Multi-Switching Combination Synchronization of Three Fractional-Order Delayed Systems

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Abstract: Multi-switching combination synchronization of three fractional-order delayed systems is investigated. This is a generalization of previous multi-switching combination synchronization of fractional-order systems by introducing time-delays. Based on the stability theory of linear fractional-order systems with multiple time-delays, we propose appropriate controllers to obtain multi-switching combination synchronization of three non-identical fractional-order delayed systems. In addition, the results of our numerical simulations show that they are in accordance with the theoretical analysis.

Keywords: multi-switching combination synchronization; time-delay; fractional-order; stability

1. Introduction

Fractional calculus has attracted researchers from various fields due to fractional dimensions widely existing in nature and engineering fields [1–3]. Compared to the integer-order dynamical systems, the fractional-order counterparts can exhibit more complex dynamical behaviors. Some of the researches on integer-order dynamical systems can be generalized to fractional-order dynamical systems. Fractional-order dynamical systems have been widely investigated, such as synchronization [4], identification [5], stabilization [6] and approximate entropy analysis [7,8]. Time-delay is a frequently encountered phenomenon in real applications, such as physical, communication, economical, pneumatic and biological systems [9]. Introducing time-delay into a system can enrich its dynamic characteristics and describe a real-life phenomenon more precisely. Thus, the fractional-order delayed differential equation (FDDE) is becoming a hot topic for scientists and engineers, and it has many theoretical and practical applications [10]. Nowadays, the chaotic behavior and synchronization of FDDE attract intensive research interests. In [11], Bhalekar et al. introduced the fractional-order delayed Liu system. The fractional-order delayed financial system was presented in [12], and hybrid projective synchronization between the aforementioned two systems was achieved in [13]. The fractional-order delayed Chen system was proposed in [14], while its adaptive synchronization was investigated in [15]. The fractional-order delayed porous media was proposed in [16]. In [17], a fractional-order delayed Newton–Leipnik system was taken as an example to present intermittent synchronizing delayed fractional nonlinear system.

Due to its wide applications in secure communication, synchronization of fractional-order delayed chaotic systems are extensively investigated [18–21]. However, all the above-mentioned and other synchronization schemes are in traditional drive–response ways, which only have unique drive and response systems. Recently, Luo et al. [22] extended the traditional drive–response synchronization to combination synchronization, which has two drive systems and one response system. Compared to the drive–response synchronization, combination synchronization has stronger

anti-decode and anti-attack abilities in image encryption and secure communication, in which origin messages are split into two parts and each part can be embedded into two separate drive systems. There are many works on combination synchronization [23–26]. To further strengthen the security in secure communication, Vincent et al. [27] proposed multi-switching combination synchronization scheme, in which the two drive systems are synchronized with the response system in different states. Based on the nonlinear control technique, Zheng [28] studied multi-switching combination synchronization of three non-identical chaotic systems. Khan [29] investigated adaptive multi-switching combination synchronization among three non-identical chaotic systems. In [30], Ahmad et al. proposed globally exponential multi-switching combination synchronization scheme, and applied it to secure communications. Multi-switching combination synchronization was applied in encrypted audio communication in [31]. The previous work on multi-switching combination synchronization schemes are based on integral-order chaotic systems. To improve the security in these synchronization schemes, based on fractional-order chaotic systems, Bhat et al. [32] extended the work in [27] to study multi-switching combination synchronization among three non-identical fractional-order chaotic systems. Khan et al. [33] investigated multi-switching combination-combination synchronization among a class of four non-identical fractional-order chaotic systems. In multi-switching combination-combination synchronization scheme, the state variables of two drive systems synchronize with different state variables of two response systems simultaneously, which makes the security of this scheme higher than that in [32].

Since time-delay is a frequently encountered phenomenon in real applications, and the time-delay can be used as an additional parameter in synchronization to increase security in secure communication, we consider the work in [32] to investigate multi-switching combination synchronization scheme for non-identical fractional-order delayed systems by introducing time-delays in fractional-order systems.

The rest of this paper is organized as follows. In Section 2, the concept of fractional calculus and the stability theory of linear fractional-order systems with multiple time-delays are briefly introduced. The multi-switching combination synchronization scheme of three non-identical fractional-order delayed systems is analyzed in Section 3. In Section 4, numerical simulations performed using MATLAB are presented. Finally, conclusions are drawn in Section 5.

2. Preliminaries

Fractional calculus is a generalization of integration and differentiation to non-integer order fundamental operator $_aD_t^r$, which is defined as

$$
_aD_t^r = \begin{cases} \frac{d^r}{dt^r} & : r > 0, \\ 1 & : r = 0, \\ \int_a^t (d\tau)^{-r} & : r < 0. \end{cases} \tag{1}
$$

There are several different definitions for the fractional-order differential operator [34]. Because the Caputo definition is easy to understand and is frequently used in the literature, we apply this definition in this paper, which is

$$
_aD_t^r f(t) = \frac{1}{\Gamma(n-r)} \int_a^t \frac{f(\tau)^{r-n+1}}{t-\tau} d\tau \tag{2}
$$

where $1 < r < n$.

Lemma 1 ([35]). *Suppose $f(t) \in C_a^\alpha([a,b]), D_a^\alpha f(t) \in C_a^\beta([a,b]), \alpha > 0, \beta > 0, m-1 < \beta < m, n-1 < \alpha < n$, then*

$$
{}_a^C D_t^\beta \left({}_a^C D_t^\alpha f(t) \right) = {}_a^C D_t^\alpha ({}_a^C D_t^\beta f(t)) = {}_a^C D_t^{\alpha+\beta} f(t) \tag{3}
$$

Lemma 2 ([35]). *Consider*

$$\,^{C}_{a}D^{\alpha}_{t}x^{n} = \frac{\Gamma(n+1)x^{n-\alpha}}{\Gamma(n+1-\alpha)}\,^{C}_{a}D^{\alpha}_{t}x \tag{4}$$

Lemma 3 ([35]). *Let $\phi(t) \in R$ be a continuous and derivable function. Then, for any time instant $t \geq t_0$*

$$\frac{1}{2}D^{q}\phi^{2}(t) = \phi(t)D^{q}\phi(t), \; \forall q \in (0,1) \tag{5}$$

Given the following n-dimensional linear fractional-order system with multiple time-delays [36]:

$$\begin{cases} D^{q_1}y_1(t) = \alpha_{11}y_1(t-\tau_{11}) + \alpha_{12}y_2(t-\tau_{12}) \\ \qquad\qquad + \cdots + \alpha_{1n}y_n(t-\tau_{1n}), \\ D^{q_2}y_2(t) = \alpha_{21}y_1(t-\tau_{21}) + \alpha_{22}y_2(t-\tau_{22}) \\ \qquad\qquad + \cdots + \alpha_{2n}y_n(t-\tau_{2n}), \\ \qquad\qquad \vdots \\ D^{q_n}y_n(t) = \alpha_{n1}y_1(t-\tau_{n1}) + \alpha_{n2}y_2(t-\tau_{n2}) \\ \qquad\qquad + \cdots + \alpha_{nn}y_n(t-\tau_{nn}), \end{cases} \tag{6}$$

where $q_i \in (0,1)$ is the fractional-derivative order, $y_i(t)$ is the state, and $\tau_{ij} > 0$ is the time-delay, the initial value $y_i(t) = \phi_i(t)$ is given by $-max\tau_{ij} = -\tau_{max} \leq t \leq 0$, $A = [a_{ij}] \in R_{n \times n}$ is the coefficient matrix.

