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Pythagorean Fuzzy Interaction Muirhead Means with Their Application to Multi-Attribute Group Decision-Making

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Abstract: Due to the increased complexity of real decision-making problems, representing attribute values correctly and appropriately is always a challenge. The recently proposed Pythagorean fuzzy set (PFS) is a powerful and useful tool for handling fuzziness and vagueness. The feature of PFS that the square sum of membership and non-membership degrees should be less than or equal to one provides more freedom for decision makers to express their assessments and further results in less information loss. The aim of this paper is to develop some Pythagorean fuzzy aggregation operators to aggregate Pythagorean fuzzy numbers (PFNs). Additionally, we propose a novel approach to multi-attribute group decision-making (MAGDM) based on the proposed operators. Considering the Muirhead mean (MM) can capture the interrelationship among all arguments, and the interaction operational rules for PFNs can make calculation results more reasonable, to take full advantage of both, we extend MM to PFSs and propose a family of Pythagorean fuzzy interaction Muirhead mean operators. Some desirable properties and special cases of the proposed operators are also investigated. Further, we present a novel approach to MAGDM with Pythagorean fuzzy information. Finally, we provide a numerical instance to illustrate the validity of the proposed model. In addition, we perform a comparative analysis to show the superiorities of the proposed method.

Keywords: Pythagorean fuzzy set; Muirhead mean; interaction operational laws; multi-attribute group decision-making

1. Introduction

As one of the most important branches of modern decision-making theory, multi-attribute group decision-making (MAGDM) has been widely investigated and successfully applied to many fields, owing to its high capacity of modelling the process of real decision-making problems [1–6]. With the development of management and economics, actual decision-making problems are becoming more and more diversified and complicated. Thus, one of the most significant issues is representing and denoting attribute values appropriately. Zadeh [7] originally introduced the fuzzy set (FS) theory, which makes it possible to describe vagueness and uncertainty. However, the shortcoming of the FS is that it only has a membership degree, making it insufficient to express fuzziness comprehensively. Recently, Atanassov [3] put forward the concept of an intuitionistic fuzzy set (IFS), which can express the complex fuzzy information effectively as it simultaneously has a membership degree and a non-membership degree. Considering its effective vagueness information processing capabilities, IFS has been widely investigated and applied to so many fields since its appearance. For instance, Liu and Ren [8] proposed a novel intuitionistic fuzzy entropy and based on which a novel approach to MAGDM was proposed. Ren and Wang [9] proposed a new similarity measure for interval-valued IFSs, which considers not only the impacts of membership and membership degrees but also the median point of interval-valued IFSs. Kaur and Garg [10] extended IFSs and proposed cubic intuitionistic fuzzy sets as well as their aggregation operators. P. Liu and X. Liu [11] proposed the concept of linguistic intuitionistic fuzzy sets based on the combination of IFSs and linguistic terms sets and applied them to MAGDM. Liu and Wang [12] extend partitioned Heronian mean operator to linguistic intuitionistic fuzzy sets and applied it to MAGDM. Lakshmana et al. [13] proposed a total order on the entire class of intuitionistic fuzzy numbers using an upper lower dense sequence in the interval [1]. Liu and Teng [14] proposed the concept of normal interval-valued intuitionistic fuzzy numbers and applied it to decision-making. Liu and Chen [15] introduced some intuitionistic fuzzy Heronian mean operators based on the Archimedean t-conorm and t-norm and applied them to dealing with MAGDM problems.

Recently, as an extension of the IFS, the Pythagorean fuzzy set (PFS) [16], which is also characterized by a membership degree and a non-membership degree, has been proposed. The prominent feature of the PFS that the sum of membership and non-membership degrees may be greater than one and their square sum should be less than or equal to one, makes the PFS more powerful and useful than the IFS. Since its appearance, it has drawn much attention. For example, Zhang [17] proposed a novel similarity measure for PFSs and based on which a new method to Pythagorean fuzzy MAGDM problems was developed. Zhang and Xu [18] and Ren et al. [19] respectively extended the traditional TOPSIS (technique for order preference by similarity to ideal solution) method and the TODIM (an acronym in Portuguese for interactive multi-criteria decision-making) approach to solve MAGDM in a Pythagorean fuzzy context. Aggregation operators are a central topic in MAGDM, as they can ingrate individual input data into collective ones, and rank the alternatives based on the collective value. In the past years, quite a few Pythagorean fuzzy operators have been proposed and been applied to MAGDM successfully [20–26]. However, the main shortcomings of these operators are:

(1) They cannot consider the interrelationship between Pythagorean fuzzy numbers (PFNs). In other words, these aggregation operators assume that the attributes are independent, signifying that the correlations among attribute values are not taken into consideration when aggregating them. Generally, the Bonferroni mean (BM) [27], Heronian mean (HM) [28], and Maclaurin symmetric mean (MSM) [29] are aggregation technologies that consider the interrelationships among arguments. Thus, in order to overcome the shortcoming of the aforementioned aggregation operators, some other Pythagorean fuzzy aggregation operators have been proposed. Liang et al. [30,31] proposed some Pythagorean fuzzy Bonferroni mean and geometric Bonferroni mean operators, respectively. Zhang et al. [32] investigated the generalized Bonferroni mean to aggregate Pythagorean fuzzy information and proposed a family of Pythagorean fuzzy generalized Bonferroni means. Wei and Lu [33], and Qin [34] proposed some Pythagorean fuzzy Maclaurin symmetric mean operators, respectively. These operators consider the interrelationships between any two or among multiple arguments, however, they fail to capture the interrelationships among all arguments. The Muirhead mean (MM) [35] is a useful and powerful aggregation technology that captures the interrelationships among all arguments. Moreover, it has a parameter vector that leads to flexible aggregation processes. Quite a few existing aggregation operators are some special cases of MM. The MM was introduced for crisp numbers and, up to now, MM has been investigated in intuitionistic fuzzy [36] and 2-tuple linguistic environments [37]. However, to the best of our knowledge, nothing has been done about MM in a Pythagorean fuzzy environment. Thus, in order to aggregate Pythagorean fuzzy information, it is necessary to extend the MM to a Pythagorean fuzzy environment

(2) The aforementioned aggregation operators are based on the traditional Pythagorean fuzzy operational rules introduced in [18]. However, these operations cannot be used to deal with some situations. For instance, let $p_1 = (\mu_1, v_1)$ and $p_2 = (\mu_2, v_2)$ be two PFNs, if $\mu_1 = 0$ and $\mu_2 \neq 0$, then according to the operational laws proposed by Zhang and Xu [18], we can obtain $\mu_{p_1 \oplus p_2} = 0$. It is noted that μ_2 is not accounted for at all. Similarly, if $v_1 = 0$ and $v_2 \neq 0$, then and v_2 is not accounted for at all. It is not consistent with our intuition and the reality. To overcome the drawback of the proposed operations, Wei [38] proposed the interaction operations for PFNs.

Therefore, to take full advantages of MM and Wei' [38] Pythagorean fuzzy interaction operations, we propose a family of Pythagorean fuzzy interaction Muirhead mean operators. Thus, the proposed operators not only capture the interrelationships among all input arguments, but also effectively handle situations in which a membership or non-membership degree of an attribute value is equal to one. It is worth pointing out that in [39], Zhu and Li also proposed some Pythagorean fuzzy Muirhead mean operators. However, the proposed operators in this paper are different from those proposed by Zhu and Li. The main difference is that Zhu and Li's [39] operators are based on the basic operational laws proposed in [18]. Therefore, Zhu and Li's [39] operators do not work for situations in which one membership degree or one non-membership degree is equal to one. Our operators are based on the interaction operational rules of PFNs, so that the proposed operators in this study are more powerful and flexible than Zhu and Li's operators. Further, based on the proposed aggregation operators, we propose a novel approach to MAGDM in which attribute values take the form of PFNs. The main aims and motivations of this paper are: (1) to develop a family of Pythagorean fuzzy Muirhead mean operations based on interaction operational laws; and (2) to propose a novel approach to MAGDM with Pythagorean fuzzy information. The rest of the paper is organized as follows. Section 2 recalls some basic concepts, such as PFS, MM, and the interaction operations of PFNs. Section 3 extends the MM to Pythagorean fuzzy environment and proposes the Pythagorean fuzzy interaction Muirhead mean (PFIMM) operator and the Pythagorean fuzzy interaction weighted Muirhead mean (PFIWMM) operator. Section 4 extends the DMM to aggregating Pythagorean fuzzy information and develops the Pythagorean fuzzy interaction dual Muirhead mean (PFIDMM) operator and the Pythagorean fuzzy interaction weighted dual Muirhead mean (PFIDWMM) operator. Section 5 develops a novel approach to MAGDM with Pythagorean fuzzy information based on the proposed operators. Section 6 provides a numerical example to illustrate the performance of the proposed method and the final section summarizes the whole paper.

2. Basic Concepts

In this section, we briefly review the concepts of IFS, PFS, and MM.

2.1. IFS and PFS

Definition 1 [3]. *An intuitionistic fuzzy set A with an object X is defined as follows:*

$$A = \{\langle x, \mu A(x), vA(x)\rangle \,|x \in X\} \tag{1}$$

where $\mu_A(x)$ and $v_A(x)$ represent the membership and non-membership degrees respectively, satisfying $\mu A(x) \in [0,1]$, $vA(x) \in [0,1]$ and $\mu A(x) + vA(x) \in [0,1]$, $\forall x \in X$. For convenience, $(\mu A(x), vA(x))$ is called an intuitionistic fuzzy number (IFN), which can be denoted by $\alpha = (\mu, v)$.

Yager [16] extended Atanassov's IFS and proposed the PFS.

Definition 2 [16]. *A Pythagorean fuzzy set P with an object X is defined as follows:*

$$P = \{\langle x, \mu_p(x), v_p(x)\rangle |x \in X\}, \tag{2}$$

where $\mu_p(x)$ and $v_p(x)$ are the membership degree the non-membership degree respectively, satisfying $\mu_p(x) \in [0,1]$, $v_p(x) \in [0,1]$ and $(\mu_P(x))^2 + (v_P(x))^2 \leq 1$, $\forall x \in X$. Then the hesitancy degree of P is defined as $\pi_P(x) = \sqrt{1 - (\mu_P(x))^2 - (v_P(x))^2}$, $\forall x \in X$. For convenience, $(\mu_p(x), v_p(x))$ is called a PFN, which can be denoted by $p = (\mu_P, v_P)$.

