CDOM spatiotemporal variability in the Mediterranean Sea: a modelling study.

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Supplementary

1 The biogeochemical model state variables

BFM variables are reported in the Table 1. For each entry in parenthesis is reported the number of elements used to describe the specific variable. Dissolved organic matter is expressed both in terms of its transparent and colored components.

Name	Quota components
oxygen[1]	O[mmolO2/m3]
phosphate[1]	P[mmol P/m3]
nitrate[1]	N[mmol N/m3]
ammonium[1]	N[mmol N/m3]
silicate[1]	Si[mmol Si/m3]
Aerobic and Anaerobic Bacteria[3]	C[mgC/m3]; N[mmol N/m3]; P[mmol P/m3]
diatom[5]	C[mgC/m3]; N[mmol N/m3]; P[mmol P/m3]; Chl[mgChl/m3]; Si[mmolSi/3]
flagellate[4]	C[mgC/m3]; N[mmol N/m3]; P[mmol P/m3]; Chl[mgChl/m3];
picophytoplankton[4]	C[mgC/m3]; N[mmol N/m3]; P[mmol P/m3]; Chl[mgChl/m3];
large phytoplankton[4]	C[mgC/m3]; N[mmol N/m3]; P[mmol P/m3]; Chl[mgChl/m3];
microzooplankton[3]	C[mgC/m3]; N[mmol N/m3]; P[mmol P/m3]
heterotrophic nanoflagellates[3]	C[mgC/m3]; N[mmol N/m3]; P[mmol P/m3]
carnivorous mesozooplankton[3]	C[mgC/m3]; N[mmol N/m3]; P[mmol P/m3]
omnivorous mesozooplankton[3]	C[mgC/m3]; N[mmol N/m3]; P[mmol P/m3]
labile dissolved organic matter[4]	C[mgC/m3]; N[mmol N/m3]; P[mmol P/m3]
semi-labile dissolved organic carbon[2]	C[mgC/m3]
semi-refractory dissolved organic carbon[2]	C[mgC/m3]
particulate organic matter[4]	C[mgC/m3]; N[mmol N/m3]; P[mmol P/m3]; Si[mmolSi/3]
dissolved inorganic carbon[1]	C[mgC/m3]
alkalinity[1]	EQ[mmol eq/m3]
calcite[1]	C[mgC/m3]

Table 1: Biogeochemical Flux Model (BFM) state variables.

2 The biogeochemical model paramaters (BFM Namelists)

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#-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-
# NAMELISTS
#-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-
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# BFM - Biogeochemical Flux Model
#-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=--
#
# COPYING
#
#   Copyright (C) 2015 BFM System Team (bfm_st@lists.cmcc.it)
#
#   This program is free software; you can redistribute it and/or modify
#   it under the terms of the GNU General Public License as published by
#   the Free Software Foundation;
#   This program is distributed in the hope that it will be useful,
#   but WITHOUT ANY WARRANTY; without even the implied warranty of
#   MERCHANTEABILITY or FITNESS FOR A PARTICULAR PURPOSE.  See the
#   GNU General Public License for more details.
#
#----------------------------------------------------------------------!



!----------------------------------------------------------------------!
!NAMELIST bfm_nml
!----------------------------------------------------------------------!
! Main initialisation and output specifications
! NAME            KIND     DESCRIPTION
! bio_calc        logical  Switch on/off BFM (for coupled configurations)
! bfm_init        integer  Initialization state
!                          0. from constant values in bfm_init_nml below
!                          1. from restart
! bfm_rstctl      logical  Save initial state of bfm in the output file.
! bio_setup       integer  BFM configuration:
!                          1. pelagic
!                          2. benthic
!                          3. pelagic and benthic
!                          4. sea ice
!                          5. pelagic and sea ice
! out_fname       string   Name of NetCDF output file
! out_dir         string   Path to the output file
! out_title       string   Name of the experiment in NetCDF file
```

```
! out_delta      integer Output is saved every out_delta timesteps
!                        Use -1 to store with real monthly frequency
! parallel_log   logical Set true in parallel jobs for a single log file
!-----------------------------------------------------------------------!
&bfm_nml
   bio_calc     = .TRUE.
   bfm_init     = 0
   bfm_rstctl   = .FALSE.
   bio_setup    = 1
   out_fname    = 'BFM_standalone_pelagic'
   out_dir      = '.'
   out_title    = 'BFM_STANDALONE_PELAGIC'
   in_rst_fname = 'in_bfm_restart'
   out_delta    = 300  !every 30 days
   parallel_log = .FALSE.
filename_nml_conf = 'BFM_General.nml',
/




!-----------------------------------------------------------------------!
! NAMELIST Param_parameters
!-----------------------------------------------------------------------!
! Global Switches : turn on/off or choose model components
! NAME                        KIND    DESCRIPTION
! CalcPelagicFlag             logical Pelagic System
! CalcBenthicFlag             numeric Benthic system
!                                 0 = No Benthic System
!                                     The following are Not Yet Activated
!                                 1 = Simple Benthic Return
!                                 2 = Benthic organisms and intermediate
!                                         complexity nutrient regeneration
!                                 3 = Benthic organisms and full nutrient
!                                         regeneration (early diagenesis)
! CalcTransportFlag           logical Compute Transport Term (when coupled
!                                     with a OGCM)
! CalcConservationFlag        logical Mass Conservation Check
! CalcPhytoPlankton           logical Pelagic Phytoplankton (vector)
! CalcPelBacteria             logical Pelagic Bacteria (vector)
! CalcMesoZooPlankton         logical Mesozooplankton (vector)
! CalcMicroZooPlankton        logical Microzooplankton (vector)
! CalcPelChemistry            logical Pelagic Hydrochemical Processes
! AssignPelBenFluxesInBFMFlag logical Benthic-pelagic fluxes are added to the
!                                     time integration
! AssignAirPelFluxesInBFMFlag logical Air-sea fluxes are added to the
!                                     time integration
```

3

