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Review

A Comprehensive Exploration of Hellwig's Taxonomic Measure of Development and Its Modifications—A Systematic Review of Algorithms and Applications

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Abstract: This paper presents an original and comprehensive investigation into the Taxonomic Measure of Development (TMD), introduced by Hellwig in 1968, enriching both its theoretical foundations and practical applications. It provides an overview of various variants of the Hellwig method, including their extensions and applications, while also exploring recent trends across multiple research domains. Primarily developed as a method for multidimensional analysis, TMD has evolved into a pivotal tool in multi-criteria decision-making. It is widely used for evaluating and ranking alternatives, particularly in the analysis of complex socio-economic phenomena and decision-making scenarios involving multiple criteria. This study systematically reviews the original algorithm and its subsequent extensions and modifications, including adaptations for fuzzy sets, intuitionistic fuzzy sets, and interval-valued fuzzy sets. Furthermore, it explores an integrated multi-criteria approach based on Hellwig's method and its practical applications across various domains. This paper introduces an original approach by conducting a detailed, step-by-step analysis of the TMD framework. This process-oriented analysis is a novel contribution that sets this study apart from typical reviews based on statistical or bibliometric data. By examining key steps in the TMD framework—such as data collection, criterion weighting, data normalization, ideal value determination, distance calculation, and normalization factor—this paper highlights the method's versatility in addressing complex, real-world decision-making problems. Although similar to the widely used Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) method in its reliance on distance to evaluate alternatives, Hellwig's approach is unique in focusing exclusively on proximity to an ideal solution, without considering distance from a negative ideal. This distinctive emphasis has led to numerous adaptations and extensions that address specific issues such as criterion dependencies, uncertainty, and rank reversal. The findings underscore the continued relevance of the Hellwig method, its recent extensions, and its growing international recognition.

Keywords: MCDM; FMCDM; Taxonomic Measure of Development; Hellwig-based method; ideal point; pattern of development; anti-ideal point

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1. Introduction

Multi-criteria decision-making (MCDM) methods, rooted in operations and management science, comprise a set of techniques designed to tackle complex problems involving the assessment, ranking, classification, and clustering of a set of alternatives [1–3]. These methods address problems by considering multiple options and multiple, often conflicting criteria, analyzing both quantitative and qualitative parameters, and taking into account various interests and perspectives. The goal is to provide decision-makers (DMs) with a systematic and structured approach to making choices when faced with multiple alternatives.

Among the various MCDM methods, one that stands out is Hellwig's Taxonomic Measure of Development (TMD), introduced in 1968 [4], which is based on reference points. This method has gained notable recognition for its ability to address socio-economic challenges by linearly ranking alternatives based on their distance from an ideal point, referred to as the pattern of development. Hellwig's concept of a pattern of development is built around the most favorable values for each criterion, along with a synthetic measure that calculates the distance of each object from this ideal pattern using the Euclidean metric.

The TMD, commonly known as Hellwig's method, shares similarities with the widely known multi-criteria methods in theoretical analysis and practical implementations such as the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) [5] method but focuses solely on the distance to the ideal solution. However, the TOPSIS emerged over a decade after Hellwig's initial proposal.

The TMD method has become a particularly popular linear ordering technique in the field of economic research in Poland. Due to a significant increase in publications utilizing this method in English, it is gaining greater recognition in the international academic community, both for analyzing complex socio-economic phenomena and as a multi-criteria method based on reference points. Hellwig's method has been adapted to tackle a range of issues involving data represented by real numbers [6–9]. It has also been adapted to fuzzy sets [6,10,11]. Recent modifications have focused on incorporating intuitionistic fuzzy sets [12–15], interval-valued fuzzy sets [16], and oriented fuzzy sets [17].

The primary aim of this paper is to present a comprehensive investigation into the Taxonomic Measure of Development proposed by Hellwig [4]. This investigation systematically reviews various algorithms associated with TMD, including their extensions and modifications. This study contributes by systematically reviewing TMD's applications in different fields, particularly focusing on fuzzy and intuitionistic fuzzy adaptations. This paper will explore the integrated multi-criteria approach based on Hellwig's method and its application in addressing complex real-life problems. The investigation will be grounded in a detailed examination of several key steps integral to the TMD framework. These steps include the collection of relevant data, determination of criterion weights, definition of ideal values, and the normalization of data. Furthermore, this paper will analyze the calculation of distances between normalized values and ideal values and the factor normalization of aggregated measures. Each of these elements contributes to the overall ranking of alternatives and the practical utility of the TMD approach. By exploring these methodological aspects, this paper aims to provide a thorough understanding of Hellwig's method and its practical applications in various domains.

In summary, this paper makes a significant contribution by offering an original and comprehensive investigation into the Taxonomic Measure of Development, enriching both its theoretical foundations and practical applications. The main contributions are as follows.

Comprehensive review of TMD: This study systematically reviews the original Taxonomic Measure of Development method and its extensions, including data represented by real numbers or linguistic evaluation and fuzzy and intuitionistic fuzzy sets, and provides a thorough analysis of its evolution and diverse applications.

Novel, author-driven analytical approach: Unlike existing reviews, this paper introduces an original approach by conducting a detailed, step-by-step analysis of the TMD framework—focusing on data collection, criterion weighting, normalization, and distance calculation. This process-oriented analysis is a novel contribution, which allows for a deeper understanding of the methodology, distinguishing this paper from typical literature reviews that focus on statistical or bibliometric analyses. It also provides a clear mapping of the various fields where Hellwig-based methods can be applied, such as socio-economic analyses and multi-criteria decision-making.

Practical applicability: This paper highlights the versatility of TMD, particularly in solving complex, real-world decision-making problems, by illustrating its use in socio-economic analysis, sustainability assessments, and other complex decision-making scenarios. It underscores its unique focus on proximity to ideal solutions, distinguishing it from other methods like TOPSIS and enhancing its utility in certain decision contexts.

This paper is structured into five sections. Section 2 presents the background of the problem and the literature review and outlines the research methodology used in analyzing the Taxonomic Measure of Development and its extensions. In Section 3, the original Hellwig's TMD is presented, along with a series of practical applications in various fields, mainly socio-economic analysis. Section 4 explores extensions and modifications to the classic Hellwig method, including approaches using data represented by real numbers, fuzzy sets, intuitionistic fuzzy sets, and linguistic evaluations, with examples of real-world applications provided for each. Finally, this paper concludes with a summary in Section 5. The conclusion summarizes the findings, contributions of this paper, and limitations, offering suggestions for future research, particularly regarding the further development of the Taxonomic Measure of Development.

2. Background of the Problem and Research Methodology

In this section, we present the background of the problem and conduct a literature review, providing context for the Taxonomic Measure of Development and its relevance in contemporary research. Additionally, we will outline our author-driven research methodology, which focuses on a systematic analysis of algorithms and applications related to Hellwig-based methods.

2.1. Background of the Problem and Literature Review

The MCDM framework encompasses a wide variety of methods, each tailored to different decision-making contexts and each with its unique approach and specific algorithms [2,3,18,19]. However, classical MCDM models often struggle to handle uncertainty due to incomplete or imprecise information, which is why Fuzzy Multi-Criteria Decision-Making (FMCDM) methods are highly relevant. These methods handle uncertainties and vague information by incorporating fuzzy logic, which captures the imprecision inherent in human judgment [18,20]. Intuitionistic fuzzy sets allow for the consideration of hesitation or uncertainty in the decision-making process [21], while interval-valued fuzzy sets provide a way to represent uncertainty more comprehensively by considering ranges of possible values rather than fixed points [22]. Fuzzy logic facilitates the representation of uncertain or ambiguous information through the use of linguistic variables such as "high", "medium", or "low" [23,24]. This approach is particularly advantageous in decision-making processes where obtaining precise data may be challenging or unattainable. MCDM and FMCDM methods have found extensive applications across various domains [25], including business [26], engineering [27], environmental science [28], economics [29], circular economy [30], finance [31], service quality [32], sustainability [33,34], bioenergy systems [35], healthcare [36], waste management [37], and transport [38], among others.

Słowiński et al. [39] identified three main approaches in multi-criteria analysis: utility function, outranking relation, and sets of decision rules. The utility-based theory includes methods for synthesizing information into a singular parameter to facilitate the ranking of alternatives [40]. This category encompasses techniques such as SMART (Simple Multi-Attribute Rating Technique) [41], SAW (Simple Additive Weighting) [42], TOPSIS [43], MACBETH (Measuring Attractiveness by a Categorical Based Evaluation Technique) [44] and AHP (Analytic Hierarchy Process) [45], among others. The outranking relation theory involves methods that compare pairs of options to assess whether "alternative A is at least as good as alternative B" [3]. Key techniques that utilize a preference relational system to compare alternatives and establish preference relations include ELECTRE (Elimination and Choice Expressing Reality) [46] and PROMETHEE (Preference Ranking Organization Method for Enrichment Evaluations) [47]. Finally, the decision rule approach enables the development of a preference model by classifying or comparing decision examples [48].

A recent and growing application of MCDM is in the construction of composite indicators. As indicated by Lindén et al. [49], the objective of constructing a synthetic measure is "to condense and summarize the information contained in a number of underlying indicators, in a way that accurately reflects the underlying concept". A literature review by Greco et al. [50] highlights the substantial expansion of this research field, emphasizing its relevance and practical significance within the realm of MCDM. Within these methods, there is a class of techniques based on aggregation formulas that incorporate reference solutions. These include TOPSIS [43], VIKOR (VIseKriterijumska Optimizacija I Kompromisno Resenje) [51], DARP (Distances to Aspiration Reference Points) method [52], BWM (Best-Worst Method) [53], BIPOLAR [54], and the Taxonomic Measure of Development (TMD) [4].

2.2. Research Methodology

The following steps can be applied to the class of methods based on the utility theory approach, with specific details varying depending on the selected MCDM technique. The same principles also apply to TMD and their extensions. In the realm of MCDM, various methods grounded in utility theory provide structured frameworks for analyzing complex decision problems. The specific details of these methods may differ based on the selected technique, yet they all share fundamental principles that guide their implementation. This is also true for the Taxonomic Measure of Development and its extensions.

The general process for algorithms based on the TMD begins with a crucial first step: data collection. This phase entails gathering relevant information for each alternative in accordance with predefined criteria, setting the stage for subsequent analysis and decision-making. This data can be qualitative or quantitative and should be sourced from reliable references or expert opinions, represented as real numbers, fuzzy or intuitionistic fuzzy numbers, or in linguistic terms [2,20,55]. Next, weights are determined where the significance of each criterion in the decision-making process is assessed. This can be undertaken using methods ranging from objective to subjective, or a combination of both [56–59]. Subsequently, the ideal solution is defined as the ideal value for each criterion. These values, which may be maximum, minimum, or target values, are set according to the decision maker's preferences and are used to evaluate how well each alternative meets the criteria [52,60]. Similarly, the anti-ideal solution is determined if needed. Following this, normalization is applied to standardize the collected data, ensuring that all criteria are on a common scale. This step employs methods such as standardization, min-max, vector, or linear normalization [61–63].

