applied sciences

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An NHPP Software Reliability Model with S-Shaped Growth Curve Subject to Random Operating Environments and Optimal Release Time

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Received: 26 October 2017; Accepted: 12 December 2017; Published: 16 December 2017

Abstract: The failure of a computer system because of a software failure can lead to tremendous losses to society; therefore, software reliability is a critical issue in software development. As software has become more prevalent, software reliability has also become a major concern in software development. We need to predict the fluctuations in software reliability and reduce the cost of software testing: therefore, a software development process that considers the release time, cost, reliability, and risk is indispensable. We thus need to develop a model to accurately predict the defects in new software products. In this paper, we propose a new non-homogeneous Poisson process (NHPP) software reliability model, with S-shaped growth curve for use during the software development process, and relate it to a fault detection rate function when considering random operating environments. An explicit mean value function solution for the proposed model is presented. Examples are provided to illustrate the goodness-of-fit of the proposed model, along with several existing NHPP models that are based on two sets of failure data collected from software applications. The results show that the proposed model fits the data more closely than other existing NHPP models to a significant extent. Finally, we propose a model to determine optimal release policies, in which the total software system cost is minimized depending on the given environment.

Keywords: software reliability; non-homogeneous Poisson process; optimal release time; mean squared error

1. Introduction

'Software' is a generic term for a computer program and its associated documents. Software is divided into operating systems and application software. As new hardware is developed, the price decreases; thus, hardware is frequently upgraded at low cost, and software becomes the primary cost driver. The failure of a computer system because of a software failure can cause significant losses to society. Therefore, software reliability is a critical issue in software development. This problem requires finding a balance between meeting user requirements and minimizing the testing costs. It is necessary to know in the planning cycle the fluctuation of software reliability and the cost of testing, in order to reduce costs during the software testing stage, thus a software development process that considers the release time, cost, reliability, and risk is indispensable. In addition, it is necessary to develop a model to predict the defects in software products. To estimate reliability metrics, such as the number of residual faults, the failure rate, and the overall reliability of the software, various non-homogeneous Poisson process (NHPP) software reliability models have been developed using a fault intensity rate function and mean value function within a controlled testing environment. The purpose of many

NHPP software reliability models is to obtain an explicit formula for the mean value function, m(t), which is applied to the software testing data to make predictions on software failures and reliability in field environments [1]. A few researchers have evaluated a generalized software reliability model that captures the uncertainty of an environment and its effects on the software failure rate, and have developed a NHPP software reliability model when considering the uncertainty of the system fault detection rate per unit of time subject to the operating environment [2–4]. Inoue et al. [5] developed a bivariate software reliability growth model that considers the uncertainty of the change in the software failure-occurrence phenomenon at the change-point for improved accuracy. Okamura and Dohi [6] introduced a phase-type software reliability model and developed parameter estimation algorithms using grouped data. Song et al. [7,8] recently developed an NHPP software reliability model to consider a three-parameter fault detection rate, and applied a Weibull fault detection rate function during the software development process. They related the model to the error detection rate function by considering the uncertainty of the operating environment. In addition, Li and Pham [9] proposed a model accounting for the uncertainty of the operating environment under the condition that the fault content function is a linear function of the testing time, and that the fault detection rate is based on the testing coverage.

In this paper, we discuss a new NHPP software reliability model with S-shaped growth curve applicable to the software development process and relate it to the fault detection rate function when considering random operating environments. We examine the goodness-of-fit of the proposed model and other existing NHPP models that are based on several sets of software failure data, and then determine the optimal release times that minimize the expected total software cost under given conditions. The explicit solution of the mean value function for the new NHPP software reliability model is derived in Section 2. Criteria for the model comparisons and the selection of the best model are discussed in Section 3. The optimal release policy is discussed in Section 4, and the results of a model analysis and the optimal release times are discussed in Section 5. Finally, Section 6 provides some concluding remarks.

2. A New NHPP Software Reliability Model

2.1. Non-Homogeneous Poisson Process

The software fault detection process has been formulated using a popular counting process. The counting process $\{N(t),\ t \geq 0\}$ is a non-homogeneous Poisson process (NHPP) with an intensity function $\lambda(t)$, if it satisfies the following condition.

(I) $N(0) = 0$
(II) Independent increments
(III) $\int_{t_1}^{t_2} \lambda(t)dt$, $(t_2 \geq t_1)$: the average of the number of failures in the interval $[t_1,\ t_2]$

Assuming that the software failure/defect conforms to the NHPP condition, $N(t)(t \geq 0)$ represents the cumulative number of failures up to the point of execution, and $m(t)$ is the mean value function. The mean value function $m(t)$ and the intensity function $\lambda(t)$ satisfy the following relationship.

$$m(t) = \int_0^t \lambda(s)ds, \frac{dm(t)}{dt} = \lambda(t). \tag{1}$$

$N(t)$ is a Poisson distribution involving the mean value function, $m(t)$, and can be expressed as:

$$\Pr\{N(t) = n\} = \frac{\{m(t)\}^n}{n!} \exp\{-m(t)\},\ n = 0, 1, 2, 3 \ldots. \tag{2}$$

2.2. General NHPP Software Reliability Model

Pham et al. [10] formalized the general framework for NHPP-based software reliability and provided analytical expressions for the mean value function $m(t)$ using differential equations.

The mean value function $m(t)$ of the general NHPP software reliability model with different values for $a(t)$ and $b(t)$, which reflects various assumptions of the software testing process, can be obtained with the initial condition $N(0) = 0$.

$$\frac{d\,m(t)}{dt} = b(t)[a(t) - m(t)]. \tag{3}$$

The general solution of (1) is

$$m(t) = e^{-B(t)}\left[m_0 + \int_{t_0}^{t} a(s)b(s)e^{B(s)}bs\right] \tag{4}$$

where $B(t) = \int_{t_0}^{t} b(s)ds$, and $m(t_0) = m_0$ is the marginal condition of (2).

2.3. New NHPP Software Reliability Model

Pham [3] formulated a generalized NHPP software reliability model that incorporated uncertainty in the operating environment as follows:

$$\frac{d\,m(t)}{dt} = \eta[b(t)][N - m(t)], \tag{5}$$

where η is a random variable that represents the uncertainty of the system fault detection rate in the operating environment with a probability density function g; $b(t)$ is the fault detection rate function, which also represents the average failure rate caused by faults; N is the expected number of faults that exists in the software before testing; and, m(t) is the expected number of errors detected by time t (the mean value function).

Thus, a generalized mean value function, m(t), where the initial condition m(0) = 0, is given by

$$m(t) = \int_{\eta} N\left(1 - e^{-\eta \int_0^t b(x)dx}\right) dg(\eta). \tag{6}$$

The mean value function [11] from (4) using the random variable η has a generalized probability density function g with two parameters $\alpha \geq 0$ and $\beta \geq 0$ and is given by

$$m(t) = N\left(1 - \frac{\beta}{\beta + \int_0^t b(s)ds}\right)^{\alpha}, \tag{7}$$

where b(t) is the fault detection rate per fault per unit of time.

We propose an NHPP software reliability model including the random operating environment using Equations (3)–(5) and the following assumptions [7,8]:

(a) The occurrence of a software failure follows a non-homogeneous Poisson process.
(b) Faults during execution can cause software failure.
(c) The software failure detection rate at any time depends on both the fault detection rate and the number of remaining faults in the software at that time.
(d) Debugging is performed to remove faults immediately when a software failure occurs.
(e) New faults may be introduced into the software system, regardless of whether other faults are removed or not.
(f) The fault detection rate $b(t)$ can be expressed by (6).
(g) The random operating environment is captured if unit failure detection rate $b(t)$ is multiplied by a factor η that represents the uncertainty of the system fault detection rate in the field

In this paper, we consider the fault detection rate function $b(t)$ to be as follows:

$$b(t) = \frac{a^2 t}{1 + at},\ a > 0,\ a, b > 0, \tag{8}$$

We obtain a new NHPP software reliability model with S-shaped growth curve subject to random operating environments, m(t), that can be used to determine the expected number of software failures detected by time t by substituting function b(t) above into (5) so that:

$$m(t) = N\left(1 - \frac{\beta}{\beta + at - \ln(1 + at)}\right)^{\alpha}. \quad (9)$$

3. Criteria for Model Comparisons

Theoretically, once the analytical expression for mean value function $m(t)$ is derived, then the parameters in $m(t)$ can be estimated using parameter estimation methods (MLE: the maximum likelihood estimation method, LSE: the least square estimation method); however, in practice, accurate estimates may not be obtained by the MLE, particularly under certain conditions where the mean value function $m(t)$ is too complex. The model parameters to be estimated in the mean value function $m(t)$ can then be obtained using a MATLAB program that is based on the LSE method. Six common criteria; the mean squared error (MSE), Akaike's information criterion (AIC), the predictive ratio risk (PRR), the predictive power (PP), the sum of absolute errors (SAE), and R-square (R^2) will be used for the goodness-of-fit estimation of the proposed model, and to compare the proposed model with other existing models, as listed in Table 1. These criteria are described as follows.

