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Review

Entanglement and Phase-Mediated Correlations in Quantum Field Theory. Application to Brain-Mind States

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Abstract: The entanglement phenomenon plays a central role in quantum optics and in basic aspects of quantum mechanics and quantum field theory. We review the dissipative quantum model of brain and the role of the entanglement in the brain-mind activity correlation and in the formation of assemblies of coherently-oscillating neurons, which are observed to appear in different regions of the cortex by use of EEG, ECoG, fNMR, and other observational methods in neuroscience.

Keywords: the dissipative quantum model of brain; entanglement; phase-mediated correlations; brain-mind functional activity

1. Introduction

In this paper, we review the general features of the dissipative quantum model of brain and the role played by entanglement and quantum field theory (QFT) phase correlations in modeling brain functional activity.

We start by recalling that in 1967, Umezawa and Ricciardi proposed to study the brain as a many-body problem [1,2]. In those years, QFT was achieving great successes in the study of condensed matter and elementary particle physics. The theoretical and experimental successful developments were indeed due in a substantial way to the structure of QFT, which is characterized by the existence of infinitely many unitarily inequivalent representations of the canonical commutation relations (CCR) [3,4]. These representations describe physically different state spaces or phases of the system, and their existence allows the possibility of the spontaneous breakdown of symmetry, which in turn implies the dynamical formation of long-range correlations [5–8].

In their paper [1], Umezawa and Ricciardi were observing: "First of all, at which level should the brain be studied and described? In other words, is it essential to know the behavior in time of any single neuron in order to understand the behavior of natural brains? Probably the answer is negative. The behavior of any single neuron should not be significant for the functioning of the whole brain, otherwise higher and higher degree of malfunctioning should be observed, ... the activity of any single neuron is not significant, but rather the patterns of activity of clusters of them; ... the existence of similar and almost simultaneous responses in several regions of the brain (a kind of long-range correlation) to a particular stimulation technique does not find any explanation in terms of activity of the single nerve cells: new non-classical mechanisms have to be looked for ... it is strongly suggestive of a quantum model".

One of the motivations underlying the proposal by Umezawa and Ricciardi was in the fact that although brain studies were improving and there were many progresses in molecular and cellular biology, the understanding of brain functional activity was still a fully-open problem. Neuroscientists were considering with great interest what was going on in other disciplines in the hope of finding possible help in their own field of study. For example, in the search for an explanation for the observed long-range correlations among neurons, Karl Pribram [9,10] suggested to apply to brain studies the concepts of the hologram and coherence developed in laser physics in those years [11]. The one posed by Karl Lashley in the Nineteen Forties was in fact still an unsolved dilemma [12]: "... Here is the dilemma. Nerve impulses are transmitted ... form cell to cell through definite intercellular connections. Yet, all behavior seems to be determined by masses of excitation ... within general fields of activity, without regard to particular nerve cells ... What sort of nervous organization might be capable of responding to a pattern of excitation without limited specialized path of conduction? The problem is almost universal in the activity of the nervous system".

The concepts of "field" and "masses of excitation" in Lashley's remarks were taken to be central ones in the successive years by Walter Freeman, who established a comprehensive quantitative treatment of brain activity in his celebrated book "Mass action in the nervous system" of 1975 [13]. Referring to his studies and to the state of neuroscience research in those years, he observed in 1991 [14] that "My own group's studies, carried out over more than 30 years at the University of California at Berkeley, suggest that perception cannot be understood solely by examining properties of individual neurons, a microscopic approach that currently dominates neuroscience research. We have found that perception depends on the simultaneous, cooperative activity of millions of neurons spread throughout expanses of the cortex. Such global activity can be identified, measured and explained only if one adopts a macroscopic view alongside the microscopic one".

As reported by Lashley, observations showed that neuronal patterns of excitation are formed "without limited specialized path of conduction", thus posing the question of what is the "sort of nervous organization" capable of responding to such excitation activity. The dilemma arises since, on the contrary, the "anatomical" neuronal connections form the extremely dense cortical neuropil structure [15]; the packing of neurons in the cortex is estimated to be of the order of $10^5/\ \mathrm{mm}^3$, each neuron on average being connected to about 10^4 other neurons by its axonal tree synapses and receiving synapses' connections on its dendritic trees by another 10^4 neurons. Lashley observations suggest, however, that such a network of anatomical neuronal links appears to be not fully responsible for the observed patterns of neuronal oscillation, which seem to be generated instead by some other "sort" of nervous organization, by "simultaneous, cooperative activity of millions of neurons spread throughout expanses of the cortex", in Freeman's words [14]. Cutting or damaging those anatomical links, "higher and higher degree of malfunctioning should be observed" [1], which in fact is not commonly observed: "Bioelectrical waves in the brain can be stopped by treatment with cold, electric shock, or drugs, without loss of memory after recovery, and moreover, memory is not lost after many ablation experiments or when a brain is sliced in many directions so that certainly some pre-existent networks are destroyed. These facts suggest that memories are not "wired" into individual neuronal nets, but are instead diffused in the brain" [1,16].

Many interesting efforts have been produced in modern neurophysiology and neuroscience towards the understanding and the solution of the Lashley dilemma, also measuring long-range zero-lag correlations in brain activity, by using multiple array electrodes and a number of advanced techniques, allowing observations within functional neuronal time windows; see, e.g., [17–22]. In the following, we limit ourselves to the review of the main features of the many-body model of the brain and of its extension to include the dissipative distinctive character of brain dynamics. We discuss the model features related to the entanglement phenomenon and show that it is the coherent structure of the system ground state that promotes long-range correlation among neurons. We also remark that many of the results we obtain have their root in the unitary inequivalence between the entangled state and the non-entangled one, which is a distinctive feature of QFT. This marks a difference between preceding studies on the entanglement phenomenon (see, e.g., [23–25] and the references therein quoted) and the analysis presented in this work.

The plan of the paper is the following. In Section 2, we describe briefly the transition from the many-body model to the dissipative quantum model. In Section 3, we review general features of the dissipative quantum model of brain [26–29]. Classical chaotic trajectories in the memory space are discussed in Section 4. Concluding remarks and comments on the fluctuating random force in the system–environment coupling are presented in Section 5. Mathematical results needed for our discussion are reported in Appendix A, where general features of the entanglement in QFT are also reviewed for completeness and the reader's convenience. The formalism presented in Appendix A makes explicit the mentioned difference between the QFT analysis and other preceding studies.

2. From the Many-Body Model to the Dissipative Quantum Model

From observations carried out with EEG, ECoG, fNMR, and other techniques, it appears that myriads of neurons undergo "in phase", i.e., coherent, oscillations with amplitude modulations (AM) and phase modulations (PM) [13,30–35]. These synchronized patterns form in few ms, have a finite size, persist in a time interval of 80–120 ms, have carrier frequencies in the range of 12–80 Hz (the beta-gamma range), and re-synchronize at frame rates in the theta-alpha range (3–12 Hz). They cover domains of 19 cm in linear size in humans and much of the hemisphere in rabbits and cats [13,30–35] (see also [17–22]).

The analysis of the observed time scale and space extension of the AM patterns leads to excluding that they might be generated by propagation of chemicals, which would be too slow. Moreover, cortex patterns are observed [31] to jump abruptly from a receiving state to an active transmitting state. Long-range correlations are not created by the exchange of neurotransmitters, whose propagation is therefore not the cause, rather the effect of the formation of AM patterns; it is by them facilitated. On the other hand, magnetic field and electric current densities sustained by extracellular dendritic trees are too weak to be responsible for the observed rich texture of the AM patterns [13]. Neuronal radio waves (combined electric and magnetic field propagation) are also excluded due to the unbalance between the electric permittivity and magnetic permeability (80:1) of neural tissue. They are excluded also because of the low frequency (<100 Hz) and km wavelengths of em radiation at frequencies of EEG [36].