```
! ChlDynamicsFlag    numeric Choose the dynamics of Chl-a
!                            1 = diagnostic, optimal light property
!                                in phytoplankton
!                                (Ebenhoeh et al 1995, ERSEM-II)
!                            2 = state variable, constituent of
!                                phytoplankton
!
! Global Parameters : used throughout the model and not related
!                     to a specific component
! NAME          UNIT          DESCRIPTION
! p_small       [-]           Smallest numeric value (the model "zero")
! slp0          [mbar]        Reference sea level pressure
! p_pe_R1c      [-]           Fractional content of C in cytoplasm
! p_pe_R1n      [-]           Fractional content of N in cytoplasm
! p_pe_R1p      [-]           Fractional content of P in cytoplasm
! p_qro         [mmolHS-/     Stoichiometric coefficient for
!               mmolO2]       anaerobic reactions
! p_qon_dentri  [mmolO2/      Stoichiometric coefficient for
!               mmolN]        denitrification
! p_qon_nitri   [mmolO2/      Stoichiometric coefficient for
!               mmolN]        nitrification
!-----------------------------------------------------------------------!
&Param_parameters
! Switches :
           CalcPelagicFlag = .TRUE.
           CalcBenthicFlag = 0
      CalcConservationFlag = .FALSE.
         CalcTransportFlag = .FALSE.
      CalcPhytoPlankton(1) = .TRUE.
      CalcPhytoPlankton(2) = .TRUE.
      CalcPhytoPlankton(3) = .TRUE.
      CalcPhytoPlankton(4) = .TRUE.
        CalcPelBacteria(1) = .TRUE.
   CalcMicroZooPlankton(1) = .TRUE.
   CalcMicroZooPlankton(2) = .TRUE.
    CalcMesoZooPlankton(1) = .TRUE.
    CalcMesoZooPlankton(2) = .TRUE.
           CalcPelChemistry = .TRUE.
AssignPelBenFluxesInBFMFlag = .FALSE.
AssignAirPelFluxesInBFMFlag = .TRUE.
          ChlDynamicsFlag = 2
! Parameters :
                   p_small = 1.0e-20
                    p_atm0 = 1013.25E0
                  p_pe_R1c = 0.60
                  p_pe_R1n = 0.72
```

```
                          p_pe_R1p = 0.832
                              p_qro = 0.5
                      p_qon_dentri = 1.25
                       p_qon_nitri = 2.0
filename_nml_conf = 'BFM_General.nml',
/


!-----------------------------------------------------------------------------!
! NAMELIST bfm_init_nml
!-----------------------------------------------------------------------------!
!Pelagic initialisation of standard variables
!<variablename>0 = <realvalue>
!-----------------------------------------------------------------------------!
&bfm_init_nml
   O2o0 = 300.0,
   N1p0 = 1.0,
   N3n0 = 5.0,
   N4n0 = 1.0,
   N5s0 = 8.0,
   N6r0 = 1.0,
   O3c0 = 27060.00,
   O3h0 = 2660.0,
   O4n0 = 200.0,
   P1c0 = 1.0,
   P2c0 = 1.0,
   P3c0 = 1.0,
   P4c0 = 1.0,
   Z3c0 = 1.0,
   Z4c0 = 1.0,
   Z5c0 = 1.0,
   Z6c0 = 1.0,
   B1c0 = 1.0,
   R1c0 = 1.0,
   R2c0 = 0.1,
   R3c0 = 1.0,
   R6c0 = 1.0,
   filename_nml_conf = 'BFM_General.nml',
/


!-----------------------------------------------------------------------------!
!NAMELIST bfm_save_nml
!-----------------------------------------------------------------------------!
! Stored variables
!ave_save: average values over the output interval
!var_save: instantaneous value at the output interval
!-----------------------------------------------------------------------------!
```

```
&bfm_save_nml
   var_save = ''
   ave_save = ''

filename_nml_conf = 'BFM_General.nml',
/


!-------------------------------------------------------------------------!
!NAMELIST Settling_parameters
!-------------------------------------------------------------------------!
! BURIAL VELOCITIES into the sediment
! NAME           [UNIT]/KIND          DESCRIPTION
! p_burvel_R6    [m/d]                Bottom Burial Velocity for detritus
! p_burvel_R2    [m/d]                Bottom Burial Velocity for dissolved
! p_burvel_PI    [m/d]                Bottom Burial Velocity for plankton
! p_burvel_O5    [m/d]                Bottom Burial Velocity for calcite
!-------------------------------------------------------------------------!
&Settling_parameters
  p_burvel_R6 = 1.5,
  p_burvel_O5 = 30.0,
  p_burvel_R2 = 0.0,
  p_burvel_PI = 0.0
filename_nml_conf = 'Benthic_Environment.nml',
/


!-------------------------------------------------------------------------!
!NAMELIST CSYS_parameters
!-------------------------------------------------------------------------!
! CARBONATE SYSYEM SETTING
! NAME            [UNIT]/KIND            DESCRIPTION
! AtmCO20         [ppmv]   Initial atmospheric concentration of CO2
! MaxIterPHsolver integer  Maximum number of iterations for SolveSAPHE (default 50)
! CalcBioAlk      logical  Compute biological processes corrections on total alkalinity
! Co2fluxfac      real     Multiplier to enhance air-sea co2 exchange rate
!           --------- Parameters for calcium and calcite ---------
! p_kdca          [d-1]              Calcite dissolution rate constant
! p_nomega        [-]                Order of the dissolution rate dependence on Omega
!           --------- EXTERNAL DATA INPUT STRUCTURES -----------
! AtmCO2_N        structure       Read external data for atmospheric CO2 values
! AtmSLP_N        structure       Read external data for atmospheric sea level pressure
! Example of general input structure for the data structure:
!         ! Read !    File                                      ! NetCDF ! Var    !
!         ! Input !   name                                      ! Logical ! name  !
! AtmCO2_N =    0  , 'CMIP5_Historical_GHG_1765_2005.dat' , .FALSE.  , 'CO2'  ,
!         ! RefTime           ! Input      !   Time    !
!         ! yyyymmdd          ! Frequency  !  interp   !
```

```
!              '1764-07-01 00:00' , 'yearly'  ,  .TRUE.
!
! Convention for Input reading : 0 = use constant value
!                                    (default if struct is not initialized)
!                                1 = read timeseries file ( e.g. CO2 mixing ratios)
!                                2 = read 2D fields using NEMO fldread
!                                3 = field from a coupled model
!                                    (e.g. atmospheric SLP from OGCM)
! NOTE: The file "CMIP5_Historical_GHG_1765_2005.dat" is located in
!"$BFMDIR/tools" folder
!-----------------------------------------------------------------------------!
&CSYS_parameters
    AtmCO20         =  370.0E0
    MaxIterPHsolver = 50
    CalcBioAlk      =  .TRUE.
    AtmCO2_N        =  0    'CMIP5_Historical_GHG_1765_2005.dat' .FALSE.
                      'CO2'    '1764-07-01 00:00' 'yearly'  .TRUE.
    AtmSLP_N        =  0    'AtmSLP.nc'                        .TRUE.
                      'AtmSLP' '1764-07-01 00:00' 'dummy'   .TRUE.
    p_kdca          =  10.9
    p_nomega        =  1
    Co2fluxfac      =  1.0
filename_nml_conf = 'Carbonate_Dynamics.nml',
/