To compare two PFNs, Zhang and Xu [18] proposed a comparison law.

Definition 3 [18]. *Let $p = (\mu, v)$ be a PFN, then the score function of p is defined as $S(p) = \mu^2 - v^2$. For any two PFNs, $p_1 = (\mu_1, v_1)$ and $p_2 = (\mu_2, v_2)$, if $S(p_1) > S(p_2)$, then $p_1 > p_2$; if $S(p_1) = S(p_2)$, then $p_1 = p_2$.*

Moreover, Zhang and Xu [18] proposed some operations for PFNs.

Definition 4 [18]. *Let $p = (\mu, v)$, $p_1 = (\mu_1, v_1)$ and $p_2 = (\mu_2, v_2)$ be any three PFNs, and λ be a positive real number, then*

(1) $p_1 \oplus p_2 = \left(\sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2}, v_1 v_2\right)$,

(2) $p_1 \otimes p_2 = \left(\mu_1\mu_2, \sqrt{v_1^2 + v_2^2 - v_1^2 v_2^2}\right)$,

(3) $\lambda p = \left(\sqrt{1 - (1-\mu^2)^\lambda}, v^\lambda\right)$,

(4) $p^\lambda = \left(\mu^\lambda, \sqrt{1 - (1-v^2)^\lambda}\right)$.

However, the operational laws shown above cannot reflect the correlations between membership degrees and non-membership degrees. Thus, Wei [38] proposed some interaction operations for PFNs that are shown as the following.

Definition 5 [38]. *Let $p = (\mu, v)$,$p_1 = (\mu_1, v_1)$ and $p_2 = (\mu_2, v_2)$ be any of the three PFNs, and λ* be any positive real number, then

(1) $p_1 \oplus p_2 = \left(\sqrt{1 - (1-\mu_1^2)(1-\mu_2^2)}, \sqrt{(1-\mu_1^2)(1-\mu_2^2) - (1-\mu_1^2-v_1^2)(1-\mu_2^2-v_2^2)}\right)$,

(2) $p_1 \otimes p_2 = \left(\sqrt{(1-v_1^2)(1-v_2^2) - (1-\mu_1^2-v_1^2)(1-\mu_2^2-v_2^2)}, \sqrt{1 - (1-v_1^2)(1-v_2^2)}\right)$,

(3) $\lambda p = \left(\sqrt{1 - (1-\mu^2)^\lambda}, \sqrt{(1-\mu^2)^\lambda - (1-\mu^2-v^2)^\lambda}\right)$,

(4) $p^\lambda = \left(\sqrt{(1-v^2)^\lambda - (1-\mu^2-v^2)^\lambda}, \sqrt{1 - (1-v^2)^\lambda}\right)$.

2.2. The Muirhead Mean

The MM was introduced by Muirhead [35] for crisp numbers. The prominent advantage of the MM is that it can capture interrelationships among all of the aggregated arguments.

Definition 6 [35]. *Let $a_i (i = 1, 2, \cdots, n)$ be a collection of crisp numbers and $R = (r_1, r_2, \ldots, r_n) \in R^n$ be a vector of parameters, then the MM can be defined as*

$$MM^R(a_1, a_2, \ldots, a_n) = \left(\frac{1}{n!}\sum_{\vartheta \in S_n}\prod_{j=1}^{n} a_{\vartheta(j)}^{r_j}\right)^{\frac{1}{\sum_{j=1}^{n} r_j}} \tag{3}$$

where $\vartheta(j)(j = 1, 2, \cdots, n)$ is any permutation of $(1, 2, \ldots, n)$, S_n is the collection of $\vartheta(j)(j = 1, 2, \cdots, n)$.

Liu and Li [36] proposed the dual operator of MM, which is called the DMM operator.

Definition 7 [36]. *Let $a_i (i = 1, 2, \cdots, n)$ be a collection of crisp numbers and $P = (p_1, p_2, \cdots, p_n) \in R^n$ be a vector of parameters. If*

$$DMM^P(a_1, a_2, \ldots, a_n) = \frac{1}{\sum_{j=1}^{n} p_j}\left(\prod_{\vartheta \in S_n}\sum_{j=1}^{n}\left(p_j a_{\vartheta(j)}\right)\right)^{\frac{1}{n!}} \tag{4}$$

Then DMM^P is called the DMM, where $\vartheta(j)(j = 1, 2, \cdots, n)$ is any permutation of $(1, 2, \ldots, n)$ and S_n is the collection of $\vartheta(j)(j = 1, 2, \cdots, n)$.

3. The Pythagorean Fuzzy Interaction Muirhead Mean and the Pythagorean Fuzzy Interaction Weighted Muirhead Mean

In this section, we extend the MM to Pythagorean fuzzy environment and propose some new Pythagorean fuzzy aggregation operators.