The very small time scales and the extremely large number of brain states spanned with the observed high efficiency in the brain functional activity led Umezawa and Ricciardi to suggest that "new non-classical mechanisms have to be looked for ... it is strongly suggestive of a quantum model" [1], as already mentioned in Section 1. They proposed that the neuronal correlations could be generated by the mechanism of the spontaneous breakdown of symmetry (SBS) in QFT in a similar way as long-range correlations arise in condensed matter physics. General theorems in QFT imply indeed that as a consequence of SBS, long-range correlations are dynamically produced in the system [5–8,37]. The system's lowest energy state (the vacuum or ground state) is then a coherent condensate of the quanta, called Nambu–Goldstone (NG) modes, associated with such correlation waves. These NG quanta are massless so that the associated correlation waves may span the whole system volume, taken to be very large (infinite in the QFT limit). For example, the magnetic order in a magnet, measured by the "order parameter" magnetization, is generated by the "spin wave" correlations, whose associated quanta, coherently condensed in the system vacuum, are called magnons (coherent boson condensation occurs in a wide range of temperatures in many systems, such as ferromagnets, crystals, superfluids, and superconductors; for example, the iron magnetization is lost at 770 °C; diamond crystal melts at 3545 °C; kitchen salt (sodium chloride crystal) melts at 804 °C; the superconductivity of compounds of niobium is lost at −153 °C, and at −252 °C for copper and bismuth compounds; coherence phenomena are observed in photosynthesis at ordinary temperatures).

We note that the order parameter, which provides a measure of the ordering at the quantum level, behaves, however, as a classical field, in the sense that it is not affected by quantum fluctuations. This is a property due to the coherence of the system ground states, and we see that it allows the transition to the macroscopic classical behavior of the system. We stress that it is not the "classical limit" obtained for $\hbar \to 0$. It is the macroscopic manifestation of the coherent dynamics at the quantum level.

Umezawa, referring in particular to the problem of memory recording, observed [38] that "in any material in condensed matter physics any particular information is carried by certain ordered pattern maintained by certain long range correlation mediated by massless quanta. It looked to me that this is the only way to memorize some information; memory is a printed pattern of order supported by long range correlations . . . ". In the many-body model of brain, memory storage is thus described by the condensation process of the NG quanta in the system vacuum state (the lowest energy state or ground state). The external input received by the brain triggers the SBS and the consequent dynamical generation of the long-range correlations, whose NG quanta coherently condense in the vacuum. Memory is the printed ordered pattern generated in this way. Excitation from the ground state of condensed NG mode due to a stimulus similar to the one causing that memory recording describes memory recollection. During the excitation time, the brain "consciously" feels the stored memory [1,39].

In the many-body model, there was no treatment of the finite temperature of the brain, and the symmetry undergoing spontaneous breakdown was not specified. One shortcoming of the model is the very limited memory capacity. Indeed, any subsequent external input reaching brain triggers the chain of events just described, from the SBS to the NG vacuum condensation; the new, most recent memory is therefore overprinted on the pre-existing one, thus erasing it.

In 1995, one of the authors (G.V.) [26] and Jibu and Yasue [40,41], also inspired by other existing works in biology and physics [42–45], suggested that in the many-body brain model, the QFT mechanism of SBS could apply to the rotational spherical symmetry of the electrical dipoles of the water molecules and other bio-molecules in the brain. The quantum degrees of freedom were identified with the quanta of these vibrational electric dipole modes. Water molecules constitute indeed the great majority of molecules in the brain (above 90% in number), namely the bath in which neurons, glia cells, and any other biological entities are embedded and can be active only if there embedded, as shown by biochemistry.

A crucial property of brains is their being "open" on their environment, which means that any description of their functional activity and microscopic cellular structure must account for such a characterizing feature, i.e., it must be a dissipative dynamics description. In 1995, the many-body model was thus extended by G.V. to include the dissipative dynamics, and the dissipative quantum model of brain was thus formulated [26,27] (see also [2]).

In the dissipative model, consciousness finds its root and resides in the persisting "dialog" between the system and its environment, which constitutes its *Double*, described as its "time-mirror image" according to the doubling of the degrees of freedom formalism of the model (see below).

In the model, neurons, glia cells, and other biological units are considered to be classical entities. In this respect, the dissipative model is different from other quantum models discussed in the literature (see, e.g., [46]).

The long-range correlations generated by the SBS and the NG associated quanta are identified, respectively, with the dipole waves and the associated dipole wave quanta (DWQ), call them A_κ, with κ generically denoting their quantum numbers, e.g., their momentum. The condition for the SBS is the non-vanishing value of the polarization density P in the ground state: $\langle D^3 \rangle = P \neq 0$, with D^3 denoting the molecular dipole moment in the third direction. As a dynamical consequence of the SBS, long-range correlations are then generated, whose associated quanta (the NG modes) are the DWQ denoted by the A_κ operators (details of the formal derivation in [8,42,43]).

Neurons, axons, dendrites, and glia cells are embedded in the medium of water dipoles and form the densely-populated cortical neuropil tissue. The coherence of the long-range dipole waves, resulting from the coherent condensation of the DWQ, facilitates synaptic and ephaptic communication among neuron populations. The DWQ fields and their condensation act as the medium interconnecting the cortical neurons and sustain the formation of the AM patterns promoting the ephaptic neuronal excitability [47–50]. The linear size of the neuronal correlated regions goes from a few mm, covers the primary sensory cortices, up to the entire limbic system, and then to the entire neocortex.

The transmembrane ionic gradients continuously fed by metabolism supply the needed energy in the ephaptic neuronal communication. The neuropil dynamic behavior goes through phase transitions, which are irreversible, and manifests criticality in the neuronal mutual excitations (see the details in [33,34,51]).

3. Entanglement in the Dissipative Quantum Model of Brain

In the dissipative quantum model, brain states are considered to be finite temperature states with irreversible time evolution. Thermo field dynamics (TFD) and the irreversible time evolution formalism in QFT [5–7,26] require the "doubling" of the system degrees of freedom A_k: $A_\kappa \to A_\kappa \times \tilde{A}_\kappa$, for any κ. $\tilde{A}_\kappa$ describes the thermal bath or the environment in which the system is embedded.

The doubling operation has a well-defined algebraic structure [6,7] (Hopf algebra [7]). It describes the system–environment interaction, and it leads to considering the $\{A, \tilde{A}\}$ system as a closed system. Thus, it allows the use of the QFT canonical formalism, which is indeed limited to closed systems. We refer to the $\tilde{A}$ modes as the *Double*. As we will see, also related to the algebraic structure of the doubling operation is the dynamic origin of the system–environment entanglement.

At this stage of the modeling, the Hamiltonian of non-interacting DWQ has the form:

$$H_0 = \sum_\kappa \hbar\Omega_\kappa \left(A_\kappa^\dagger A_\kappa - \tilde{A}_\kappa^\dagger \tilde{A}_\kappa\right) , \tag{1}$$

where Ω_κ is the frequency, and the minus sign accounts for the fact that energy fluxes outgoing from the A system are ingoing fluxes for the $\tilde{A}$ system, and vice versa. Because of the in/out exchange role between the A and $\tilde{A}$ modes, they are, in this specific sense (not as backward causation), "time-reversed mirror modes"; we thus say that $\tilde{A}_\kappa$ is the "time-reversed image" of A_κ.