Next, the distance between the normalized values and the ideal values, anti-ideal values, or both of them is calculated. Various metrics can be used for this purpose, including the Euclidean, Manhattan, and Mahalanobis distances or other measures. Afterward, aggregation combines the normalized values, weights, and distance measures to determine an overall preference value for each alternative. This step involves using an aggregation formula, such as a weighted sum, weighted product, or another method depending on the specific MCDM technique employed. Following this step, normalization aggregation standardizes the multi-criteria measure to a common scale, typically within the interval [0, 1]. Finally, ranking alternatives orders the alternatives based on their relative closeness values, where higher values indicate better alternatives.

After the original Hellwig's method was introduced, various modifications and extensions were proposed to address different conditions. These adaptations focus on aspects such as the type of data, ideal construction, weight system, normalization factor, distance measures, and aggregation formulas. For example, to address uncertainty, a fuzzy Hellwig's approach was developed [6,10,11], and more recent developments have included intuitionistic approaches [12–15]. To provide a comprehensive review of techniques based on Hellwig's method and their practical applications, Figure 1 will be employed.

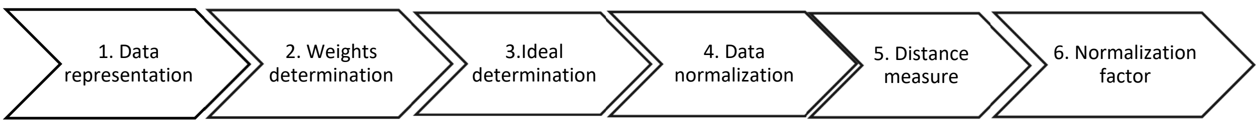

Figure 1. The framework of a comprehensive review of Hellwig-based methods.

3. Classic Hellwig's Method and Its Applications

In this section, we present the classic Hellwig algorithm alongside a comprehensive review of its applications. By examining real-world case studies, we will illustrate how the Taxonomic Measure of Development method can be effectively applied to complex decision-making scenarios, including assessing regional development levels, conducting socio-economic analyses, and evaluating environmental sustainability. Furthermore, we will complement the TMD algorithm with the procedure for selecting diagnostic variables, which enhances the robustness of the analysis by identifying the most relevant indicators for evaluation.

3.1. The Taxonomic Measure of Development

The concept of the Taxonomic Measure of Development, introduced by Professor Zdzisław Hellwig, was published in 1968 in the Polish journal, Statistical Review [4]. It is noteworthy that this idea was initially presented at the UNESCO conference in Warsaw and discussed in English in an unpublished UNESCO working paper [64]. Hellwig's work has motivated many Polish researchers to investigate linear ordering methods, resulting in adaptations of his approach and the creation of new measures, as outlined in [65–68].

Below, we will present the classic TMD algorithm, hereafter referred to as the Hellwig method. Let $O = \{O_1, \ldots, O_m\}$, where $i = 1, 2, \ldots, m$ is the set of objects under assessment, and $C = \{C_1, \ldots, C_n\}$, where $j = 1, 2, \ldots, n$ is the set of diagnostic variables (indicators) comprising a complex phenomenon. Additionally, consider P and N as the sets of stimulating (positive) and destimulating (negative) indicators, respectively, influencing the complex phenomenon. The original Hellwig's method involves the following steps [4]:

Step 1. Definition of a decision matrix

$$D = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}, \tag{1}$$

where x_{ij} is the assessment of ith object concerning the jth diagnostic variable (indicator) $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$.

Step 2. Identification of the pattern of development, $I = \left[x_1^+, x_2^+, \ldots, x_n^+\right]$, following the principle

$$x_j^+ = \begin{cases} \max\limits_i x_{ij} & \text{if} \quad x_{ij} \in P, \\ \min\limits_i x_{ij} & \text{if} \quad x_{ij} \in N. \end{cases} \tag{2}$$

for $j = 1, 2, \ldots, n$.

Step 3. Construction of the normalized decision matrix

$$\overline{D} = \left[\overline{x}_{ij}\right] \tag{3}$$

using the standardization formula

$$\overline{x}_{ij} = \frac{x_{ij} - \overline{x}_j}{S_j}, \tag{4}$$

where $\overline{x}_j = \frac{1}{m}\sum_{i=1}^m x_{ij}$, $S_j = \sqrt{\frac{1}{m}\sum_{i=1}^m \left(x_{ij} - \overline{x}_j\right)^2}$, $i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n$.

Step 4. Calculation of the distance of the ith object from the pattern of development using the Euclidean distance

$$d_i^+(O_i, I) = \sqrt{\sum_{j=1}^n \left(\overline{x}_{ij} - \overline{x}_j^+\right)^2}, \tag{5}$$

where $\overline{x}_j^+ = \frac{x_j^+ - \overline{x}_j}{S_j}$ is the normalized values of the pattern of development, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

Step 5. Computation of the relative closeness of the *i*th object to the pattern of development using the following formula

$$H(O_i) = 1 - \frac{d_i^+}{d_0}, \tag{6}$$

where $d_0 = \overline{d} + 2S$, $\overline{d} = \frac{1}{m}\sum_{i=1}^{m} d_i^+$, $S = \sqrt{\frac{1}{m}\sum_{i=1}^{m}\left(d_i^+ - \overline{d}\right)^2}$,$i = 1, 2, \ldots, m$.

Step 6. Ranking the objects according to the decreasing values of $H(O_i)$

The synthetic measure H usually takes the values from the interval [0, 1]. The higher the values of the measure, the closer the object is to the pattern of development.

In the subsequent stage, the objects can be categorized into four distinct classes. Class 1 consists of objects within the range: $\left[\overline{H} + S_H; \max\{H_i\}\right]$ which indicates a very high level of the TMD measure. Class 2 includes objects that fall within the interval: $\left[\overline{H}; \overline{H} + S_H\right)$, representing a medium-high level of the TMD measure. Class 3 is defined by the range $\left[\overline{H} - S_H; \overline{H}\right)$, indicating a medium-low level of the TMD measure. Lastly, Class 4 encompasses objects within the interval: $\left[\min\{H_i\}; \overline{H} - S_H\right)$, which signifies a very low level of the TMD measure. Here, $\overline{H}$ represents the average value, and S_H denotes the standard deviation of the synthetic measure.

It is important to highlight that the TMD can be applied not only in taxonomic analysis but also in decision-making contexts, serving as a multi-criteria decision-making method. In this framework, specific terms are used: an "object" or "option under consideration" refers to an alternative; a "diagnostic variable" is a criterion utilized in the analysis; a "stimulant" denotes a benefit criterion; a "destimulant" signifies a cost criterion; a "pattern of development" represents an ideal solution.

3.2. The Examples of Application Classical Taxonomic Measure of Development

The TMD method provides a structured approach for a wide range of socio-economic analyses across multidimensional concepts that are not directly measurable or clearly defined, and it has been particularly popular among Polish researchers. A review of the literature shows that the majority of studies using the TMD method focus on Poland, often involving statistical analyses of issues such as sustainable development, with an emphasis on the socio-economic potential of Polish regions in the years 2005, 2009, and 2013 [69], regional disparities in development regarding demographic potential, social and economic development, and technical infrastructure for the years 2006, 2010, and 2015 [70], spatial diversity in socio-economic development across Polish voivodeships from 2005 to 2017 [71], the level of socio-economic development in Poland from 2013 to 2019 [72], socio-economic sustainability goals in Polish regions in 2019 and 2020 [73], and the level of socio-economic development of Poland's regions in the years 2005, 2010, and 2017 [74].

Other studies have analyzed selected socio-economic issues such as environmental governance in 2010 and 2015 [75], demographic growth in Polish voivodeships between 1999 and 2006 [76], the agricultural market in Poland [77], the diversity of Polish regions in terms of technical infrastructure development from 2005 to 2018 [78], the readiness of Polish regions for business tourism development [79], poverty in Polish regions within the framework of European Union policy between 2011 and 2021 [80], and the relationship between social capital levels and the green economy in Polish voivodeships [81].

Additionally, some research focused on the municipal or city level includes a comparison of socio-economic development levels between semi-urban and rural gminas in Poland [82], the demographic potential and socio-economic development of urban functional areas [83], the level of development in social, economic, and environmental dimensions of functional areas of voivodship capitals in Poland between 2005 and 2015 [84], the socio-economic development of rural municipalities of various demographic and functional types in 2021 [85], and the development of cultural institutions in Poland in 2018 [86].

All the studies presented so far have included statistical analyses based on data collected from the Local Data Bank or the Central Statistical Office of Poland.

Another example is a study by Gałecka and Smolny [87] evaluating theatre activity in Poland using data from the Central Statistical Office and individual inquiries into the financial statements and substantive reports of cultural institutions, while Ziemba [88] used Hellwig's method to assess the adoption of information and communication technologies (ICT) in enterprises from the Silesian Province and sustainability within these organizations with data collected through a Computer-Assisted Web Interview via SurveyMonkey. A different study by Kowalik and Markowicz [89] examined the economic efficiency of Polish hospitals before and after ownership transformations in EU countries using statistical data from the financial statements of the entities involved, focusing on profitability, liquidity, debt, and efficiency.

The classic Hellwig's method has also been successively applied in capital market analyses, as seen in the work of Tarczyńska-Łuniewska and Tarczyński [90]. The research by Węgrzyn [91] analyzed stock selection based on financial ratios from the Warsaw Stock Exchange between 2001 and 2010, Kompa and Witkowska [92] compared the development of capital markets in 19 European stock exchanges from 2002 to 2011, and Witkowska and Kuźnik [93] applied the Hellwig approach to estimate the fundamental strength of 27 non-financial companies listed on the Warsaw Stock Exchange from 2012 to 2017.

In subsequent group papers, the TMD method has been applied outside of Poland in various socio-economic analyses within EU countries. These studies cover topics such as public health in Poland compared to selected European countries between 2004 and 2009, based on medical, economic, and social indicators [94], the quality of human capital [95], tourism competitiveness [96], agricultural performance [97], innovation outcomes [98], the development of e-commerce [99], and the competitive position of the food industry within the EU between 2005 and 2017 [100]. Other studies include research on economic development [101], addressing socio-economic inequalities [102], the progress of the Visegrad countries in meeting EU energy objectives within the Europeanization context [103], smart growth in EU countries from 2000 to 2017 [104], national innovation systems across EU countries [105], renewable energy sources in the context of the European Green Deal [106], farmland abandonment in EU countries [107], living standards in rural EU areas [108], and evaluation of Sustainable Development Goal (SDG) 9 in EU countries between 2015 and 2020 [109]. A common feature of these studies is their reliance on Eurostat databases.

The final group presents examples of papers that apply the classic Hellwig method to analyze various problems using data from diverse sources, not limited to the Local Data Bank, the Central Statistical Office of Poland, or the Eurostat database. Di Domizio [110] applied the method to study the Italian football league. Similarly, Jurkowska [111] used Hellwig's approach to assess the socio-economic development of Germany's federal states in comparison to the entire country, relying on data from the Statistisches Jahrbuch 2011 für die Bundesrepublik. In another study, Kudełko and Rynio [112] evaluated the impact of the COVID-19 pandemic on unemployment dynamics across regions with varying levels of socio-economic development in Germany and Poland, drawing from sources such as Eurostat, the German Bundesagentur für Arbeit, and the Central Statistical Office of Poland. Stanimir [113] also utilized the method to explore Generation Y's perceptions of work values and job quality in the EU labor market, using data from Eurostat and the European Social Survey. Moreover, Paduszyńska and Lesiak [114] focused on the stability of the banking sector across EU countries, utilizing data from the World Bank. Janusz and Kowalczyk [115] investigated the implementation of the Smart City concept in selected cities of the Visegrad Group in 2018, with data from Eurostat, the World Bank, and Wifimap.io. Finally, Smarżewska et al. [116] conducted a comparative study of the Polish healthcare system and those of other OECD countries, using insights from the OECD Health at a Glance 2021 report.