Performing Laplace transform on the system in Equation (6) yields

$$\Delta(s) \cdot Y(s) = b(s), \tag{7}$$

where $Y(s) = (Y_1(s), \ Y_2(s), \ ..., \ Y_n(s))^T$ is the Laplace transform of $y(t) = (y_1(t), \ y_2(t), \ ..., \ y_n(t))^T$, $b(s) = (b_1(s), \ b_2(s), \ ..., \ b_n(s))^T$ is the remaining non-linear part, the characteristic matrix of the system in Equation (6) is

$$\Delta(s) = \begin{pmatrix} s^{q_1} - \alpha_{11}e^{-s\tau_{11}} & -\alpha_{12}e^{-s\tau_{12}} & \cdots & -\alpha_{1n}e^{-s\tau_{1n}} \\ -\alpha_{21}e^{-s\tau_{21}} & s^{q_2} - \alpha_{22}e^{-s\tau_{22}} & \cdots & -\alpha_{2n}e^{-s\tau_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{n1}e^{-s\tau_{n1}} & -\alpha_{n2}e^{-s\tau_{n2}} & \cdots & s^{q_n} - \alpha_{nn}e^{-s\tau_{nn}} \end{pmatrix}. \tag{8}$$

Theorem 1 ([36]). *If all the roots of the characteristic equation $\det(\Delta(s)) = 0$ have negative real parts, then the zero solution of the system in Equation (6) is Lyapunov globally asymptotically stable.*

Corollary 1 ([36]). *If $q_1 = q_2 = \cdots = q_n = \beta \in (0,1)$, all the eigenvalues λ of the coefficient matrix A satisfy $|\arg(\lambda)| > \beta\pi/2$, and the characteristic equation $\det(\Delta(s)) = 0$ has no purely imaginary roots for any $\tau_{ij} > 0$, $i,j = 1,2,\ldots,n$, then the zero solution of the system in Equation (6) is Lyapunov globally asymptotically stable.*

3. Multi-Switching Combination Synchronization Scheme

Multi-switching combination synchronization among three non-identical fractional-order delayed systems is investigated in this section.

The two drive systems are

$$\begin{aligned} D^{\alpha}x(t) &= x(t) + x(t-\tau) + A(x(t), x(t-\tau)), \\ x(t) &= x(0), t \in [-\tau, 0], \end{aligned} \tag{9}$$

and

$$D^\alpha y(t) = y(t) + y(t - \tau) + B(y(t), y(t - \tau)),$$
$$y(t) = y(0), t \in [-\tau, 0]. \tag{10}$$

The response system is

$$D^\alpha z(t) = z(t) + z(t - \tau) + C(z(t), z(t - \tau)) + U,$$
$$z(t) = z(0), t \in [-\tau, 0], \tag{11}$$

in which, $\alpha \in (0,1)$ is the fractional order, $\tau > 0$ is the time-delay, $U = (U_1, ..., U_n)$ is the controller vector, $x = (x_1, ..., x_n)^T \in R^n$, $y = (y_1, ..., y_n)^T \in R^n$ and $z = (z_1, ..., z_n)^T \in R^n$ are state vectors, and $A : R^{2n} \to R^n$, $B : R^{2n} \to R^n$ and $C : R^{2n} \to R^n$ are continuous vector functions.

Define the error state as $e_{klm} = f_k z_k - g_l x_l - h_m y_m (k, l, m = 1, ..., n)$. Then, we have the error state vector

$$e(t) = Fz - Gx - Hy, \tag{12}$$

where $e(t)$ is the vector form of e_{klm}, $F = diag\{f_1, f_2, \ldots, f_n\} \in R^{n \times n}$, $G = diag\{g_1, g_2, \ldots, g_n\} \in R^{n \times n}$ and $H = diag\{h_1, h_2, \ldots, h_n\} \in R^{n \times n}$ are real scaling matrix. Accordingly, $e_{klm}(t - \tau) = f_k z_k(t - \tau) - g_l x_l(t - \tau) - h_m y_m(t - \tau)$.

Definition 1 ([27]). *The systems in Equations (9) and (10) and the system in Equation (11) are defined to be multi-switching combination synchronization if F, G, H are non-zeros, and $k \neq l \neq m$, $k = l \neq m$, $k \neq l = m$, $k = m \neq l$, such that:*

$$\lim_{t \to +\infty} \| e(t) \| = \lim_{t \to +\infty} \| Fz - Gx - Hy \| = 0 \tag{13}$$

where $\| \cdot \|$ represents the matrix norm.

Remark 1. *If $k = l = m$, the systems in Equations (9) and (10) and the system in Equation (11) are defined to be combination synchronization [22].*

Remark 2. *If the scaling matrix $F \neq 0$, $G = 0$ or $H = 0$, the multi-switching combination synchronization mentioned above is simplified to multi-switching hybrid projective synchronization.*

From the systems in Equations (9)–(11), we have the error system as follows

$$D^\alpha e(t) = FD^\alpha z(t) - GD^\alpha x(t) - HD^\alpha y(t) \tag{14}$$

To achieve multi-switching combination synchronization among the above systems, a non-linear controller is constructed:

$$U = \tilde{K}e(t) + GA(x(t), x(t - \tau)) + HB(y(t), y(t - \tau)) - FC(z(t), z(t - \tau)), \tag{15}$$

where $\tilde{K} = K - I$, I is an n-dimensional unit matrix, $K = diag\{k_1, k_2, \ldots, k_n\}$ is a feedback gain matrix.

Substituting the systems in Equations (9)–(11) and (15) into the system in Equation (14), we have

$$D^\alpha e(t) = (\tilde{k} + I)e(t) + e(t - \tau) = Ke(t) + e(t - \tau). \tag{16}$$

Thus, the multi-switching combination synchronization between the systems in Equations (9) and (10) and the system in Equation (11) is changed into the analysis of the asymptotical stability of the system in Equation (16).

In light of Corollary 1, a sufficient condition to achieve multi-switching combination synchronization between the systems in Equations (9) and (10) and the system in Equation (11) is obtained as follows.

Proposition 1. *Multi-switching combination synchronization between the systems in Equations (9) and (10) and the system in Equation (11) can be achieved if there exists a matrix $K = diag\{k_1, k_2, \ldots, k_n\}$ in the system in Equation (16) such that $k_i < -1/\sin(\alpha\pi/2)$, $(i = 1, 2, \ldots, n)$.*

Proof. For the system in Equation (16), $A = K + I$ is the coefficient matrix. Since $k_i < -1/\sin(\alpha\pi/2)$, $\alpha \in (0, 1)$, the eigenvalues of A are $\lambda_i = k_i + 1 < 0$, $(i = 1, 2, \ldots, n)$. Then, $|\arg(\lambda)| > \pi/2 > \alpha\pi/2$ holds.

Performing Laplace transform on the system in Equation (16) yields

$$\Delta(s) \cdot E(s) = s^{\alpha-1}e(0) + e(0)e^{-s\tau} \int_{-\tau}^{0} e^{-s\tau}\, dx, \tag{17}$$

where $E(s)$ is the Laplace transform of $e(t)$, $e(0) = Fz(0) - Gx(0) - Hy(0)$, $\Delta(s) = s^{\alpha}I - K - e^{-s\tau}I$ is the characteristic matrix. Then,

$$det(\Delta(s)) = |s^{\alpha}I - K - e^{-s\tau}I| = (s^{\alpha} - k_1 - e^{-s\tau})(s^{\alpha} - k_2 - e^{-s\tau})\ldots(s^{\alpha} - k_n - e^{-s\tau}) = 0. \tag{18}$$

Assume

$$(s^{\alpha} - k_i - e^{-s\tau}) = 0, i = 1, 2, \ldots, n. \tag{19}$$

has a root $s = wi = |w|(\cos(\pi/2) + i\sin(\pm\pi/2))$. Thus,

$$|w|^{\alpha}(\cos(\alpha\pi/2) + i\sin(\pm\alpha\pi/2)) - k_i - \cos(\omega\tau) + i\sin(\omega\tau) = 0. \tag{20}$$

From the above equation, we can get

$$|w|^{\alpha}\cos(\alpha\pi/2) - k_i = \cos(\omega\tau),$$
$$|w|^{\alpha}\sin(\pm\alpha\pi/2) = -\sin(\omega\tau). \tag{21}$$

Hence,

$$|w|^{2\alpha} - 2k_i\cos(\alpha\pi/2)|w|^{\alpha} + k_i^2 - 1 = 0. \tag{22}$$

Since $k_i < -1/\sin(\alpha\pi/2)$, $\alpha \in (0, 1)$, the discriminant of the roots satisfies

$$\begin{aligned} \Delta &= (-2k_i\cos(\alpha\pi/2))^2 - 4(k_i^2 - 1) \\ &= 4(1 - k_i^2\sin^2(\alpha\pi/2)) \\ &< 0. \end{aligned} \tag{23}$$

Then, Equation (22) has no real solutions, and Equation (18) has no purely imaginary roots.