Let us denote by $\{|\mathcal{N}_{A_\kappa}, \mathcal{N}_{\tilde{A}_\kappa}\rangle\}$ the set of simultaneous eigenvectors of the number operators $\hat{N}_{A_\kappa} \equiv A_\kappa^\dagger A_\kappa$ and $\hat{N}_{\tilde{A}_\kappa} \equiv \tilde{A}_\kappa^\dagger \tilde{A}_\kappa$, where $\mathcal{N}_{A_\kappa}$ and $\mathcal{N}_{\tilde{A}_\kappa}$ are non-negative integers, and let $|0\rangle_0 \equiv |\mathcal{N}_{A_\kappa} = 0, \mathcal{N}_{\tilde{A}_\kappa} = 0\rangle$ be the vacuum annihilated by A_κ and $\tilde{A}_\kappa$: $A_\kappa|0\rangle_0 = 0 = \tilde{A}_\kappa|0\rangle_0$ for any κ.

The eigenvalue of H_0 on the state $|0\rangle_0$ and on any state that is the condensate of equal number of A_κ and $\tilde{A}_\kappa$ for any κ is thus zero, which expresses the balance of in/out energy fluxes and, therefore, that the couple $\{A, \tilde{A}\}$ is a closed system. This also means that anyone of these states with an equal number of tilde and non-tilde modes may be taken as a ground state.

Consider one of the infinitely many zero-energy states at some initial time $t_0 = 0$. The memory state is defined [26] to be $|0\rangle_\mathcal{N}$, with $\mathcal{N} \equiv \{\mathcal{N}_{A_\kappa} = \mathcal{N}_{\tilde{A}_\kappa}, \forall\kappa, at \;\; t_0 = 0\}$ the set of integers defining the "initial value" of the condensate. $\mathcal{N}$ is thus the "code" associated with the information recorded at time $t_0 = 0$. It turns out [26] that $|0\rangle_\mathcal{N}$ is the two-mode $SU(1,1)$ generalized squeezed coherent state:

$$|0\rangle_\mathcal{N} = \prod_\kappa \frac{1}{\cosh\theta_\kappa} \exp\left(-\tanh\theta_\kappa A_\kappa^\dagger \tilde{A}_\kappa^\dagger\right) |0\rangle_0 , \tag{2}$$

which is normalized to one, ${}_\mathcal{N}\langle 0|0\rangle_\mathcal{N} = 1$ for all $\mathcal{N}$ and where the parameter θ_κ is actually temperature dependent, $\theta_\kappa = \theta_\kappa(\beta)$. For notational simplicity, we will avoid to use the temperature dependence and simply write θ_κ, for any κ.

We realize that the algebraic structure of the model is the one of TFD, and we also see that formal structures introduced in Appendix A are here re-obtained. The number $\mathcal{N}_{A_\kappa}$ is given by:

$$\mathcal{N}_{A_\kappa} = {}_\mathcal{N}\langle 0|A_\kappa^\dagger A_\kappa|0\rangle_\mathcal{N} = \sinh^2\theta_\kappa , \tag{3}$$

which shows how the $\mathcal{N}$-set, $\mathcal{N} \equiv \{\mathcal{N}_{A_\kappa} = \mathcal{N}_{\tilde{A}_\kappa}, \forall\kappa, at \;\; t_0 = 0\}$, is related to the θ-set, $\theta \equiv \{\theta_\kappa, \forall\kappa, at \;\; t_0 = 0\}$. For simplicity of notation, we use $\mathcal{N}_{A_\kappa} \equiv \mathcal{N}_{A_\kappa}(\theta)$ and $|0\rangle_\mathcal{N} \equiv |0(\theta)\rangle_\mathcal{N}$.

We remark that for different codes $\mathcal{N} \neq \mathcal{N}'$, $\{|0\rangle_{\mathcal{N}}\}$ and $\{|0\rangle_{\mathcal{N}'}\}$ are each unitarily inequivalent state spaces in the infinite volume limit:

$$ {}_{\mathcal{N}}\langle 0|0\rangle_{\mathcal{N}'} \underset{V\to\infty}{\longrightarrow} 0 \quad \forall\, \mathcal{N} \neq \mathcal{N}' \,. \tag{4} $$

We conclude that representations $\{|0\rangle_{\mathcal{N}}\}$ denote different, non-overlapping "points" with different labels $\mathcal{N}$, meaning that $\mathcal{N}$ is a good code. The space of the representations is the "memory space".

We thus see that the possibility offered by QFT of having infinitely many unitarily-inequivalent representations $\{|0\rangle_{\mathcal{N}}\}$, for all $\mathcal{N}$'s, of the CCR, provides a solution to the overprinting problem in the many-body model. Now, we have indeed that infinitely many vacua $|0\rangle_{\mathcal{N}}$ for all $\mathcal{N}$'s are independently accessible, and memories of code $\mathcal{N}$, for all $\mathcal{N}$'s, can be recorded, coexisting without destructive interference.

We recognize that the state $|0\rangle_{\mathcal{N}}$ is of the same type of the state in Equation (A7) in Appendix A, and in particular, it is an entangled state that cannot be factorized into two single-mode states (cf. Equation (A9)):

$$ |0\rangle_{\mathcal{N}} = \left(\prod_{\kappa} \frac{1}{\cosh\theta_\kappa}\right)\left(|0\rangle_0 \otimes |\tilde{0}\rangle_0 - \sum_{\kappa} \tanh\theta_\kappa \left(|A_\kappa\rangle \otimes |\tilde{A}_\kappa\rangle\right) + \dots\right) . \tag{5} $$

The conclusion at this point of the discussion is that the whole set of results listed in Equations (A8)–(A15) also holds in the present case. As stressed also in Appendix A, the features of the entanglement and the related results there presented are indeed of general validity, not limited to the special case of two photon entanglement, discussed there for the interested reader's convenience.

Since the $\tilde{A}$ modes represent the environment, we see that the brain–environment entanglement is a specific dynamical feature of the brain functional activity. The origin of the entanglement is dissipation. The entanglement robustness, rooted in the fact that in QFT there is no unitary operator able to disentangle the brain from its environment, ensures the stability and, at the same time, the persistence of the openness of brain on the world.

All the above observations and remarks express the strict relationship between the formalism of the dissipative quantum model of brain and the entanglement phenomenon. As we see from the specific mathematical aspects discussed above, entanglement plays a central, essential role in the formal structure on which the dissipative model is constructed.

We observe that, however, the robustness of the entanglement may be spoiled by finite volume effects. In such cases, the A_κ and $\tilde{A}_\kappa$ modes may acquire non-zero effective masses, due to such finite volume effects, loosing their property of being massless [26,27,52]. Then, the non-unitary relation among different representations may be lost, and disentanglement processes may be activated by some unitary operator. This particular feature of the model may be related to some pathological behavior, for example with depression states arising from loss of interest in (disentanglement from) the surrounding world [27,52]. However, we will not discuss these problems, which are out of the scope of our presentation.