3.1. The Pythagorean Fuzzy Interaction Muirhead Mean

Definition 8 . *Let $p_i(i = 1, 2, \ldots, n)$ be a collection of PFNs and $R = (r_1, r_2, \ldots, r_n) \in R^n$ be a vector of parameters. If*

$$PFIMM^R(p_1, p_2, \ldots, p_n) = \left(\frac{1}{n!} \sum_{\vartheta \in S_n} \prod_{j=1}^{n} p_{\vartheta(j)}^{r_j} \right)^{\frac{1}{\sum_{j=1}^{n} r_j}} \tag{5}$$

then $PFIMM^R$ is called the PFIMM, where $\vartheta(j)(j = 1, 2, \cdots, n)$ is any a permutation of $(1, 2, \cdots, n)$, and S_n is the collection of $\vartheta(j)(j = 1, 2, \cdots, n)$.

According to the interaction operations for PFNs presented in Definition 5, the following theorem can be obtained.

Theorem 1. *Let $p_i = (\mu_i, v_i)(i = 1, 2, \cdots, n)$ be a collection of PFNs, the aggregated value by using the PFIMM is still a PFN and*

$$\begin{gathered} PFIMM^R(p_1, p_2, \ldots, p_n) = \\ \left(\left(\begin{array}{l} \left(1 - \prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - v_{\vartheta(j)}^2\right)^{r_j} + \prod_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{r_j} \right)^{\frac{1}{n!}} + \prod_{\vartheta \in S_n} \prod_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{\frac{r_j}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} r_j}} \\ - \prod_{\vartheta \in S_n} \prod_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{\frac{r_j}{n! \sum_{j=1}^{n} r_j}} \end{array} \right)^{\frac{1}{2}}, \right. \\ \left. \left(1 - \left(1 - \prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - v_{\vartheta(j)}^2\right)^{r_j} + \prod_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{r_j} \right)^{\frac{1}{n!}} + \prod_{\vartheta \in S_n} \left(\prod_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{r_j} \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} r_j}} \right)^{\frac{1}{2}} \right) \end{gathered} \tag{6}$$

Proof. According to the Definition 5, we have

$$p_{\vartheta(j)}^{r_j} = \left(\sqrt{\left(1 - v_{\vartheta(j)}^2\right)^{r_j} - \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{r_j}}, \sqrt{1 - \left(1 - v_{\vartheta(j)}^2\right)^{r_j}} \right) \tag{7}$$

and,

$$\prod_{j=1}^{n} p_{\vartheta(j)}^{r_j} = \left(\sqrt{\prod_{j=1}^{n} \left(1 - v_{\vartheta(j)}^2\right)^{r_j} - \prod_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{r_j}}, \sqrt{1 - \prod_{j=1}^{n} \left(1 - v_{\vartheta(j)}^2\right)^{r_j}} \right) \tag{8}$$

Then,

$$\sum_{\vartheta\in S_n}\prod_{j=1}^{n}p_{\vartheta(j)}^{r_j}=\left(\sqrt{1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^2\right)^{r_j}+\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{r_j}\right)},\right.$$
$$\left.\sqrt{\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^2\right)^{r_j}+\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{r_j}\right)-\prod_{\vartheta\in S_n}\left(\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{r_j}\right)}\right) \tag{9}$$

Further,

$$\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}p_{\vartheta(j)}^{r_j}=\left(\sqrt{1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^2\right)^{r_j}+\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{r_j}\right)^{\frac{1}{n!}}},\right.$$
$$\left.\sqrt{\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^2\right)^{r_j}+\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{r_j}\right)^{\frac{1}{n!}}-\prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{\frac{r_j}{n!}}}\right) \tag{10}$$

Moreover,

$$\left(\frac{1}{n!}\sum_{\vartheta\in S_n}\prod_{j=1}^{n}p_{\vartheta(j)}^{r_j}\right)^{\frac{1}{\sum_{j=1}^{n}r_j}}=$$
$$\left(\left(\begin{array}{l}\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^2\right)^{r_j}+\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{r_j}\right)^{\frac{1}{n!}}+\prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{\frac{r_j}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}r_j}}\\ -\prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{\frac{r_j}{n!\sum_{j=1}^{n}r_j}}\end{array}\right)^{\frac{1}{2}},\right.$$
$$\left.\left(1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^2\right)^{r_j}+\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{r_j}\right)^{\frac{1}{n!}}+\prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{\frac{r_j}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}r_j}}\right)^{\frac{1}{2}}\right) \tag{11}$$

Hence, Equation (6) is maintained.
For convenience, let

$$\mu=\left(\begin{array}{l}\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^2\right)^{r_j}+\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{r_j}\right)^{\frac{1}{n!}}+\prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{\frac{r_j}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}r_j}}\\ -\prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{\frac{r_j}{n!\sum_{j=1}^{n}r_j}}\end{array}\right)^{\frac{1}{2}}$$

and

$$v=\left(1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^2\right)^{r_j}+\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{r_j}\right)^{\frac{1}{n!}}+\prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{\frac{r_j}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}r_j}}\right)^{\frac{1}{2}}$$

Evidently,

$$0\leq\mu_{\vartheta(j)}\leq 1,\ 0\leq v_{\vartheta(j)}\leq 1,\ 0\leq\mu_{\vartheta(j)}^2+v_{\vartheta(j)}^2\leq 1, \tag{12}$$

and,

$$0\leq\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^2\right)^{r_j}\leq 1,\ \text{and}\ 0\leq\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{r_j}\leq 1. \tag{13}$$

Then,

$$0 \leq 1-\left(\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^{2}\right)^{r_j}-\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^{2}-v_{\vartheta(j)}^{2}\right)^{r_j}\right) \leq 1 \tag{14}$$