!-----------------------------------------------------------------------------!
!NAMELIST PelBac_parameters
!-----------------------------------------------------------------------------!
!   PELAGIC BACTERIA
!
! NAME           [UNIT]/KIND          DESCRIPTION
! p_version   integer          Switch for bacteria parameterization
!                                 1 : Baretta-Bekker et al. 1995;
!                                     Vichi et al., 2007
!                                 2 : Vichi et al., 2004
!                                 3 : Polimene et al., 2006
! p_q10       [-]              Q10-value (temperature dependency)
! p_chdo      [mmol/m3]        Half-saturation constant for O2 limitation
! p_sd        [1/d]            Specific mortality rate
! p_sd2       [1/d]            Density dependent specific mortality rate
! p_suhR1     [1/d]            Specific potential uptake for nutrient-rich DOM
! p_sulR1     [1/d]            Specific potential uptake for nutrient-poor DOM
! p_suR2      [1/d]            Specific potential uptake for semi-labile DOC
! p_suR3      [1/d]            Specific potential uptake for semi-refractory DOC
! p_suR6      [1/d]            Specific potential uptake for POM (1/d)
! p_sum       [1/d]            Potential specific growth rate
```

```
! p_pu_ra      [-]              Activity respiration fraction
! p_pu_ra_o    [-]              Additional respiration fraction at low O2 conc
! p_srs        [1/d]            Specific rest respiration
! p_qncPBA     [mmolN/mgC]      Optimal N/C ratio
! p_qpcPBA     [mmolP/mgC]      Optimal P/C ratio
! p_qlnc       [mmolN/mgC]      Minimal N/C ratio
! p_qlpc       [mmolP/mgC]      Minimal P/C ratio
! p_qun        [mmolN/mgC/day]  Membrane affinity for N
! p_qup        [mmolP/mgC/day]  Membrane affinity for N
! p_chn        [mmolN/m3]       Half saturation ammonium conc. for uptake
! p_chp        [mmolP/m3]       Half saturation phosphate conc. for uptake
! p_ruen       [1/d]            Relaxation timescale for N uptake/remin.
! p_ruep       [1/d]            Relaxation timescale for P uptake/remin.
! p_rec        [1/d]            Relaxation timescale for semi-labile excretion
! p_pu_ea_R3   [-]              Excretion of semi-refractory DOC
!--------------------------------------------------------------------!
&PelBacteria_parameters
!                 B1
    p_version = 2
        p_q10 = 2.95
       p_chdo = 30.0
         p_sd = 0.0
        p_sd2 = 0.0
      p_suhR1 = 0.5
      p_sulR1 = 0.0
       p_suR2 = 0.25
       p_suR3 = 0.0
       p_suR6 = 0.1
        p_sum = 8.38
      p_pu_ra = 0.6
    p_pu_ra_o = 0.2
        p_srs = 0.01
     p_qncPBA = 0.017
     p_qpcPBA = 0.0019
       p_qlnc = 0.0085
       p_qlpc = 0.00095
        p_qun = 0.05
        p_qup = 0.005
        p_chn = 0.05
        p_chp = 1.00
        p_rec = 1.0
       p_ruen = 1.0
       p_ruep = 1.0
    p_pu_ea_R3 = 0.0
filename_nml_conf = 'Pelagic_Ecology.nml',
/
```

```
!-------------------------------------------------------------------------!
!NAMELIST Phyto_parameters, Phyto_parameters_iron
!-------------------------------------------------------------------------!
!  PELAGIC PHYTOPLANKTON
! NAME          [UNIT]/KIND         DESCRIPTION
!      :        --------- Physiological parameters -----------------
!  p_q10        [-]            Characteristic Q10 coefficient
!  p_qtemp      [-]            Cut-off threshold for temperature factor
!  p_sum        [1/d]          Maximal productivity at 10 degrees C
!  p_srs        [1/d]          Respiration rate at 10 degrees C
!  p_sdmo       [1/d]          Max.specific nutrient-stress lysis rate
!  p_thdo       [-]            Half saturation constant for nutrient stress lysis
!  p_seo        [1/d]          Extra lysis rate (biomass density-dependent)
!  p_sheo       [mgC/3]        Half saturation constant for extra lysis
!  p_pu_ea      [-]            Excreted fraction of primary production
!  p_pu_ra      [-]            Activity respiration fraction
!  p_switchDOC  [1-3]          Switch for the type of DOC excretion
!                              This choice must be consistent with bacteria
!                              1. All DOC is released as R1c (Vichi et al., 2007)
!                              2. Activity DOC is released as R2c (Vichi et al., 2004)
!                                 (there is no nutrient-stress excretion)
!                              3. All DOC is released as R2c (Polimene et al., 2006)
!
!              --------- Nutrient parameters in phytoplankton -----------------
!  p_netgrowth  [T or F]       Logical switch for nutrient-limited growth
!                              .T. nutrient-balanced growth (Vichi et al.2004)
!                              .F. nutrient-stress carbon excretion
!                                (Baretta-Bekker et al.1995 and Vichi et al.2007)
!  p_limnut     [1-3]          Switch for N-P co-limitation
!                              0. Geometric mean
!                              1. Threshold (Liebig-like)
!                              2. Combined
!                     ---- N limitation control ----
!  p_qun        [m3/mgC/d]     Membrane affinity for N
!  p_lN4        [mmolN/m3]     Half saturation constant for NH4 uptake preference over NO3
!  p_qnlc       [mmolN/mgC]    Minimum quotum Si:C
!  p_qncPPY     [mmolN/mgC]    Reference quotum Si:C
!  p_xqn        [-]            Multiplication factor for luxury storage
!                     ---- P limitation control ----
!  p_qup        [m3/mgC/d]     Membrane affinity for P
!  p_qplc       [mmolP/mgC]    Minimum quotum Si:C
!  p_qpcPPY     [mmolP/mgC]    Reference quotum Si:C
!  p_xqp        [-]            Multiplication factor for luxury storage
!                     ---- Si limitation control ----
!  p_switchSi   [1-2]          Switch for Silica limitation
```