We also remark that the memory code $\mathcal{N}$ is a macroscopic observable in the sense mentioned above, namely it is not affected by quantum fluctuations as a result of the coherence of the DWQ condensation. We have thus the "change of scale", from micro to macro: the memory state $|0\rangle_{\mathcal{N}}$ is a "macroscopic quantum state". We may then define the "brain (ground) state" as the full set of entangled memory states $|0\rangle_{\mathcal{N}}$, for all $\mathcal{N}$.

We recall that the $\tilde{A}_\kappa$ modes are the brain "imaging" of the environment, and the ordered patterns in the condensate of the $\{A, \tilde{A}\}$ couples facilitate the formation of neuronal AM synchronized assemblies. As such, the $\tilde{A}_\kappa$ modes have been related [29,53,54] to the mental activity of the brain, the "mind". The $A - \tilde{A}$ entanglement thus results in the brain and mind activity entanglement. We will comment further on this point in the following (cf. Section 5).

Let us now consider the time evolution of the memory states. The Hamiltonian is [26]:

$$H = H_0 + H_I = \sum_\kappa \hbar\Omega_\kappa \left(A_\kappa^\dagger A_\kappa - \tilde{A}_\kappa^\dagger \tilde{A}_\kappa\right) + i\sum_\kappa \hbar\Gamma_\kappa \left(A_\kappa^\dagger \tilde{A}_\kappa^\dagger - A_\kappa \tilde{A}_\kappa\right) . \tag{6}$$

Γ_κ is the damping constant, and the H_I term accounts for the dissipative time evolution. We immediately remark that the number $(\mathcal{N}_{A_\kappa} - \mathcal{N}_{\tilde{A}_\kappa})$ is a constant of motion for any κ, since H_0 commutes with H_I, $[H_0, H_I] = 0$. H_0 is the Casimir operator of the $SU(1,1)$ group structure of H.

We find [26] that the memory state $|0\rangle_\mathcal{N}$ evolves in time as:

$$|0(t)\rangle_\mathcal{N} = \exp\left(-it\frac{H}{\hbar}\right)|0\rangle_\mathcal{N} = \prod_\kappa \frac{1}{\cosh\left(\Gamma_\kappa t - \theta_\kappa\right)} \exp\left(\tanh\left(\Gamma_\kappa t - \theta_\kappa\right) A_k^\dagger \tilde{A}_k^\dagger\right)|0\rangle_0 , \tag{7}$$

and:

$$\mathcal{N}_{A_\kappa}(\theta, t) = {}_\mathcal{N}\langle 0(t)|A_\kappa^\dagger A_\kappa|0(t)\rangle_\mathcal{N} = \sinh^2\left(\Gamma_\kappa t - \theta_\kappa\right) , \tag{8}$$

and a similar expression is obtained for the $\tilde{A}_\kappa$ modes.

Again, $|0(t)\rangle_\mathcal{N}$ is a normalized $SU(1,1)$ generalized coherent state, ${}_\mathcal{N}\langle 0(t)|0(t)\rangle_\mathcal{N} = 1$, and it is an entangled state. Thus, all previous conclusions for the state $|0\rangle_\mathcal{N}$ at the initial time extend to the time-evolved states $|0(t)\rangle_\mathcal{N}$ at any t.

We remark that since the condensate of the modes A and $\tilde{A}$ promotes the synaptic and ephaptic neuronal communication, the entanglement between A and $\tilde{A}$ produces neuronal synchronicity. Neurons, which in the model together with other cells are considered to be classical units, embedded in the medium of DWQ act in some sense as "observers" (myriads of Alice and Bob) of the collective background of entangled A and $\tilde{A}$ modes. Phase-mediated correlations for the couples $(A_\kappa, \tilde{A}_\kappa)$ and their coherent dynamics are thus "transferred" to neuronal populations.

In the infinite volume limit, we also have, for $\int d^3\kappa\, \Gamma_\kappa$ finite and positive,

$$ {}_\mathcal{N}\langle 0(t)|0\rangle_\mathcal{N} \underset{V\to\infty}{\longrightarrow} 0 , \qquad {}_\mathcal{N}\langle 0(t)|0(t')\rangle_\mathcal{N} \underset{V\to\infty}{\longrightarrow} 0 , \qquad \forall\, t, t' \quad , \quad t \neq t' \tag{9}$$

We thus recognize that the non-unitary time evolution implied by dissipation is rooted in the unitary inequivalence, in the infinite volume limit, of the states $|0(t)\rangle_\mathcal{N}$, each one minimizing the free energy at different times $t \neq t'$.

In its time evolution, the memory state $|0\rangle_\mathcal{N}$ undergoes (continuous) transitions through the set of $\{|0(t)\rangle_\mathcal{N}\}$ state spaces (representations) at different t's. These transitions may be described as "trajectories" through the "points" $\{|0(t)\rangle_\mathcal{N}\}$ in the "space of the representations", with the $\mathcal{N}$-set specifying the initial condition at $t_0 = 0$. These trajectories can be shown to be classical chaotic trajectories [55]. We will comment on this point in the following Section.

4. Chaos in the Dissipative Quantum Model of Brain

We recall that the set of $SU(1,1)$ coherent states forms a Kählerian manifold $\mathbf{H}$, which has a symplectic structure. It may thus represent the phase space for the classical dynamics generated by the $SU(1,1)$ group action [56].

The $SU(1,1)$ generalized coherent states, i.e., the memory states $|0(t)\rangle_\mathcal{N}$ considered above, are thus recognized to be "points" in $\mathbf{H}$, and transitions among these points induced by the group action are therefore classical trajectories [56] in $\mathbf{H}$ with the $\mathcal{N}$-set specifying the initial condition at $t_0 = 0$.

In the dissipative model, such a conclusion plays a crucial role; indeed, it confirms the already observed transition from the quantum dynamics to the classical level.

The trajectories in the memory space can be shown to satisfy the requirements characterizing the chaotic behavior in non-linear dynamics [57], namely:

(a) the trajectories are bounded, and each trajectory does not intersect itself (they are not periodic).

(b) trajectories specified by different initial conditions do not cross each other.
(c) different initial conditions lead to diverging trajectories.

The boundedness of the trajectories follows from the finiteness of the norm of the state vectors at any time t, ${}_{\mathcal{N}}\langle 0(t)|0(t)\rangle_{\mathcal{N}} = 1$. As time evolves, the trajectory of given $\mathcal{N}$ never crosses itself; the "points" $|0(t)\rangle_{\mathcal{N}}$ and $|0(t')\rangle_{\mathcal{N}}$ for any t and t', with $t \neq t'$, never coincide, as shown by Equations (9). The requirement (a) is thus satisfied.

One can show [55,58,59] that in the infinite volume limit:

$$ {}_{\mathcal{N}}\langle 0(t)|0\rangle_{\mathcal{N}'} \underset{V\to\infty}{\longrightarrow} 0 \quad \forall t \quad , \ \forall \mathcal{N} \neq \mathcal{N}' \tag{10} $$

$$ {}_{\mathcal{N}}\langle 0(t)|0(t')\rangle_{\mathcal{N}'} \underset{V\to\infty}{\longrightarrow} 0 \quad \forall t, t' \quad , \ \forall \mathcal{N} \neq \mathcal{N}' \, . \tag{11} $$

Equation (11) holds also for $t = t'$ for any $\mathcal{N} \neq \mathcal{N}'$. Equations (10) and (11) imply that trajectories specified by different initial conditions, $\mathcal{N} \neq \mathcal{N}'$, never cross each other. Requirement (b) is thus satisfied, which implies that at any time t, there is no interference, i.e., no confusion, among different memories.