The MSE is

$$MSE = \frac{\sum_{i=0}^{n}(\hat{m}(t_i) - y_i)^2}{n - m}. \quad (10)$$

AIC [12] is

$$AIC = -2\log L + 2m. \quad (11)$$

The PRR [13] is

$$PRR = \sum_{i=0}^{n}\left(\frac{\hat{m}(t_i) - y_i}{\hat{m}(t_i)}\right)^2. \quad (12)$$

The PP [13] is

$$PP = \sum_{i=0}^{n}\left(\frac{\hat{m}(t_i) - y_i}{y_i}\right)^2. \quad (13)$$

The SAE [8] is

$$SAE = \sum_{i=0}^{n}|\hat{m}(t_i) - y_i|. \quad (14)$$

The correlation index of the regression curve equation (R^2) [9] is

$$R^2 = 1 - \frac{\sum_{i=0}^{n}(\hat{m}(t_i) - y_i)^2}{\sum_{i=0}^{n}(y_i - \overline{y_i})^2}. \quad (15)$$

Here, $\hat{m}(t_i)$ is the estimated cumulative number of failures at t_i for $i = 1, 2, \cdots, n$; y_i is the total number of failures observed at time t_i; n is the actual data which includes the total number of observations; and, m is the number of unknown parameters in the model.

The MSE measures the distance of a model estimate from the actual data that includes the total number of observations and the number of unknown parameters in the model. AIC is measured to compare the capability of each model in terms of maximizing the likelihood function (L), while considering the degrees of freedom. The PRR measures the distance of the model estimates from the actual data against the model estimate. The PP measures the distance of the model estimates from the actual data. The SAE measures the absolute distance of the model. For five of these criteria, i.e., MSE, AIC, PRR, PP, and SAE, the smaller the value is, the closer the model fits relative to other models run on the same dataset. On the other hand, R^2 should be close to 1.

We use (8) below to obtain the confidence interval [13] of the proposed NHPP software reliability model. The confidence interval is described as follows;

$$\hat{m}(t) \pm Z_{\alpha/2}\sqrt{\hat{m}(t)}, \quad (16)$$

where, $Z_{\alpha/2}$ is $100(1-\alpha)$, the percentile of the standard normal distribution.

Table 1 summarizes the different mean value functions of the proposed new model and several existing NHPP models. Note that models 9 and 10 consider environmental uncertainty.

Table 1. NHPP software reliability models.

No.	Model	$\mathbf{m(t)}$
1	GO Model [14]	$m(t) = a\left(1 - e^{-bt}\right)$
2	Delayed S-shaped Model [15]	$m(t) = a\left(1 - (1+bt)e^{-bt}\right)$
3	Inflection S-shaped Model [16]	$m(t) = \frac{a\left(1-e^{-bt}\right)}{1+\beta e^{-bt}}$
4	Yamada ImperfectDebugging Model [17]	$m(t) = a\left[1 - e^{-bt}\right]\left[1 - \frac{\alpha}{b}\right] + \alpha at$
5	PNZ Model [10]	$m(t) = \frac{a\left[1-e^{-bt}\right]\left[1-\frac{\alpha}{b}\right]+\alpha at}{1+\beta e^{-bt}}$
6	PZ Model [18]	$m(t) = \frac{\left((c+a)\left[1-e^{-bt}\right]-\left[\frac{ab}{b-\alpha}\left(e^{-\alpha t}-e^{-bt}\right)\right]\right)}{1+\beta e^{-bt}}$
7	Dependent Parameter Model [19]	$m(t) = m_0\left(\frac{\gamma t+1}{\gamma t_0+1}\right)e^{-\gamma(t-t_0)} + \alpha(\gamma t+1)(\gamma t - 1 + (1-\gamma t_0)e^{-\gamma(t-t_0)}$
8	Testing Coverage Model [4]	$m(t) = N\left[1 - \left(\frac{\beta}{\beta+(at)^b}\right)^{\alpha}\right]$
9	Three parameter Model [7]	$m(t) = N\left[1 - \left(\frac{\beta}{\beta - \frac{a}{b}\ln\left(\frac{(1+c)e^{-bt}}{1+ce^{-bt}}\right)}\right)\right]$
10	Proposed New Model	$m(t) = N\left(1 - \frac{\beta}{\beta+at-\ln(1+at)}\right)^{\alpha}$

4. Optimal Software Release Policy

In this section, we next discuss the use of the software reliability model under varying situations to determine the optimal software release time, and to determine the optimal software release time, T*, which minimizes the expected total software cost. Many studies have been conducted on the optimal software release time and its related problems [20–24]. The quality of the system will normally depend on the testing efforts, such as the testing environment, times, tools, and methodologies. If testing is short, the cost of the system testing is lower, but the consumers may face a higher risk e.g., buying an unreliable system. This also involves the higher costs of the operating environment because it is much more expensive to detect and correct a failure during the operational phase than during the testing phase. In contrast, the longer the testing time, the more faults that can be removed, which leads to a more reliable system; however, the testing costs for the system will also increase. Therefore, it is very important to determine when to release the system based on test cost and reliability. Figure 1 shows the system development lifecycle considered in the following cost model: the testing phase before release time T, the warranty period, and the operational life in the actual field environment, which is usually quite different from the testing environment [24].

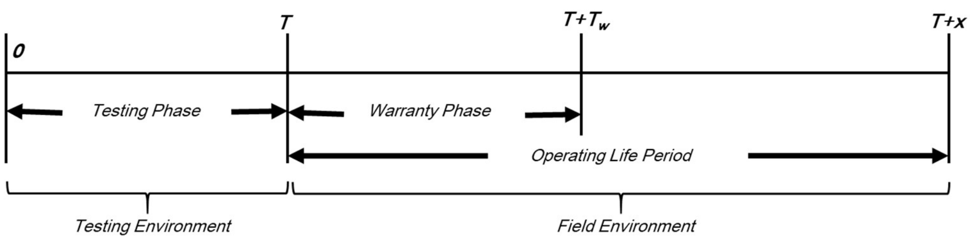

Figure 1. System cost model infrastructure.

The expected total software cost $C(T)$ [24] can be expressed as

$$C(T) = C_0 + C_1T + C_2m(T)\mu_y + C_3(1 - R(x|T)) + C_4[m(T + T_w) - m(T)]\mu_w \tag{17}$$

where, C_0 is the set-up cost of testing, C_1T is the cost of testing, $C_2m(T)\mu_y$ is the expected cost to remove all errors detected by time T during the testing phase, $C_3(1 - R(x|T))$ is the penalty cost owing to failures that occurs after the system release time T, and $C_4[m(T + T_w) - m(T)]\mu_w$ is the expected cost to remove all of the errors that are detected during the warranty period $[T,\ T + T_w]$. The cost that is required to remove faults during the operating period is higher than during the testing period, and the time that is needed is much longer.

Finally, we aim to find the optimal software release time, T*, with the expected minimum in the environment as follows:

$$\text{Minimize } C(T). \tag{18}$$

5. Numerical Examples

5.1. Data Information

Dataset #1 (DS1), presented in Table 2, was reported by Musa [25] based on software failure data from a real time command and control system (RTC&CS), and represents the failures that were observed during system testing (25 hours of CPU time). The number of test object instructions delivered for this system, which was developed by Bell Laboratories, was 21,700.

Table 2. Dataset #1 (DS1) : real time command and control system (RTC&CS) data set.

Hour Index	Failures	Cumulative Failures	Hour Index	Failures	Cumulative Failures
1	27	27	14	5	111
2	16	43	15	5	116
3	11	54	16	6	122
4	10	64	17	0	122
5	11	75	18	5	127
6	7	83	19	1	128
7	2	84	20	1	129
8	5	89	21	2	131
9	3	92	22	1	132
10	1	93	23	2	134
11	4	97	24	1	135
12	7	104	25	1	136
13	2	106	-	-	-

Dataset #2 (DS2), as shown in Table 3, is the second of three releases of software failure data collected from three different releases of a large medical record system (MRS) [26], consisting of 188 software components. Each component contains several files. Initially, the software consisted of 173 software components. All three releases added new functionality to the product. Between three and seven new components were added in each of the three releases, for a total of 15 new components. Many other components were modified during each of the three releases as a side effect of the added functionality. Detailed information of the dataset can be obtained in the report by Stringfellow and Andrews [26].

Dataset #3 (DS3), as shown in Table 4, is from one of four major releases of software products at Tandom Computers (TDC) [27]. There are 100 failures that are observed within testing CPU hours. Detailed information of the dataset can be obtained tin the report by Wood [27].

Table 3. DS2: medical record system (MRS) data set.

Week Index	Failures	Cumulative Failures	Week Index	Failures	Cumulative Failures
1	90	90	10	0	190
2	17	107	11	2	192
3	19	126	12	0	192
4	19	145	13	0	192
5	26	171	14	0	192
6	17	188	15	11	203
7	1	189	16	0	203
8	1	190	17	1	204
9	0	190	-	-	-

Table 4. DS3: Tandom Computers (TDC) data set.

Time Index (CPU hours)	Cumulative Failures	Time Index (CPU hours)	Cumulative Failures	Time Index (CPU hours)	Cumulative Failures
519	16	4422	58	8205	96
968	24	5218	69	8564	98
1430	27	5823	75	8923	99
1893	33	6539	81	9282	100
2490	41	7083	86	9641	100
3058	49	7487	90	10,000	100
3625	54	7846	93	-	-

5.2. Model Analysis

Tables 5–7 summarize the results of the estimated parameters of all 10 models in Table 1 using the LSE technique and the values of the six common criteria: MSE, AIC, PRR, PP, SAE, and R^2. We obtained the six common criteria at $t = 1, 2, \cdots, 25$ from DS1 (Table 2), at $t = 1, 2, \cdots, 17$ from DS2 (Table 3), and at cumulative testing CPU hours from DS3 (Table 4). As can be seen in Table 5, when comparing all of the models, the MSE and AIC values are the lowest for the newly proposed model, and the PRR, PP, SAE, and R^2 values are the second best. The MSE and AIC values of the newly proposed model are 7.361, 114.982, respectively, which are significantly less than the values of the other models. In Table 6, when comparing all of the models, all criteria values for the newly proposed model are best. The MSE value of the newly proposed model is 60.623, which is significantly lower than the value of the other models. The AIC, PRR, PP, and SAE values of the newly proposed model are 151.156, 0.043, 0.041, and 98.705, respectively, which are also significantly lower than the other models. The value of R^2 is 0.960 and is the closest to 1 for all of the models. In Table 7, when comparing all of the models, all the criteria values for the newly proposed model are best. The MSE value of the newly proposed model is 6.336, which is significantly lower than the value of the other models. The PRR, PP, and SAE values of the newly proposed model are 0.086, 0.066, and 36.250, respectively, which are also significantly lower than the other models. The value of R^2 is 0.9940 and is the closest to 1 for all of the models.