3.3. The Procedure of Selecting Diagnostic Variables

When applying the TMD method, one critical aspect should be discussed, namely the selection of diagnostic variables [117,118]. The proper selection of variables determines the final results of the study and influences the accuracy of assessments, the precision of predictions, and consequently, the accuracy of decisions made based on them. The selection of variables encompasses both substantive and formal aspects, including the measurement of variability and the verification of the degree of correlation between variables.

First of all, the substantive selection of variables involves preparing potential diagnostic features that, in light of the substantive knowledge about the phenomenon under investigation, are deemed most important for conducting a comparative analysis of the subjects. Next, the diagnostic features should be chosen based on the criterion of universality. They should be characterized by substantive suitability from the perspective of the studied phenomenon and possess economic significance. Diagnostic variables should capture the most significant properties of the phenomena being considered and accurately represent them, formulated in a straightforward, clear, and precise manner. The selected features must be measurable, preferably expressed as structural or intensity indicators. When selecting variables, one should also consider the availability of the most recent and up-to-date data.

Finally, it is sometimes necessary to reduce the dataset by eliminating features with a low diagnostic value, characterized by low variability and high correlation. Variables describing objects in linear ordering should effectively discriminate between objects (exhibit high variability), be weakly correlated with each other, and be strongly correlated with the rejected variables. Selected features should exhibit sufficient spatial variability and differentiate the examined objects. The degree of variability is verified based on the value of the coefficients of variation (Vs) determined for individual characteristics. It is commonly assumed that variables with a coefficient of variation below 10% are eliminated. Generally, it is assumed that strongly correlated variables carry similar information. In the case of multivariate analyses, the goal is to build a set in which the risk of duplicating information by strongly correlated variables is minimized. When faced with two strongly correlated features, the selection of the representative one should be based on substantive premises and the level of variability described above.

The procedures for determining variables have been refined in several papers, including [73,83,95,115,119], among others.

4. The Modifications and Extensions of Classic Hellwig's Method

In this section of the paper, we present the modifications and extensions to Hellwig's method, carefully considering different types of data. This includes numerical data, linguistic evaluations, fuzzy numbers, and intuitionistic fuzzy numbers. We will provide a detailed presentation of the algorithmic steps associated with these modifications, highlighting how each adaptation enhances the method's applicability. Additionally, we will explore the potential applications of Hellwig-based methods in various practical scenarios, demonstrating their versatility and effectiveness in addressing complex decision-making challenges.

4.1. The Hellwig-Based Methods with Data Represented by Real Numbers or Linguistic Evaluation

First, we posit the representation of data through real numbers or linguistic terms with a numerical scale. The modification of Hellwig's method extends across all aspects delineated in Figure 1. To illustrate the universality of these methods in solving various complex problems, we utilized the terminology of MCDM analyses. Let $A = \{A_1, A_2, \ldots, A_m\}$, $i = 1, 2, \ldots, m$ be the set of alternatives, $C = \{C_1, C_2, \ldots, C_n\}$, $j = 1, 2, \ldots, n$ the set of criteria, and $C = P \cup N$, where P is the set of benefit criteria and N, is the set of cost criteria. The alternative A_i is represented as a vector, $A_i = [x_{i1}, x_{i2}, \ldots, x_{in}]$, $i = 1, 2, \ldots, m$. The general framework rooted in Hellwig's principles involves the following steps.

Step 1. Definition of a decision matrix

We assume that alternatives are represented by real numbers or assessed using an ordinal scale. Let $OS = \{1, 2, \ldots, k\}$ represent the numerical values of the linguistic scale, where a higher number indicates a greater preference. The linguistic scale should include 5, 7, 9, 11, or 13 distinct levels. As highlighted by Herrera and Herrera-Viedma [23] and Xu [120], the scale must be sufficiently granular to effectively differentiate the performances of the evaluated options. An example of a seven-point scale is shown in Table 1.

Table 1. Linguistic terms for the options rating—seven-point linguistic scale.

Numerical Representation Linguistic Scale (Rating)	Linguistic Term	Label
1	Very poor	VP
2	Poor	P
3	Medium-poor	MP
4	Fair	F
5	Medium-good	MG
6	Good	G
7	Very-good	VG

Source: see [23,120].

The decision matrix D is defined as

$$D = [x_{ij}], \tag{7}$$

where x_{ij} represents the assessment of the ith alternative with respect to the jth criterion $(i = 1, 2, \ldots, m\,; j = 1, 2, \ldots, n)$. The value x_{ij} can be either a real number $(x_{ij} \in R)$ or a numerical representation of the linguistic scale.

Step 2. Determination of the system of weights

The system of weights is represented as

$$w = [w_1, \ldots, w_n], \tag{8}$$

where $w_j > 0$ $(j = 1, \ldots, n)$ denotes the weight of the criterion C_j, and $\sum_{j=1}^{n} w_j = 1$.

The original Hellwig's approach did not assign specific weights to the criteria. Maggino and Ruviglioni [56] observed that equal weights were commonly used in many applications. Greco et al. [50] noted that equal weights are often preferred due to reasons such as the simplicity of construction, lack of a theoretical basis for a differential weighting system, disagreements among decision-makers, and insufficient statistical or empirical evidence.

The literature presents various methods for determining weights [57,59,121,122], each with its own advantages and limitations. These methods are typically categorized into three main types [58]: subjective, objective, and integrated. Subjective weights are assigned based on expert judgment. Objective weights are derived from the statistical properties of the available data, employing theoretical or mathematical statistical methods. Integrated weight systems combine elements of both subjective and objective approaches. Table 2 outlines the objective weighting methods used in Hellwig's approach, including those based on the coefficient of variation, correlation matrix, and entropy [43,121–123].

A higher weight coefficient is assigned to criteria with a higher average coefficient of variation or those that show a stronger correlation with the values of other criteria. The entropy-based weighting method operates on the principle that criteria with higher entropy (greater variability) are considered more important, while criteria with lower entropy are perceived as less influential in the decision-making process.

Table 2. Objective weighting methods used in Hellwig's method.

Weights Method	Formula
Equal weights	$w_j = \frac{1}{n},\ j = 1,2,\ldots,n.$
Coefficients of variation	$w_j = \frac{\lvert V_j \rvert}{\sum_{j=1}^{n}\lvert V_j\rvert},\ V_j = \frac{\bar{x}_j}{S_j},\ \bar{x}_j = \frac{1}{m}\sum_{i=1}^{m} x_{ij},\ S_j = \sqrt{\frac{1}{m}\sum_{i=1}^{m}\left(x_{ij} - \bar{x}_j\right)^2},\ j = 1,2,\ldots,n$
Correlation matrix	$w_j = \frac{\sum_{k=1}^{n}\lvert r_{kj}\rvert}{\sum_{k=1}^{n}\sum_{j=1}^{n}\lvert r_{kj}\rvert},\ r_{kj} = \frac{\sum_{i=1}^{m}(x_{ik}-\bar{x}_k)\left((x_{ij}-\bar{x}_j\right)}{\sqrt{\frac{1}{m}\sum_{i=1}^{m}(x_{ik}-\bar{x}_k)^2}\sqrt{\frac{1}{m}\sum_{i=1}^{m}\left(x_{ij}-\bar{x}_j\right)^2}};\ k,j = 1,2,\ldots,n$
Entropy method	$w_j = \frac{1-E_j}{\sum_{j=1}^{n}(1-E_j)} = \frac{1-E_j}{n-\sum_{j=1}^{n}E_j};\ E_j = -\frac{1}{lnm}\sum_{i=1}^{m}\frac{x_{ij}}{\sum_i^m x_{ij}} ln \frac{x_{ij}}{\sum_i^m x_{ij}},\ j = 1,2,\ldots,n.$

Source: own study.

Step 3. Identification of the ideal and anti-ideal points (solutions)

In the classical Hellwig procedure, the ideal I is defined as an internal point, determined by combining the performances of all alternatives under consideration using maximum and minimum relations (see Formula (2)). The ideal point or ideal solution is represented by the maximum values of the benefit criteria and the minimum values of the cost criteria. Roszkowska and Filipowicz-Chomko [9] introduced the concept of common and individual patterns of development. The common pattern of development (ideal) is defined externally as common benchmarks arbitrarily and subjectively defined by decision-makers or experts:

$$I = \left[x_1^+, x_2^+, \ldots, x_n^+\right]. \tag{9}$$

The internal and external ideal points are common for all objects. Additionally, Roszkowska and Filipowicz-Chomko [9] introduced individual patterns of development (individual ideals).

$$I_i = \left[x_{i1}^+, x_{i2}^+, \ldots, x_{ik}^+\right]. \tag{10}$$

These are defined separately for the ith alternative, with benchmarks as coordinates. This approach effectively addresses the challenges posed by the implementation of EU and/or national targets when constructing a developmental pattern within Hellwig's approach, facilitating the comparison of the sustainable performance of EU countries at the national level, particularly in the area of education.

If the analysis of a socio-economic phenomenon involves comparing objects across several periods, a common ideal point (development pattern) for each feature, such as the maximum and minimum values for the entire study period, is used. This is referred to as a dynamic approach to Hellwig's measure of development [124].

In the linguistic approach, the ideal is determined as a vector with the maximum linguistic evaluation from the scale:

$$I = [k, k, \ldots, k]. \tag{11}$$

It is worth noting that in the classical Hellwig method, only the ideal point is used in the algorithm. However, one modification [125] introduces the concept of the anti-ideal. In this approach, the distance between the ideal and the anti-ideal is utilized for normalizing the aggregated value. The anti-ideal point, which can be either internal—represented by the minimum values of the benefit criteria and the maximum values of the cost criteria—or external, is denoted as

$$AI = \left[x_1^-, x_2^-, \ldots, x_n^-\right]. \tag{12}$$

In the linguistic approach, the anti-ideal is determined as a vector with the minimum linguistic evaluation from the scale:

$$AI = [1, 1, \ldots, 1]. \tag{13}$$

The ideal and anti-ideal can also be represented as vectors, corresponding to the maximum and minimum linguistic values obtained for evaluated objects [126].

Step 4. Construction of the normalized decision matrix

The normalized decision matrix has the form

$$\overline{D} = [\overline{x}_{ij}], \tag{14}$$

where $\overline{x}_{ij}$ is the normalized value of $x_{ij}(i = 1,2,\ldots,m;\ j = 1,2,\ldots,n)$.

Data normalization is a crucial step in Hellwig's method, as it standardizes the data related to the criteria onto a uniform scale. Various approaches for data normalization are discussed in the literature [62,63,127]. However, in Hellwig-based methods, max-min normalization, vector normalization, and sum normalization have sometimes been utilized instead of Formula (4) [128,129]. When individual ideals are considered (Formula (10)), two scenarios must be accounted for [9]:

For benefit criteria,

- if the ideal is reached, i.e., $x_j^- < x_{ij}^+ \leq x_{ij}$ then $\frac{x_{ij}-x_j^-}{x_{ij}^+-x_j^-} \geq 1$;
- if the ideal is not reached, i.e., $x_j^- < x_{ij} < x_{ij}^+$ then $\frac{x_{ij}-x_j^-}{x_{ij}^+-x_j^-} < 1$.