However, finite volume effects may allow non-zero inner products between memory states with different codes. In such cases, at the crossing points between two different trajectories, one might switch from one to another trajectory, which can be experienced as an "association" of memories.

Let us now consider the time evolution of the difference between the codes $\mathcal{N} \neq \mathcal{N}'$ ($\theta \neq \theta'$), specifying the initial time conditions, at $t_0 = 0$, of two trajectories. This gives us the time evolution of the "distance" between the trajectories. For a very small difference $\delta\theta_\kappa$, for any κ, in the initial conditions of the two memory states, we obtain:

$$ \Delta\mathcal{N}_{A_\kappa}(t) \equiv \mathcal{N}'_{A_\kappa}(\theta', t) - \mathcal{N}_{A_\kappa}(\theta, t) \approx \sinh\big(2(\Gamma_\kappa t - \theta_\kappa)\big)\delta\theta_\kappa \, , \tag{12} $$

with $\delta\theta_\kappa \equiv \theta_\kappa - \theta'_\kappa$, assumed in full generality to be positive, and:

$$ \frac{\partial}{\partial t}\Delta\mathcal{N}_{A_\kappa}(t) = 2\Gamma_\kappa \cosh\big(2(\Gamma_\kappa t - \theta_\kappa)\big)\delta\theta_\kappa \, . \tag{13} $$

The modulus of the difference $\Delta\mathcal{N}_{A_\kappa}(t)$ and its time derivative thus diverge as $\exp(2\Gamma_\kappa t)$, for all κ's, for large t, suggesting that $2\Gamma_\kappa$, for each κ, acts like the Lyapunov exponent in chaos theory [57].

Suppose that two perceptive experiences differ solely by very tiny details (very small $\Delta\mathcal{N}_{A_\kappa}(t)$, for all κ). These slightly different inputs correspond to slightly different initial conditions for two trajectories in the memory space. In the absence of the trajectories' exponential divergence, one might be induced into the "confusion" of considering the two inputs as being identical ones, a low level of "perceptive resolution". On the contrary, the divergence of the trajectories, according to their chaotic nature, allows almost immediate distinction between the two inputs. Chaos thus provides high resolution in perceptual inputs.

We remark that the codes $\mathcal{N}$ and $\mathcal{N}'$ contain a very large number, approaching infinity in the continuum limit, of $\mathcal{N}_{A_\kappa}(\theta, t)$ components. Thus, in the case that a finite number of these components are equal, which as shown by Equation (12) happens at time $t_\kappa = \theta_\kappa/\Gamma_\kappa$ for the κ–components, $\mathcal{N}$ and $\mathcal{N}'$ are still different codes. However, let τ_{min} and τ_{max} be the minimum and the maximum, respectively, of $t_\kappa = \theta_\kappa/\Gamma_\kappa$, for all κ's, then for $\delta\theta_\kappa \equiv \theta_\kappa - \theta'_\kappa$ very small, if $\Delta t = \tau_{max} - \tau_{min}$ is also very small, the codes $\mathcal{N}$ and $\mathcal{N}'$ are recognized to be "almost" equal in such Δt. Such a time interval Δt then provides a measure of the "recognition time", and we see that the recognition (or recall) processes of a given memory code $\mathcal{N}$ may be triggered by "slightly different" $\mathcal{N}'$ codes (corresponding to new inputs, slightly different from the one originating the memory $\mathcal{N}$).

In conclusion, the requirement (iii) is also satisfied, and thus, trajectories in the memory space are classical chaotic trajectories. Moreover, one can also show [51,60–63] that the memory states have a self-similar fractal structure. These results are in agreement with experimental observations of scaling behavior in human brain oscillations reported since 1987 by Walter Freeman and his collaborators [14,64,65], also confirmed in other works [66–70], and the object of rigorous mathematical analysis [71]. Freeman has stressed [14] that "The brain transforms sensory messages into conscious perceptions almost instantly. Chaotic, collective activity involving millions of neurons seems essential for such rapid recognition."... "Our studies have led us as well to the discovery in the brain of chaos-complex behavior that seems random but actually has some hidden order. The chaos is evident in the tendency of vast collections of neurons to shift abruptly and simultaneously from one complex activity pattern to another in response to the smallest of inputs ... This changeability is a prime characteristic of many chaotic systems ... In fact, we propose it is the very property that makes perception possible. We also speculate that chaos underlies the ability of the brain to respond flexibly to the outside world and to generate novel activity patterns, including those that are experienced as fresh ideas ...".

5. Concluding Remarks

We reviewed the main features of the dissipative quantum model of brain. The embedding of the brain in its environment and the reciprocal unavoidable interaction was described in terms of the entangled degrees of freedom A_κ and $\tilde{A}_\kappa$ (the system and its *Double*). These modes are the dipole wave quanta (DWQ), and their condensation in the ground state describes the memory recording process. The density of the condensate is the code of the recorded memory. States with different codes are unitarily inequivalent states.

Many results in agreement with laboratory observations have been derived from the dissipative model. A partial list of them, as reported e.g., in [63], includes the coexistence of physically-distinct AM patterns in distinct frequency bands correlated with categories of conditioned stimuli, the rapid onset of these AM patterns into (irreversible) sequences, the very low energy required to excite them, their large diameters with respect to the small sizes of the component neurons, their duration, size, and power as decreasing functions of their carrier wave number k, the lack of their invariance with invariant stimuli, but constancy with the unchanging meaning of the stimuli, self-similarity in brain background activity showing power-law distributions of power spectral densities derived from ECoGs data, heat dissipation at (almost) constant-in-time temperature, the occurrence of near-zero down-spikes in phase transitions, the whole phenomenology of phase gradients and phase singularities in the vortices formation, the constancy of the phase field within the frames, the insurgence of a phase singularity associated with the abrupt decrease of the order parameter and the concomitant increase of the spatial variance of the phase field, the occurrence of phase cones and random variation of the sign (implosive and explosive) at the apex, random spatial locations of the phase cone apices, their occurrence between frames (during phase transitions), not within frames, and the "classicality" (not derived as the classical limit, but as a dynamical output) of functionally self-regulated and self-organized background activity of the brain.

Concerning the fluctuating random force in the system–environment coupling, we only mention that the modes $\tilde{A}_\kappa$ account for the quantum noise generated by such forces, and that this manifests itself through the coherent structure of the ground state $|0\rangle_{\mathcal{N}}$. By resorting to the Schwinger [72] and Feynman and Vernon [73] description of dissipative systems in the frame of the quantum Brownian motion, we can show [74,75] that noise effects can be treated by the use of the density matrix $\langle x_+|\rho(t)|x_-\rangle = \psi^*(x_+,t)\,\psi(x_-,t) \equiv W(x,y,t)$, where $W(x,y,t)$ is the Wigner function and $x_\pm \equiv x \pm (1/2)y$. The y variable can be shown to account for noise effects. Such coordinate representation may be related through canonical quantization procedures to the field operators A_κ and $\tilde{A}_\kappa$ in the dissipative model formalism [58,59,74,75].

From Equation (8), we see that $t = \tau$, with τ the largest of the values $\tau_\kappa \equiv \theta_\kappa/\Gamma_\kappa$, can be taken as the life-time of the memory of code $\mathcal{N}$: at $t = \tau$, the memory state is reduced to the "empty" vacuum $|0\rangle_0$, which thus acts as an attractor state, and the information has been forgotten. DWQ modes with higher momentum have been found to possess a longer life-time [52]. Since the mode can propagate over distances proportional to the reciprocal of the momentum, longer life-time modes propagate over shorter distances and form smaller coherent domains. On the contrary, shorter life-time modes form larger coherent domains. We have thus domains with different sizes and different stability, smaller domains surviving longer than larger domains. The dissipative model thus describes memories with different life-times.