Table 5. Model parameter estimation and comparison criteria from RTC&CS data set (DS1). Least-squares estimate (LSE); mean squared error; Akaike's information criterion (AIC); predictive ratio risk (PRR); predictive power (PP), sum absolute error (SAE), correlation index of the regression curve equation (R^2).

Model	LSE's	MSE	AIC	PRR	PP	SAE	R^2
GOM	$\hat{a}$ = 136.050, $\hat{b}$ = 0.138	33.822	121.878	0.479	0.262	118.530	0.972
DSM	$\hat{a}$ = 124.665, $\hat{b}$ = 0.356	134.582	210.287	12.787	1.181	239.335	0.889
ISM	$\hat{a}$ = **136.050**, $\hat{b}$ = **0.138** $\hat{\beta}$ = 0.0001	35.363	123.878	0.479	0.262	118.532	0.972
YIDM	$\hat{a}$ = 81.252, $\hat{b}$ = 0.340 $\hat{\alpha}$ = 0.0333	9.435	116.403	0.035	0.031	60.842	0.993
PNZM	$\hat{a}$ = 81.562, $\hat{b}$ = 0.337 $\hat{\alpha}$ = 0.033, $\hat{\beta}$ = 0.00	9.888	118.388	0.037	0.032	60.877	0.993
PZM	$\hat{a}$ = 0.01, $\hat{b}$ = 0.138 $\hat{\alpha}$ = **800.0**, $\hat{\beta}$ = **0.00**, $\hat{c}$ = 136.04	38.895	127.878	0.479	0.262	118.530	0.972
DPM	$\hat{\alpha}$ = 28650, $\hat{\beta}$ = 0.003 t_0 = 0.00, m_0 = 71.8	274.911	382.143	0.857	3.568	304.212	0.792
TCM	$\hat{a}$ = 0.000035, $\hat{b}$ = 0.734, $\hat{\alpha}$ = **0.29**, $\hat{\beta}$ = **0.002**, $\hat{N}$ = 427	7.640	116.932	0.019	0.019	47.304	0.995
3PFDM	$\hat{a}$ = 1.696, $\hat{b}$ = 0.001 $\hat{c}$ = **6.808**, $\hat{\beta}$ = **1.574** $\hat{N}$ = 173.030	17.827	119.523	0.137	0.100	81.313	0.987
New Model	$\hat{a}$ = 0.277, $\hat{\alpha}$ = 0.328 $\hat{\beta}$ = 17.839, $\hat{N}$ = 228.909	7.361	114.982	0.022	0.022	47.869	0.994

Table 6. Model parameter estimation and comparison criteria from MRS data set (DS2).

Model	LSE's	MSE	AIC	PRR	PP	SAE	R^2
GOM	$\hat{a}$ = 197.387, $\hat{b}$ = 0.399	80.678	184.331	0.170	0.101	104.403	0.939
DSM	$\hat{a}$ = 192.528, $\hat{b}$ = 0.882	232.628	331.857	1.291	0.333	142.544	0.823
ISM	$\hat{a}$ = 197.354, $\hat{b}$ = 0.399 $\hat{\beta}$ = 0.000001	86.440	186.334	0.171	0.101	104.370	0.939
YIDM	$\hat{a}$ = 182.934, $\hat{b}$ = 0.464 $\hat{\alpha}$ = 0.0071	78.837	157.825	0.128	0.087	100.617	0.944
PNZM	$\hat{a}$ = 183.124, $\hat{b}$ = 0.463 $\hat{\alpha}$ = 0.007, $\hat{\beta}$ = 0.00	84.902	159.873	0.128	0.087	100.608	0.944
PZM	$\hat{a}$ = 195.990, $\hat{b}$ =0.3987 $\hat{\alpha}$ = **1000.00**, $\hat{\beta}$ = **0.00**, $\hat{c}$ = 1.390	100.989	190.332	0.172	0.102	104.354	0.939
DPM	$\hat{\alpha}$ = 26124.0, $\hat{\gamma}$ = 0.0044 t_0 = 0.00, m_0 = 147.00	769.282	480.341	0.415	0.712	334.128	0.494
TCM	$\hat{a}$ = 0.053, $\hat{b}$ = 0.774, $\hat{\alpha}$ = **181.0**, $\hat{\beta}$ = **38.6**, $\hat{N}$ = 204.1	72.283	158.933	0.052	0.048	103.196	0.956
3PFDM	$\hat{a}$ = 0.028, $\hat{b}$ = 0.210 $\hat{c}$ = 9.924, $\hat{\beta}$ = 0.005 $\hat{N}$ = 206.387	81.090	163.797	0.073	0.061	106.341	0.951
New Model	$\hat{a}$ = 0.008, $\hat{\alpha}$ = 0.275, $\hat{\beta}$ = 0.001, $\hat{N}$ = 207.873	60.623	151.156	0.043	0.041	98.705	0.960

Figures 2–4 show the graphs of the mean value functions for all 10 models for DS1, DS2, and DS3, respectively. Figures 5–7 show the graphs of the 95% confidence limits of the newly proposed model for DS1, DS2, and DS3. Tables A1–A3 in Appendix A list the 95% confidence intervals of all 10 NHPP software reliability models for DS1, DS2, and DS3. In addition, the relative error value of the proposed software reliability model confirms its ability to provide more accurate predictions as it remains closer to zero when compared to the other models (Figures 8–10).

Table 7. Model parameter estimation and comparison criteria from MRS data set (DS3).

Model	LSE's	MSE	AIC	PRR	PP	SAE	R^2
GOM	$\hat{a}$ = 133.835, $\hat{b}$ = 0.000146	8.620	86.136	0.556	0.242	42.166	0.991
DSM	$\hat{a}$ = 101.918, $\hat{b}$ = 0.000507	45.783	117.316	22.692	1.318	101.659	0.951
ISM	$\hat{a}$ = 133.835, $\hat{b}$ = 0.000146 $\hat{\beta}$ = 0.000001	9.127	88.136	0.556	0.242	42.166	0.991
YIDM	$\hat{a}$ = 130.091, $\hat{b}$ = 0.00015 $\hat{\alpha}$ = 0.000003	9.084	88.267	0.561	0.243	42.052	0.991
PNZM	$\hat{a}$ = 121.178, $\hat{b}$ = 0.000163 $\hat{\alpha}$ = 0.000009, $\hat{\beta}$ = 0.00	9.532	90.326	0.530	0.234	41.538	0.991
PZM	$\hat{a}$ = 122.259, = 0.0002 $\hat{\alpha}$ = 9955.597, $\hat{\beta}$ = 0.305 $\hat{c}$ = 0.569	11.491	92.020	0.643	0.268	44.848	0.990
DPM	$\hat{\alpha}$ = 123.193, $\hat{\gamma}$ = 0.0001 t_0 = 0.0001, m_0 = 38.459	156.480	212.867	0.917	2.879	196.360	0.851
TCM	$\hat{a}$ = 0.000013, $\hat{b}$ = 0.78, $\hat{\alpha}$ = 141.399, $\hat{\beta}$ = 54.71, $\hat{N}$ = 254.707	7.090	90.758	0.091	0.068	37.880	0.9937
3PFDM	$\hat{a}$ = 0.016, $\hat{b}$ = 0.07 $\hat{c}$ = 0.00001, $\hat{\beta}$ = 157.458 $\hat{N}$ = 205.025	9.410	92.360	0.420	0.200	39.909	0.992
New Model	$\hat{a}$ = 0.064, $\hat{\alpha}$ = 0.731, $\hat{\beta}$ = 2509.898, $\hat{N}$ = 337.765	6.336	88.885	0.086	0.066	36.250	0.9940

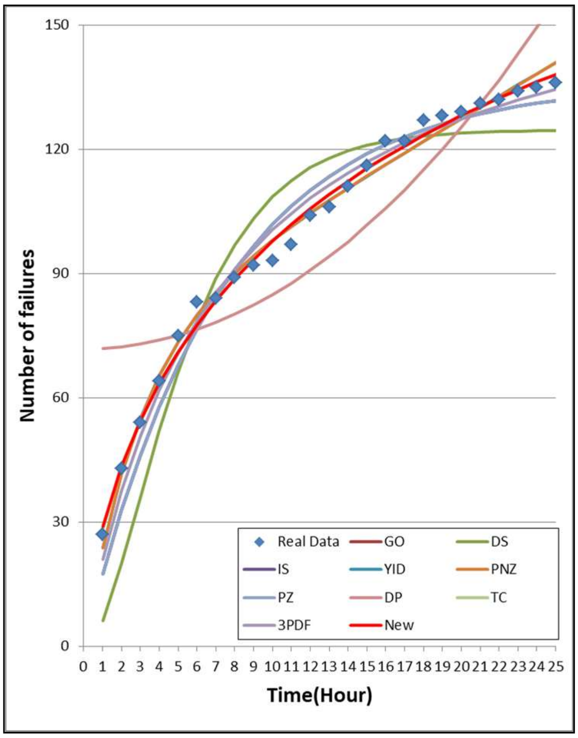

Figure 2. Mean value function of the ten models for DS1.