In light of Corollary 1, the zero solution of the system in Equation (16) is globally asymptotically stable, i.e., multi-switching combination synchronization is obtained between the systems in Equations (9) and (10) and the system in Equation (11). □

4. Numerical Examples

Numerical simulations were carried out to illustrate the above proposed multi-switching combination synchronization scheme. We used the same systems as in [32] with time-delays, which are fractional-order delayed Lorenz, Chen, and Rössler systems, and the numerical simulations were carried out in MATLAB.

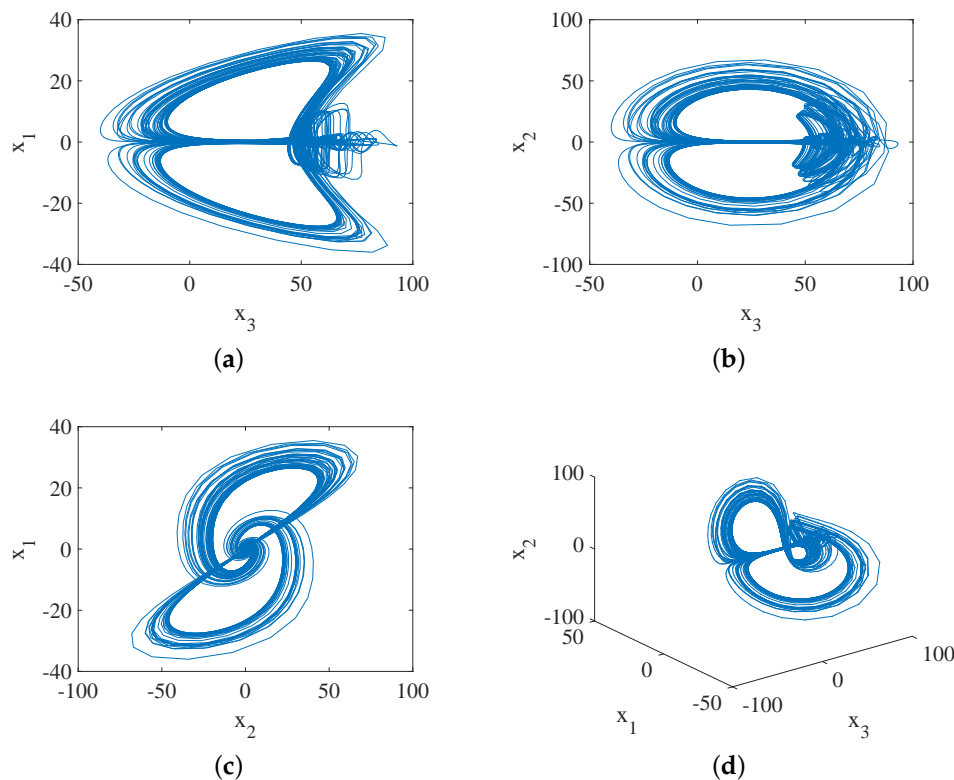The fractional-order delayed Lorenz system [37] was considered as the first drive system

$$\begin{cases} D^\alpha x_1 = a_1(x_2 - x_1), \\ D^\alpha x_2 = c_1 x_1 - x_2 - x_1 x_3, \\ D^\alpha x_3 = x_1 x_2 - b_1 x_3(t - \tau). \end{cases} \tag{24}$$

The system in Equation (24) exhibits a chaotic attractor, as illustrated in Figure 1. The system in Equation (24) can be rewritten as

$$\begin{aligned} D^\alpha x(t) &= x(t) + x(t - \tau) + A(x(t), x(t - \tau)), \\ x(t) &= x(0), t \in [-\tau, 0], \end{aligned} \tag{25}$$

where

$$A(x(t), x(t - \tau)) = \begin{pmatrix} a_1 x_2 - (a_1 + 1)x_1 - x_1(t - \tau) \\ c_1 x_1 - 2x_2 - x_1 x_3 - x_2(t - \tau) \\ x_1 x_2 - (b_1 + 1)x_3(t - \tau) - x_3 \end{pmatrix}. \tag{26}$$

Figure 1. Chaotic attractor of Lorenz system with $\alpha = 0.95$, $\tau = 0.4$: (**a**) $x_3 - x_1$ plane; (**b**) $x_3 - x_2$ plane; (**c**) $x_2 - x_1$ plane; and (**d**) $x_3 - x_1 - x_2$ space.

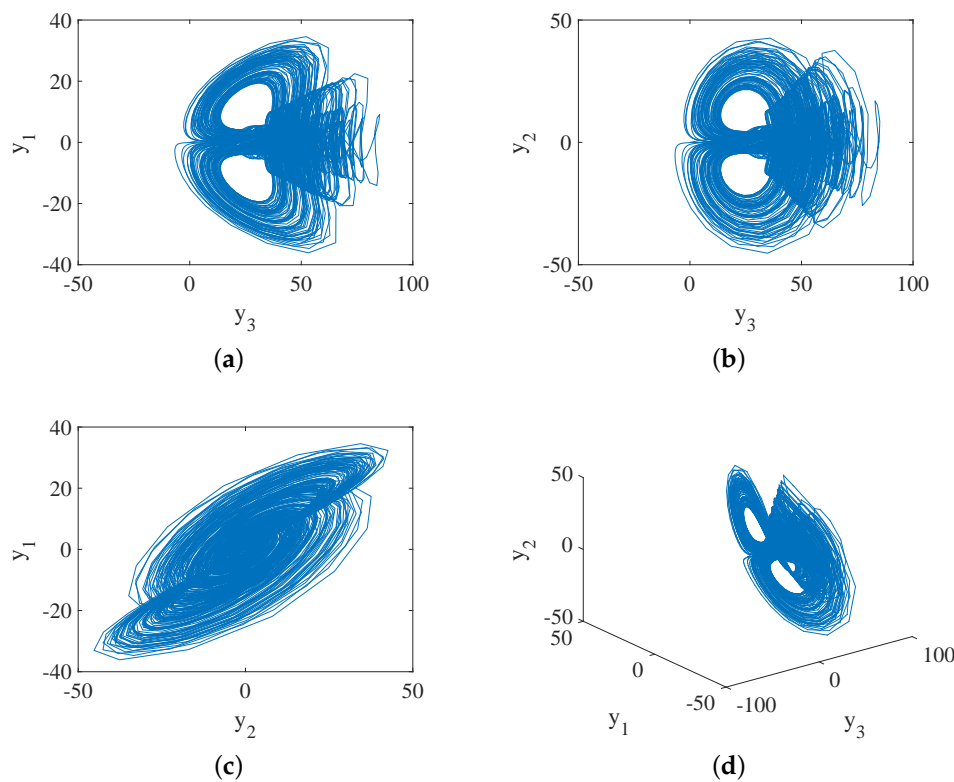The fractional-order delayed Chen system [14] is the second drive system

$$\begin{cases} D^\alpha y_1 = a_2(y_2 - y_1), \\ D^\alpha y_2 = (c_2 - a_2)y_1 - y_1 y_3 + c_2 y_2, \\ D^\alpha y_3 = y_1 y_2 - b_2 y_3(t - \tau). \end{cases} \tag{27}$$