```
!                                  1. Si limitation is controlled by external Si conc.
!                                  2. Si limitation is controlled by internal quota
!  p_chPs      [mmolSi/m3]   Half saturation conc. for dissolved Si limitation
!  p_Contois   [>=0]         If >0, use Contois formulation
!  p_qus       [m3/mgC/d]    Membrane affinity for Si
!  p_qslc      [mmolSi/mgC]  Minimum quotum Si:C
!  p_qscPPY    [mmolSi/mgC]  Reference quotum Si:C
!                            Brzezinski (1985) 0.13 mol/mol
!                    ---- nutrient stressed sinking ----
!  p_esNI      [-]           Nutrient stress threshold for sinking
!  p_res       [m/d]         Maximum Sinking vel city (m/d)
!                    ---- Calcification ----
!  p_caco3r    [-]           Reference PIC:POC (rain) ratio
!                --------- Chlorophyll parameters -----------
!  p_switchChl [1-4]         Switch for Chla-a synthesis
!  p_sdchl     [1/d]         Specific turnover rate for Chla
!  p_alpha_chl [mgC s m2/    Initial slope of the P-E curve
!               mgChl/uE]
!  p_qlcPPY    [mgChla/mgC]  Reference quotum Chla:C
!  p_epsChla   [m2/mgChla]   Chla-specific extinction coefficient
!  p_tochl_relt [1/d]        Relaxation rate towards maximum Chla:C
!  p_EpEk_or   [-]           Optimal value of E_PAR/E_K
!                --------- Iron parameters -----------
!  p_quf       [m3/mgC/d]    Membrane affinity for Fe
!  p_qflc      [umolFe/mgC]  Minimum quotum Fe:C derived from 3 umol Fe/mol C
!                            Sunda & Huntsman (1997), Nature, 390, p 389-392
!  p_qfcPPY    [umolFe/mgC]  Reference quotum Fe:C
!  p_xqf       [-]           Multiplication factor for luxury storage
!                --------- Light parameters ERSEM-II -----------
!  p_iswLtyp   [0-6]         Shape of the productivity function
!  p_chELiPPY  [W/m2]        Maximum Iopt
!  p_clELiPPY  [W/m2]        Minimum Iopt
!  p_ruELiPPY  [1/d]         Maximum daily shift in Iopt (1/d)
!  p_addepth   [m]           Adaptation depth. Meaningless with high-res models
!                --------- Sinking parameters -----------
!  p_rPIm      [m/d]         Phytoplankton background sinking rate
!----------------------------------------------------------------------!

&Phyto_parameters
!             P1      P2      P3      P4
     p_q10 = 2.0,    2.0,    2.0,    2.0
    p_temp = 0.0,    0.0,    0.75,   0.0
     p_sum = 2.5,    3.0,    3.5,    1.5
     p_srs = 0.1,    0.05,   0.1,    0.1
    p_sdmo = 0.0,    0.0,    0.0,    0.0
    p_thdo = 0.0,    0.0,    0.0,    0.0
     p_seo = 0.0,    0.0,    0.0,    0.0
```

```
      p_sheo = 0.0,     0.0,     0.0,      100.0
     p_pu_ea = 0.05,    0.1,     0.1,     0.15
     p_pu_ra = 0.1,     0.1,     0.2,    0.1
  p_switchDOC = 2,       2,       2,        2
  p_netgrowth = .TRUE.,.TRUE.,.TRUE.,.TRUE.
    p_limnut = 1,         1,        1,         1
       p_qun = 0.025,   0.025,   0.25,    0.025
       p_lN4 = 1.0,     0.5,     0.1,      1.0
      p_qnlc = 0.00687, 0.00687, 0.00687, 0.00687
    p_qncPPY = 1.26e-2,1.26e-2,1.26e-2,1.26e-2
       p_xqn = 2.0,     2.0,     2.0,      2.0
       p_qup = 0.0025,   0.0025,   0.0025,   0.0025
      p_qplc = 0.0004288,0.0004288,0.0004288,0.0004288
    p_qpcPPY = 7.86e-4,7.86e-4,7.86e-4,7.86e-4
       p_xqp = 2.0,     2.0,     2.0,      2.0
  p_switchSi = 2,         0,        0,         0
      p_chPs = 0.3,     0.0,     0.0,      0.0
   p_Contois = 0.0,     0.0,     0.0,      0.0
       p_qus = 0.0025,    0.0,     0.0,      0.0
      p_qslc = 0.007,     0.0,     0.0,      0.0
    p_qscPPY = 0.01,    0.0,     0.0,      0.0
      p_esNI = 0.7,     0.75,    0.75,     0.75
      p_res  = 5.0,     0.0,     0.0,      2.5
    p_caco3r = 0.0,     0.3,     0.0,      0.0
  p_switchChl = 2,        2,        2,         2
      p_sdchl = 0.2,     0.2,     0.2,      0.2
  p_alpha_chl = 1.38e-5, 0.46e-5, 1.52e-5, 0.68e-5
    p_qlcPPY = 0.02,    0.02,    0.02,     0.02
    p_epsChla = 0.03,     0.03,    0.03,     0.03
    p_EpEk_or = 0.0,     0.0,     0.0,      0.0
 p_tochl_relt = 0.0,     0.0,     0.0,      0.0
    p_iswLtyp = 5,        5,        5,        5
    p_addepth = 50.0,    50.0,    50.0,     50.0
  p_chELiPPY  = 100.0,   100.0,   100.0,   100.0
  p_clELiPPY  = 8.0,     10.0,    6.0,      12.0
  p_ruELiPPY  = 0.2,     0.25,    0.3,      0.15
       p_rPIm = 0.0,     0.0,     0.0,      0.0
filename_nml_conf = 'Pelagic_Ecology.nml',
/

&Phyto_parameters_iron
filename_nml_conf = 'Pelagic_Ecology.nml',
/

!-------------------------------------------------------------------------------!
!NAMELIST PAR_parameters
```

```
!--------------------------------------------------------------------------!
! LightPeriodFlag    numeric Choose the light averaging period
!                            1 = Instantanous irradiance
!                            2 = Daily average
!                            3 = Daylight average with explicit
!                                 photoperiod
! LightLocationFlag  numeric Choose the parameterization of light
!                            location in the discrete grid
!                            1 = Light at the top of the cell
!                            2 = Light in the middle of the cell
!                            3 = Average Light in the cell
! ChlAttenFlag       numeric Choose the PAR attenuation due to Chl
!                            1 = broadband linear attenuation
!                            2 = 3-band tabulated attenuation coefficients
!                                 (Morel, 1988; Lengaigne et al, 2007)
! p_PAR         [-]          Fraction of Photosynthetically Available Radiation
! p_eps0        [1/m]        Background extinction coefficient
! p_epsIR       [1/m]        Infrared extinction coefficient
!                            (to use with ChlAttenFlag=2)
! p_epsESS      [m2/g]       Specific attenuation coefficient of
!                            suspended sediments
! p_epsR6       [m2/mgC]     Specific attenuation coefficient of particulate
!                            detritus
!--------------------------------------------------------------------------!
&PAR_parameters
  LightPeriodFlag    =  1
  LightLocationFlag  =  2
  ChlAttenFlag       =  1
  p_PAR        =  0.40
  p_eps0       =  0.0435
  p_epsIR      =  2.857
  p_epsESS     =  0.04d-3
  p_epsR6      =  0.1d-3
filename_nml_conf = 'Pelagic_Environment.nml',
/


!--------------------------------------------------------------------------!
!NAMELIST MicroZoo_parameters
!--------------------------------------------------------------------------!
!  MICRO-ZOOPLANKTON
!
! NAME          [UNIT]/KIND          DESCRIPTION
! p_q10         [-]          Q10 value for physiological rates
! p_srs         [1/d]        Respiration rate at 10 degrees Celsius
! p_sum         [1/d]        Potential growth rate
! p_sdo         [1/d]        Mortality rate due to oxygen limitation
```