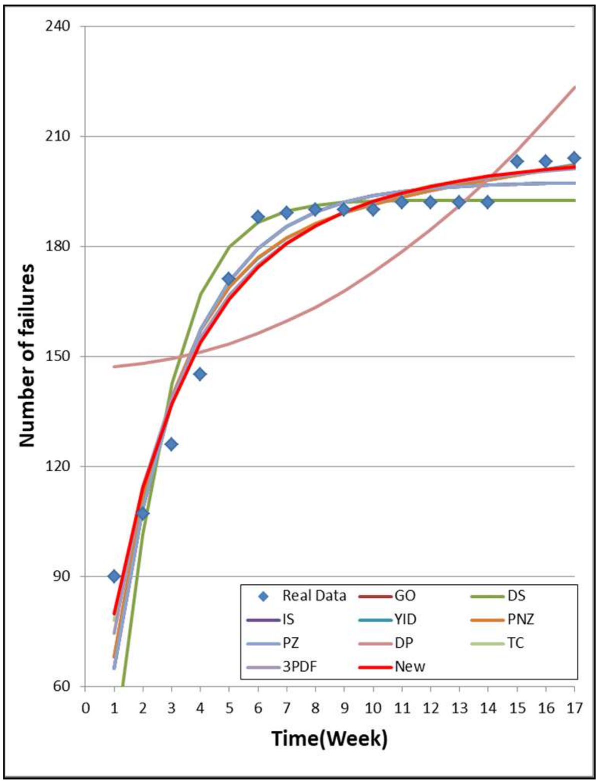

Figure 3. Mean value function of the ten models for DS2.

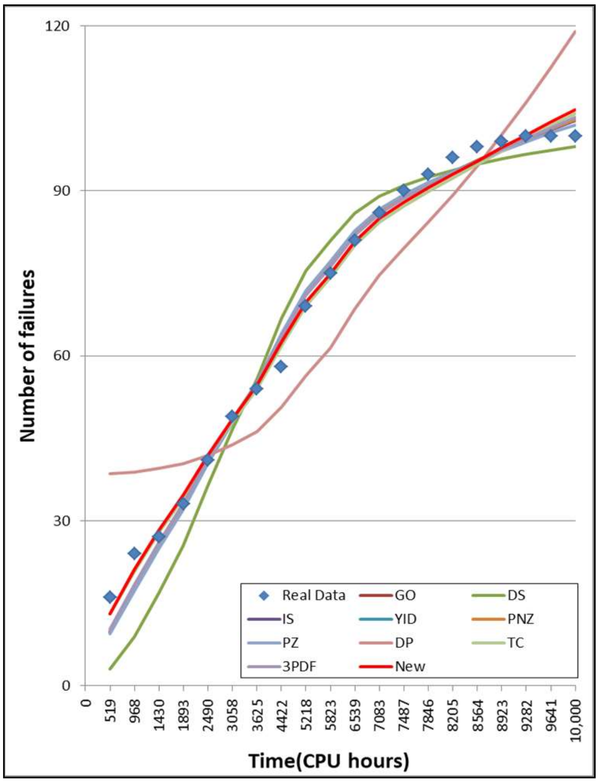

Figure 4. Mean value function of the ten models for DS3.

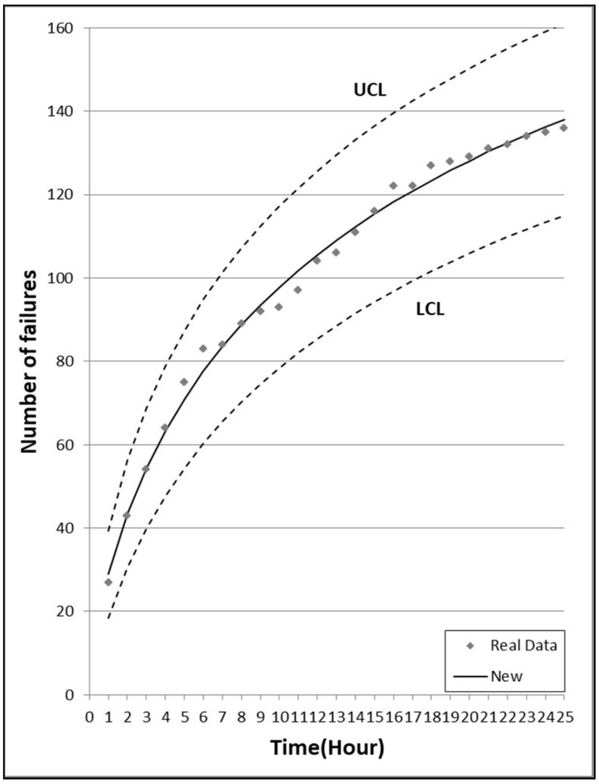

Figure 5. 95% confidence limits of the newly proposed model for DS1.

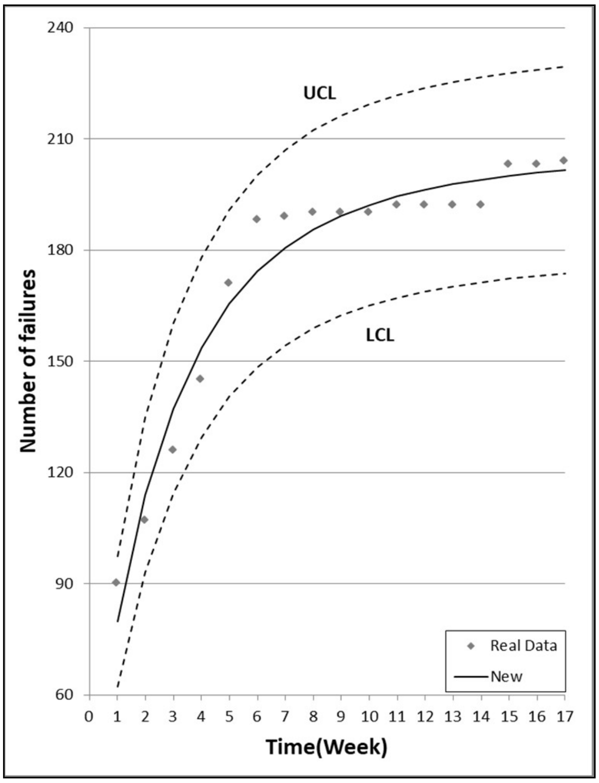

Figure 6. 95% confidence limits of the newly proposed model for DS2.

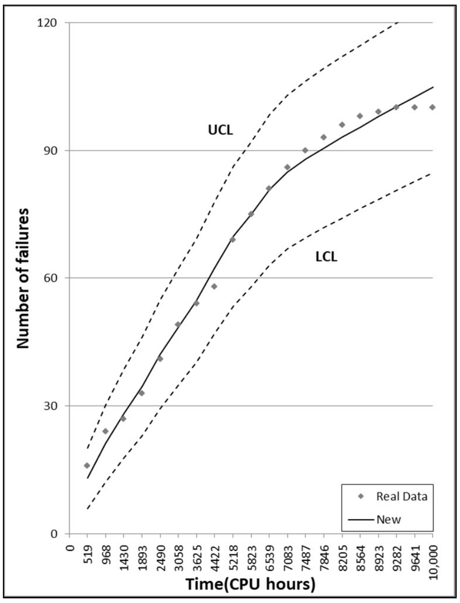

Figure 7. 95% confidence limits of the newly proposed model for DS3.

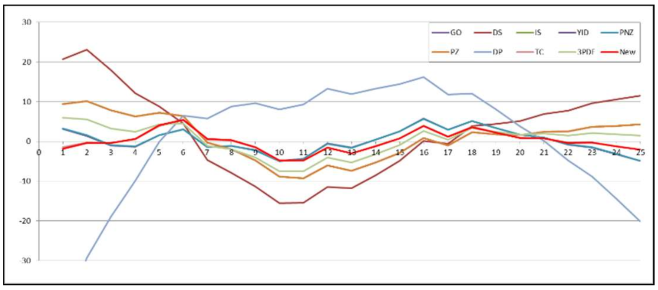

Figure 8. Relative error of the ten models for DS1.

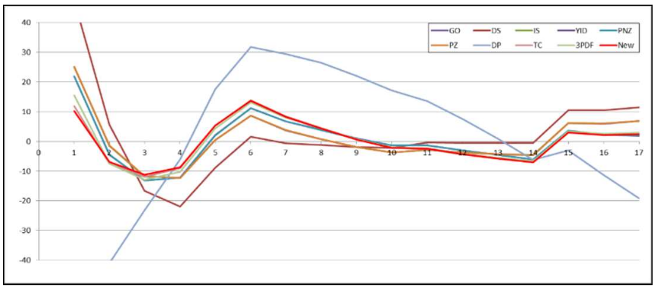

Figure 9. Relative error of the ten models for DS2.