For $t > \tau$, one can show that $|0(t)\rangle_{\mathcal{N}}$ "runs away" from the attractor state $|0\rangle_0$ with the exponential law. If the memory code $\mathcal{N}$ is not decayed, but only "corrupted", since some of the code components $\mathcal{N}_\kappa$ have been lost for some κ's, at $t_\kappa = \theta_\kappa / \Gamma_\kappa$, then it can be "restored", or "refreshed" by brushing up the subject. The code components may be recovered in a process of "external stimuli maintained memory". In some sense, one resets the "memory clock" updating initial time t_0 in the register of the memory recording. This avoids to fall in the attractor $|0\rangle_0$.

We remark that since H_0 is the $SU(1,1)$ Casimir operator, thus commuting with H_I, the memory code $\mathcal{N}$ (the initial conditions of the trajectory) can be "measured" (recalled) at any time $t < \tau$ of the time evolution (persistence of the information code $\mathcal{N}$).

As is well known, entanglement is at the heart of the quantum measurement problem. In our discussion, we have used the QFT formalism where the entanglement phenomenon is described by taking advantage of the existence of infinitely-many unitarily-inequivalent representations of the CCR, a feature usually not considered in preceding studies of entanglement. In general, one describes the apparently "realistic" outcome of a measurement process in terms of a von Neumann chain of entangled interacting systems, starting with the observed microscopic system and culminating in the macroscopic measuring apparatus (or in the observer's brain). One of the authors (S.A.S) showed [76] that a realistic outcome of a quantum measurement is conditional upon information about the measurement process persisting in at least one of the interacting systems of the von Neumann chain, which always happens in actual measurements, since part of the von Neumann chain is macroscopic. It is interesting to note that, however, the result does not depend on the macroscopic nature of the information, as confirmed in various experiments (quantum beats [77], quantum eraser [78,79]). It is an interesting question to ask whether the understanding of entanglement as imprints on the structure of the vacuum along the lines of the present work could be extended to the above-mentioned "persistence of information" approach to the quantum measurement problem. We leave such a challenge for a future work.

The dissipative model describes features of the brain functional activity that are in agreement with experimental data showing a continuous field of collective neural activity coexisting with discrete neural firing of pulses. There is then the observation of modulation patterns of phase cones, which exhibit outward or inward pulsations with converging (imploding) and diverging (exploding) phase correlations, in the form of wave packets, with or without rotational gradients (vortices). The exploding gradient is explained by conventional neurodynamics. However, there is no explanation of imploding gradients. The dissipative model explains instead both of them as reciprocal time-reversed images, with factors $\exp(\pm\Gamma t)$ (thus not in the sense of backward causation) [80,81]. Forward and backward in time modes play both an important role. It is the *Double* that, going backward in time, in the reconstruction of a past perceptual experience, "provides the imagination that construct the hypothesis to be tested by the action". The forward-in-time neuronal activity guides the "intentional" action that follows the perceptual experience, in the action-perception cycle, and it is planned on the basis of the hypothesis provided by the *Double*. "It is the Double that imagines the world outside, free from the shackles of thermodynamic reality". Brains test the hypothesis [29] "by use of the action-perception cycle. Since the experience of the body in action on repeated trials is infallibly followed by the experience of changes in the sensorium, action creates the perception of time and simultaneously of causation". The openness of the brain on the world also means openness to "other brains", involvement in shaping cultural nets, and sharing common views and aesthetic tastes, a "social brain" activity [82–84] (see also [85]). The social dimension, a higher level of phase-mediated correlations among brains, is intrinsic to the brain functional activity in the dissipative model.

We stress that there is no possibility of separation between mental activity and brain activity: "the brain modes and the mental (Double) modes appear entangled in the coherent states through which the activity evolves" [29]. There are not two entities, not a dual level of existence, matter and mind, but only one undividable entity. The dissipative model describes the brain activity in the relation with its environment; it does not describe mental states separated by brain. Consciousness arises and resides in the dialog of the self with its *Double* [26,27]. In this dialog, perceptual experiences acquire a meaning [28,53,86–89] within the landscape of attractors constructed in previous experiences and continuously reshaping at any new input. Memory is not memory of information, it is memory of meanings.

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Appendix A. Phase-Mediated Entanglement in Quantum Field Theory

Results and general features of the entanglement in QFT that have been used in our discussion are reported in this Appendix. In order to obtain these results, we consider the example of a couple of entangled photons generated in the parametric down-conversion process, well known in quantum optics (there is a vast literature on entanglement and all the related issues; we limit ourselves to [23–25] (see also [76–79]) and to the more specialized references mentioned there, which we do not need for our presentation). We stress, however, that the general features of the entanglement here reviewed are of general validity, not limited to the special example of two photon entanglement.

The simultaneous creation of two photons by a neutral spin-zero source, in the center-of-momentum frame, is described by the Hamiltonian [5,7,23]:

$$H = \sum_{\mu} \int d^3k \left[\omega_k (a^{\dagger}_{\mu,\mathbf{k}} a_{\mu,\mathbf{k}} + b^{\dagger}_{\mu,\mathbf{k}} b_{\mu,\mathbf{k}}) + \nu_k (b^{\dagger}_{-\mu,-\mathbf{k}} a^{\dagger}_{\mu,\mathbf{k}} + a_{\mu,\mathbf{k}} b_{-\mu,-\mathbf{k}}) \right] . \tag{A1}$$

The interacting field operators (also called the Heisenberg field operators) $(a^{\dagger}_{\mu,\mathbf{k}}, b^{\dagger}_{-\mu,-\mathbf{k}})$ $((a_{\mu,\mathbf{k}}, b_{-\mu,\mathbf{k}}))$ are the creation (annihilation) of the two photons in the pair, each one with energy ω_k; $\mu = \pm 1$ is the spin index. We assume $\nu_k^2 < \omega_k^2$, both real positive. We use natural units $\hbar = 1 = c$. The canonical commutation relations are $[a_{\mu,\mathbf{k}}, a^{\dagger}_{\sigma,\mathbf{l}}] = \delta_{\mu\sigma}\delta(\mathbf{k}-\mathbf{l})$, $[b_{\mu,\mathbf{k}}, b^{\dagger}_{\sigma,\mathbf{l}}] = \delta_{\mu\sigma}\delta(\mathbf{k}-\mathbf{l})$, and all other commutators are zero. The vacuum state is $|0\rangle \equiv |0\rangle_a \otimes |0\rangle_b$: $a_{\mu,\mathbf{k}}|0\rangle = 0$, $b_{\mu,\mathbf{k}}|0\rangle = 0$. Let $\mathcal{H}$ denote the space where Heisenberg field operators are defined. As is well known [3–7,90], this space is not accessible to observations, which would be otherwise interfering with the interaction to be studied.