```
! p_sd          [1/d]         Temperature independent mortality rate
! p_pu          [-]           Assimilation efficiency
! p_pu_ea       [-]           Fraction of activity excretion
! p_chro        [mmolO2/m3]   Half-saturation oxygen concentration
! p_chuc        [mgC/m3]      Half-saturation Food concentration for Type II
! p_minfood     [mgC/m3]      Half-saturation food concentration for
!                             preference factor
! p_pecaco3     [-]           Portion of egested calcified shells during grazing
! p_qncMIZ      [mmolN/mgC]   Maximum quotum P:C
! p_qpcMIZ      [mmolN/mgC]   Maximum quotum N:C
! p_paPBA(z,b)  [-]           Availability of pelagic Bacteria group b
!                             to Zooplankton group z
! p_paPPY(z,p)  [-]           Availability of PhytoPlankton group p
!                             to Zooplankton group z
! p_paMIZ(z,m)  [-]           Availability of MicroZooplankton group m
!                             to Zooplankton group z
!-------------------------------------------------------------------!
&MicroZoo_parameters
!               Z5      Z6
      p_q10 =  2.0,     2.0
      p_srs =  0.02,    0.02
      p_sum =  2.0,     5.0
      p_sdo =  0.05,    0.05
       p_sd =  0.0,    0.0
       p_pu =  0.5,     0.3
    p_pu_ea =  0.5,     0.5
     p_chro =  8,       8
     p_chuc =  30.0,    100.0
   p_minfood =  50.0,    50.0
   p_pecaco3 =  0.5,     0.5
    p_qpcMIZ =  1.85d-3,1.85d-3
    p_qncMIZ =  1.67d-2,1.67d-2
! Food matrix parameters: take care of the notation
!               Z5      Z6
    p_paPBA =  0.1,     1.0
!               P1      P2       P3       P4
!       Z5
p_paPPY(1,:) =  0.7,     1.0,     0.1,     0.1
!       Z6
p_paPPY(2,:) =  0.0,     0.2,     1.0,     0.0
!               Z5      Z6
!       Z5
p_paMIZ(1,:) =  1.0,     1.0
!       Z6
p_paMIZ(2,:) =  0.0,     0.2
filename_nml_conf = 'Pelagic_Ecology.nml',
```

```
/

!-----------------------------------------------------------------------!
!NAMELIST MesoZoo_parameters
!-----------------------------------------------------------------------!
!   MESO-ZOOPLANKTON
! NAME          [UNIT]/KIND          DESCRIPTION
! p_q10         [-]           Q10 value for physiological rates
! p_srs         [1/d]         Respiration rate at 10 degrees C
! p_sum         [1/d]         Maximal productivity at 10 degrees C
! p_sd          [1/d]         Background natural mortality
! p_vum         [m3/mgC/d]    Specific search volume
! p_puI         [-]           Assimilation efficiency
! p_peI         [-]           Fraction of Faeces production
! p_sdo         [m3/mgC/d]    Specific density-dependent mortality
! p_sds         [-]           Exponent of density-dependent mortality
! p_pecaco3     [-]           Portion of egested calcified shells during grazing
! p_qpcMEZ      [mmolP/mgC]   Maximum quotum P:C
! p_qncMEZ      [mmolN/mgC]   Maximum quotum N:C
! p_clO2o       [mmolO2/m3]   Half-saturation oxygen concentration
! p_paPPY(z,p)  [-]           Availability of PhytoPlankton group p
!                             to Zooplankton group z
! p_paMIZ(z,m)  [-]           Availability of MicroZooplankton group m
!                             to Zooplankton group z
! p_paMEZ(z,m)  [-]           Availability of MesoZooplankton group m
!                             to Zooplankton group z
!-----------------------------------------------------------------------!
&MesoZoo_parameters
!               Z3      Z4
      p_q10 =  2.0,     2.0
      p_srs =  0.01,    0.02
      p_sum =  2.0,     2.0
      p_vum =  0.008,   0.02
      p_puI =  0.6,     0.6
      p_peI =  0.3,     0.35
      p_sdo =  0.01,    0.01
       p_sd =  0.02,    0.02
      p_sds =  2.0,     2.0
  p_pecaco3 =  0.75,    0.75
   p_qpcMEZ =  1.67d-3,1.67d-3
   p_qncMEZ =  0.015,   0.015
    p_clO2o =  30.0,    30.0
! Food matrix parameters: take care of the notation
!               P1      P2      P3      P4
!       Z3
p_paPPY(1,:) =  0.0,    0.0,    0.0,    1.0
```

```
!        Z4
p_paPPY(2,:) =  1.0,     0.75,    0.0,     1.0
!               Z5       Z6
!        Z3
p_paMIZ(1,:) =  0.0,     0.0
!        Z4
p_paMIZ(2,:)=   1.0,     0.0,
!               Z3       Z4
!        Z3
p_paMEZ(1,:)=   1.0,     1.0
!        Z4
p_paMEZ(2,:)=   0.0,     1.0
filename_nml_conf = 'Pelagic_Ecology.nml',
/


!-------------------------------------------------------------------------!
!NAMELIST PelChem_parameters, PelChem_parameters_iron
!-------------------------------------------------------------------------!
!  Pelagic Chemistry parameters
! NAME          [UNIT]/KIND      DESCRIPTION
! p_q10N4N3     [-]              Q10 factor for nitrification/denit
! p_sN4N3       [1/d]            Specific nitrification rate at 10 degC
! p_clO2o       [mmolO2/m3]      Half-saturation O2 concentration for
!                                nitrification and reoxidation
! p_rOS         [1/d]            Specific reoxidation rate of reduction
!                                equivalents
! p_sN3O4n      [1/d]            Specific denitrification rate
! p_clN6r       [mmolHS/m3]      Half-saturation concentration of
!                                reduction equivalents for denitrification
! p_rPAo        [mmolO2/m3/d]    Reference anoxic mineralization rate
! p_q10R6N5     [-]              Q10 factor for biogenic silica
! p_sR6N5       [1/d]            Specific remineralization rate of
!                                biogenic silica
!               --------- Iron parameters ----------
! p_q10R6N7     [-]              Q10 temperature dependence
! p_sR6N7       [1/d]            Specific remineralization rate of particulate
! p_sR1N7       [1/d]            Specific remineralization rate of dissolved
! p_scavN7f     [1/d]            Specific scavenging rate
! p_N7fsol      [umolFe/m3]      Solubility concentration
!-------------------------------------------------------------------------!
&PelChem_parameters
  p_q10N4N3 = 2.367
    p_sN4N3 = 0.01
    p_clO2o = 10.0
      p_rOS = 0.05
   p_sN3O4n = 0.35
```

