left(1+\frac{1}{\alpha}\right)^4\cdot\Gamma(\beta+1)^4}{\Gamma\left(1+\beta+\frac{1}{\alpha}\right)^4}}{\left(\frac{\Gamma\left(1+\frac{2}{\alpha}\right)\cdot\Gamma(\beta+1)}{\Gamma\left(1+\beta+\frac{2}{\alpha}\right)} - \frac{\Gamma\left(1+\frac{1}{\alpha}\right)^2\cdot\Gamma(\beta+1)^2}{\Gamma\left(1+\beta+\frac{1}{\alpha}\right)^2}\right)^2} \tag{48}$$

The equations needed to estimate the parameters with L-moments have the following expressions [7,8]:

$$L_1 = \gamma + \lambda \cdot \frac{\Gamma\left(1+\frac{1}{\alpha}\right)\cdot\Gamma(\beta+1)}{\Gamma\left(1+\beta+\frac{1}{\alpha}\right)} \tag{49}$$

$$L_2 = \beta\cdot\lambda\cdot\left(\frac{\Gamma\left(1+\frac{1}{\alpha}\right)\cdot\Gamma(\beta)}{\Gamma\left(1+\beta+\frac{1}{\alpha}\right)} - 2\cdot\frac{\Gamma\left(1+\frac{1}{\alpha}\right)\cdot\Gamma(2\cdot\beta)}{\Gamma\left(1+2\cdot\beta+\frac{1}{\alpha}\right)}\right) \tag{50}$$

$$L_3 = \beta\cdot\lambda\cdot\left(\frac{\Gamma\left(1+\frac{1}{\alpha}\right)\cdot\Gamma(\beta)}{\Gamma\left(1+\beta+\frac{1}{\alpha}\right)} + 6\cdot\frac{\Gamma\left(1+\frac{1}{\alpha}\right)\cdot\Gamma(3\cdot\beta)}{\Gamma\left(1+3\cdot\beta+\frac{1}{\alpha}\right)} - 6\cdot\frac{\Gamma\left(1+\frac{1}{\alpha}\right)\cdot\Gamma(2\cdot\beta)}{\Gamma\left(1+2\cdot\beta+\frac{1}{\alpha}\right)}\right) \tag{51}$$

$$L_4 = \beta\cdot\lambda\cdot\left(\frac{\Gamma\left(1+\frac{1}{\alpha}\right)\cdot\Gamma(\beta)}{\Gamma\left(1+\beta+\frac{1}{\alpha}\right)} - 20\cdot\frac{\Gamma\left(1+\frac{1}{\alpha}\right)\cdot\Gamma(4\cdot\beta)}{\Gamma\left(1+4\cdot\beta+\frac{1}{\alpha}\right)} + 30\cdot\frac{\Gamma\left(1+\frac{1}{\alpha}\right)\cdot\Gamma(3\cdot\beta)}{\Gamma\left(1+3\cdot\beta+\frac{1}{\alpha}\right)} - 12\cdot\frac{\Gamma\left(1+\frac{1}{\alpha}\right)\cdot\Gamma(2\cdot\beta)}{\Gamma\left(1+2\cdot\beta+\frac{1}{\alpha}\right)}\right) \tag{52}$$

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