For cost criteria,

- if the ideal is reached, i.e., $x_{ij} \leq x_{ij}^+ < x_j^-$ then $\frac{x_{ij}-x_j^-}{x_{ij}^+-x_j^-} \geq 1$;
- if the ideal is not reached, i.e., $x_{ij}^+ \leq x_{ij} < x_j^-$ then $\frac{x_{ij}-x_j^-}{x_{ij}^+-x_j^-} < 1$.

As a result, the adopted max-min normalization (see Table 3, last method) is applied. In the Hellwig method, based on Weber's median, the Weber standardization is applied [6,130,131]. A summary of the normalization approaches used in Hellwig's modified procedures is presented in Table 3.

Table 3. Normalization used in Hellwig-based methods.

Normalization Procedure	Formula
Standardization	$\overline{x}_{ij} = \frac{x_{ij}-\overline{x}_j}{S_j}, \overline{x}_j = \frac{1}{m}\sum_{i=1}^m x_{ij}, S_j = \sqrt{\frac{1}{m}\sum_{i=1}^m \left(x_{ij} - \overline{x}_j\right)^2}, i = 1,2,\ldots,m;\ j = 1,2,\ldots,n$
Max-Min method	$\overline{x}_{ij} = \frac{x_{ij}-\min_i x_{ij}}{\max_i x_{ij}-\min_i x_{ij}}; i = 1,\ldots,m,\ j = 1,\ldots,n$
Vector normalization	$\overline{x}_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^m (x_{ij})^2}}; i = 1,\ldots,m,\ j = 1,\ldots,n$
Sum method	$\overline{x}_{ij} = \frac{x_{ij}}{\sum_{i=1}^m x_{ij}}; i = 1,\ldots,m,\ j = 1,\ldots,n$
Max-Min method adapted tothe common or individual ideal concept	$\overline{\overline{x}}_{ij} = \begin{cases} 1 \text{if} \frac{x_{ij}-x_j^-}{x_{ij}^+-x_j^-} \geq 1 \\ \frac{x_{ij}-x_j^-}{x_{ij}^+-x_j^-} \text{if} \frac{x_{ij}-x_j^-}{x_{ij}^+-x_j^-} < 1 \end{cases}$ x_{ij}^+ is the ideal of ith object, which is individually defined; $x_j^- = \begin{cases} \max_i x_{ij} \text{ if } x_{ij} \in N \\ \min_i x_{ij} \text{ if } x_{ij} \in P \end{cases} i = 1,\ldots,m,\ j = 1,\ldots,n.$
Weber standardization	$\overline{x}_{ij} = \frac{x_{ij}-Me_j}{1.4826 MAD_j}$; Me_j- jth value of a Weber median vector; MAD_j-median absolute deviation of the jth variable; 1.4826−constant coefficient estimated in empirical studies; $i = 1,\ldots,m, j = 1,\ldots,n$

Source: own study.

Let us observe that in the case of linguistic evaluation, this step can be omitted.

Step 5. Construction of the weighted normalized decision matrix

The weighted normalized matrix has the form

$$\tilde{D} = \left[\tilde{x}_{ij}\right], \tag{15}$$

where

$$\tilde{x}_{ij} = w_j \overline{x}_{ij} \tag{16}$$

for $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$.

Step 6. Calculation of the distance of the ith alternative from the ideal point

In the classic Hellwig's method, distances are computed using the Euclidean norm, with the implicit assumption that the criteria are independent. However, in practical scenarios, achieving independence among criteria is seldom the case. To address this limitation, an extension of Hellwig's method has been suggested, involving the incorporation of the Mahalanobis distance [132,133]. The Mahalanobis distance considers the covariance structure of the data, offering a more accurate representation of distances when criteria exhibit correlations, particularly in situations involving asymmetrical relationships between them. The distance measures used in Hellwig-based methods for data represented by real numbers [11,133,134] are presented in Table 4.

Table 4. Distance measures used in Hellwig-based methods for data represented by real numbers.

Distance Measures	Formula
Euclidean	$d_i(A_i, I) = \mathrm{E}\left(\tilde{A}_i, \tilde{I}\right) = \sum_{j=1}^{n}\left(\tilde{x}_{ij} - \tilde{x}_j^{+}\right)^2, i = 1, \ldots, m$ $\tilde{x}_{ij}, \tilde{x}_j^{+}$ are weighted normalized values x_{ij} and x_j^{+}, respectively.
Mahalanobis	$dM_i(A_i, \mathrm{I}) = \mathrm{M}(\overline{A}_i, \overline{I}) = \sqrt{(\overline{A}_i - \overline{I})WC^{-1}W^T(\overline{A}_i - \overline{I})^T}$, C is the variance-covariance matrix of the data matrix $\overline{D}$, $W = \mathrm{diag}\left(\sqrt{w_1}, \ldots, \sqrt{w_n}\right)$ is the diagonal matrix, where $w_1, w_2, \ldots, w_n$ are the weights assigned to the criteria.

Source: own study.

Practically, the selection of a distance measure hinges on the characteristics of the data and the details of the multi-criteria analysis. The Euclidean distance is chosen when variables are assumed to be independent, but if the dataset exhibits correlations or asymmetrical distributions, the Mahalanobis distance becomes a more suitable option. In the dynamic approach to Hellwig's measure of development for calculating the distance, the values of the common pattern were taken in the form of the minimum and maximum from the entire period under study [124].

The ordinal nature of the data required the use of a specialized metric to measure the distance between objects. Object evaluation is based on the numerical relationships of "equal to", "greater than", and "less than". In the linguistic Hellwig's method the distance metric for ordinal data proposed by Walesiak [135,136] can be applied.

Let $OS = \{1, 2, \ldots, k\}$ be a numerical representation of the ordinal scale, where the higher number means "more preferable", and let $O = \{O_1, \ldots O_m\}$ be a set of objects evaluated on the ordinal scale OS with respect to n criteria. Let $O_i = \left[x_{i1}, x_{2j}, \ldots, x_{in}\right], O_k = \left[x_{k1}, x_{k2}, \ldots, x_{kn}\right]$ represent the ith (kth) object, where $x_{ij}\left(x_{kj}\right)$ are the evaluations of ith (kth) object with respect to jth criterion, and $x_{ij}, x_{kj} \in OS(j = 1, \ldots n;\ i, k = 1, 2, \ldots, m)$. The GDM2 distance between objects O_i and O_k are characterized by n criteria and has the following form [135,136]:

$$GDM2_{ik} = \frac{1}{2} - \frac{\sum_{j=1}^{n} \mathrm{w}_j a_{ikj} b_{kij} + \sum_{j=1}^{n} \sum_{l=1;\ l \neq i,k}^{m} \mathrm{w}_j a_{ilj} b_{klj}}{2\left[\left(\sum_{j=1}^{n} \mathrm{w}_j a_{ikj}^2 + \sum_{j=1}^{n} \sum_{l=1;\ l \neq i,k}^{m} \mathrm{w}_j a_{ilj}^2\right)\left(\sum_{j=1}^{n} \mathrm{w}_j b_{kij}^2 + \sum_{j=1}^{n} \sum_{l=1;\ l \neq i,k}^{m} \mathrm{w}_j b_{klj}^2\right)\right]^{\frac{1}{2}}} \tag{17}$$

where $i, k, l = 1, \ldots, m$ is the number of objects, $j = 1, \ldots, n$ is the number of criteria, w_j is the weights of the jth criterion, and $x_{ij}\left(x_{kj}, x_{lj}\right)$ is the evaluation of the ith (kth, lth) object with respect jth criterion.

For the ordinal scale, a_{ipj}, b_{krj} are given as

$$a_{ipj}\left(b_{krj}\right) = \begin{cases} 1 & \text{if} \quad x_{ij} > x_{pj}\left(x_{kj} > x_{rj}\right) \\ 0 & \text{if} \quad x_{ij} = x_{pj}\left(x_{kj} = x_{rj}\right), \text{ for } p = k, l;\ r = i, l \\ -1 & \text{if} \quad x_{ij} < x_{pj}\left(x_{kj} < x_{rj}\right) \end{cases} \tag{18}$$

The distances $GDM2_i^+$ between alternative A_i (i= $1, 2, \ldots, m$) and the ideal I are later calculated using the GDM2 measure.

Step 7. Calculation of Hellwig's measure for the ith alternative.

The following formula is used to calculate the aggregation of Hellwig's measure:

$$H(A_i) = 1 - \frac{d_i^+}{N} \tag{19}$$

where N is a normalization factor of the procedure.

Instead of the classical approach (see Formula (6)), other normalization factors are used. Hellwig [125] proposed the distance between the pattern (ideal point) and anti-pattern (anti-ideal point), while Fura and Wang [137] suggested using $max(d_i)$. The normalization factors used in Hellwig-based methods are presented in Table 5.

Table 5. Normalization factors used in Hellwig-based methods for data represented by real numbers.

Normalization Factor	Formula
Classical factor	$N = d_0 = \overline{d} + 2S, \overline{d} = \frac{1}{m}\sum_{i=1}^{m} d_i^+, S = \sqrt{\frac{1}{m}\sum_{i=1}^{m}\left(d_i^+ - \overline{d}\right)^2}, i = 1, 2, \ldots, m$
Max value of measure	$N = max(d_i)$
Distance between ideal and anti-ideal	$N = \sqrt{\sum_{j=1}^{n}\left(\tilde{x}_j^- - \tilde{x}_j^+\right)^2}$, where $\tilde{x}_j^-, \tilde{x}_j^+$ −normalized weighted values of anti-ideal and ideal for jth criterion

Source: own study.

For data represented on an ordinal scale, $N = GDM2^{+-}$ where $GDM2^{+-}$ is the generalized distance measure between the ideal and anti-ideal points calculated using Formula (17).

Step 8. Ranking of alternatives according to the decreasing values of Hellwig-based algorithm

Synthetic measure H typically range between 0 and 1. Higher values indicate that the object is closer to the patern of development.

Finally, it is worth mentioning other methods derived from Hellwig's approach, each offering a different way of evaluating alternatives. The first modification is the positional Hellwig method, which uses positional measures to normalize input variables. This version, based on Weber's spatial median [6,130,138], generalizes the classic median concept to multiple dimensions by minimizing the sum of Euclidean distances to the data points, thereby positioning itself centrally and remaining robust against outliers.

The Weber normalization is defined by the formula (see Table 3)

$$\overline{x}_{ij} = \frac{x_{ij} - \theta_j}{1.4826MAD_j} \tag{20}$$

where θ_j is the value of a particular component of the multidimensional median vector, known as the Weber median, and MAD_j is the median absolute deviation of the jth criterion, calculated as $MAD_j = \underset{i=1,\ldots,m}{med}\left|x_{ij} - \theta_j\right|$, and *med* is the median. The constant 1.4826 is an empirically derived coefficient, and x_{ij} is the value of the ith alternative for the jth criterion $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$.