Further,

$$0 \leq 1-\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^{2}\right)^{r_j}+\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^{2}-v_{\vartheta(j)}^{2}\right)^{r_j} \leq 1 \tag{15}$$

and,

$$0 \leq \prod_{\vartheta \in S_n}\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^{2}\right)^{r_j}+\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^{2}-v_{\vartheta(j)}^{2}\right)^{r_j}\right)^{\frac{1}{n!}} \leq 1,\ 0 \leq \prod_{\vartheta \in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^{2}-v_{\vartheta(j)}^{2}\right)^{\frac{r_j}{n!}} \leq 1. \tag{16}$$

Moreover,

$$0 \leq \left(1-\prod_{\vartheta \in S_n}\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^{2}\right)^{r_j}+\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^{2}-v_{\vartheta(j)}^{2}\right)^{r_j}\right)^{\frac{1}{n!}}+\prod_{\vartheta \in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^{2}-v_{\vartheta(j)}^{2}\right)^{\frac{r_j}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} r_j}} \leq 1 \tag{17}$$

and,

$$0 \leq \prod_{\vartheta \in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^{2}-v_{\vartheta(j)}^{2}\right)^{\frac{r_j}{n!\sum_{j=1}^{n} r_j}} \leq 1 \tag{18}$$

Therefore,

$$0 \leq \left(\begin{array}{l}\left(1-\prod_{\vartheta \in S_n}\left(1-\prod_{j=1}^{n}\left(1-v_{\vartheta(j)}^{2}\right)^{r_j}+\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^{2}-v_{\vartheta(j)}^{2}\right)^{r_j}\right)^{\frac{1}{n!}}+\prod_{\vartheta \in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^{2}-v_{\vartheta(j)}^{2}\right)^{\frac{r_j}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} r_j}} \\ -\prod_{\vartheta \in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^{2}-v_{\vartheta(j)}^{2}\right)^{\frac{r_j}{n!\sum_{j=1}^{n} r_j}}\end{array}\right)^{\frac{1}{2}} \leq 1$$

Therefore, $0 \leq \mu \leq 1$. Similarly, we can get $0 \leq v \leq 1$.
Then,

$$\mu^2+v^2=1-\prod_{\vartheta \in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^{2}-v_{\vartheta(j)}^{2}\right)^{\frac{r_j}{n!\sum_{j=1}^{n} r_j}}$$

We have proved that

$$0 \leq \prod_{\vartheta \in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^{2}-v_{\vartheta(j)}^{2}\right)^{\frac{r_j}{n!}} \leq 1$$

Thus,

$$0 \leq \prod_{\vartheta \in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^{2}-v_{\vartheta(j)}^{2}\right)^{\frac{r_j}{n!\sum_{j=1}^{n} r_j}} \leq 1, \text{ and } 0 \leq 1-\prod_{\vartheta \in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^{2}-v_{\vartheta(j)}^{2}\right)^{\frac{r_j}{n!\sum_{j=1}^{n} r_j}} \leq 1$$

Therefore, $0 \leq \mu^2+v^2 \leq 1$, which completes the proof. □

Moreover, the PFIMM has the following properties.

Theorem 2. *(Idempotency) If all of the* $p_i (i = 1, 2, \cdots, n)$ *are equal, i.e.,* $p_i = p = (\mu, v)$*, then*