The system in Equation (27) displays a chaotic attractor, as shown in Figure 2. We rewrite the system in Equation (27) as

$$
\begin{aligned}
D^\alpha y(t) &= y(t) + y(t - \tau) + B(y(t), y(t - \tau)), \\
y(t) &= y(0), t \in [-\tau, 0],
\end{aligned}
\tag{28}
$$

where

$$
B(y(t), y(t - \tau)) = \begin{pmatrix} a_2(y_2 - y_1) - y_1 - y_1(t - \tau) \\ (c_2 - a_2)y_1 - y_1 y_3 + (c_2 - 1)y_2 - y_2(t - \tau) \\ y_1 y_2 - (b_2 + 1)y_3(t - \tau) - y_3 \end{pmatrix}.
\tag{29}
$$

Figure 2. Chaotic attractor of Chen system with $\alpha = 0.95$, $\tau = 0.4$: (**a**) $y_3 - y_1$ plane; (**b**) $y_3 - y_2$ plane; (**c**) $y_2 - y_1$ plane; and (**d**) $y_3 - y_1 - y_2$ space.

The fractional-order delayed Rössler system is the response system, given by

$$
\begin{cases}
D^\alpha z_1 = -(z_2 + z_3) + 0.2z_1(t - \tau) + U_1, \\
D^\alpha z_2 = z_1 + a_3 z_2 + U_2, \\
D^\alpha z_3 = z_3(z_1 - m_3) + b_3 + U_3.
\end{cases}
\tag{30}
$$

where U_1, U_2 and U_3 are determined afterwards. Without the controllers, the system in Equation (30) exhibits a chaotic attractor, as illustrated in Figure 3. The system in Equation (30) is rewritten as

$$
\begin{aligned}
D^\alpha z(t) &= z(t) + z(t - \tau) + C(z(t), z(t - \tau)) + U, \\
z(t) &= z(0), t \in [-\tau, 0],
\end{aligned}
\tag{31}
$$

where

$$C(z(t), z(t-\tau)) = \begin{pmatrix} -(z_2 + z_3) - 0.8z_1(t-\tau) - z_1 \\ z_1 + (a_3 - 1)z_2 - z_2(t-\tau) \\ z_3(z_1 - m_3) + b_3 - z_3 - z_3(t-\tau) \end{pmatrix}. \tag{32}$$
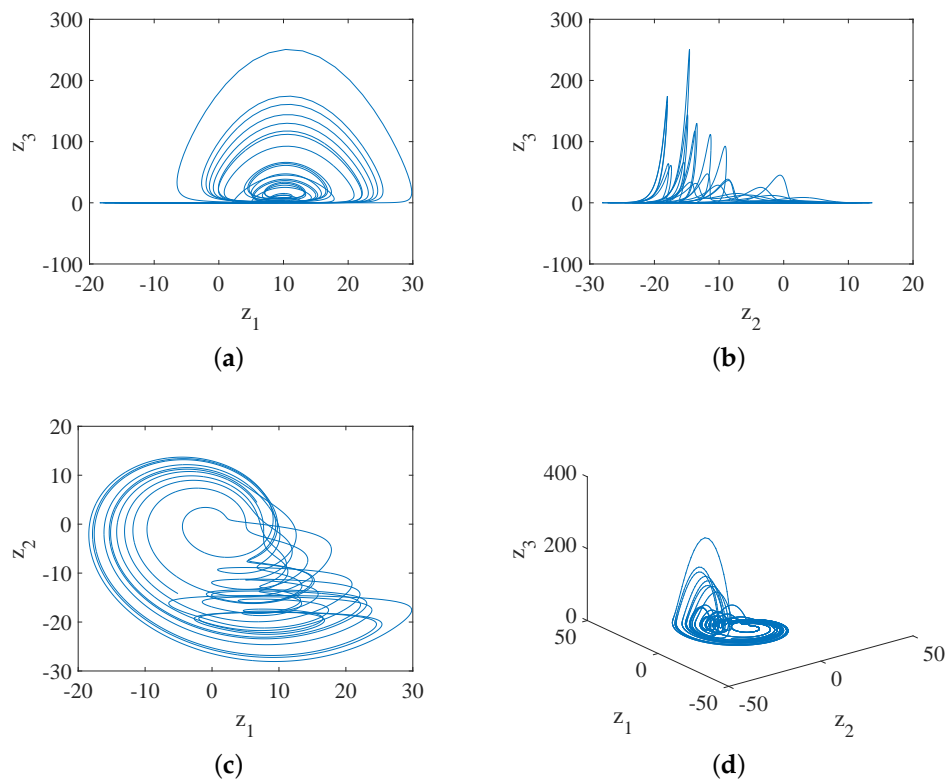
For the systems in Equations (24), (27) and (30), there are eight possible switch combination synchronization cases.

When $k \neq l \neq m$, we have e_{123}, e_{231}, e_{312} and e_{132}, e_{213}, e_{321}.

When $k = m \neq l$, we have e_{121}, e_{232}, e_{313} and e_{131}, e_{212}, e_{323}.

When $k \neq l = m$, we have e_{122}, e_{233}, e_{311} and e_{133}, e_{211}, e_{322}.

When $k = l \neq m$, we have e_{112}, e_{223}, e_{331} and e_{113}, e_{221}, e_{332}.

Figure 3. Chaotic attractor of Rössler system with $\alpha = 0.95$, $\tau = 0.4$: (**a**) $z_1 - z_3$ plane; (**b**) $z_2 - z_3$ plane; (**c**) $z_1 - z_2$ plane; and (**d**) $z_2 - z_1 - z_3$ space.

We randomly pick two cases

$$\begin{cases} e_{123} = f_1 z_1 - g_2 x_2 - h_3 y_3, \\ e_{231} = f_2 z_2 - g_3 x_3 - h_1 y_1, \quad \textbf{case 1} \\ e_{312} = f_3 z_3 - g_1 x_1 - h_2 y_2. \end{cases} \tag{33}$$

and

$$\begin{cases} e_{112} = f_1 z_1 - g_1 x_1 - h_2 y_2, \\ e_{223} = f_2 z_2 - g_2 x_2 - h_3 y_3, \quad \textbf{case 2} \\ e_{331} = f_3 z_3 - g_3 x_3 - h_1 y_1. \end{cases} \tag{34}$$

In the following, we analyze these two cases in detail.