The positional Hellwig measure of normalized variables (Formula (20)) is calculated as follows:

$$HP(A_i) = 1 - \frac{d_i}{med(D) + 2.5mad(D)} \quad (21)$$

where d_i represents the individual values of the distance vector D, $D = [d_1, \ldots, d_n]$ is the distance vector, $med(D)$ is the median of the distance vector D, $mad(D)$ is the median absolute deviation of the distance vector D, and 2.5 is a constant value (immune threshold value).

The distance vector d_i is calculated using the Manhattan distance formula

$$d_i = \sum_{j=1}^{n}\left|\overline{x}_{ij} - \overline{x}_j^{+}\right|, \quad (22)$$

where $\overline{x}_j^{+}$ is the jth coordinate of the ideal point constituted of the maximum values of the normalized features $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$.

Antczak [139] modified the classical development measure by introducing the Taxonomic Spatial Measure of Development (TSMD), which enables simultaneous analysis across three dimensions: section, time, and space. This modification involves adding a spatial weight matrix to the classic TMD [4]. Pietrzak [140] suggested a modification spatial development measure considering the potential strength of interactions among regions, rather than modifying the standardization formula, while Sobolewski et al. [141] included the locational component in the analysis.

In a different vein, Roszkowska and Filipowicz-Chomko [142] introduced the Multi-Criteria Method Integrating Distances to Ideal and Anti-Ideal Points (MIDIA), which serves as an extension of Hellwig's approach. MIDIA enhances the original method by incorporating a weighted system that accounts for both balance and asymmetry in evaluating alternatives based on their distances from ideal and anti-ideal points. This approach allows for a more nuanced ranking of alternatives, focusing on their relative distances from these reference points, rather than relying solely on positional normalization.

To understand the developments and applications of Hellwig's method for real data, it is essential to review several notable and representative contributions in the field.

In a series of five papers, the Hellwig method with max-min normalization was utilized. Jóźwik and Gawroński [143] employed it to assess changes in the level of socio-economic development of communes in the Lubelskie Voivodeship for the years 2005 and 2015. Mazur-Wierzbicka [119] used this approach for a multidimensional comparative analysis of the implementation of the circular economy by EU countries for the years 2010–2018. Bartniczak and Raszkowski [144] adopted the method to analyze sustainable cities and communities in the European Union, using EUROSTAT data from 2010–2020. Kalinowska et al. [145] applied it to evaluate the economic, social, and environmental dimensions of sustainable agricultural development in EU countries. Finally, Mikuła and Raczkowska [146] employed this method to examine the level of inequality in the context of Sustainable Development Goal 10 in EU countries in the year 2022. All studies utilized data from EUROSTAT and [145] additionally data from FADN, Statistics Poland, EUROSTAT, and FAOSTAT for the years 2011 and 2018. In turn, the work of Dębkowska and Jarocka [128] analyzes the impact of data normalization methods on the final rankings obtained using Hellwig's method. The study examines methods such as standardization, Weber's standardization, max-min normalization, and vector normalization, among others. The research uses data on innovation in European Union regions from Eurostat.

In the next group of papers, different weighting systems were applied in the Hellwig method. Korzeb and Niedziółka [147] assessed the performance of commercial banks in Poland during the COVID-19 pandemic. Their research utilized both Hellwig's method and the TOPSIS method with various weighting systems, including equal weights, expert weights based on the coefficient of variation, and correlation coefficients. Tarczyńska-Łuniewska et al. [148] utilize the Fundamental Power Index (FPI), a measure based on Hellwig's concept, to evaluate the fundamental strength of energy companies listed on the Warsaw Stock Exchange. The FPI was determined using a dynamic approach, with variables weighted over time, where the earliest period received the lowest weight and the most recent period the highest. The study analyzes the connection between fundamental strength and market value in the context of Environmental, Social, and Corporate Governance (ESG) factors from 2013 to 2020, offering insights into the growing importance of sustainability in the financial performance of the energy sector. Korzeb et al. [149] used the Hellwig method with equal and expert weights to identify groups of countries with similar levels of supervisory stringency based on financial penalties as a supervisory action. From a database of over 300 European banks, 53 banks (including the 50 largest European banks and three additional G-SIIs designated by the EBA) were identified with penalties imposed between 2005 and 2022. Finally, Roszkowska and Wachowicz [129] examined how different normalization methods affect entropy-based weights and rankings obtained by Hellwig's methods in the context of sustainable development in the EU education sector in 2021 year. Their simulation study, which involved adjusting Eurostat data, explored how various normalization relationships among criteria influence entropy-based weights and the outcomes of Hellwig's method. The results were then compared to rankings generated by Hellwig's method using equal weights.

The next two studies utilize a modified concept of the ideal solution in the Hellwig method. Roszkowska and Filipowicz-Chomko [9] proposed a framework to compare the sustainable performance of EU countries in education on a national level. Their extended Hellwig's method incorporates EU and national targets to build a development pattern and applies three variants of Hellwig's method—classical, with EU targets, and with national targets—using a max-min normalization formula. In study [124], Antczak and Wiaderny applied Hellwig's dynamic measure of development to assess sustainable urban mobility in selected EU countries during the period 2011–2020. This approach facilitated the ranking of countries and provided a comparative analysis of the magnitude and direction of changes in sustainable urban mobility development over both individual years and the entire study period.

The following group of papers, while focusing on different topics, applied various distance metrics in the Hellwig method, sometimes incorporating modifications to normalization or weighting systems. The results obtained using the Hellwig method were also compared with those generated by the TOPSIS method. In study [134], Dymtrów compared four methods: Taxonomic Measure of Location's Attractiveness based on Hellwig's approach, Generalised Distance Measure, the TOPSIS method using Euclidean distances, and the TOPSIS method using GDM distances. This comparison aimed to evaluate their effectiveness in selecting locations for the order-picking process. The study applied vector normalization and seven combinations of expert weights to problem with three criteria for each method. Łuczak and Kalinowski [11] compared methodological approaches for constructing a synthetic measure of subjective household poverty. They examined the use of TOPSIS and Hellwig's method, incorporating GDM2 for the classical approach and vertex distance for the fuzzy approach as distance measures. Both the positional TOPSIS method and the positional Hellwig method were employed to estimate the standard of living in administrative districts in Wielkopolska Voivodeship in 2010. Roszkowska et al. [126] applied the Synthetic Measure for Ordinal Data (SMOD), based on the Hellwig framework, to rank overall satisfaction with airlines using reviews from the TripAdvisor website. The SMOD technique utilizes the GDM2 distance measure to handle data represented on an ordinal scale.

In another paper, Roszkowska [132] proposed extending Hellwig's method by incorporating the Mahalanobis distance, which handles correlations among criteria, particularly in asymmetrical relationships. The study also examined how different normalization formulas affect rankings, revealing that both the distance measure and normalization formulas influence the final rankings. Additionally, Roszkowska et al. [133] introduced an enhanced Hellwig's method integrating entropy-based weighting and the Mahalanobis distance, applied to evaluate progress toward Sustainable Development Goal 4 (education) in EU countries for 2021. Entropy is used to determine weights based on the informational content of matrix data, while the Mahalanobis distance accounts for interdependencies among criteria.

The next series of papers explores various modifications to the Hellwig method, including multidimensional scaling, normalization factors based on the distance between the ideal and anti-ideal, and maximum values. For instance, Walesiak [8] examined the concept of isoquants and the development path—defined as the shortest route connecting a pattern and an anti-pattern object proposed in paper [125]—using the Euclidean distance in the Hellwig procedure. Dehnel and Walesiak [150] and Walesiak and Dehnel [151] measured social cohesion at the provincial level in Poland, employing metric and interval-valued data to rank provinces using an enhanced Hellwig method that incorporated both patterns and anti-patterns. Similarly, Dehnel et al. [152] assessed changes in population aging in the V4 countries by applying multidimensional scaling, a composite indicator based on Hellwig's approach and Theil decomposition.

Fura and Wang [137] utilized the Hellwig method with a normalization factor based on maximum values to analyze the level of socio-economic development of EU countries and the state of International Organization for Standardization (ISO) 14001 environmental management system certification. In a related study, Hajduk [153] applied Hellwig's method with the same normalization factor to evaluate the diversity of smart cities in Europe, using data from the World Council on City Data and adhering to the International Organization for Standardization (ISO) 37120 standard for sustainable community development. The study revealed that cities with higher certification levels demonstrated better management and greater civic engagement, with significant disparities in smartness levels among the four ranked categories.

Bieszk-Stolorz and Dmytrów [154] assessed the attractiveness of large Polish cities for marketplace trade development between 2008 and 2019 using the Hellwig method with the maximum value as the normalization factor and based on data from the Local Data Bank. Additionally, Gierusz-Matkowska et al. [155] introduced the Fundamental Power of the City Index, focusing on 18 voivodeship capital cities in Poland and incorporating variables related to sustainable, smart, and resilient city indicators. Their research applied six linear ordering methods, including the Hellwig measure with a normalization factor based on distances between ideal and anti-ideal solutions. Inspired by the classic Hellwig's method, Roszkowska [52] proposed a new multi-criteria approach called Distances to Aspiration Reference Points (DARP) for evaluating alternatives with varying aspiration levels. This approach was used to measure the smart growth of EU countries in 2017 within the framework of the Europe 2020 Strategy.

The next series of papers utilizes a positional variant of the Hellwig method based on Weber's median. Frequently, the results obtained using this method are compared with those from the classic Hellwig method. For example, Wysocki [6] evaluated the standard of living in the Wielkopolska Voivodeship of Poland using both the classical Hellwig method and the positional Hellwig method based on Weber's median. The findings demonstrated the effectiveness of the positional method, particularly when dealing with datasets characterized by strong asymmetry or outliers. Similarly, Cheba and Szopik-Depczyńska [156] applied both the classical and positional Hellwig methods to analyze the spatial differentiation of EU countries and European regions in terms of their competitive capacity. Cheba and Bąk [157] used these methods to explore the relationship between sustainable development and the green economy within the EU. Czech et al. [158] employed these methods to assess urban transport development in selected Polish voivodeships using data from the Central Statistical Office for 2013–2016. In another study, Czech et al. [159] conducted a dynamic evaluation of sustainable transportation development in EU countries, using the positional Hellwig method with data sourced from Eurostat databases. Additionally, Gostkowski et al. [77] assessed the agricultural market in Poland for the years 2006, 2009, 2013, and 2016, applying various methods including the classic Hellwig method, the positional Hellwig method, TOPSIS, and SAW

The final series of papers highlights the potential applications of a variant of the Hellwig method known as spatial measures of development. Antczak [139] utilized both classical and spatial taxonomic measures to describe the level of sustainable development in Poland's voivodeships for the years 2000 and 2010. Kuc [7] applied spatial measures of development to analyze the convergence in living standards across EU countries from 1995 to 2012. In a subsequent study, Kuc [160] used various spatial measures based on the Hellwig approach, alongside the classical Hellwig method, to assess living standards across 67 NUTS-3 regions in Nordic countries, using data from national statistical offices. Additionally, Majdzińska [161] developed a typology of Polish regions based on demographic potential, employing spatial measures of development and data from the Central Statistical Office in Poland. This typology focused on age structure and trends in natural movement and migration within counties for the years 2005 and 2016.