$$PFIMM^R(p_1, p_2, \cdots, p_n) = p \tag{19}$$

Proof. According to Theorem 1, we can get

$$\begin{aligned}
&PFIMM^R(p, p, \cdots, p) \\
&= \left(\left(\begin{array}{l} \left(1 - \prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - v^2\right)^{r_j} + \prod_{j=1}^{n} \left(1 - \mu^2 - v^2\right)^{r_j}\right)^{\frac{1}{n!}} + \prod_{\vartheta \in S_n} \prod_{j=1}^{n} \left(1 - \mu^2 - v^2\right)^{\frac{r_j}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} r_j}} \\ - \prod_{\vartheta \in S_n} \prod_{j=1}^{n} \left(1 - \mu^2 - v^2\right)^{\frac{r_j}{n! \sum_{j=1}^{n} r_j}} \end{array} \right)^{\frac{1}{2}}, \right. \\
&\quad \left. \left(1 - \left(1 - \prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - v^2\right)^{r_j} + \prod_{j=1}^{n} \left(1 - \mu^2 - v^2\right)^{r_j}\right)^{\frac{1}{n!}} + \prod_{\vartheta \in S_n} \prod_{j=1}^{n} \left(1 - \mu^2 - v^2\right)^{\frac{r_j}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} r_j}}\right)\right)^{\frac{1}{2}} \right) \\
&= \left(\sqrt{\left(1 - \prod_{\vartheta \in S_n} \left(1 - \left(1 - v^2\right)^{\sum_{j=1}^{n} r_j} + \left(1 - \mu^2 - v^2\right)^{\sum_{j=1}^{n} r_j}\right)^{\frac{1}{n!}} + \prod_{\vartheta \in S_n} \left(1 - \mu^2 - v^2\right)^{\frac{\sum_{j=1}^{n} r_j}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} r_j}} - \prod_{\vartheta \in S_n} \left(1 - \mu^2 - v^2\right)^{\frac{1}{n!}}}, \right. \\
&\quad \left. \sqrt{1 - \left(1 - \prod_{\vartheta \in S_n} \left(1 - \left(1 - v^2\right)^{\sum_{j=1}^{n} r_j} + \left(1 - \mu^2 - v^2\right)^{\sum_{j=1}^{n} r_j}\right)^{\frac{1}{n!}} + \prod_{\vartheta \in S_n} \left(1 - \mu^2 - v^2\right)^{\frac{\sum_{j=1}^{n} r_j}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} r_j}}} \right) \\
&= \left(\sqrt{\left(1 - \left(1 - \left(1 - v^2\right)^{\sum_{j=1}^{n} r_j} + \left(1 - \mu^2 - v^2\right)^{\sum_{j=1}^{n} r_j}\right) + \left(1 - \mu^2 - v^2\right)^{\sum_{j=1}^{n} r_j}\right)^{\frac{1}{\sum_{j=1}^{n} r_j}} - \left(\left(1 - \mu^2 - v^2\right)^{\sum_{j=1}^{n} r_j}\right)^{\frac{1}{\sum_{j=1}^{n} r_j}}}, \right. \\
&\quad \left. \sqrt{1 - \left(1 - \left(1 - \left(1 - v^2\right)^{\sum_{j=1}^{n} r_j} + \left(1 - \mu^2 - v^2\right)^{\sum_{j=1}^{n} r_j}\right) + \left(1 - \mu^2 - v^2\right)^{\sum_{j=1}^{n} r_j}\right)^{\frac{1}{\sum_{j=1}^{n} r_j}}} \right) \\
&= \left(\sqrt{\left(\left(1 - v^2\right)^{\sum_{j=1}^{n} r_j}\right)^{\frac{1}{\sum_{j=1}^{n} r_j}} - \left(\left(1 - \mu^2 - v^2\right)^{\sum_{j=1}^{n} r_j}\right)^{\frac{1}{\sum_{j=1}^{n} r_j}}}, \sqrt{1 - \left(\left(1 - v^2\right)^{\sum_{j=1}^{n} r_j}\right)^{\frac{1}{\sum_{j=1}^{n} r_j}}} \right) \\
&= \left(\sqrt{(1 - v^2) - (1 - \mu^2 - v^2)}, \sqrt{1 - (1 - v^2)} \right) = \left(\sqrt{\mu^2}, \sqrt{v^2} \right) = (\mu, v).
\end{aligned}$$

The parameter vector R of PFIMM plays an important role in the final result. In the following, we explore some special cases of PFIMM. □

Case 1: If $R = (1, 0, \ldots, 0)$, then the PFIMM is reduced to the following

$$PFIMM^{(1,0,0,\ldots,0)}(p_1, p_2, \cdots, p_n) = \left(\sqrt{1 - \prod_{j=1}^{n} \left(1 - \mu_i^2\right)^{\frac{1}{n}}}, \sqrt{\prod_{j=1}^{n} \left(1 - \mu_i^2\right)^{\frac{1}{n}} - \prod_{j=1}^{n} \left(1 - \mu_i^2 - v_i^2\right)^{\frac{1}{n}}} \right) = \frac{1}{n} \sum_{i=1}^{n} p_i \tag{20}$$

which is the Pythagorean fuzzy interaction averaging (PFIA) operator.
Case 2: If $R = (\lambda, 0, \ldots, 0)$, then the PFIMM is reduced to the following

$$\begin{aligned}
PFIMM^{(\lambda,0,0,\ldots,0)}(p_1, p_2, \ldots, p_n) = &\left(\sqrt{\left(1 - \left(1 - \prod_{j=1}^{n} \left(1 - v_i^2\right)^{\lambda} + \prod_{j=1}^{n} \left(1 - \mu_i^2 - v_i^2\right)^{\lambda}\right)^{\frac{1}{n}}\right)^{\frac{1}{\lambda}} - \prod_{j=1}^{n} \left(1 - \mu_i^2 - v_i^2\right)^{\frac{1}{n}}}, \right. \\
&\left. \sqrt{1 - \left(1 - \left(1 - \prod_{j=1}^{n} \left(1 - v_i^2\right)^{\lambda} + \prod_{j=1}^{n} \left(1 - \mu_i^2 - v_i^2\right)^{\lambda}\right)^{\frac{1}{n}} + \prod_{j=1}^{n} \left(1 - \mu_i^2 - v_i^2\right)^{\frac{\lambda}{n}}\right)^{\frac{1}{\lambda}}} \right) = \left(\frac{1}{n} \sum_{i=1}^{n} p_i^{\lambda}\right)^{\frac{1}{\lambda}},
\end{aligned} \tag{21}$$

which is the generalized Pythagorean fuzzy interaction averaging (GPFIA) operator.
Case 3: If $R = (1, 1, 0, 0, \cdots, 0)$, then the PFIMM is reduced to the following