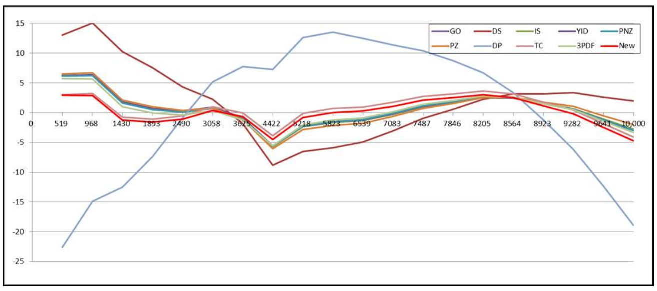

Figure 10. Relative error of the ten models for DS3.

5.3. Optimal Software Release Time

Factor η captures the effects of the field environmental factors based on the system failure rate as described in Section 2. System testing is commonly carried out in a controlled environment, where we can use a constant factor η equal to 1. The newly proposed model becomes a delayed S-shaped model when $\eta = 1$ in (7). Thus, we apply different mean value functions $m(t)$ to the cost model $C(T)$ of (8) when considering the three conditions described below. We apply the cost model to these three conditions using DS1 (Table 2). Using the LSE method, the parameters of the delayed S-shaped model and the newly proposed model are obtained, as described in Section 5.2.

(1) The expected total software cost with controlled environmental factor ($\eta = 1$) is

$$C_1(T) = C_0 + C_1 T + C_2 m(T)\mu_y + C_3(1 - R(x|T)) + C_4[m(T+T_w) - m(T)]\mu_w \tag{19}$$

where

$$m(T) = a\left(1 - (1+bT)e^{-bT}\right), m(T+T_w) = a\left(1 - (1+b(T+T_w))e^{-b(T+T_w)}\right). \tag{20}$$

(2) The expected total software cost with a random operating environmental factor ($\eta = f(x)$) is

$$C_2(T) = C_0 + C_1 T + C_2 m(T)\mu_y + C_3(1 - R(x|T)) + C_4[m(T+T_w) - m(T)]\mu_w \tag{21}$$

where

$$m(T) = N\left(1 - \frac{\beta}{\beta + aT - \ln(1+aT)}\right)^{\alpha}, m(T+T_w) = N\left(1 - \frac{\beta}{\beta + a(T+T_w) - \ln(1+a(T+T_w))}\right)^{\alpha}. \tag{22}$$

(3) The expected total software cost between the testing environment ($\eta = 1$) and field environment ($\eta = f(x)$) is

$$C_3(T) = C_0 + C_1 T + C_2 m_1(T)\mu_y + C_3(1 - R(x|T)) + C_4[m_2(T+T_w) - m_1(T)]\mu_w \tag{23}$$

where

$$m_1(T) = a\left(1 - (1+bT)e^{-bT}\right), m_2(T+T_w) = N\left(1 - \frac{\beta}{\beta + a(T+T_w) - \ln(1+a(T+T_w))}\right)^{\alpha}. \tag{24}$$

We consider the following coefficients in the cost model for the baseline case:

$$C_0 = 100, C_1 = 20, C_2 = 50, C_3 = 2000, C_4 = 400, T_w = 10, x = 20, \mu_y = 0.1, \mu_w = 0.2 \tag{25}$$

The results of the baseline case are listed in Table 8, and the expected total cost for the three conditions above is 1338.70, 2398.24, and 2263.33, respectively. For the second condition, the expected total cost and the optimal release time are high. The expected total cost is the lowest for the first condition, and the optimal release time is shortest for the third condition.

Table 8. Optimal release time T* subject to the warranty period.

Warrnaty Period	$C_1(T)$	T*	$C_2(T)$	T*	$C_3(T)$	T*
$T_w = 2$	1173.41	14.2	1403.78	11.6	599.88	10.5
$T_w = 5$	1286.95	14.9	1928.63	22.8	1334.72	11.3
$T_w = 10$(basic)	1338.70	15.1	2398.24	34.7	2263.33	12.3
$T_w = 15$	1348.88	15.2	2702.33	42.7	2969.88	13.0

To study the impact of different coefficients on the expected total cost and the optimal release time, we vary some of the coefficients and then compare them with the baseline case. First, we evaluate the impact of the warranty period on the expected total cost by changing the value of the corresponding warranty time and comparing the optimal release times for each condition. Here, we change the values of T_w from 10 h to 2, 5, and 15 h, and the values of the other parameters remain unchanged. Regardless of the warranty period, the optimal release time for the third condition is the shortest, and the expected total cost for the first condition is the lowest overall. Figure 11 shows the graph of the expected total cost for the baseline case. Figures 12–14 show the graphs of the expected total cost subject to the warranty period for the three conditions.

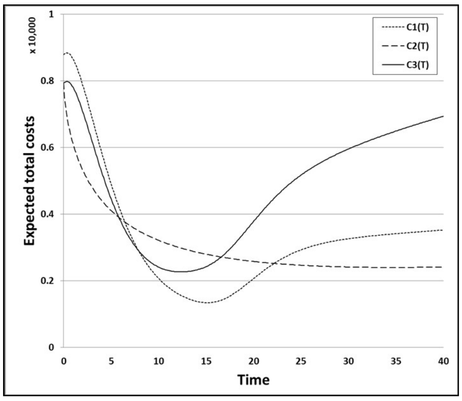

Figure 11. Expected total cost for the baseline case.

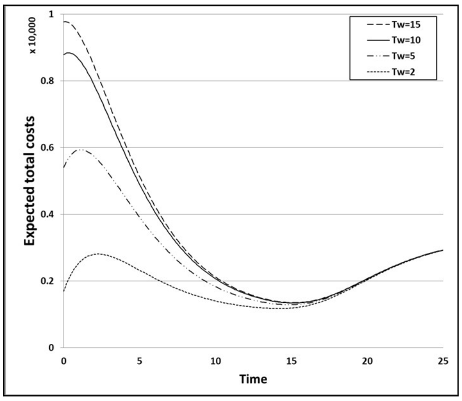

Figure 12. Expected total cost subject to the warranty period for the 1st condition.

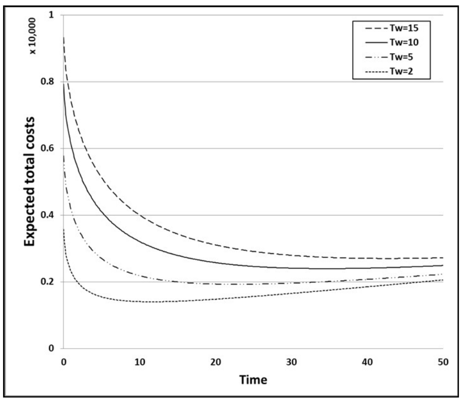

Figure 13. Expected total cost subject to the warranty period for the 2nd condition.

Next, we examine the impact of the cost coefficients, C_1, C_2, C_3, and C_4 on the expected total cost by changing their values and comparing the optimal release times. Without loss of generality, we change only the values of C_2, C_3, and C_4, and keep the values of the other parameters C_0 and C_1

unchanged, because different values of C_0 and C_1 will certainly increase the expected total cost. When we change the values of C_2 from 50 to 25 and 100, the optimal release time is only changed significantly for the second condition. As can be seen from Table 9, the optimal release time T* is 37.5 when the value of C_2 is 25, and 29.1 when the value of C_2 is 100. When we change the value of C_3 from 2000 to 500 and 4000, the optimal release time is only changed significantly for the first condition. As Table 10 shows, the optimal release time T* is 16.5 when the value of C_3 is 500, and 14.6 when the value of C_3 is 4000. When we change the value of C_4 from 400 to 200 and 1000, the optimal release time is changed for all of the conditions. As can be seen from Table 11, the optimal release time T* is 14.3 for the first condition when the value of C_4 is 200, and 16.3 when the value of C_4 is 1000. In addition, the optimal release time T* is 20.0 for the second condition when the value of C_4 is 200, and 61.0 when the value of C_4 is 1000. The optimal release time T* is 11.6 for the third condition when the value of C_4 is 200, and 12.8 when the value of C_4 is 1000. Thus, the second condition has a much greater variation in optimal release time than the other conditions. As a result, we can confirm that the cost model of the first condition does not reflect the influence of the operating environment, and that the cost model of the second condition does not reflect the influence of the test environment. Figure 15 shows the graph of the expected total cost according to the cost coefficient C_2 in the 2nd condition. Figures 16–18 show the graphs of the expected total cost according to cost coefficient C_4 in the three conditions.

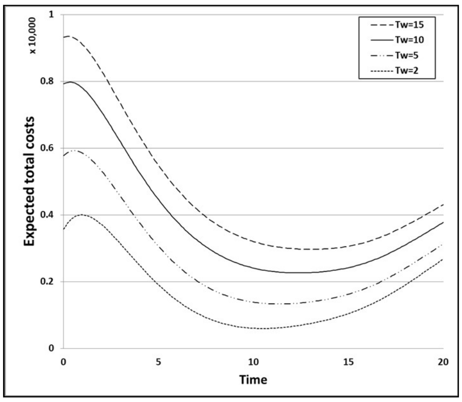

Figure 14. Expected total cost subject to the warranty period for the 3rd condition.

Table 9. Optimal release time T* according to cost coefficient C_2.

Cost Coefficient C_2	C_1(T)	T*	C_2(T)	T*	C_3(T)	T*
C_2 = 25	1036.02	15.2	2013.25	37.5	1972.06	12.5
C_2 = 50 (basic)	1338.70	15.1	2398.24	34.7	2263.33	12.3
C_2 = 100	1943.64	15.1	3141.20	29.1	2843.35	12.1

Table 10. Optimal release time T* according to cost coefficient C_3.