Once the pair creation has occurred (the ν_k-term in Equation (A1)), the two photons reach the space-time asymptotic region, far away from the creation vertex in the interaction region. They are spatially separated from each other and can be detected by the observers, Alice and Bob, as physical fields. Let $\alpha_{\mu,\mathbf{k}}$ and $\beta_{\mu,\mathbf{k}}$ ($\alpha^{\dagger}_{\mu,\mathbf{k}}$ and $\beta^{\dagger}_{\mu,\mathbf{k}}$) be their annihilation (creation) operators, and let, e.g., Alice detect the α-beam and Bob the β-beam. The canonical commutation relations of the $\alpha_{\mu,\mathbf{k}}$ and $\beta_{\mu,\mathbf{k}}$ operators and of their Hermitian conjugates are the usual ones. These outgoing photon field operators act on the Hilbert space $\mathcal{H}_{\mathcal{F}}$ of the physical states, which is the space accessible to observations. The realization of the Hamiltonian (A1) in such a space has to provide the total energy E_k of the observed α- and β-modes. Thus, when it is written in terms of these physical fields, it must have the form:

$$H = H_0 + W_0 = \sum_{\mu} \int d^3k E_k (\alpha^{\dagger}_{\mu,\mathbf{k}} \alpha_{\mu,\mathbf{k}} + \beta^{\dagger}_{\mu,\mathbf{k}} \beta_{\mu,\mathbf{k}}) + W_0 . \tag{A2}$$

One can show that Equation (A1) can be written in the form of Equation (A2) when E_k and W_0 are given by:

$$E_k = \sqrt{\omega_k^2 - \nu_k^2}\,, \qquad W_0 = \int d^3k \left(\sqrt{\omega_k^2 - \nu_k^2} - \omega_k \right)\,, \tag{A3}$$

and α and β are expressed in terms of a and b as:

$$\alpha_{\mu,\mathbf{k}} \equiv \alpha_{\mu,\mathbf{k}}(\theta) = a_{\mu,\mathbf{k}} \cosh\theta_k - b^{\dagger}_{-\mu,-\mathbf{k}} \sinh\theta_k \tag{A4}$$

$$\beta_{\mu,\mathbf{k}} \equiv \beta_{\mu,\mathbf{k}}(\theta) = b_{\mu,\mathbf{k}} \cosh\theta_k - a^{\dagger}_{-\mu,-\mathbf{k}} \sinh\theta_k \tag{A5}$$

and θ_k given by:

$$\cosh 2\theta_k = \frac{\omega_k}{\sqrt{\omega_k^2 - \nu_k^2}}\,, \qquad \sinh 2\theta_k = -\frac{\nu_k}{\sqrt{\omega_k^2 - \nu_k^2}}\,. \tag{A6}$$

Clearly, $\alpha_{\mu,\mathbf{k}}$ and $\beta_{\mu,\mathbf{k}}$ do not annihilate $|0\rangle$. The state annihilated by $\alpha_{\mu,\mathbf{k}}$ and $\beta_{\mu,\mathbf{k}}$ is $|0(\theta)\rangle$, $\alpha_{\mu,\mathbf{k}}|0(\theta)\rangle = 0$, $\beta_{\mu,\mathbf{k}}|0(\theta)\rangle = 0$, given, at finite volume V, by:

$$|0(\theta)\rangle = \prod_{\mathbf{k},\mu} \frac{1}{\cosh\theta_k} \exp\left(\tanh\theta_k\, a^{\dagger}_{\mu,\mathbf{k}} b^{\dagger}_{-\mu,-\mathbf{k}} \right) |0\rangle, \tag{A7}$$

with $\langle 0(\theta)|0(\theta)\rangle = 1$. The state $|0(\theta)\rangle$ is thus the physical vacuum state in the space $\mathcal{H}_{\mathcal{F}}$, which now we denote as $\mathcal{H}_{\mathcal{F}}(\theta)$. It is known [5–7,23,56,91] to be a generalized $SU(1,1)$ two-mode squeezed coherent state, with the phase θ_k being related to the squeezing parameter.

By using the continuous limit relation $\sum_{\mathbf{k}} \to [V/(2\pi)^3] \int d^3k$ in Equation (A7), the factor $\exp\left[(-V/2(2\pi)^3) \int d^3k\, \mathcal{F}(\theta_k\right]$, with $\mathcal{F}(\theta_k) = \ln\cosh^2\theta_k$, appears, and for $\int d^3k\, \mathcal{F}(\theta_k)$ non-vanishing and finite, $\mathcal{H}$ turns out to be unitarily inequivalent to $\mathcal{H}_{\mathcal{F}}(\theta)$, for any $\theta \equiv \{\theta_k, \forall\, k\}$, for $V \to \infty$, i.e.,

$$\langle 0|0(\theta)\rangle \to 0 \quad \text{for} \quad V \to \infty\,, \tag{A8}$$

and similarly for any couple of states of $\mathcal{H}$ and $\mathcal{H}_{\mathcal{F}}(\theta)$; all the states of $\mathcal{H}$ are orthogonal to those of $\mathcal{H}_{\mathcal{F}}(\theta)$, unless $\theta_k = 0$ for all k. The spaces $\mathcal{H}_{\mathcal{F}}(\theta)$ and $\mathcal{H}_{\mathcal{F}}(\theta')$ are also unitarily inequivalent, $\langle 0(\theta')|0(\theta)\rangle \to 0$ for $V \to \infty$, for any $\theta \neq \theta'$. This is a characteristic feature of QFT, called foliation, not present in QM (the von Neumann theorem [92]).

The unitary inequivalence expressed by Equation (A8) means that in the infinite volume limit, the state $|0(\theta)\rangle$ cannot be expressed in terms of the state $|0\rangle$. We have the transition to non-perturbative physics. In the $V \to \infty$ limit, Equation (A7) is to be understood as a formal expression.

Remarkably, the unitary inequivalence between $|0(\theta)\rangle$ and $|0\rangle$ arises since the contributions of the order of $(1/V)_{V\to\infty}$ are neglected. These infrared(or $(1/V)_{V\to\infty}$) contributions are necessarily missing due to the "locality" of any laboratory observations, where the volume V can be always considered to be extremely large (infinite) with respect to the finiteness of the spatial region covered by the observations. Therefore, the locality of the QFT formalism is exactly the origin of the "difference" (unitary inequivalence, thus observable physical diversity) between the entangled $|0(\theta)\rangle$ state and the non-entangled one $|0\rangle$ (for the role of the infrared contributions in QFT, see, e.g., [93–96]). These specific formal points represent distinctive features of QFT, and they mark the differences between the present discussion and preceding studies of the entanglement phenomenon. As already mentioned, these points and the results listed below are direct consequences of the existence in QFT of infinitely-many unitarily-inequivalent representations of the CCR.

For brevity, we simply list the results obtained from Equation (A7) without reporting their derivation (details can be found in [5–7,56,91]). They are of general validity for such kinds of coherent states.

The state (A7) is an entangled state for $a_{\mu,\mathbf{k}}$ and $b_{-\mu,-\mathbf{k}}$. In fact, it can be written as:

$$|0(\theta)\rangle = \prod_k \frac{1}{\cosh\theta_k}\left[|0\rangle\otimes|0\rangle + \sum_{\mathbf{k},\mu}\tanh\theta_k\left(|a_{\mu,\mathbf{k}}\rangle\otimes|b_{-\mu,-\mathbf{k}}\rangle\right)+\ldots\right], \tag{A9}$$

which shows that it cannot be factorized into the product of two single-mode states. In the summation term in Equation (A9), there are states of the Bell type, e.g., $|\psi_\pm\rangle \propto \left(|a_{\mu,\mathbf{k}}\rangle\,|b_{-\mu,-\mathbf{k}}\rangle \pm |a_{-\mu,\mathbf{k}}\rangle|b_{\mu,-\mathbf{k}}\rangle\right)$.