$$\begin{gathered} PFIMM^{(1,1,0,0,\ldots,0)}(p_1, p_2, \ldots, p_n) = \\ \left(\begin{array}{l} \left(1 - \prod\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - v_i^2)\left(1 - v_j^2\right) + \left(1 - \mu_i^2 - v_j^2\right)\left(1 - \mu_j^2 - v_j^2\right) \right)^{\frac{1}{n(n-1)}} + \prod\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(1 - \mu_i^2 - v_j^2\right)\left(1 - \mu_j^2 - v_j^2\right) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}} \\ - \prod\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\left(1 - \mu_i^2 - v_j^2\right)\left(1 - \mu_j^2 - v_j^2\right) \right)^{\frac{1}{2n(n-1)}} \end{array} \right)^{\frac{1}{2}}, \\ \left(1 - \left(1 - \prod\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - v_i^2)\left(1 - v_j^2\right) + (1 - \mu_i^2 - v_i^2)\left(1 - \mu_j^2 - v_j^2\right) \right)^{\frac{1}{n(n-1)}} + \prod\limits_{\substack{i,j=1 \\ i \neq j}}^{n} \left((1 - \mu_i^2 - v_i^2)\left(1 - \mu_j^2 - v_j^2\right) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\ = \left(\frac{1}{n(n-1)} \sum\limits_{\substack{i,j=1 \\ i \neq j}}^{n} p_i p_j \right)^{\frac{1}{2}}, \end{gathered} \tag{22}$$

which is the Pythagorean fuzzy interaction BM (PFIBM) operator.
Case 4: If $R = \left(\overbrace{1, 1, \cdots, 1}^{k}, \overbrace{0, 0, \cdots, 0}^{n-k} \right)$, then the PFIMM is reduced to the following

$$\begin{gathered} PFIMM^{\left(\overbrace{1, 1, \cdots, 1}^{k}, \overbrace{0, 0, \cdots, 0}^{n-k} \right)}(p_1, p_2, \ldots, p_n) = \\ \left(\left(\begin{array}{l} \left(1 - \prod\limits_{1 \leq i_1 \prec \cdots \prec i_k \leq n} \left(1 - \prod\limits_{j=1}^{k} \left(1 - v_{i_j}\right)^2 + \prod\limits_{j=1}^{n} \left(1 - \mu_{i_j} - v_{i_j}\right)^2 \right)^{\frac{1}{C_n^k}} + \prod\limits_{1 \leq i_1 \prec \cdots \prec i_k \leq n} \prod\limits_{j=1}^{n} \left(1 - \mu_{i_j} - v_{i_j}\right)^{\frac{2}{C_n^k}} \right)^{\frac{1}{k}} \\ - \prod\limits_{1 \leq i_1 \prec \cdots \prec i_k \leq n} \prod\limits_{j=1}^{n} \left(1 - \mu_{i_j} - v_{i_j}\right)^{\frac{2}{kC_n^k}} \end{array} \right)^{\frac{1}{2}}, \right. \\ \left. \left(1 - \left(1 - \prod\limits_{1 \leq i_1 \prec \cdots \prec i_k \leq n} \left(1 - \prod\limits_{j=1}^{k} \left(1 - v_{i_j}\right)^2 + \prod\limits_{j=1}^{n} \left(1 - \mu_{i_j} - v_{i_j}\right)^2 \right)^{\frac{1}{C_n^k}} + \prod\limits_{1 \leq i_1 \prec \cdots \prec i_k \leq n} \prod\limits_{j=1}^{n} \left(1 - \mu_{i_j} - v_{i_j}\right)^{\frac{2}{C_n^k}} \right)^{\frac{1}{k}} \right)^{\frac{1}{2}} \right) \\ = \left(\frac{\bigoplus\limits_{1 \leq i_1 \prec \ldots \prec i_k \leq n} \bigotimes\limits_{j=1}^{k} p_{i_j}}{C_n^k} \right)^{\frac{1}{k}}, \end{gathered} \tag{23}$$

which is the Pythagorean fuzzy interaction Maclaurin symmetric mean (PFIMSM) operator.
Case 5: If $R = (1, 1, \cdots, 1)$, then the PFIMM is reduced to the following

$$PFIMM^{(1,1,\cdots,1)}(p_1, p_2, \cdots, p_n) = \left(\sqrt{\prod_{i=1}^{n} (1 - v_i^2)^{\frac{1}{n}} - \prod_{i=1}^{n} (1 - \mu_i^2 - v_i^2)^{\frac{1}{n}}}, \sqrt{1 - \prod_{i=1}^{n} (1 - v_i^2)^{\frac{1}{n}}} \right) = \left(\prod_{i=1}^{n} p_i \right)^{\frac{1}{n}} \tag{24}$$

which is the Pythagorean fuzzy interaction geometric averaging (PFIGA) operator.
Case 6: If $R = (1/n, 1/n, \ldots, 1/n)$, then the PFIMM is reduced to the PFIGA operator, which is shown as Equation (24).

3.2. The Pythagorean Fuzzy Interaction Weighted Muirhead Mean

Evidently, the main drawback of the PFIMM is that it cannot take the weights of arguments into consideration. Therefore, we propose the PFIWMM.