Case 1

From the systems in Equations (24), (27) and (30), we have the error dynamical system

$$
\begin{cases}
D^{\alpha} e_{123} = f_1 D^{\alpha} z_1 - g_2 D^{\alpha} x_2 - h_3 D^{\alpha} y_3, \\
D^{\alpha} e_{231} = f_2 D^{\alpha} z_2 - g_3 D^{\alpha} x_3 - h_1 D^{\alpha} y_1, \\
D^{\alpha} e_{312} = f_3 D^{\alpha} z_3 - g_1 D^{\alpha} x_1 - h_2 D^{\alpha} y_2.
\end{cases}
\tag{35}
$$

such that

$$
\begin{cases}
\lim\limits_{t \to +\infty} \| f_1 z_1 - g_2 x_2 - h_3 y_3 \| = 0, \\
\lim\limits_{t \to +\infty} \| f_2 z_2 - g_3 x_3 - h_1 y_1 \| = 0, \\
\lim\limits_{t \to +\infty} \| f_3 z_3 - g_1 x_1 - h_2 y_2 \| = 0.
\end{cases}
\tag{36}
$$

Substituting the systems in Equations (24), (27) and (30) into the system in Equation (35) yields

$$
\begin{cases}
D^{\alpha} e_{123} = f_1(-(z_2 + z_3) + 0.2 z_1(t - \tau) + U_1) - g_2(c_1 x_1 - x_2 - x_1 x_3) \\
\quad - h_3(y_1 y_2 - b_2 y_3(t - \tau)), \\
D^{\alpha} e_{231} = f_2(z_1 + a_3 z_2 + U_2) - g_3(x_1 x_2 - b_1 x_3(t - \tau)) \\
\quad - h_1(a_2(y_2 - y_1)), \\
D^{\alpha} e_{312} = f_3(z_3(z_1 - m_3) + b_3 + U_3) - g_1(a_1(x_2 - x_1)) \\
\quad - h_2((c_2 - a_2)y_1 - y_1 y_3 + c_2 y_2).
\end{cases}
\tag{37}
$$

Here, we obtain the following results.

Theorem 2. *Multi-switching combination synchronization between the systems in Equations (24) and (27) and the system in Equation (30) can be achieved with the following controllers*

$$
\begin{cases}
U_1 = \dfrac{1}{f_1} \{ (k_1 - 1)(f_1 z_1 - g_2 x_2 - h_3 y_3) + g_2(c_1 x_1 - 2 x_2 - x_1 x_3 - x_2(t - \tau)) \\
\quad + h_3(y_1 y_2 - (b_2 + 1)y_3(t - \tau) - y_3) - f_1(-(z_2 + z_3) - 0.8 z_1(t - \tau) - z_1) \}, \\
U_2 = \dfrac{1}{f_2} \{ (k_2 - 1)(f_2 z_2 - g_3 x_3 - h_1 y_1) + g_3(x_1 x_2 - (b_1 + 1)x_3(t - \tau) - x_3) \\
\quad + h_1(a_2(y_2 - y_1) - y_1 - y_1(t - \tau)) - f_2(z_1 + (a_3 - 1)z_2 - z_2(t - \tau)) \}, \\
U_3 = \dfrac{1}{f_3} \{ (k_3 - 1)(f_3 z_3 - g_1 x_1 - h_2 y_2) + g_1(a_1 x_2 - (a_1 + 1)x_1 - x_1(t - \tau)) \\
\quad + h_2((c_2 - a_2)y_1 - y_1 y_3 + (c_2 - 1)y_2 - y_2(t - \tau)) \\
\quad - f_3(z_3(z_1 - m_3) + b_3 - z_3 - z_3(t - \tau)) \}.
\end{cases}
\tag{38}
$$

Supposing $F \neq 0$ and $G = 0$ or $H = 0$, we have the following results.

Corollary 2. *(i) Supposing that $f_i \neq 0$, $g_i = 0$ and $h_i \neq 0$ ($i = 1, 2, 3$), multi-switching hybrid projective synchronization between the systems in Equations (27) and (30) can be achieved with the following controllers*

$$
\begin{cases}
U_1 = \dfrac{1}{f_1}\{(k_1 - 1)(f_1 z_1 - h_3 y_3) + h_3(y_1 y_2 - (b_2 + 1)y_3(t - \tau) - y_3) \\
\quad - f_1(-(z_2 + z_3) - 0.8z_1(t - \tau) - z_1)\}, \\
U_2 = \dfrac{1}{f_2}\{(k_2 - 1)(f_2 z_2 - h_1 y_1) + h_1(a_2(y_2 - y_1) - y_1 - y_1(t - \tau)) \\
\quad - f_2(z_1 + (a_3 - 1)z_2 - z_2(t - \tau))\}, \\
U_3 = \dfrac{1}{f_3}\{(k_3 - 1)(f_3 z_3 - h_2 y_2) + h_2((c_2 - a_2)y_1 - y_1 y_3 + (c_2 - 1)y_2 - y_2(t - \tau)) \\
\quad - f_3(z_3(z_1 - m_3) + b_3 - z_3 - z_3(t - \tau))\}.
\end{cases}
\tag{39}
$$

(ii) Similarly, supposing that $f_i \neq 0$, $g_i \neq 0$ and $h_i = 0$ ($i = 1, 2, 3$), multi-switching hybrid projective synchronization between the systems in Equations (24) and (30) can be achieved with the following controllers

$$
\begin{cases}
U_1 = \dfrac{1}{f_1}\{(k_1 - 1)(f_1 z_1 - g_2 x_2) + g_2(c_1 x_1 - 2x_2 - x_1 x_3 - x_2(t - \tau)) \\
\quad - f_1(-(z_2 + z_3) - 0.8z_1(t - \tau) - z_1)\}, \\
U_2 = \dfrac{1}{f_2}\{(k_2 - 1)(f_2 z_2 - g_3 x_3) + g_3(x_1 x_2 - (b_1 + 1)x_3(t - \tau) - x_3) \\
\quad - f_2(z_1 + (a_3 - 1)z_2 - z_2(t - \tau))\}, \\
U_3 = \dfrac{1}{f_3}\{(k_3 - 1)(f_3 z_3 - g_1 x_1) + g_1(a_1 x_2 - (a_1 + 1)x_1 - x_1(t - \tau)) \\
\quad - f_3(z_3(z_1 - m_3) + b_3 - z_3 - z_3(t - \tau))\}.
\end{cases}
\tag{40}
$$

Corollary 3. *Supposing that $f_i \neq 0$, $g_i = 0$ and $h_i = 0$ ($i = 1, 2, 3$), the system in Equation (30) can be stabilized to its equilibrium $O(0, 0, 0)$ with the following controllers*

$$
\begin{cases}
U_1 = \dfrac{1}{f_1}\{(k_1 - 1)(f_1 z_1) - f_1(-(z_2 + z_3) - 0.8z_1(t - \tau) - z_1)\}, \\
U_2 = \dfrac{1}{f_2}\{(k_2 - 1)(f_2 z_2) - f_2(z_1 + (a_3 - 1)z_2 - z_2(t - \tau))\}, \\
U_3 = \dfrac{1}{f_3}\{(k_3 - 1)(f_3 z_3) - f_3(z_3(z_1 - m_3) + b_3 - z_3 - z_3(t - \tau))\}.
\end{cases}
\tag{41}
$$

Case 2

From the systems in Equations (24), (27) and (30), we have

$$
\begin{cases}
D^\alpha e_{112} = f_1 D^\alpha z_1 - g_1 D^\alpha x_1 - h_2 D^\alpha y_2, \\
D^\alpha e_{223} = f_2 D^\alpha z_2 - g_2 D^\alpha x_2 - h_3 D^\alpha y_3, \\
D^\alpha e_{331} = f_3 D^\alpha z_3 - g_3 D^\alpha x_3 - h_1 D^\alpha y_1.
\end{cases}
\tag{42}
$$

such that

$$
\begin{cases}
\lim\limits_{t \to +\infty} \| f_1 z_1 - g_1 x_1 - h_2 y_2 \| = 0, \\
\lim\limits_{t \to +\infty} \| f_2 z_2 - g_2 x_2 - h_3 y_3 \| = 0, \\
\lim\limits_{t \to +\infty} \| f_3 z_3 - g_3 x_3 - h_1 y_1 \| = 0.
\end{cases}
\tag{43}
$$

Substituting the systems in Equations (24), (27), and (30) into the system in Equation (42) yields:

$$
\begin{cases}
D^\alpha e_{112} = f_1(-(z_2 + z_3) + 0.2z_1(t - \tau) + U_1) - g_1(a_1(x_2 - x_1)) \\
\qquad - h_2((c_2 - a_2)y_1 - y_1y_3 + c_2y_2), \\
D^\alpha e_{223} = f_2(z_1 + a_3z_2 + U_2) - g_2(c_1x_1 - x_2 - x_1x_3) \\
\qquad - h_3(y_1y_2 - b_2y_3(t - \tau)), \\
D^\alpha e_{331} = f_3(z_3(z_1 - m_3) + b_3 + U_3) - g_3(x_1x_2 - b_1x_3(t - \tau)) \\
\qquad - h_1(a_2(y_2 - y_1)).
\end{cases}
\tag{44}
$$

Here, we have the following similar results.