Next, we introduce the entropy operator $\mathcal{S}_a$, which provides a measure of the degree of entanglement:

$$\mathcal{S}_a \equiv -\sum_{\mathbf{k},\mu}\left\{a^\dagger_{\mu,\mathbf{k}}a_{\mu,\mathbf{k}}\ln\sinh^2\theta_k - a_{\mu,\mathbf{k}}a^\dagger_{\mu,\mathbf{k}}\ln\cosh^2\theta_k\right\}. \tag{A10}$$

A similar expression for $\mathcal{S}_b$ is obtained by replacing $a_{\mu,\mathbf{k}}$ and $a^\dagger_{\mu,\mathbf{k}}$ with $b_{-\mu,-\mathbf{k}}$ and $b^\dagger_{-\mu,-\mathbf{k}}$, respectively. We write $\mathcal{S}$ for either $\mathcal{S}_a$ or $\mathcal{S}_b$. We obtain:

$$\langle 0(\theta)|\mathcal{S}|0(\theta)\rangle = -\sum_{n=0}^{+\infty} W_n(\theta)\ln W_n(\theta)\,, \tag{A11}$$

where $W_n(\theta)$ is a decreasing monotonic function of n, given by:

$$W_n(\theta) = \prod_{k,\mu}\frac{\sinh^{2n_{\mu,k}}\theta_k}{\cosh^{2(n_{\mu,k}+1)}\theta_k}, \qquad 0 < W_n < 1 \quad \text{and} \quad \sum_{n=0}^{+\infty} W_n = 1\,. \tag{A12}$$

Equation (A11) shows that W_n gives the probability of having entanglement of the two sets of $\{n\}$ modes a and b and confirms that $\mathcal{S}$ provides a measure of the degree of entanglement.

A further result is that the linear correlation coefficient $J(\hat{n}_a,\hat{n}_b)$ acquires in $|0(\theta)\rangle$ the maximal value for the θ-phase-mediated correlation between the a and b modes, i.e.,

$$J(\hat{n}_a,\hat{n}_b) = \frac{cov(\hat{n}_a,\hat{n}_b)}{(\langle(\Delta\hat{n}_a)^2\rangle)^{1/2}\,(\langle(\Delta\hat{n}_b)^2\rangle)^{1/2}} = 1, \tag{A13}$$

where for simplicity, the indexes $\pm\mu$ and $\pm\mathbf{k}$ are omitted, $\hat{n}_a = a^\dagger a$, $\hat{n}_b = b^\dagger b$ are the number operators, the variance is $\langle(\Delta\hat{n})^2\rangle \equiv \langle(\hat{n}-\langle\hat{n}\rangle)^2\rangle = \langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2$, and the covariance is denoted by $cov(\hat{n}_a,\hat{n}_b) \equiv \langle\hat{n}_a\hat{n}_b\rangle - \langle\hat{n}_a\rangle\langle\hat{n}_b\rangle$. The symbol $\langle * \rangle$ denotes the expectation value in $|0(\theta)\rangle$.

For non-correlated modes $\langle\hat{n}_a\hat{n}_b\rangle = \langle\hat{n}_a\rangle\langle\hat{n}_b\rangle$, the covariance is zero. On the contrary, the strong (a,b)-pair correlation is due to the coherent structure of the vacuum $|0(\theta)\rangle$.

In deriving Equation (A13), we have used the relation $(a^\dagger_{\mu,\mathbf{k}}a_{\mu,\mathbf{k}} - b^\dagger_{-\mu,-\mathbf{k}}b_{-\mu,-\mathbf{k}})|0(\theta)\rangle = 0$ and $n_{a_{\mu,k}}(\theta) \equiv \langle\hat{n}_{a_{\mu,\mathbf{k}}}\rangle = \sinh^2\theta_k = \langle\hat{n}_{b_{-\mu,-\mathbf{k}}}\rangle \equiv n_{b_{-\mu,-k}}(\theta)$. Note that $(a^\dagger_{\mu,\mathbf{k}}a_{\mu,\mathbf{k}} - b^\dagger_{-\mu,-\mathbf{k}}b_{-\mu,-\mathbf{k}})$ is proportional to the Casimir operator of $SU(1,1)$, and thus, it is a conserved quantity in all the state spaces. We finally remark that:

$$n_{a_{\mu,k}}(\theta) = \langle 0(\theta)|a^\dagger_{\mu,\mathbf{k}}a_{\mu,\mathbf{k}}|0(\theta)\rangle = \langle 0(\theta)|\beta_{-\mu,-\mathbf{k}}\beta^\dagger_{-\mu,-\mathbf{k}}|0(\theta)\rangle\sinh^2\theta_k = \sinh^2\theta_k, \tag{A14}$$

which shows that the measurement of the a-mode number performed by Alice is in fact determined by Bob's β-modes: the strong phase correlation between the photon modes is dynamically built in in the $|0(\theta)\rangle$ vacuum. In the experiments with, e.g., calcite crystals separating Alice's and Bob's respective photon beams into orthogonally-polarized beams, Bob's observation of the photon in one of the two polarization states determines Alice's observation of the (other) orthogonal polarization photon state, even if their observations occur at a space-like distance. These correlations are phase correlations, not-mediated by messenger particles, and therefore, there is no violation of relativity postulates.

We also note that a single measurement made by Bob determines only a probability distribution for the result observed by Alice. Such a statistical feature is also built in in the condensate structure of $|0(\theta)\rangle$. In fact, the a-mode (and b-mode) number is given by the Bose–Einstein distribution function:

$$n_{a_{\mu,k}}(\theta) = \sinh^2\theta_k = \frac{1}{e^{\beta\omega_k} - 1}\,, \tag{A15}$$

where $\beta^{-1} = k_B T$, k_B is the Boltzmann constant and T the temperature. Equation (A15) is obtained by the minimization condition of the free energy F_a, $\partial F_a/\partial\theta_k = 0$, for all k. The minimization of the free energy thus plays a crucial role in the entanglement phenomenon. The density matrix operator is given by $\hat{\rho}_{\mathbf{k},\mu} = f_k^{a^\dagger_{\mu,\mathbf{k}} a_{\mu,\mathbf{k}}}$, with $f_k \equiv e^{-\beta\omega_k}$.

Summing up, the observations by Alice and Bob are "embedded" in the collective mode vacuum background $|0(\theta)\rangle$ controlled by the Bose–Einstein condensation. The "element of reality" in the entanglement problem is thus of the same degree as the one attributed to condensate states in condensed matter physics. Bob's measurements have a probabilistic effect on Alice's measurements at space-like separation, not induced by (spooky) forces, but due to the coherence of the vacuum condensate.

Equation (A15) shows that $\theta_k = \theta_k(\beta)$, i.e., it is temperature dependent. We thus realize that $|0(\theta)\rangle$ is actually a thermal state, and this leads us to consider the finite temperature QFT in the thermo field dynamics (TFD) formalism [5,6], where the computation of thermal averages is performed in terms of the $|0(\theta)\rangle$ pure state representation.

In conclusion, phase-mediated long-range correlations over space-like distances find their origin in the coherent dynamical structure of the collective mode vacuum background. When entanglement is studied in the QFT formalism, in a natural way there are no "spooky forces at a distance" and violations of special relativity bounds.

We do not discuss entanglement in the $SU(2)$ coherent state, in the case of superconductors, in the neutrino mixing and oscillation, neither in the case of cosmology, nor in theoretical computer science [7,91,97–101], since these problems are out of the scope of the present paper.

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