Cost Coefficient C_3	C_1(T)	T*	C_2(T)	T*	C_3(T)	T*
C_3 = 500	1270.65	16.5	2398.24	34.7	2262.96	12.4
C_3 = 2000 (basic)	1338.70	15.1	2398.24	34.7	2263.33	12.3
C_3 = 4000	1376.26	14.6	2398.24	34.7	2263.77	12.3

Table 11. Optimal release time T* according to cost coefficient C_4.

Cost Coefficient C_4	C_1(T)	T*	C_2(T)	T*	C_3(T)	T*
C_4 = 200	1183.14	14.3	1859.29	20	1590.02	11.6
C_4 = 400 (basic)	1338.70	15.1	2398.24	34.7	2263.33	12.3
C_4 = 1000	1680.99	16.3	3272.23	61	4253.45	12.8

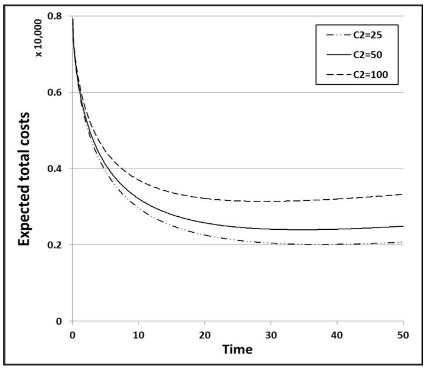

Figure 15. Expected total cost according to cost coefficient C_2 for the 2nd condition.

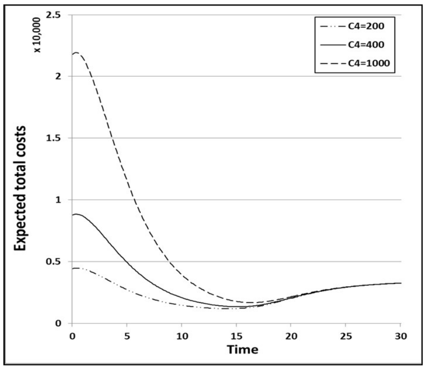

Figure 16. Expected total cost according to cost coefficient C_4 for the 1st condition.

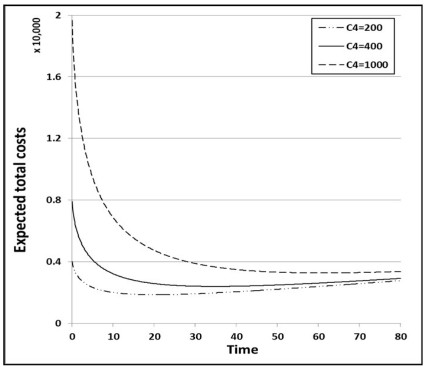

Figure 17. Expected total cost according to cost coefficient C_4 for the 2nd condition.

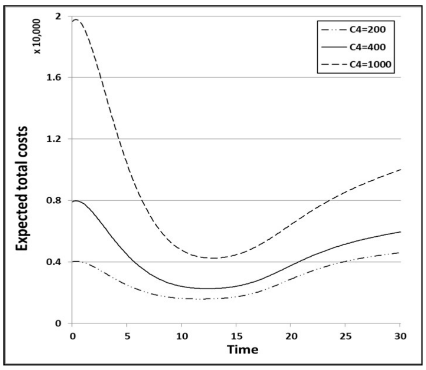

Figure 18. Expected total cost according to cost coefficient C_4 for the 3rd condition.

6. Conclusions

Existing well-known NHPP software reliability models have been developed in a test environment. However, a testing environment differs from an actual operating environment, so we considered random operating environments. In this paper, we discussed a new NHPP software reliability model, with S-shaped growth curve that accounts for the randomness of an actual operating environment. Tables 5–7 summarize the results of the estimated parameters of all ten models that are applied using the LSE technique and six common criteria (MSE, AIC, PRR, PP, SAE, and R^2) for the DS1, DS2, and DS3 datasets. As can be seen from Tables 5–7, the newly proposed model displays a better overall fit than all of the other models when compared, particularly in the case of DS2. In addition, we provided optimal release policies for various environments to determine when the total software system cost is minimized. Using a cost model for a given environment is beneficial as it provides a means for determining when to stop the software testing process. In this paper, faults are assumed to be removed immediately when a software failure has been detected, and the correction process is assumed to not introduce new faults. Obviously, further work in revisiting these assumptions is worth the effort as our future study. We hope to present some new results on this aspect in the near future.

Acknowledgments: This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2015R1D1A1A01060050). We are pleased to thank the Editor and the Referees for their useful suggestions.

Author Contributions: The three authors equally contributed to the paper.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Table A1. 95% Confidence interval of all 10 models (DS1).

Model	Time Index	1	2	3	4	5	6	7	8	9
GOM	LCL	9.329	21.586	32.810	42.823	51.671	59.452	66.277	72.253	77.479
	$\hat{m}(t)$	17.537	32.814	46.121	57.713	67.811	76.607	84.269	90.944	96.758
	UCL	25.745	44.041	59.431	72.602	83.950	93.761	102.261	109.635	116.037
DSM	LCL	1.352	11.192	24.289	37.792	50.271	61.120	70.191	77.571	83.458
	$\hat{m}(t)$	6.253	19.945	36.058	51.914	66.220	78.484	88.644	96.861	103.387
	UCL	11.154	28.698	47.827	66.035	82.170	95.847	107.097	116.150	123.316
ISM	LCL	9.328	21.584	32.808	42.820	51.668	59.449	66.274	72.250	77.476
	$\hat{m}(t)$	17.535	32.811	46.118	57.709	67.807	76.603	84.266	90.941	96.755
	UCL	25.743	44.038	59.428	72.599	83.947	93.758	102.258	109.631	116.034
YIDM	LCL	14.263	28.936	40.449	49.466	56.637	62.468	67.333	71.506	75.185
	$\hat{m}(t)$	23.831	41.573	54.982	65.305	73.433	79.998	85.451	90.111	94.209
	UCL	33.399	54.211	69.515	81.144	90.228	97.528	103.568	108.717	113.232
PNZM	LCL	14.191	28.840	40.360	49.400	56.598	62.455	67.343	71.534	75.227
	$\hat{m}(t)$	23.741	41.460	54.879	65.230	73.389	79.984	85.462	90.143	94.255
	UCL	33.291	54.080	69.399	81.059	90.179	97.513	103.581	108.752	113.283
PZM	LCL	9.329	21.586	32.810	42.823	51.671	59.452	66.277	72.253	77.479
	$\hat{m}(t)$	17.537	32.813	46.121	57.713	67.811	76.607	84.269	90.944	96.758
	UCL	25.745	44.041	59.431	72.602	83.950	93.761	102.261	109.635	116.037
DPM	LCL	55.306	55.649	56.223	57.028	58.068	59.344	60.859	62.614	64.611
	$\hat{m}(t)$	71.929	72.316	72.964	73.874	75.047	76.485	78.189	80.162	82.403
	UCL	88.551	88.984	89.706	90.720	92.026	93.626	95.520	97.710	100.195
TCM	LCL	17.974	30.408	39.981	47.851	54.561	60.419	65.621	70.302	74.555
	$\hat{m}(t)$	28.423	43.306	54.443	63.465	71.086	77.695	83.535	88.768	93.508
	UCL	38.872	56.204	68.905	79.080	87.611	94.971	101.449	107.234	112.461
3PFDM	LCL	12.011	25.458	36.751	46.227	54.252	61.123	67.065	72.251	76.816
	$\hat{m}(t)$	20.991	37.452	50.708	61.611	70.737	78.487	85.151	90.942	96.021
	UCL	29.970	49.447	64.665	76.995	87.221	95.851	103.237	109.633	115.227
New	LCL	18.358	30.526	39.950	47.745	54.426	60.283	65.501	70.208	74.492
	$\hat{m}(t)$	28.893	43.444	54.407	63.344	70.933	77.542	83.401	88.663	93.438
	UCL	39.428	56.363	68.864	78.944	87.440	94.801	101.300	107.118	112.384

Table A1. *Cont.*

Model	Time Index	10	11	12	13	14	15	16	17
GOM	LCL	82.045	86.033	89.514	92.552	95.201	97.512	99.527	101.284
	$\hat{m}(t)$	101.823	106.235	110.078	113.426	116.342	118.882	121.095	123.023
	UCL	121.600	126.436	130.641	134.300	137.483	140.253	142.663	144.762
DSM	LCL	88.083	91.672	94.431	96.534	98.127	99.326	100.225	100.895
	$\hat{m}(t)$	108.498	112.457	115.494	117.807	119.557	120.874	121.861	122.596
	UCL	128.914	133.241	136.557	139.080	140.988	142.423	143.497	144.298
ISM	LCL	82.043	86.031	89.512	92.550	95.200	97.511	99.526	101.283
	$\hat{m}(t)$	101.820	106.232	110.076	113.424	116.340	118.881	121.094	123.022
	UCL	121.597	126.434	130.639	134.298	137.481	140.251	142.662	144.761
YIDM	LCL	78.512	81.588	84.485	87.257	89.939	92.558	95.133	97.678
	$\hat{m}(t)$	97.905	101.316	104.523	107.586	110.546	113.433	116.267	119.064
	UCL	117.298	121.044	124.561	127.916	131.153	134.307	137.401	140.451
PNZM	LCL	78.562	81.642	84.540	87.310	89.987	92.600	95.168	97.703
	$\hat{m}(t)$	97.960	101.376	104.584	107.645	110.600	113.479	116.305	119.092
	UCL	117.359	121.110	124.628	127.980	131.212	134.358	137.442	140.481
PZM	LCL	82.045	86.033	89.514	92.552	95.201	97.512	99.527	101.284
	$\hat{m}(t)$	101.823	106.235	110.078	113.426	116.342	118.882	121.095	123.023
	UCL	121.600	126.436	130.641	134.300	137.483	140.253	142.663	144.762
DPM	LCL	66.855	69.346	72.087	75.081	78.331	81.838	85.606	89.636
	$\hat{m}(t)$	84.916	87.701	90.759	94.093	97.704	101.593	105.762	110.212
	UCL	102.977	106.055	109.432	113.105	117.078	121.348	125.918	130.788
TCM	LCL	78.453	82.050	85.387	88.500	91.416	94.157	96.744	99.191
	$\hat{m}(t)$	97.840	101.828	105.521	108.959	112.174	115.193	118.038	120.726
	UCL	117.227	121.606	125.654	129.418	132.933	136.229	139.332	142.261
3PFDM	LCL	80.863	84.475	87.718	90.647	93.303	95.724	97.940	99.974
	$\hat{m}(t)$	100.512	104.512	108.096	111.326	114.253	116.917	119.352	121.586
	UCL	120.162	124.549	128.473	132.006	135.203	138.110	140.764	143.198
New	LCL	78.423	82.051	85.417	88.555	91.491	94.248	96.844	99.296
	$\hat{m}(t)$	97.806	101.829	105.554	109.019	112.257	115.293	118.148	120.841
	UCL	117.189	121.607	125.690	129.484	133.024	136.338	139.452	142.387