Theorem 3. *Multi-switching combination synchronization between the systems in Equations (24) and (27) and the system in Equation (30) can be achieved with the following controllers*

$$
\begin{cases}
U_1 = \dfrac{1}{f_1}\{(k_1 - 1)(f_1z_1 - g_1x_1 - h_2y_2) + g_1(a_1x_2 - (a_1 + 1)x_1 - x_1(t - \tau)) \\
\quad + h_2((c_2 - a_2)y_1 - y_1y_3 + (c_2 - 1)y_2 - y_2(t - \tau)) - f_1(-(z_2 + z_3) - 0.8z_1(t - \tau) - z_1)\}, \\
U_2 = \dfrac{1}{f_2}\{(k_2 - 1)(f_2z_2 - g_2x_2 - h_3y_3) + g_2(c_1x_1 - 2x_2 - x_1x_3 - x_2(t - \tau)) \\
\quad + h_3(y_1y_2 - (b_2 + 1)y_3(t - \tau) - y_3) - f_2(z_1 + (a_3 - 1)z_2 - z_2(t - \tau))\}, \\
U_3 = \dfrac{1}{f_3}\{(k_3 - 1)(f_3z_3 - g_3x_3 - h_1y_1) + g_3(x_1x_2 - (b_1 + 1)x_3(t - \tau) - x_3) \\
\quad + h_1(a_2(y_2 - y_1) - y_1 - y_1(t - \tau)) - f_3(z_3(z_1 - m_3) + b_3 - z_3 - z_3(t - \tau))\}.
\end{cases}
\tag{45}
$$

Corollary 4. *(i) Supposing that $f_i \neq 0$, $g_i = 0$ and $h_i \neq 0$ $(i = 1, 2, 3)$, multi-switching hybrid projective synchronization between the systems in Equations (27) and (30) can be achieved with the following controllers*

$$
\begin{cases}
U_1 = \dfrac{1}{f_1}\{(k_1 - 1)(f_1z_1 - h_2y_2) + h_2((c_2 - a_2)y_1 - y_1y_3 + (c_2 - 1)y_2 - y_2(t - \tau)) \\
\quad - f_1(-(z_2 + z_3) - 0.8z_1(t - \tau) - z_1)\}, \\
U_2 = \dfrac{1}{f_2}\{(k_2 - 1)(f_2z_2 - h_3y_3) + h_3(y_1y_2 - (b_2 + 1)y_3(t - \tau) - y_3) \\
\quad - f_2(z_1 + (a_3 - 1)z_2 - z_2(t - \tau))\}, \\
U_3 = \dfrac{1}{f_3}\{(k_3 - 1)(f_3z_3 - h_1y_1) + h_1(a_2(y_2 - y_1) - y_1 - y_1(t - \tau)) \\
\quad - f_3(z_3(z_1 - m_3) + b_3 - z_3 - z_3(t - \tau))\}.
\end{cases}
\tag{46}
$$

(ii) Supposing that $f_i \neq 0$, $g_i \neq 0$ and $h_i = 0$ $(i = 1, 2, 3)$, hybrid projective synchronization between the systems in Equations (24) and (30) can be achieved with the following controllers

$$
\begin{cases}
U_1 = \dfrac{1}{f_1}\{(k_1 - 1)(f_1z_1 - g_1x_1) + g_1(a_1x_2 - (a_1 + 1)x_1 - x_1(t - \tau)) \\
\quad - f_1(-(z_2 + z_3) - 0.8z_1(t - \tau) - z_1)\}, \\
U_2 = \dfrac{1}{f_2}\{(k_2 - 1)(f_2z_2 - g_2x_2) + g_2(c_1x_1 - 2x_2 - x_1x_3 - x_2(t - \tau)) \\
\quad - f_2(z_1 + (a_3 - 1)z_2 - z_2(t - \tau))\}, \\
U_3 = \dfrac{1}{f_3}\{(k_3 - 1)(f_3z_3 - g_3x_3) + g_3(x_1x_2 - (b_1 + 1)x_3(t - \tau) - x_3) \\
\quad - f_3(z_3(z_1 - m_3) + b_3 - z_3 - z_3(t - \tau))\}.
\end{cases}
\tag{47}
$$

Corollary 5. *Supposing that $f_i \neq 0$, $g_i = 0$ and $h_i = 0$ ($i = 1, 2, 3$), the system in Equation (30) can be stabilized to its equilibrium $O(0, 0, 0)$ with the following controllers*

$$
\begin{cases}
U_1 = \dfrac{1}{f_1}\{(k_1 - 1)(f_1 z_1) - f_1(-(z_2 + z_3) - 0.8 z_1(t - \tau) - z_1)\}, \\[2mm]
U_2 = \dfrac{1}{f_2}\{(k_2 - 1)(f_2 z_2) - f_2(z_1 + (a_3 - 1)z_2 - z_2(t - \tau))\}, \\[2mm]
U_3 = \dfrac{1}{f_3}\{(k_3 - 1)(f_3 z_3) - f_3(z_3(z_1 - m_3) + b_3 - z_3 - z_3(t - \tau))\}.
\end{cases}
\tag{48}
$$

The system parameters are given as $a_1 = 10$, $b_1 = \frac{8}{3}$, $c_1 = 28$, $a_2 = 35$, $b_2 = 3$, $c_2 = 28$, $a_3 = 0.4$, $b_3 = 0.2$, $m_3 = 10$, thus the systems in Equations (24), (27) and (30) exhibit chaotic behaviors, respectively. We assume $f_1 = f_2 = f_3 = 1$, $g_1 = g_2 = g_3 = 1$ and $h_1 = h_2 = h_3 = 1$, and the initial values are $(x_1(0), x_2(0), x_3(0)) = (-20, 2, 3)$, $(y_1(0), y_2(0), y_3(0)) = (7, 4.04, 20)$ and $(z_1(0), z_2(0), z_3(0)) = (1, 2, 40)$, respectively. Multi-switching combination synchronization between the systems in Equations (24), (27) and (30) can be realized with $K = diag\{-10, -10, -10\}$. Figure 4 illustrates synchronization errors e_{123}, e_{231}, e_{312}. Figure 5 shows synchronization states $x_2 + y_3$ vs. z_1, $x_3 + y_1$ vs. z_2 and $x_1 + y_2$ vs. z_3 of the drive systems in Equations (24) and (27) and the response system in Equation (30). Figure 6 displays synchronization errors e_{112}, e_{223}, and e_{331}. Figure 7 illustrates synchronization states $x_1 + y_2$ vs. z_1, $x_2 + y_3$ vs. z_2 and $x_3 + y_1$ vs. z_3 of the drive systems in Equations (24) and (27) and the response system in Equation (30). In Figures 4–7, we can see that the multi-switching combination synchronization errors converge to zero, i.e., the multi-switching combination synchronizations for Cases 1 and 2 are achieved, respectively.