```
      p_clN6r = 1.0
       p_rPAo = 1.0
   p_q10R6N5 = 1.49
     p_sR6N5 = 0.1
filename_nml_conf = 'Pelagic_Environment.nml',
/


&PelChem_parameters_iron
filename_nml_conf = 'Pelagic_Environment.nml',
/


!-------------------------------------------------------------------------!
!NAMELIST PelGlobal_parameters
!-------------------------------------------------------------------------!
! Sinking rates of Pelagic Variables
!  : for mem_PelGlobal filled by InitPelGlobal
! NAME            UNIT      DESCRIPTION
! p_rR6m         [m/d]    detritus sinking rate
! KSINK_rPPY      [m]      prescribe sinking rate for phytoplankton below this
!                          depth threshold to p_rR6m value. Use 0.0 to disable.
! AggregateSink  logic    use aggregation = true to enhance the sink rate
!                          and bypass the prescribed sinking
! depth_factor    [m]      depth factor for aggregation method
!-------------------------------------------------------------------------!
&PelGlobal_parameters
  p_rR6m         = 5.0
  KSINK_rPPY     = 150.0
  AggregateSink  = .FALSE.
  depth_factor   = 2000.0
filename_nml_conf = 'Pelagic_Environment.nml',
/


!-------------------------------------------------------------------------!
!NAMELIST standalone_nml
!-------------------------------------------------------------------------!
!NAME KIND DESCRIPTION
!nboxes   integer Number of water volumes (boxes)
!indepth  real  Depth of each box (m)
!latitude  real  Latitude of each box
!longitude  real  Longitude of each box
!maxdelt  real  Maximum timestep duration (s)
!mindelt  real  Minimum timestep duration (s)
!method   integer Integration method
! 1. Euler forward
! 2. Runge-Kutta 2nd order
! 3. Leap-frog
```

```
!-----------------------------------------------------------------------
&standalone_nml
   nboxes      = 1,
   indepth     = 5.0,
   latitude    = 45.0,
   longitude   = 13.5,
   maxdelt     = 8640.0,
   mindelt     = 1.0,
   method      = 2,
filename_nml_conf = 'Standalone.nml',
/


!------------------------------------------------------------------------!
!NAMELIST time_nml
!------------------------------------------------------------------------!
!Specify time related formats and variables here.
!timefmt integer implicitly uses timestep=maxdelt
!                 1- MaxN only - fake start time used.
!                 2- start and stop - MaxN calculated.
!                 3- start and MaxN - stop calculated.
!                 4- simdays - fake start time used
!     and MaxN calculated.
!MaxN integer do loop from n=1,MaxN
!start string Initial time: YYYY/MM/DD HH:MM:SS
!stop string Final   time: YYYY/MM/DD HH:MM:SS
!------------------------------------------------------------------------!
 &time_nml
  timefmt=        2,
  MaxN=           144,
  simdays=        5760,
  start=          "2000-01-01 00:00:00",
  stop=           "2010-01-01 00:00:00",
filename_nml_conf = 'Standalone.nml',
/


!------------------------------------------------------------------------!
!NAMELIST forcings_nml
!------------------------------------------------------------------------!
!forcing_method integer Choice of the external forcing functions
! 1 = analytical forcings
! 2 = from file
! 3 = interactive fluxes (not yet implemented)
!
!Method 1: Analytical forcing functions
!lw real Sinusoidal light intensity (winter) W m^-2
!ls real Sinusoidal light intensity (summer) W m^-2
```

```
!sw real Sinusoidal salinity (winter)
!ss real Sinusoidal salinity (summer)
!tw real Sinusoidal temperature (winter) degC
!ts real Sinusoidal temperature (summer) degC
!tde real Sinusoidal temperature daily excursion degC
!ww real Sinusoidal wind (winter) m/s
!ws real Sinusoidal wind (summer) m/s
!CO2inc        real    Linear increase in CO2 air partial pressure [% per year]
!
!Method 2: data file
!forcing_file char Filename for external forcings
!
!use_external_data logical Read external data (user defined)
!data_file    char    Filename for external data
!
!-----------------------------------------------------------------------!
&forcings_nml
   forcing_method   = 1,
   ltype            = 1,
   lw               = 20.0,
   ls               = 300.0,
   sw               = 37.0,
   ss               = 34.0,
   tw               = 8.0,
   ts               = 28.0,
   tde              = 1.0,
   ww               = 20.0,
   ws               = 10.0,
   forcing_file     = '',
   use_external_data = .false.,
   data_file        = '',
filename_nml_conf = 'Standalone.nml',
/