Model	Time Index	18	19	20	21	22	23	24	25
GOM	LCL	102.8153	104.15	105.3133	106.3271	107.2106	107.9804	108.6512	109.2357
	$\hat{m}(t)$	124.7022	126.165	127.4392	128.5491	129.516	130.3582	131.0919	131.731
	UCL	146.5892	148.1799	149.565	150.7711	151.8214	152.736	153.5326	154.2263
DSM	LCL	101.3931	101.7621	102.0346	102.2353	102.3828	102.4909	102.57	102.6278
	$\hat{m}(t)$	123.1427	123.5475	123.8463	124.0664	124.2281	124.3466	124.4333	124.4967
	UCL	144.8924	145.3328	145.658	145.8975	146.0734	146.2023	146.2967	146.3656
ISM	LCL	102.8143	104.1492	105.3126	106.3265	107.21	107.9799	108.6508	109.2353
	$\hat{m}(t)$	124.7012	126.164	127.4383	128.5484	129.5154	130.3577	131.0914	131.7306
	UCL	146.588	148.1789	149.5641	150.7703	151.8207	152.7354	153.5321	154.2259
YIDM	LCL	100.2015	102.7107	105.2103	107.7037	110.1934	112.6811	115.1679	117.6547
	$\hat{m}(t)$	121.8354	124.5875	127.3263	130.0555	132.7779	135.4956	138.2097	140.9215
	UCL	143.4693	146.4644	149.4423	152.4073	155.3625	158.31	161.2516	164.1883
PNZM	LCL	100.2169	102.7154	105.2037	107.6855	110.1632	112.6385	115.1129	117.587
	$\hat{m}(t)$	121.8523	124.5927	127.3191	130.0356	132.7449	135.4491	138.1497	140.8477
	UCL	143.4877	146.47	149.4345	152.3856	155.3266	158.2597	161.1866	164.1084
PZM	LCL	102.8153	104.15	105.3133	106.3271	107.2106	107.9804	108.6512	109.2357
	$\hat{m}(t)$	124.7022	126.165	127.4392	128.5491	129.516	130.3582	131.0919	131.731
	UCL	146.5892	148.1799	149.565	150.7711	151.8214	152.736	153.5326	154.2263
DPM	LCL	93.93128	98.49414	103.3268	108.4316	113.8108	119.4664	125.4007	131.6157
	$\hat{m}(t)$	114.9445	119.961	125.2629	130.8518	136.7288	142.8956	149.3535	156.1038
	UCL	135.9577	141.4278	147.199	153.2719	159.6469	166.3248	173.3063	180.5919
TCM	LCL	101.5124	103.7205	105.8252	107.8352	109.7585	111.6019	113.3714	115.0726
	$\hat{m}(t)$	123.2736	125.6944	127.9996	130.1994	132.3026	134.3169	136.2493	138.1057
	UCL	145.0349	147.6682	150.174	152.5635	154.8467	157.032	159.1271	161.1389
3PFDM	LCL	101.8497	103.5836	105.1914	106.6864	108.0801	109.3824	110.6019	111.7464
	$\hat{m}(t)$	123.6436	125.5443	127.3057	128.9424	130.4673	131.8914	133.2244	134.4748
	UCL	145.4374	147.505	149.4199	151.1983	152.8544	154.4004	155.8469	157.2032
New	LCL	101.6161	103.8169	105.9085	107.8999	109.799	111.6129	113.3476	115.009
	$\hat{m}(t)$	123.3873	125.8	128.0908	130.2702	132.3469	134.3289	136.2233	138.0364
	UCL	145.1586	147.783	150.2732	152.6404	154.8947	157.045	159.099	161.0638

Table A2. 95% Confidence interval of all 10 models (DS2).

Model	Time index	1	2	3	4	5	6	7	8	9
GOM	LCL	49.147	88.100	114.753	132.788	144.944	153.123	158.620	162.312	164.791
	$\hat{m}(t)$	64.942	108.518	137.757	157.376	170.540	179.373	185.300	189.276	191.945
	UCL	80.737	128.935	160.761	181.963	196.135	205.623	211.980	216.241	219.099
DSM	LCL	29.754	81.610	119.319	141.607	153.581	159.671	162.660	164.090	164.762
	$\hat{m}(t)$	42.537	101.340	142.735	166.930	179.867	186.433	189.651	191.191	191.914
	UCL	55.320	121.071	166.151	192.253	206.153	213.194	216.643	218.291	219.066
ISM	LCL	49.138	88.084	114.731	132.764	144.918	153.095	158.591	162.282	164.761
	$\hat{m}(t)$	64.931	108.500	137.733	157.349	170.511	179.343	185.269	189.245	191.913
	UCL	80.725	128.915	160.736	181.935	196.104	205.590	211.946	216.207	219.065
YIDM	LCL	51.989	90.820	116.125	132.604	143.452	150.736	155.770	159.386	162.108
	$\hat{m}(t)$	68.171	111.517	139.254	157.176	168.926	176.797	182.228	186.125	189.057
	UCL	84.354	132.215	162.383	181.748	194.400	202.858	208.686	212.864	216.007
PNZM	LCL	51.946	90.780	116.107	132.609	143.476	150.772	155.811	159.427	162.145
	$\hat{m}(t)$	68.123	111.474	139.234	157.181	168.952	176.835	182.272	186.169	189.097
	UCL	84.300	132.168	162.361	181.753	194.428	202.899	208.733	212.912	216.049
PZM	LCL	49.064	88.017	114.677	132.724	144.892	153.080	158.585	162.284	164.769
	$\hat{m}(t)$	64.848	108.425	137.674	157.306	170.483	179.327	185.263	189.247	191.921
	UCL	80.631	128.834	160.672	181.888	196.074	205.573	211.940	216.210	219.074
DPM	LCL	123.469	124.167	125.336	126.981	129.105	131.715	134.816	138.411	142.507
	$\hat{m}(t)$	147.252	148.012	149.283	151.071	153.379	156.212	159.574	163.470	167.904
	UCL	171.036	171.857	173.230	175.161	177.652	180.709	184.333	188.529	193.301
TCM	LCL	60.749	93.551	114.901	129.743	140.446	148.355	154.305	158.844	162.346
	$\hat{m}(t)$	78.066	114.526	137.919	154.071	165.673	174.225	180.648	185.542	189.313
	UCL	95.384	135.501	160.936	178.399	190.901	200.096	206.991	212.239	216.281
3PFDM	LCL	57.549	93.474	115.850	130.807	141.307	148.941	154.636	158.970	162.319
	$\hat{m}(t)$	74.461	114.441	138.954	155.226	166.605	174.858	181.005	185.677	189.284
	UCL	91.374	135.408	162.058	179.645	191.903	200.776	207.374	212.384	216.249
New	LCL	62.346	93.042	114.265	129.443	140.426	148.466	154.433	158.930	162.375
	$\hat{m}(t)$	79.861	113.965	137.225	153.745	165.652	174.345	180.786	185.634	189.345
	UCL	97.377	134.889	160.184	178.048	190.878	200.224	207.138	212.338	216.315