The feedback gain matrix K is an important factor to affect the convergence of the error systems. With the increase of the absolute value of k_i, the convergence time will be shortened. Thus, we carried out one more simulation with $K = diag\{-40, -40, -40\}$. Figures 8 and 9 illustrate the synchronization errors for Cases 1 and 2, respectively. By comparing Figures 4 and 8, as well as Figures 6 and 9, it is easy to see that convergence times are shortened obviously.
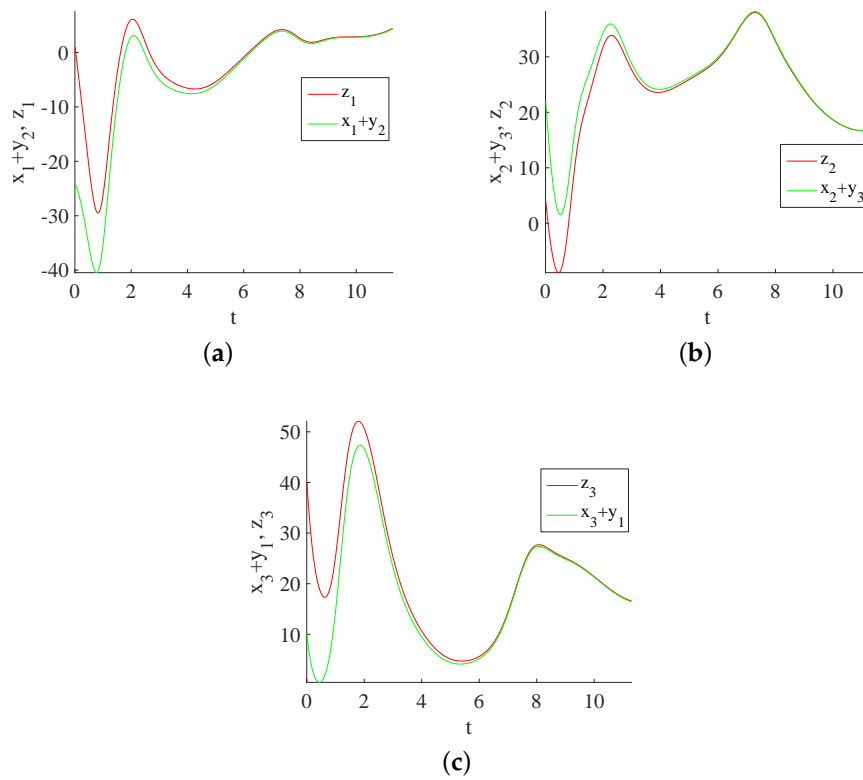

(a)

(b)

(c)
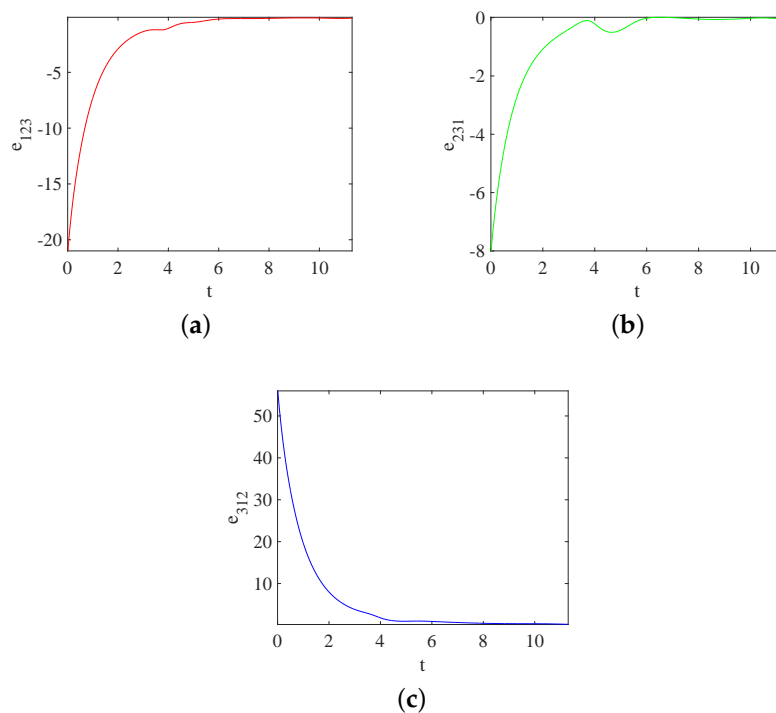
Figure 4. Synchronization errors among the systems in Equations (24), (27) and (30) with $k_1 = k_2 = k_3 = -10$: (a) e_{123}; (b) e_{231}; and (c) e_{312}.
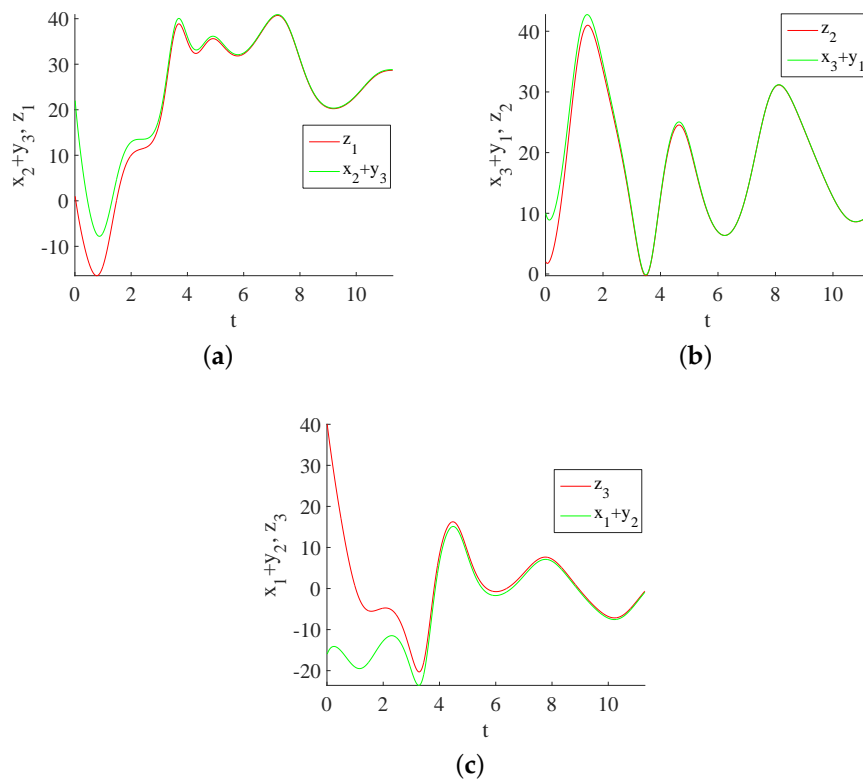
Figure 5. Responses for states between the systems in Equations (24) and (27) and the system in Equation (30): (**a**) $x_2 + y_3$ vs. z_1; (**b**) $x_3 + y_1$ vs. z_2; and (**c**) $x_1 + y_2$ vs. z_3.
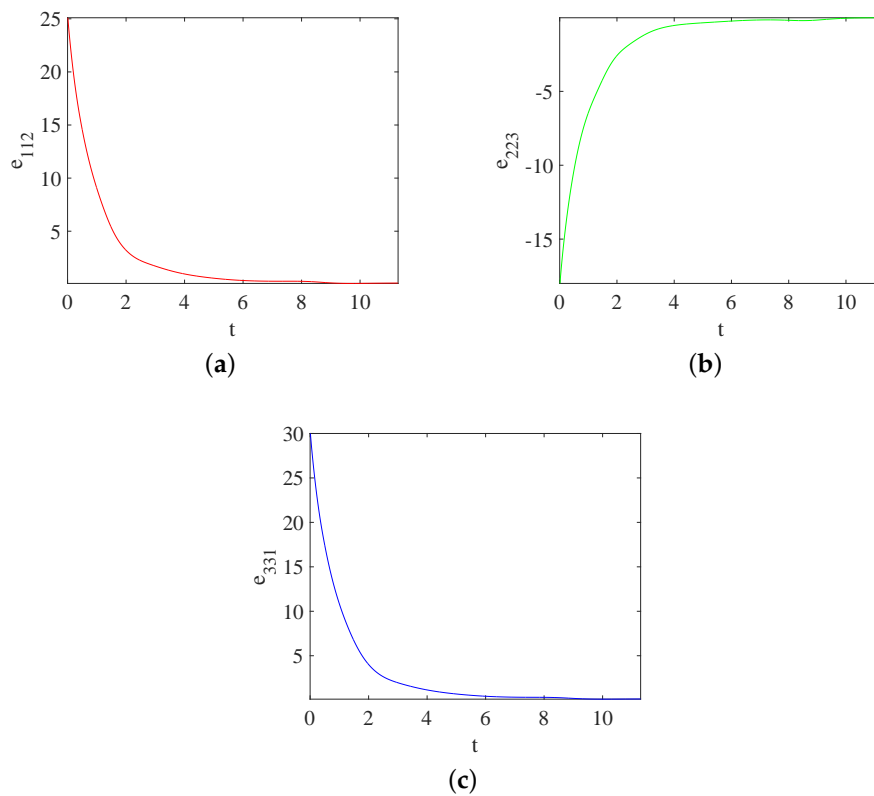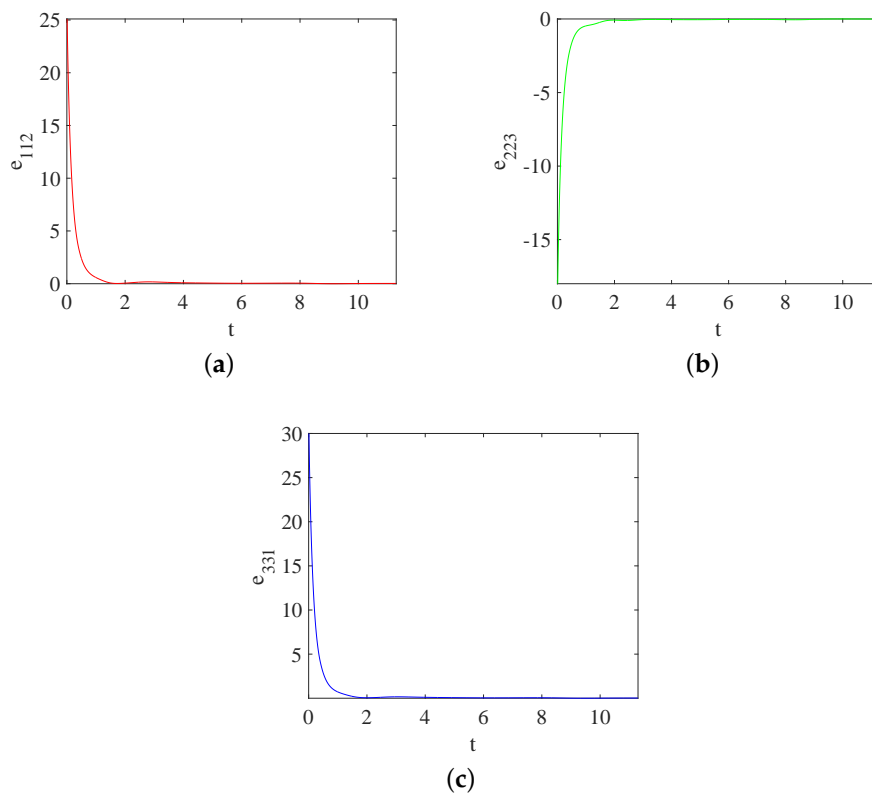