!-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-
! MODEL  BFM - Biogeochemical Flux Model
!-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-=-
```

3   The radiative transfer model (RT model)

In the following, we describe the approximate solution of the radiative transfer equations. The approach is that of [Dutkiewicz et al., 2015], given in more detail

3.1 The Equations

The irradiance is separated into three streams, namely: the direct light E_d, the scattered light E_s going downwards, and the scattered light E_u going upwards. These functions are governed by the equations

$$\frac{dE_d}{dz} = -\frac{a+b}{\bar{u}_d}E_d \tag{1}$$

$$\frac{dE_s}{dz} = -\frac{a+r_s b_b}{\bar{v}_s}E_s + \frac{r_u b_b}{\bar{v}_u}E_u + \frac{b-r_d b_b}{\bar{u}_d}E_d \tag{2}$$

$$-\frac{dE_u}{dz} = -\frac{a+r_u b_b}{\bar{v}_u}E_u + \frac{r_s b_b}{\bar{v}_s}E_s + \frac{r_d b_b}{\bar{u}_d}E_d. \tag{3}$$

The variable $z \in (0, H)$, where the depth H is sufficiently high, so that light could be considered low at this depth. Here r_s, r_u, r_d are corrections to the backscattering coefficient for three streams of radiance, and they are given by the following assumption:

$$r_d \approx 1.0, \quad r_s \approx 1.5, \quad r_u \approx 3.0,$$

and $\bar{v}_s$, $\bar{v}_u$ are average cosines of the zenith angle, defined by the assumption

$$\bar{v}_s \approx 0.83, \quad \bar{v}_u \approx 0.4.$$

The backscattering parameter b_b, the light absorption factor $a(z)$, and the average cosine of the zenith angle $\bar{u}_d$ for the direct radiance component should be estimated from data.

In this system, $a(z)$ depends on z and, therefore, it is hardly possible to solve the equations analytically.

Equation (1) can be integrated independently of the others:

$$E_d(z) = Ed(0)\exp\int_0^z \frac{-c(\zeta)}{\bar{u}_d}d\zeta, \quad c(\zeta) = a(\zeta) + b. \tag{4}$$

3.2 Boundary conditions

The components $E_d(0) = E_d^{\text{OASIM}}$ and $E_s(0) = E_s^{\text{OASIM}}$ are given, so this is the Dirichlet boundary condition. The third component is assumed to be zero at the bottom: $E_u(H) = 0$. This can be understood as no reflection. Also, we will assume $E_d(H) \approx 0$ (light decay).

3.3 Piecewise-constant parameters

To solve the other equations, let us divide the surface-bottom span $[0, H]$ into n layers by points z_k, $k = 1, \ldots, n+1$ ($z_1 = 0$, $z_{n+1} = H$). The coefficients of the the system are approximated by constant values within each layer.

Let us solve the system within a layer. To solve equations (2), (3), we rewrite the system in the matrix form:

$$\frac{d\mathbf{E}}{dz} = \mathbf{M}\mathbf{E} + \mathbf{I}$$

where

$$\mathbf{M} = \begin{bmatrix} -C_s & B_u \\ -B_s & C_u \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} E_s \\ E_u \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} F_d \\ -B_d \end{bmatrix} E_d \qquad (5)$$

and

$$C_s = \frac{a + r_s b_b}{\bar{v}_s}, \qquad B_u = \frac{r_u b_b}{\bar{v}_u}, \qquad F_d = \frac{b - r_d b_b}{\bar{u}_d}, \qquad (6)$$

$$C_u = \frac{a + r_u b_b}{\bar{v}_u}, \qquad B_s = \frac{r_s b_b}{\bar{v}_s}, \qquad B_d = \frac{r_d b_b}{\bar{u}_d}. \qquad (7)$$

To solve this linear inhomogeneous system analytically, we need the general solution of the homogeneous system and any single solution.

To find an inhomogeneous solution, we write it as

$$\mathbf{E} = E_d \begin{bmatrix} x \\ y \end{bmatrix}$$

with x, y independent of z. Substituting into the equation, we get

$$-c_d \mathbf{E} = \mathbf{M}\mathbf{E} + \mathbf{I}, \quad c_d = \frac{c}{\bar{u}_d},$$

which is reduced to the linear algebraic system with constant (within a layer) coefficients:

$$\begin{bmatrix} -C_s + c_d & B_u \\ -B_s & C_u + c_d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -F_d \\ B_d \end{bmatrix}. \qquad (8)$$

This 2×2 system is easily solved:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{(c_d - C_s)(c_d + C_u) + B_s B_u} \begin{bmatrix} C_u + c_d & -B_u \\ B_s & -C_s + c_d \end{bmatrix} \begin{bmatrix} -F_d \\ B_d \end{bmatrix} \qquad (9)$$

To compute the homogeneous solution, we need the eigenvalues of $\mathbf{M}$:

$$k^- = D - C_s \qquad (10)$$

$$-k^+ = C_u - D = -C_s + \frac{B_s B_u}{D} \qquad (11)$$

$$D = \frac{1}{2}\left(C_s + C_u + \sqrt{(C_s + C_u)^2 - 4B_s B_u}\right). \qquad (12)$$

Note that the expression under the radical is strictly positive and the eigenvalues are real and of different signs.

In an homogeneous computational layer $[z_k, z_{k+1}]$, the solution is

$$\begin{bmatrix} E_{s;k}(z) \\ E_{u;k}(z) \end{bmatrix} = c_k^+ \begin{bmatrix} 1 \\ r_k^+ \end{bmatrix} e^{-k_k^+(z - z_k)} + c_k^- \begin{bmatrix} r_k^- \\ 1 \end{bmatrix} e^{+k_k^-(z - z_{k+1})} + \begin{bmatrix} x_k \\ y_k \end{bmatrix} E_d(z) \qquad (13)$$

Here $(1, r_k^+)^T$, $(r_k^-, 1)^T$ are the eigenvectors, and $r_k^+ = B_s/D$, $r_k^- = B_u/D$.

3.4 Continuity on layers' boundaries

The constants c_k^+, c_k^-, $2n$ in total, are obtained from the boundary conditions, which are: known irradiance on the surface, no light at large depth, and continuity on the layers' boundaries.

Writing the latter for $k = 1, \ldots, n-1$, as follows:

$$E_{s;k}(z_{k+1}) = E_{s;k+1}(z_{k+1}), \quad E_{u;k}(z_{k+1}) = E_{u;k+1}(z_{k+1}),$$

we get the linear system

$$c_k^+ e^{-k_k^+(z_{k+1}-z_k)} + c_k^- r_k^- + x_k E_d(z_{k+1}) =$$
$$c_{k+1}^+ + c_{k+1}^- r_{k+1}^- e^{k_{k+1}^-(z_{k+1}-z_{k+2})} + x_{k+1} E_d(z_{k+1}),$$
$$c_k^+ r_k^+ e^{-k_k^+(z_{k+1}-z_k)} + c_k^- + y_k E_d(z_{k+1}) =$$
$$c_{k+1}^+ r_{k+1}^+ + c_{k+1}^- e^{k_{k+1}^-(z_{k+1}-z_{k+2})} + y_{k+1} E_d(z_{k+1}).$$

Now we eliminate c_{k+1}^- from the first equation by multiplying the second one on r_{k+1}^- and substracting from the first one:

$$c_k^+ \left(1 - r_k^+ r_{k+1}^-\right) e^{-k_k^+(z_{k+1}-z_k)} + c_k^- \left(r_k^- - r_{k+1}^-\right) - c_{k+1}^+ \left(1 - r_{k+1}^+ r_{k+1}^-\right) =$$
$$= \left((x_{k+1} - x_k) - r_{k+1}^-(y_{k+1} - y_k)\right) E_d(z_{k+1}).$$

In the similar way, we eliminate c_k^+ from the second equation, by multiplying the first one on r_k^+ and substracting:

$$c_k^- \left(1 - r_k^+ r_k^-\right) - c_{k+1}^+ \left(r_{k+1}^+ - r_k^+\right) - c_{k+1}^- \left(1 - r_k^+ r_k^+\right) e^{k_{k+1}^-(z_{k+1}-z_{k+2})} =$$
$$\left((y_{k+1} - y_k) - r_k^+(x_{k+1} - x_k)\right) E_d(z_{k+1}).$$

Denote $e_k^+ = e^{-k_k^+(z_{k+1}-z_k)}$, $e_k^- = e^{k_k^-(z_k-z_{k+1})}$; then the system is rewritten in the tri-diagonal form:

$$\underbrace{e_k^+(1 - r_k^+ r_{k+1}^-)}_{\alpha_k} c_k^+ + \underbrace{(r_k^- - r_{k+1}^-)}_{\beta_k} c_k^- \underbrace{-(1 - r_{k+1}^+ r_{k+1}^-)}_{\gamma_k} c_{k+1}^+ =$$
$$= \underbrace{[x_{k+1} - x_k - (y_{k+1} - y_k)r_{k+1}^-]Ed(z_{k+1})}_{\delta_k} \quad (14)$$

$$\underbrace{(1 - r_k^- r_k^+)}_{\epsilon_k} c_k^- \underbrace{-(r_{k+1}^+ - r_k^+)}_{\zeta_k} c_{k+1}^+ \underbrace{-e_{k+1}^-(1 - r_{k+1}^- r_k^+)}_{\eta_k} c_{k+1}^- =$$
$$= \underbrace{[y_{k+1} - y_k - (x_{k+1} - x_k)r_k^+]Ed(z_{k+1})}_{\theta_k} \quad (15)$$

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Remember that the boundary condition at the surface is

$$E_{s;1}(0) = E_s^{\text{OASIM}}, \quad E_d(0) = E_d^{OASIM},$$

where the right-hand side is the known values from the data. This, with (13) taken into account, is reduced to

$$E_{s;1}(0) = c_1^+ + c_1^- r_1^- e^{-k_1^- z_2} + x_1 E_d^{\text{OASIM}}.$$

The other condition, at the bottom under the assumption that the depth H is large, is "no light at H", i.e., at $z_{n+1} = H$; so, let $e^{z_n - z_{n+1}} \approx 0$, and assume that $E_u(z_{n+1}) \approx 0$ and $E_d(z_{n+1}) \approx 0$; then

$$c_n^- = 0.$$

Let us rewrite these conditions in the form of linear equations:

$$\underbrace{1}_{\zeta_0} c_1^+ + \underbrace{r_1^- e_1^-}_{\eta_0} c_1^- = \underbrace{E_s^{\text{OASIM}} - x_1 E_d^{\text{OASIM}}}_{\theta_0}, \tag{16}$$

$$\epsilon_{n-1} c_{n-1}^- + \zeta_{n-1} c_n^+ + \eta_{n-1} \cdot \underbrace{0}_{c_n^-} = \theta_n. \tag{17}$$

In the matrix form, the system appears as a $(2n - 1) \times (2n - 1)$-dimensional matrix (i.e., c_i^+ for $i = 1, \ldots, n$ and c_i^- for $i = 1, \ldots, n - 1$):

$$
\begin{bmatrix}
\zeta_0 & \eta_0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
\alpha_1 & \beta_1 & \gamma_1 & 0 & 0 & 0 & \ldots & 0 \\
0 & \epsilon_1 & \zeta_1 & \eta_1 & 0 & 0 & \ldots & 0 \\
0 & 0 & \alpha_2 & \beta_2 & \gamma_2 & 0 & \ldots & 0 \\
0 & 0 & 0 & \epsilon_2 & \zeta_2 & \eta_2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\
0 & 0 & 0 & \ldots & 0 & \alpha_{n-1} & \beta_{n-1} & \gamma_{n-1} \\
0 & 0 & 0 & \ldots & 0 & 0 & \epsilon_{n-1} & \zeta_{n-1}
\end{bmatrix}
\begin{bmatrix}
c_1^+ \\
c_1^- \\
\vdots \\
\vdots \\
\vdots \\
c_{n-1}^+ \\
c_{n-1}^- \\
c_n^+
\end{bmatrix}
=
\begin{bmatrix}
\theta_0 \\
\delta_1 \\
\theta_1 \\
\vdots \\
\vdots \\
\vdots \\
\delta_{n-1} \\
\theta_{n-1}
\end{bmatrix}
$$

The matrix is tri-diagonal, so the system is solved by the well-known Thomas algorithm.

Before presenting the details of the algorithm, let us write down the diagonals of the matrix:

$$A_{i,i} = \begin{cases} \zeta_j & \text{if} \quad i = 2j + 1 \\ \beta_j & \text{if} \quad i = 2j \end{cases}, \quad i = 1, \ldots, 2n - 1;$$

$$A_{i,i+1} = \begin{cases} \eta_j & \text{if} \quad i = 2j + 1 \\ \gamma_j & \text{if} \quad i = 2j \end{cases}, \quad i = 1, \ldots, 2n - 2;$$

$$A_{i+1,i} = \begin{cases} \epsilon_j & \text{if} \quad i = 2j + 1 \\ \alpha_j & \text{if} \quad i = 2j \end{cases}, \quad i = 2, \ldots, 2n - 1;$$

4   Solving a Tri-Diagonal System

Consider a linear algebraic system $Ax = R$ with a tri-diagonal matrix A. Denote the diagonal elements by D_i, the upper and lower diagonal elements by U_i, L_i, respectively (i is the row index). Let the right-hand side elements be R_i.

Express each unknown x_i via its right-hand neighbour:

$$x_i = \alpha_i x_{i+1} + \beta_i.$$

For $i > 1$ we have $L_i x_{i-1} + D_i x_i + U_i x_{i+1} = R_i$ and thus $L_i(\alpha_{i-1} x_i + \beta_{i-1}) + D_i x_i + U_i x_{i+1} = R_i$, which becomes

$$x_i = \underbrace{\frac{-U_i}{\alpha_{i-1}L_i + D_i}}_{\alpha_i} x_{i+1} + \underbrace{\frac{R_i - \beta_{i-1}L_i}{\alpha_{i-1}L_i + D_i}}_{\beta_i}.$$

These are recurrent formulae for α_i, β_i:

$$\alpha_i = \frac{-U_i}{\alpha_{i-1}L_i + D_i}, \quad \beta_i = \frac{R_i - \beta_{i-1}L_i}{\alpha_{i-1}L_i + D_i}.$$

Initial values come from the first equation: $D_1 x_1 + U_1 x_2 = R_1$, which produces

$$\alpha_1 = -\frac{U_1}{D_1}, \quad \beta_1 = \frac{R_1}{D_1}.$$

Note that the general formulae for α_i, β_i can be used for $i = 1$ under the formal assumption $\alpha_0 = \beta_0 = 0$.

The last equation, which is

$$L_n x_{n-1} + D_n x_n = R_n,$$

becomes, after using the link,

$$x_n = \frac{R_n - \beta_{n-1}L_n}{\alpha_{n-1}L_n + D_n},$$

which coincides with the formula for β_n.

After that, all x_i are obtained, one by one, using the link in the decreasing order.

5   The solving algorithm

Here is the algorithm of the approximate solution:

- Solve the equation (1), using the formula (4).

- Choose n layers z_i, i from 1 to $n + 1$.

- Form the tri-diagonal matrix and the right-hand side using formulae (14) and (15), also (17) and (16) for the first and last row.

- Solve the algebraic system and get all c_i^+ for $1 \leq i \leq n$ and c_i^i for $1 \leq i \leq n-1$. The value $c_n^- = 0$.

- Use (13) to get the solution in any point z.