Finally, the potential applications of the MIDIA method, which can be considered an extension of the classic Hellwig method, are explored in the work of Roszkowska and Filipowicz-Chomko [142]. In their study, they applied MIDIA, including the classic Hellwig method as one of its cases, to analyze the achievement of SDG 4 for EU member countries in 2022. They also compared these results with those obtained using the TOPSIS and VIKOR methods.

4.2. Hellwig-Based Methods with Data Represented by Fuzzy Numbers

The preliminary definitions of fuzzy sets, fuzzy numbers, fuzzy operations, and fuzzy distances are presented below.

The fuzzy set has been defined by Zadeh [162] as follows. Let X be a universe of discourse of objects. A fuzzy set $\hat{n}$ in X is given by

$$A = \{< x, \mu_{\hat{n}}(x) > | x \in X\}, \tag{23}$$

where $\mu_{\hat{n}}$: $X \to [0,1]$ is a function with the condition $0 \leq \mu_{\hat{n}}(x) \leq 1$ for every $x \in X$. The value $\mu_{\hat{n}}(x)$ denotes degree of membership of the element $x \in X$ to the set $\mu_{\hat{n}}$.

A fuzzy number is a fuzzy subset of the universe of discourse X that is both convex and normal. The membership function of a triangular fuzzy number (TFN) is expressed in the following way [55]:

$$\mu_{\hat{n}}(x) = \begin{cases} 0 \text{ for } x < a \\ \frac{x-a}{b-a} \text{ for } a \leq x \leq \ b \\ \frac{x-c}{b-c} \text{ for } b < x \leq \ c \\ 0 \text{ for } x > c \end{cases} \tag{24}$$

Then, the triangular fuzzy number $\hat{n}$ can be represented by (a, b, c), where $a$is the left threshold value, b is the midpoint, and c is the right threshold value. Note that both the crisp data a and the interval data (a, c) can be regarded as triangular fuzzy numbers of the following form (a, a, a) and $\left(a, \frac{a+c}{2}, c\right)$, respectively. A triangular fuzzy number (a, b, c) is positive if $a > 0$.

For any two positive triangular fuzzy numbers, $\hat{m} = (a_1, b_1, c_1)$ and $\hat{k} = (a_2, b_2, c_2)$ $(a_1, a_2 > 0)$, and a positive real number r, the main operations of fuzzy numbers $\hat{m}$ and $\widetilde{k}$ can be expressed as follows [55].

$$\text{Addition: } \hat{m} \oplus \hat{k} = (a_1 + a_2, b_1 + b_2, c_1 + c_2), \tag{25}$$

$$\text{Multiplication by a scalar: } \hat{m} \otimes r = (a_1 r, b_1 r, c_1 r), \tag{26}$$

$$\text{Multiplication: } \hat{m} \otimes \hat{k} \cong (a_1 a_2, b_1 b_2, c_1 c_2). \tag{27}$$

$$\text{Max of TFN:} max\left(\hat{m}, \hat{k}\right) = (max(a_1, a_2), max(b_1, b_2), max(c_1, c_2)) \tag{28}$$

$$\text{Min of TFN: } min\left(\hat{m}, \hat{k}\right) = (minx(a_1, a_2), minx(b_1, b_2), minx(c_1, c_2)) \tag{29}$$

$$\text{Vertex distance: } d\left(\hat{m}, \hat{k}\right) = \sqrt{\frac{1}{3}\left((a_1 - a_2)^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2\right)} \tag{30}$$

$$\text{Defuzzification formula : } df(\hat{m}) = \frac{a_1 + b_1 + c_1}{3}. \tag{31}$$

Let $A = \{A_1, A_2, \ldots, A_m\}, i = 1, 2, \ldots, m$ be the set of alternatives and $C = \{C_1, C_2, \ldots, C_n\}$, $j = 1, 2, \ldots, n$ the set of criteria, where $\overset{\frown}{x}_{ij} = (a_{ij}, b_{ij}, c_{ij})$ is the evaluation of the ith alternative with respect to the jth criterion represented by positive triangular fuzzy number $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$.

We will now review the fuzzy Hellwig-based methods, focusing on the steps of the algorithm (see Figure 1).

Step 1. Definition of a fuzzy decision matrix

The fuzzy decision matrix has the following form:

$$FD = \left[\overset{\frown}{x}_{ij}\right], \tag{32}$$

where $\overset{\frown}{x}_{ij} = (a_{ij}, b_{ij}, c_{ij})$ represents the evaluation of the ith alternative with respect to the jth criterion, expressed as a triangular fuzzy number $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$. If the performance ratings of alternatives on qualitative criteria are expressed by linguistic terms, using linguistic terms, these terms can be represented by triangular fuzzy numbers (TFNs) as shown in Table 6.

Table 6. Linguistic terms for the option ratings.

L	LT	FTN
1	Very poor	(0, 0, 1)
2	Poor	(0, 1, 3)
3	Medium poor	(1, 3, 5)
4	Fair	(3, 5, 7)
5	Medium good	(5, 7, 9)
6	Good	(7, 9, 10)
7	Very good	(9, 10, 10)

Source: see [6,163].

Step 2. Determination of the system of weights

The system of weights is represented as shown in Formula (8). The weights can also be expressed as positive triangular fuzzy numbers (see Table 7).

Table 7. Linguistic terms for the weight ratings.

L	LT	FTN
1	Absolutely low importance	(0.0, 0.1, 0.2)
2	Very low importance	(0.1, 0.2, 0.3)
3	Low importance	(0.2, 0.3, 0.4)

Table 7. *Cont.*

L	LT	FTN
4	Medium low importance	(0.3, 0.4, 0.5)
5	Medium importance	(0.4, 0.5, 0.6)
6	Medium high important	(0.5, 0.6, 0.7)
7	Hight importance	(0.6, 0.7, 0.8)
8	Very high importance	(0.7, 0.8, 0.9)
9	Absolutely high importance	(0.8, 0.9, 1.0)

Source: see [6,163].

Step 3. Construction of the fuzzy normalized decision matrix

The normalized fuzzy decision matrix is given by

$$\widehat{ID} = [\hat{z}_{ij}], \tag{33}$$

where $\hat{z}_{ij}$ is expressed as

$$\widehat{z}_{ij} = \left(\frac{a_{ij}}{c_j^{max}}, \frac{b_{ij}}{c_j^{max}}, \frac{c_{ij}}{c_j^{max}}\right) \text{ if } j \text{ is } a \text{ benefit criterion} \tag{34}$$

$$\widehat{z}_{ij} = \left(\frac{a_j^{min}}{c_{ij}}, \frac{a_j^{min}}{b_{ij}}, \frac{a_j^{min}}{a_{ij}}\right) \text{ if } j \text{ is } a \text{ cost criterion.} \tag{35}$$

Here, $a_j^{min} = \underset{i}{min} a_{ij}$, $c_j^{max} = \underset{i}{max} c_{ij}$, and values a_{ij}, b_{ij}, c_{ij} represent the descriptors of the fuzzy option $\widehat{x}_{ij} = (a_{ij}, b_{ij}, c_{ij})$.

Step 4. Construction of the weighted fuzzy normalized decision matrix

The weighted normalized fuzzy decision matrix is expressed as

$$\widehat{V} = \left[\widehat{r}_{ij}\right], \tag{36}$$

where

$$\widehat{r}_{ij} = w_j \otimes \widehat{z}_{ij} \text{ for } i= 1, \ldots, m; j= 1, \ldots, n. \tag{37}$$

Step 5. Identification of the fuzzy ideal

The fuzzy positive ideal ($\hat{F}I$) is defined as

$$\hat{F}I = \left(\hat{v}_1^+, \hat{v}_2^+, \ldots, \hat{v}_n^+\right) = \left(\max_i \widehat{r}_{i1}, \max_i \widehat{r}_{i2}, \ldots, \max_i \widehat{r}_{in}\right) \tag{38}$$

where $\max_i \widehat{r}_{ij}$ represents the maximum of the triangular fuzzy numbers (see Formula (28)).

Step 6. Calculation of the distance of the ith alternative from the ideal point

The distance of each alternative from the ideal solution is calculated using the formula:

$$d_i^+ = \sum_{j=1}^{n} d(\widehat{r}_{ij}, \widehat{v}_j^+), \quad i = 1, 2, \ldots, m \tag{39}$$

where $d(\hat{x}, \hat{y})$ represents the vertex distance between two triangular fuzzy numbers $\hat{x}$, $\hat{y}$ (see Formula (30)).

Step 7. Calculation the fuzzy Hellwig's measure for the ith alternative

The following formula is used to calculate the aggregation of the fuzzy Hellwig's measure:

$$FH_i = 1 - \frac{d_i^+}{d_o} \tag{40}$$

where d_o is determined as in Formula (6).

Step 8. Ranking of alternatives according to the decreasing values of fuzzy Hellwig's measure

All alternatives A_i $(i = 1, \ldots, m)$ are ranked in descending order based on $FH(A_i)$.

Below, we will discuss an overview of approaches and applications of the Hellwig-based method in a fuzzy environment.

Łuczak and Wysocki [131] introduced modifications to the classical Hellwig method by incorporating fuzzy numbers into the synthetic measure calculations, allowing for the aggregation of both quantitative and qualitative criteria. Additionally, it integrates weights for criteria and features derived from the Analytic Hierarchy Process (AHP). This approach was demonstrated through an evaluation of socio-economic development levels in districts of the Wielkopolska Poland Voivodeship, utilizing survey and statistical data from all 31 county administrations in the region.

Subsequently, Wysocki [6] advanced the classical Hellwig's method by developing a fuzzy version. In this approach, values from the ordinal scale are converted into triangular fuzzy numbers, defined by three parameters: a for the pessimistic evaluation, b for the most probable evaluation, and c for the optimistic evaluation. After this conversion, the fuzzy numbers are normalized using Formulas (34) and (35). Next, a weighted normalized data matrix is established, incorporating the influence of a system of weights. The triangular fuzzy numbers are then converted into real numbers, typically using a simple arithmetic mean (see Formula (31)). The procedure continues with the application of the remaining steps of the classical Hellwig method, as outlined in Formulas (5) and (6). Finally, the objects are ranked based on the non-increasing values of the fuzzy Hellwig's measure.

Jefmański [10] assessed the quality of Poland's Central Statistical Office website using data from the WebQual survey questionnaire, which gathered opinions from students at the Wrocław University of Economics and the Cracow University of Economics in 2012. This questionnaire enabled the transformation of measurement results from linguistic values to numerical ones using fuzzy sets. The classical fuzzy TOPSIS and fuzzy Hellwig methods, as proposed by Wysocki [6], were employed to rank the website's quality features. In a later paper, Jefmański and Dudek [164] further explored Hellwig's measure of development for triangular fuzzy numbers.

Łuczak and Kalinowski [11] employed fuzzy Hellwig and fuzzy TOPSIS methods to evaluate subjective household poverty levels in Poland. Their approach, based on fuzzy methodologies, inspired by Hellwig [4], and further developed by Łuczak and Wysocki [6,131], involved transforming ordinal variables into triangular fuzzy numbers. They calculated both the fuzzy positive ideal solution (ideal point) and the fuzzy negative ideal solution (anti-ideal point). Additionally, they used the vertex method (see Formula (30)) to determine the distances between these solutions and the normalized fuzzy values of the assessed households.