Model	Time index	10	11	12	13.	14	15	16	17
GOM	LCL	166.455	167.572	168.321	168.824	169.162	169.389	169.541	169.643
	$\hat{m}(t)$	193.735	194.937	195.743	196.284	196.647	196.890	197.054	197.163
	UCL	221.016	222.302	223.164	223.743	224.132	224.392	224.567	224.684
DSM	LCL	165.073	165.216	165.280	165.309	165.322	165.328	165.331	165.332
	$\hat{m}(t)$	192.249	192.402	192.472	192.503	192.517	192.523	192.526	192.527
	UCL	219.424	219.588	219.663	219.696	219.711	219.718	219.721	219.722
ISM	LCL	166.425	167.542	168.291	168.794	169.131	169.358	169.510	169.612
	$\hat{m}(t)$	193.703	194.904	195.710	196.251	196.614	196.857	197.021	197.130
	UCL	220.981	222.267	223.129	223.708	224.096	224.357	224.532	224.649
YIDM	LCL	164.269	166.076	167.661	169.107	170.464	171.767	173.035	174.281
	$\hat{m}(t)$	191.383	193.328	195.033	196.587	198.047	199.446	200.809	202.147
	UCL	218.498	220.580	222.405	224.068	225.629	227.126	228.583	230.014
PNZM	LCL	164.298	166.095	167.669	169.101	170.445	171.733	172.986	174.217
	$\hat{m}(t)$	191.415	193.349	195.041	196.581	198.025	199.410	200.756	202.078
	UCL	218.531	220.602	222.413	224.061	225.606	227.087	228.526	229.940
PZM	LCL	166.437	167.557	168.309	168.814	169.152	169.380	169.532	169.635
	$\hat{m}(t)$	193.716	194.921	195.729	196.272	196.636	196.881	197.045	197.155
	UCL	220.995	222.285	223.150	223.731	224.120	224.382	224.558	224.675
DPM	LCL	147.109	152.223	157.853	164.006	170.687	177.901	185.654	193.951
	$\hat{m}(t)$	172.880	178.401	184.474	191.101	198.286	206.034	214.349	223.235
	UCL	198.650	204.580	211.094	218.195	225.885	234.167	243.045	252.519
TCM	LCL	165.073	167.214	168.906	170.251	171.327	172.192	172.889	173.455
	$\hat{m}(t)$	192.249	194.552	196.371	197.818	198.974	199.903	200.653	201.260
	UCL	219.424	221.890	223.837	225.384	226.621	227.614	228.416	229.065
3PFDM	LCL	164.940	167.013	168.668	170.001	171.083	171.968	172.697	173.303
	$\hat{m}(t)$	192.105	194.335	196.115	197.548	198.711	199.662	200.446	201.097
	UCL	219.270	221.658	223.563	225.096	226.340	227.357	228.195	228.891
New	LCL	165.057	167.175	168.872	170.249	171.380	172.319	173.107	173.773
	$\hat{m}(t)$	192.231	194.510	196.335	197.815	199.031	200.040	200.886	201.602
	UCL	219.405	221.845	223.797	225.381	226.682	227.761	228.666	229.431

Table A3. 95% Confidence interval of all 10 models (DS3).

Model	Time Index	519	968	1430	1893	2490	3058	3625	4422	5218	5823
GOM	LCL	3.641	9.407	15.376	21.176	28.274	34.589	40.464	48.022	54.803	59.482
	$\hat{m}(t)$	9.767	17.639	25.218	32.318	40.791	48.196	55.000	63.660	71.359	76.641
	UCL	15.892	25.870	35.060	43.460	53.309	61.803	69.535	79.298	87.916	93.799
DSM	LCL	−0.409	3.059	8.744	15.540	24.802	33.366	41.233	50.791	58.514	63.256
	$\hat{m}(t)$	2.966	8.910	16.770	25.422	36.671	46.770	55.885	66.811	75.550	80.883
	UCL	6.342	14.760	24.796	35.305	48.540	60.174	70.537	82.832	92.586	98.510
ISM	LCL	3.641	9.407	15.376	21.176	28.273	34.589	40.464	48.022	54.802	59.482
	$\hat{m}(t)$	9.767	17.639	25.218	32.318	40.791	48.196	55.000	63.660	71.359	76.641
	UCL	15.892	25.870	35.060	43.460	53.309	61.803	69.535	79.298	87.916	93.799
YIDM	LCL	3.631	9.384	15.337	21.122	28.202	34.503	40.367	47.916	54.698	59.387
	$\hat{m}(t)$	9.751	17.608	25.171	32.254	40.707	48.096	54.888	63.539	71.241	76.534
	UCL	15.871	25.832	35.004	43.385	53.213	61.689	69.409	79.162	87.784	93.680
PNZM	LCL	3.701	9.506	15.493	21.294	28.373	34.657	40.493	47.993	54.724	59.378
	$\hat{m}(t)$	9.853	17.767	25.363	32.460	40.909	48.275	55.033	63.627	71.271	76.523
	UCL	16.005	26.028	35.234	43.627	53.445	61.893	69.573	79.261	87.817	93.668
PZM	LCL	3.458	9.130	15.085	20.933	28.150	34.608	40.627	48.354	55.238	59.940
	$\hat{m}(t)$	9.499	17.277	24.856	32.024	40.646	48.218	55.187	64.039	71.852	77.157
	UCL	15.539	25.424	34.628	43.116	53.141	61.828	69.747	79.723	88.466	94.373
DPM	LCL	26.407	26.678	27.161	27.873	29.166	30.827	32.943	36.756	41.628	46.096
	$\hat{m}(t)$	38.581	38.903	39.475	40.318	41.845	43.799	46.275	50.713	56.340	61.461
	UCL	50.754	51.128	51.789	52.763	54.523	56.770	59.608	64.671	71.051	76.827
TCM	LCL	5.929	11.851	17.436	22.634	28.871	34.406	39.605	46.448	52.816	57.386
	$\hat{m}(t)$	12.993	20.787	27.763	34.075	41.497	47.982	54.009	61.863	69.110	74.278
	UCL	20.058	29.723	38.090	45.516	54.122	61.559	68.413	77.279	85.404	91.170
3PFDM	LCL	3.990	9.962	16.016	21.804	28.789	34.941	40.631	47.938	54.522	59.105
	$\hat{m}(t)$	10.271	18.361	26.012	33.076	41.400	48.605	55.191	63.564	71.041	76.216
	UCL	16.552	26.759	36.008	44.348	54.011	62.270	69.752	79.190	87.561	93.327
New	LCL	5.981	12.096	17.800	23.076	29.375	34.946	40.168	47.027	53.403	57.974
	$\hat{m}(t)$	13.066	21.099	28.210	34.606	42.091	48.611	54.658	62.525	69.774	74.941
	UCL	20.151	30.102	38.621	46.135	54.807	62.276	69.148	78.023	86.146	91.909
Model	**Time index**	**6539**	**7083**	**7487**	**7846**	**8205**	**8564**	**8923**	**9282**	**9641**	**10,000**
GOM	LCL	64.535	68.049	70.489	72.543	74.495	76.350	78.112	79.786	81.376	82.886
	$\hat{m}(t)$	82.318	86.251	88.977	91.267	93.441	95.504	97.461	99.319	101.081	102.754
	UCL	100.101	104.454	107.465	109.992	112.387	114.658	116.810	118.851	120.786	122.621
DSM	LCL	67.773	70.526	72.248	73.576	74.736	75.748	76.627	77.391	78.053	78.626
	$\hat{m}(t)$	85.943	89.018	90.939	92.418	93.710	94.834	95.812	96.660	97.395	98.031
	UCL	104.112	107.510	109.629	111.260	112.683	113.921	114.997	115.930	116.738	117.437
ISM	LCL	64.535	68.049	70.489	72.543	74.495	76.350	78.112	79.786	81.376	82.886
	$\hat{m}(t)$	82.318	86.251	88.977	91.267	93.441	95.504	97.461	99.319	101.081	102.754
	UCL	100.100	104.454	107.465	109.992	112.387	114.658	116.810	118.851	120.786	122.621
YIDM	LCL	64.460	67.996	70.457	72.532	74.508	76.389	78.181	79.887	81.511	83.059
	$\hat{m}(t)$	82.234	86.192	88.941	91.255	93.455	95.547	97.537	99.430	101.231	102.945
	UCL	100.007	104.388	107.425	109.978	112.403	114.706	116.894	118.974	120.951	122.831
PNZM	LCL	64.417	67.936	70.389	72.462	74.440	76.328	78.131	79.854	81.500	83.074
	$\hat{m}(t)$	82.186	86.125	88.865	91.177	93.380	95.480	97.483	99.394	101.218	102.962
	UCL	99.954	104.314	107.342	109.892	112.320	114.631	116.834	118.934	120.937	122.850
PZM	LCL	64.953	68.387	70.742	72.704	74.547	76.279	77.905	79.430	80.860	82.201
	$\hat{m}(t)$	82.786	86.629	89.260	91.447	93.499	95.425	97.231	98.924	100.510	101.995
	UCL	100.619	104.872	107.777	110.189	112.451	114.571	116.558	118.418	120.160	121.789
DPM	LCL	52.288	57.681	62.083	66.287	70.771	75.541	80.600	85.954	91.607	97.562
	$\hat{m}(t)$	68.511	74.610	79.566	84.280	89.292	94.604	100.222	106.148	112.385	118.937
	UCL	84.734	91.540	97.049	102.274	107.812	113.668	119.843	126.341	133.163	140.312
TCM	LCL	62.528	66.257	68.936	71.255	73.519	75.730	77.891	80.003	82.069	84.090
	$\hat{m}(t)$	80.065	84.247	87.243	89.832	92.355	94.815	97.216	99.560	101.849	104.086
	UCL	97.603	102.237	105.550	108.408	111.190	113.900	116.541	119.116	121.629	124.082
3PFDM	LCL	64.115	67.651	70.138	72.257	74.295	76.256	78.144	79.963	81.717	83.410
	$\hat{m}(t)$	81.847	85.806	88.586	90.949	93.218	95.399	97.497	99.515	101.459	103.333
	UCL	99.578	103.961	107.033	109.641	112.142	114.543	116.849	119.067	121.202	123.257
New	LCL	63.115	66.844	69.522	71.840	74.104	76.315	78.476	80.590	82.657	84.680
	$\hat{m}(t)$	80.725	84.903	87.897	90.484	93.006	95.465	97.866	100.210	102.500	104.739
	UCL	98.334	102.963	106.272	109.128	111.907	114.615	117.255	119.830	122.343	124.797

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