Figure 6. Synchronization errors among the systems in Equations (24), (27) and (30) with $k_1 = k_2 = k_3 = -10$: (**a**) e_{112}; (**b**) e_{223}; and (**c**) e_{331}.

Figure 7. Responses for states between the systems in Equations (24) and (27) and the system in Equation (30): (**a**) $x_1 + y_2$ vs. z_1; (**b**) $x_2 + y_3$ vs. z_2; and (**c**) $x_3 + y_1$ vs. z_3.
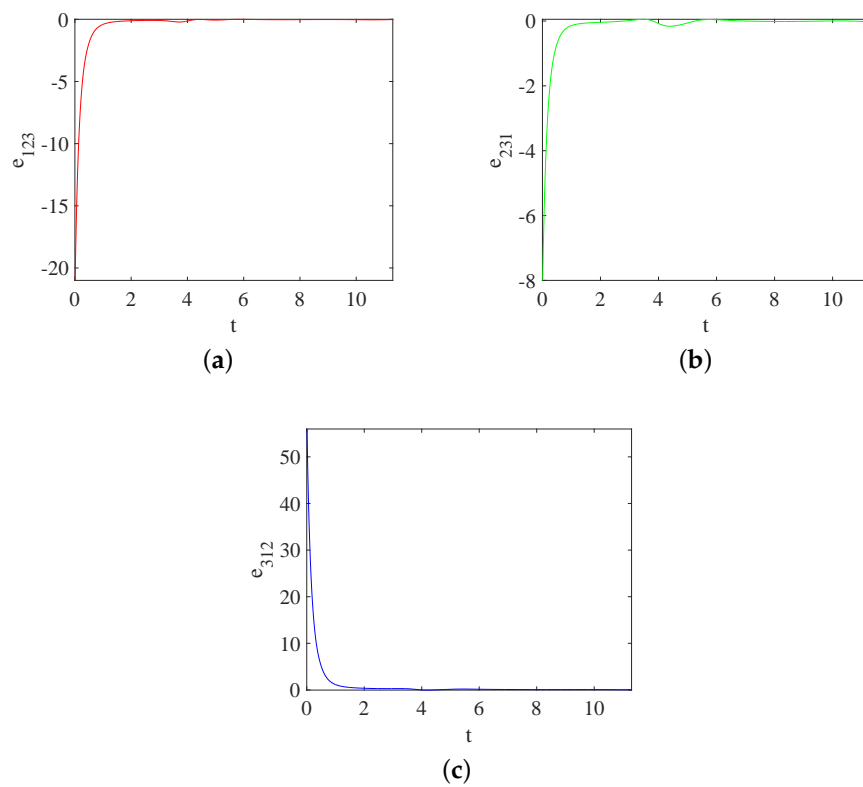
Figure 8. Synchronization errors among the systems in Equations (24), (27) and (30) with $k_1 = k_2 = k_3 = -40$: (**a**) e_{123}; (**b**) e_{231}; and (**c**) e_{312}.

Figure 9. Synchronization errors among the systems in Equations (24), (27) and (30) with $k_1 = k_2 = k_3 = -40$: (**a**) e_{112}; (**b**) e_{223}; and (**c**) e_{331}.

5. Conclusions

We extended previous work [32] to investigate multi-switching combination synchronization among three non-identical fractional-order delayed systems by introducing time-delays. Based on the stability theory for linear fractional-order systems with multiple time-delays, we designed appropriate controllers to obtain multi-switching combination synchronization among three non-identical fractional-order delayed systems. The simulations are in accordance with the theoretical analysis.

On the one hand, when applying multi-switching combination synchronization of fractional-order delayed chaotic systems in secure communications, fractional-order and time-delay can enrich systems' dynamics. On the other hand, the origin information can be separated into two parts and embedded different parts in separate drive systems via combination synchronization scheme. Besides, because the switched states are unpredictable, this synchronization scheme can increase the security of the transmitted information in secure communication. Thus, the communication security will be enhanced, which makes multi-switching combination synchronization of fractional-order delayed chaotic systems able to find better applications in security communication.

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