Finally, Roszkowska et al. [17] introduced a linguistic method based on Hellwig's approach and oriented fuzzy numbers for building a scoring system for negotiation templates. The study proposed two variants of this method and compared them with the Simple Additive Weighting (SAW) and TOPSIS methods. The results indicated that the proposed methods could serve as alternatives to SAW and TOPSIS, especially when dealing with oriented fuzzy numbers.

4.3. Hellwig-Based Methods with Data Represented by Intuitionistic Fuzzy Numbers

The basic definitions of intuitionistic fuzzy sets, intuitionistic fuzzy entropy, and intuitionistic fuzzy distances are provided below.

The intuitionistic fuzzy set (IFS) theory, introduced by Atanassov [165], extends Zadeh's Fuzzy Set (FS) theory [162] to address issues related to uncertainty by including the notion of hesitation or indeterminacy. Atanassov [165] defined intuitionistic fuzzy sets as follows. Let X be a universe of discourse of objects. An intuitionistic fuzzy set A in X is given by

$$A = \{< x, \mu_A(x), \nu_A(x) > |x \in X\} \tag{41}$$

where μ_A, ν_A: $X \rightarrow [0,1]$ are functions with the condition for every $x \in X$

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1. \tag{42}$$

The values $\mu_A(x)$ and $\nu_A(x)$ represent degrees of membership and non-membership, respectively, of the element $x \in X$ to the set A. Additionally, $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ denotes the intuitionistic fuzzy index (hesitation margin) of the element x in the set A.

When $\pi_A(x) = 0$ for $x \in X$, the intuitionistic fuzzy set A simplifies to an ordinary fuzzy set. A higher value of $\pi_A(x)$ signifies an increase in uncertainty. An intuitionistic fuzzy value (IFV) denoted as $A = (\mu_A, \nu_A)$, refers to an intuitionistic fuzzy set where the universe X consists of only one element x. The intuitionistic value (1, 0) is the largest, while (0, 1) is the smallest in the set of all intuitionistic fuzzy values (IFVs).

The entropy method for determining weights has been adapted to represent data using intuitionistic fuzzy numbers. Many researchers have explored entropy measures within the context of intuitionistic fuzzy systems [166–169]. Chen and Li [169] provides an overview of various approaches and formulas for defining entropy measures for intuitionistic fuzzy sets, with a focus on their application in determining objective weights. In this context, we present four intuitionistic entropy measures that have been employed to establish criteria weights in the Hellwig-based method within an intuitionistic fuzzy environment in Roszkowska et al. [170].

The first proposition is by Burillo and Bustince [166], who defined the entropy measure as the degree of hesitancy in intuitionistic fuzzy sets (IFS):

$$E_1(A) = \sum_{j=1}^{n} (1 - (\mu_A(x_j) + \nu_A(x_j)) = \sum_{j=1}^{n} \pi_A(x_j). \tag{43}$$

The next approach was introduced by Szmidt and Kacprzyk [167] who developed a non-probabilistic entropy measure based on a geometric interpretation of intuitionistic fuzzy sets and applied a ratio of distances between them in the following way:

$$E_2(A) = \frac{a}{b}, \tag{44}$$

where a is the distance $(A; A_{near})$ from A to the nearer point A_{near} among points (1, 0, 0) and (0, 1, 0), and b is the distance $(A; A_{far})$ from A to the farther point A_{far} among points (1, 0, 0) and (0, 1, 0).

The third definition, proposed by Vlachos and Sergiadis [171], is based on the concept of cardinality for intuitionistic fuzzy sets and utilizes an axiomatic framework. The formula is as follows:

$$E_3(A) = \frac{\sum_{j=1}^{n} \left(min\left(\mu_A(x_j), \nu_A(x_j)\right) + min\left(1 - \mu_A(x_j), 1 - \nu_A(x_j)\right)\right)}{\sum_{j=1}^{n} \left(max\left(\mu_A(x_j), \nu_A(x_j)\right) + max\left(1 - \mu_A(x_j), 1 - \nu_A(x_j)\right)\right)}. \tag{45}$$

The last approach, also proposed by Vlachos and Sergiadis [168], introduced intuitionistic entropy measures for IFSs based on the information theory. The formula is as follows:

$$E_4(A) = -\frac{1}{nln2} \sum_{j=1}^{n} [\mu_A(x_j) ln\mu_A(x_j) + \nu_A(x_j) ln\nu_A(x_j) - (1 - \pi_A(x_j)) ln(1 - \pi_A(x_j)) - \pi_A(x_j)\, ln2], \tag{46}$$

where $\frac{1}{nln2}$ —is the constant that assures $0 \leq E_4(A) \leq 1$.

Below, we present the weighted Hamming and Euclidean distances formulas between two intuitionistic fuzzy sets (IFSs), initially introduced by Xu [172]. These measures have been employed in studies utilizing the Hellwig-based method for data represented by intuitionistic fuzzy sets.

The weighted Euclidean distance between $A, B \in IFS$ with membership functions $\mu_A(x)$, $\mu_B(x)$ and non-membership functions $\nu_A(x)$, $\nu_B(x)$) is determined using the following formulas [172]:

$$e^2_{IFS}(A,B) = \sqrt{\frac{1}{2}\sum_{j=1}^{n} w_j\left[\left(\mu_A(x_j) - \mu_B(x_j)\right)^2 + \left(\nu_A(x_j) - \nu_B(x_j)\right)^2\right]} \tag{47}$$

$$e^3_{IFS}(A,B) = \sqrt{\frac{1}{2}\sum_{j=1}^{n} w_j\left[\left(\mu_A(x_j) - \mu_B(x_j)\right)^2 + \left(\nu_A(x_j) - \nu_B(x_j)\right)^2 + \left(\pi_A(x_j) - \pi_B(x_j)\right)^2\right]}, \tag{48}$$

where $\sum_{j=1}^{n} w_j = 1$.

The weighted Hamming distance between $A, B \in IFS$ with membership functions $\mu_A(x)$, $\mu_B(x)$ and non-membership functions $\nu_A(x)$, $\nu_B(x)$ is determined using the following formulas [172]:

$$h^2_{IFS}(A,B) = \frac{1}{2}\sum_{j=1}^{n} w_j\left[\left|\mu_A(x_j) - \mu_B(x_j)\right| + \left|\nu_A(x_j) - \nu_B(x_j)\right|\right] \tag{49}$$

$$h^3_{IFS}(A,B) = \frac{1}{2}\sum_{j=1}^{n} w_j\left[\left|\mu_A(x_j) - \mu_B(x_j)\right| + \left|\nu_A(x_j) - \nu_B(x_j)\right| + \left|\pi_A(x_j) - \pi_B(x_j)\right|\right] \tag{50}$$

It is noteworthy that when the weights in Formulas (47)–(50) are equal, the result is the normalized Euclidean or Hamming distance. The combination of membership degree, non-membership degree, and hesitancy degree provides a more accurate and comprehensive representation of the decision-making context. Specifically, the hesitancy degree indicates the extent of uncertainty or lack of information during the decision-making process.

The motivation to expand the classical Hellwig method through the use of intuitionistic sets was driven by the need to analyze complex problems, particularly when the data are represented in the form of surveys [12,173]. This approach incorporates ordinal scales while taking into account the lack of responses or the response "I don't know".

The preliminary stage involves representing the data collected from the questionnaire survey in the form of intuitionistic fuzzy values. Respondents' answers are collected through a questionnaire survey, which evaluates a set of objects (alternatives) and a set of questions (criteria) using an ordinal scale. Initially, the opinions of respondents regarding the i-th alternative for criterion C_j are classified into three categories: "positive opinion about the alternative", "negative opinion about the alternative", and "no opinion or no answer". Let p_{ij} represent the total number of "positive opinions", n_{ij} is the total number of "negative opinions" and h_{ij} is the total number of responses categorized as "no opinion or no answer". The total number of respondents is denoted as N_{ij}, where i corresponds to the alternative number, and j corresponds to the criterion number. It is worth noting $p_{ij} + n_{ij} + h_{ij} = N_{ij}$. Following this, the aggregated opinions are expressed as intuitionistic fuzzy values $\left(\mu_{ij}, \nu_{ij}\right)$, represented by the equations:

$$\mu_{ij} = \frac{p_{ij}}{N_{ij}},\ \nu_{ij} = \frac{n_{ij}}{N_{ij}},\ \pi_{ij} = \frac{h_{ij}}{N_{ij}}, \tag{51}$$

where i indicates the alternative number, and$j$ indicates the criterion number.

Now, we will review the intuitionistic fuzzy variants of the Hellwig-based methods, focusing on the steps of the algorithm (see Figure 1). Let $A = \{A_1, A_2, \ldots, A_m\}, i = 1, 2, \ldots, m$ represent the set of alternatives and $C = \{C_1, C_2, \ldots, C_n\}, j = 1, 2, \ldots, n$ represent the set of criteria. The representation of the ith alternative with respect to the jth criterion is given by an intuitionistic fuzzy value (μ_{ij}, v_{ij}), where $\pi_{ij} = 1 - \mu_{ij} - v_{ij}$. As a result, alternative A_i is described by the vector: $A_i = [(\mu_{i1}, v_{i1}), \ldots, (\mu_{in}, v_{in})]$, where $i = 1, 2, \ldots, m$. The general framework and intuitionistic fuzzy Hellwig-based method consist of the following steps [170,173]:

Step 1. Definition of an intuitionistic fuzzy decision matrix

The intuitionistic fuzzy decision matrix (*IFD*) is defined as follows:

$$IFD = \begin{bmatrix} (\mu_{11}, v_{11}) & (\mu_{12}, v_{12}) & \cdots & (\mu_{1n}, v_{1n}) \\ (\mu_{21}, v_{21}) & (\mu_{22}, v_{22}) & \cdots & (\mu_{2n}, v_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (\mu_{m1}, v_{m1}) & (\mu_{m2}, v_{m2}) & \cdots & (\mu_{mn}, v_{mn}) \end{bmatrix} \quad (52)$$

where (μ_{ij}, v_{ij}) is the intuitionistic fuzzy evaluation of the ith alternative in terms of the jth criterion and $\pi_{ij} = 1 - \mu_{ij} - v_{ij}$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$.

Step 2. Determination of the system of weights

Table 8 describes the weighting methods that have been be utilized in the Hellwig-based method in intuitionistic fuzzy environments [12,170,173].

Table 8. Weighting methods used in the Hellwig-based method in intuitionistic fuzzy environments.

Weights	Formula
Equal weights	$w_j = \frac{1}{n}, j = 1, 2, \ldots, n.$
Intuitionistic fuzzy Entropy-based weights	$w_j = \frac{1 - S_j}{\sum_{j=1}^{n}(1 - S_j)}$, $S_j = \frac{\sum_{i=1}^{m} E_{ij}}{\sum_{j=1}^{n}\sum_{i=1}^{m} E_{ij}}$, E_{ij} is an entropy value of IF value (μ_{ij}, v_{ij}) from the matrix IFD $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$. Entropy value E_{ij} can be calculated using one of the Formulas (43)–(46).

Source: own study.

Step 3. Identification of the intuitionistic fuzzy ideal

The determination of the intuitionistic fuzzy ideal can be approached in two ways, as outlined by [12,15,170,173]. The proposals of intuitionistic fuzzy ideals are presented in Table 9.

Table 9. Intuitionistic fuzzy ideals applied in the Hellwig-based method in intuitionistic fuzzy environments.

Ideal Description	Formula
Derived from the maximum intuitionistic fuzzy value (IFV)	$I^1_{IFI} = [(1, 0), \ldots, (1, 0)]$
Based on both maximum and minimum values form the set of data	$I^2_{IFI} = \left[\left(\max_i \mu_{i1}, \min_i v_{i1}\right), \ldots, \left(\max_i \mu_{in}, \min_i v_{in}\right)\right]$, where (μ_{ij}, v_{ij}), denote the evaluation of the ith alternative with respect to the jth criterion and $\pi_{ij} = 1 - \mu_{ij}(x) - v_{ij}(x)$.

Source: own study based on [12].

Step 4. Calculation of the distance of the ith alternative from the intuitionistic fuzzy ideal

Following the selection of the distance measure, the computation of distances between objects and the intuitionistic fuzzy ideal, chosen in step 3, is conducted using one of the Formulas (47)–(50). The expression for the distance measure from the pattern object is given [12,15,174] as

$$d_i^+(A_i, IFI) = d(I_{IFS}, A_i) \quad (53)$$

where $I_{IFS} \in \{I^1_{IFS}, I^2_{IFS}\}$, $d \in \{e^3_{IFS}, e^2_{IFS}, h^2_{IFS}, h^3_{IFS}\}$.

Table 10 presents a comparison of the proposed variants of the intuitionistic fuzzy method based on Hellwig's framework, considering the distance approaches—Hamming and Euclidean—each utilizing two or three parameters. In conjunction with the two types of intuitionistic fuzzy ideals outlined in Table 9, this results in a total of eight distinct variants of the Hellwig method within an intuitionistic fuzzy environment.

Table 10. Distance measures used in Hellwig-based method in intuitionistic fuzzy environments.

Distance Measure	Formula
Weighted Euclidean with two parameters	$d_i^+(A_i, IFI) = e_{IFS}^2(I_{IFS}, A_i)$
Weighted Euclidean with three parameters	$d_i^+(A_i, IFI) = e_{IFS}^3(I_{IFS}, A_i)$
Weighted Hamming with two parameters	$d_i^+(A_i, IFI) = h_{IFS}^2(I_{IFS}, A_i)$
Weighted Hamming with three parameters	$d_i^+(A_i, IFI) = h_{IFS}^3(I_{IFS}, A_i)$

Source: own study based on [12]. *Notes:* $I_{IFS} \in \{I_{IFS}^1, I_{IFS}^2\}$ defined in Table 9.

Step 5. Calculation of the intuitionistic fuzzy Hellwig measure for the ith alternative.

The intuitionistic fuzzy Hellwig-based measure (*IFH*) is determined using the following formula

$$IFH(A_i) = 1 - \frac{d_i^+(A_i,\ IFI)}{N}, \tag{54}$$

where N is a normalization factor.

The normalization factors used in in Hellwig's modification procedures are presented in Table 11 [12,14].

Table 11. Normalization factors used in Hellwig-based method in intuitionistic fuzzy environments.

Normalization Factor	Formula
Classical approach	$N = \bar{d} + 2S, \bar{d} = \frac{1}{m}\sum_{i=1}^m d_i^+, S = \sqrt{\frac{1}{m}\sum_{i=1}^m \left(d_i^+ - \bar{d}\right)^2}, (i = 1, 2, \ldots, m).$
Distance between intuitionistic fuzzy ideal and anti-ideal	$N = d_i^+(IFI, IFAI)$ where IFI is the intuitionistic fuzzy ideal and $IFAI$ is the intuitionistic fuzzy anti-ideal

Source: own study.

Several studies have presented various modifications of Hellwig's method, integrating intuitionistic fuzzy sets and exploring different techniques for defining ideals [13,15,173,174], developing weighting systems [14,170], applying distance measurement methods [12,174], and implementing normalization [12,14], especially for ordinal data. Jefmański [173] introduced an Intuitionistic Fuzzy Synthetic Measure (IFSM) for ordinal data, defining the ideal solution using max operators, equal weights, and the Euclidean distance. This method was applied to analyze the subjective quality of life in the Kraina Łęgów Odrzański region and was compared with both fuzzy and classical versions of Hellwig's approach. Building on this, Jefmański et al. [12] proposed an expanded IFSM framework, offering two methods for defining intuitionistic fuzzy ideals—one based on data-derived maximum values and the other on maximum intuitionistic values—along with two distance measures (Euclidean and Hamming), resulting in eight IFSM variants. The framework was successfully applied to assess satisfaction with public administration and quality of life in cities. Kusterka-Jefmańska et al. [174] used IFSM to measure subjective quality of life in selected European cities using 2019 IPSOS data, accounting for uncertainty and testing the sensitivity of results to different distance measures and ideal-defining methods. In a related study, Roszkowska et al. [170] enhanced IFSM by incorporating IF entropy-based weights to handle uncertainty in criterion importance, with four methods producing similar results and nearly identical city rankings. Kusterka-Jefmańska et al. [15] also applied IFSM, with the ideal represented as a maximum intuitionistic fuzzy number, to evaluate changes in subjective quality of life (SQoL) in selected European cities over time. The analysis revealed a systematic improvement in SQoL from 2006 to 2019, although significant disparities between cities persisted.

Roszkowska et al. [14] introduced the Double Intuitionistic Fuzzy Synthetic Measure (DIFSM), which incorporates two reference points—ideal and anti-ideal—along with entropy-based weights. An example involving the selection of an air-conditioning system demonstrated the practical application of the approach.

In another study, Roszkowska [13] proposed an intuitionistic fuzzy framework for evaluating and rank-ordering multi-issue negotiation offers using Hellwig's approach. The assessment was conducted in bipolar terms, considering both the advantages and disadvantages of each option through the lens of intuitionistic fuzzy concepts. The framework employed a weighted intuitionistic fuzzy decision matrix and a normalized distance measure to determine the distance of each alternative from the ideal point, based on the maximum intuitionistic fuzzy value. This approach effectively prevents rank reversal when new offers are introduced.

Paper [175] presents IFMCDM, an open-source R package that enables the evaluation and ranking of alternatives in an intuitionistic fuzzy context. The package includes two multi-criteria methods: Intuitionistic Fuzzy TOPSIS and Intuitionistic Fuzzy Synthetic Measure based on Hellwig's approach offering two representations of reference points and two distance measures (Hamming and Euclidean), which allow for four variants of each method.

Finally, Roszkowska and Jefmański [16] introduced a synthetic measure based on Hellwig's approach and interval-valued intuitionistic fuzzy set theory (I-VIFS) to evaluate complex social phenomena under uncertainty using ordinal scales from questionnaire surveys. They defined an optimism coefficient to set limits for intuitionistic fuzzy parameters, demonstrating the measure's usefulness through a survey on the subjective quality of life in selected Polish communes.

5. Conclusions

This paper offers a comprehensive state-of-the-art review of Hellwig's Taxonomic Measure of Development (TMD) and its various modifications, emphasizing pioneering extensions that could inspire further research in the field. The Hellwig method is widely regarded for its flexibility, simplicity, and low computational demands, making it a powerful tool for multi-criteria analysis. It provides a structured approach to analyzing a wide range of socio-economic issues, particularly those involving multidimensional concepts that are difficult to quantify.

Although Hellwig's approach shares similarities with the widely used TOPSIS method in its use of distance to evaluate alternatives, it is distinct in its exclusive focus on proximity to an ideal solution without considering the distance from a negative ideal. Since the original version of Hellwig's method was proposed, several adaptations have emerged in the literature to address specific conditions. For example, the Mahalanobis distance has been proposed to account for dependencies across criteria [132,133], the median Weber approach to address outliers [6,130], and the spatial weight matrix to allow for simultaneous analysis across three dimensions: section, time, and space [7,139–141]. Fuzzy approaches have been introduced to address uncertainty [6,10,131], and an intuitionistic approach was developed for ordered data [12,173]. Subsequent developments include other fuzzy methods [16,131,173], dynamic approaches [15,124], and frameworks designed to prevent rank reversal issues [15,17].

Through a series of preliminary searches, key topics extending the TMD method were identified, focusing on data collection, criterion weighting, data normalization, ideal value determination, distance calculation, and normalization factors. This process enabled a refined selection of relevant studies, ultimately leading to the review of about 100 papers, primarily in English, categorized by the type of modification and application areas. The findings reveal that Hellwig-based methods are frequently used in socio-economic analyses, particularly in areas such as sustainable development, regional development, quality of life and living standards, smart growth and smart cities, financial markets and institutions, human capital and education, agricultural development, sustainability, and negotiation support. Approximately half of these studies applied the classical Hellwig approach, while the remaining works involve extensions or modifications of the Hellwig method, utilizing real numbers, linguistic evaluations, fuzzy numbers, and intuitionistic fuzzy numbers. Some of these studies also incorporate comparative analyses, either applying different Hellwig-based methods or using other approaches, with the TOPSIS method being the most common. This underscores the flexibility and adaptability of the Hellwig approach across a wide range of analytical frameworks.

The main contributions of this paper can be summarized as follows:

- A comprehensive review of the original Taxonomic Measure of Development (TMD) method and its extensions, including applications with fuzzy and intuitionistic fuzzy sets.
- The introduction of an original, step-by-step analytical framework, offering a detailed examination of the TMD process and providing new insights into its practical implementations.
- Highlighting the practical versatility of the TMD method across various domains, distinguishing it from other multi-criteria methods like TOPSIS.

To the best of the author's knowledge, this paper offers the first thorough exploration of Hellwig's Taxonomic Measure of Development and its modifications. It employs an original methodology rooted in a systematic review that adheres to the steps of algorithms while also showcasing practical applications across various domains.

While this study offers an extensive overview of state-of-the-art TMD, it does have limitations that must be acknowledged. The scope of the literature reviewed is confined to existing studies available up to a certain point, which may exclude recent advancements in the field. Additionally, this paper primarily focuses on selected variants of the TMD, potentially overlooking other relevant adaptations or applications. The field of multi-criteria decision-making is rapidly evolving, and new methodologies or frameworks are continually being developed. As such, the relevance of the TMD may change over time, and the findings of this paper may need to be revisited in light of future advancements.

This study aims to enrich both the theoretical foundations and practical applications of the TMD. However, the emphasis on theoretical exploration may overshadow practical considerations, making it less applicable to practitioners seeking straightforward guidance for implementation. The practical applications discussed in this paper may not fully encapsulate the complexities and nuances of real-world decision-making scenarios. Each application context can introduce unique challenges that may not be addressed by the generalizations made in the review.

Finally, the lack of extensive comparative analyses with other multi-criteria decision-making methods may limit insights into the TMD's relative strengths and weaknesses. Acknowledging these limitations is essential for understanding the context and implications of the findings.

Future research may address these limitations and contribute to a deeper understanding of the TMD's role in multi-criteria decision-making. Future research could also explore the development of hybrid MCDA methods that integrate the Hellwig approach with other established techniques, such as AHP, DEMATEL, PROMETHEE, and ELECTRE. These hybrid methods could be further enhanced by incorporating various uncertainty theories, including fuzzy set theory, rough set theory, and cloud theory. This could expand the applicability of the Hellwig method in more complex decision-making scenarios, especially in the fuzzy or intuitionistic fuzzy context.

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