Definition 9 . *Let* $p_i = (\mu_i, v_i)(i = 1, 2, \cdots, n)$ *be a collection of PFNs,* $w = (w_1, w_2, \cdots, w_n)^T$ *be the weight vector of* $p_i(i = 1, 2, \cdots, n)$*, satisfying* $w_i \in [0, 1]$ *and* $\sum\limits_{i=1}^{n} w_i = 1$*. Let* $R = (r_1, r_2, \ldots, r_n) \in R^n$ *be a vector of parameter. If*

$$PFIWMM^R(r_1, r_2, \ldots, r_n) = \left(\frac{1}{n!} \sum_{\vartheta \in S_n} \prod_{j=1}^{n} \left(n w_{\vartheta(j)} p_{\vartheta(j)} \right)^{r_j} \right)^{\frac{1}{\sum_{j=1}^{n} r_j}} \tag{25}$$

then we call $PFIWMM^R$ *the PFIWMM operator, where* $\vartheta(j) = (j = 1, 2, \ldots, n)$ *is any a permutation of* $(1, 2, \ldots, n)$*, and* S_n *is the collection of all permutations of* $(1, 2, \ldots, n)$*.*

According to Definition 5, we can get the following theorem.

Theorem 3. *Let* $p_i = (\mu_i, v_i)(i = 1, 2, \cdots, n)$ *be a collection of PFNs, then the aggregated value by the PFIWMM is still a PFN and*

$$\begin{aligned} &PFIWMM^R(p_1, p_2, \cdots, p_n) = \\ &\left(\left(\left(1 - \prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - \mu_{\vartheta(j)}^2 \right)^{n w_{\vartheta(j)}} + \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2 \right)^{n w_{\vartheta(j)}} \right)^{r_j} + \prod_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2 \right)^{n w_{\vartheta(j)} r_j} \right)^{\frac{1}{n!}} + \prod_{\vartheta \in S_n} \prod_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2 \right)^{\frac{n w_{\vartheta(j)} r_j}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} r_j}} - \prod_{\vartheta \in S_n} \prod_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2 \right)^{\frac{n w_{\vartheta(j)} r_j}{n! \sum_{j=1}^{n} r_j}} \right)^{\frac{1}{2}}, \right. \\ &\left. \left(1 - \left(1 - \left(\prod_{\vartheta \in S_n} \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - \mu_{\vartheta(j)}^2 \right)^{n w_{\vartheta(j)}} + \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2 \right)^{n w_{\vartheta(j)}} \right)^{r_j} + \prod_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2 \right)^{n w_{\vartheta(j)} r_j} \right) \right)^{\frac{1}{n!}} + \prod_{\vartheta \in S_n} \prod_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2 \right)^{\frac{n w_{\vartheta(j)} r_j}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} r_j}} \right)^{\frac{1}{2}} \right) \end{aligned} \tag{26}$$

The proof of Theorem 3 is similar to that of Theorem 1, which is omitted here in order to save space.

4. The Pythagorean Fuzzy Interaction Dual Muirhead Mean and the Pythagorean Fuzzy Interaction Weighted Dual Muirhead Mean

4.1. The Pythagorean Fuzzy Interaction Dual Muirhead Mean Operator

Definition 10 . *Let* $p_i = (\mu_i, v_i)(i = 1, 2, \cdots, n)$ *be a collection of PFNs, and* $R = (r_1, r_2, \ldots, r_n) \in R^n$ *be a vector of parameters. If*

$$PFIDMM^R(p_1, p_2, \cdots, p_n) = \frac{1}{\sum\limits_{j=1}^{n} r_j} \left(\prod_{\vartheta \in S_n} \sum_{j=1}^{n} \left(r_j p_{\vartheta(j)} \right) \right)^{\frac{1}{n!}} \tag{27}$$

then we call $PFIDMM^R$ *the PFIDMM operator, where* $\vartheta(j) = (j = 1, 2, \ldots, n)$ *is any a permutation of* $(1, 2, \ldots, n)$ *and* S_n *is the collection of all permutations of* $(1, 2, \ldots, n)$*.*

Theorem 4. *Let* $p_i = (\mu_i, v_i)(i = 1, 2, \ldots, n)$ *be a collection of all permutations of PFNs, the aggregated value by the PFIDMM is also a PFN and*

$$PFIDMM^{R}(p_1,p_2,\cdots,p_n)=$$
$$\left(\left(1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2\right)^{r_j}+\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{r_j}\right)^{\frac{1}{n!}}+\prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{\frac{r_j}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}r_j}}\right)^{\frac{1}{2}},\right.$$
$$\left.\left(\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2\right)^{r_j}+\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{r_j}\right)^{\frac{1}{n!}}+\prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{\frac{r_j}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}r_j}}-\prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{\frac{r_j}{n!\sum_{j=1}^{n}r_j}}\right)^{\frac{1}{2}}\right) \tag{28}$$

Proof. According to the operational laws of PFNs in Definition 5, we can get

$$r_jp_{\vartheta(j)}=\left(\sqrt{1-\left(1-\mu_{\vartheta(j)}^2\right)^{r_j}},\sqrt{\left(1-\mu_{\vartheta(j)}^2\right)^{r_j}-\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{r_j}}\right) \tag{29}$$

and,

$$\sum_{j=1}^{n}(r_jp_{\vartheta(j)})=\left(\sqrt{1-\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2\right)^{r_j}},\sqrt{\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2\right)^{r_j}-\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{r_j}}\right) \tag{30}$$

Therefore,

$$\prod_{\vartheta\in S_n}\sum_{j=1}^{n}(r_jp_{\vartheta(j)})=\left(\sqrt{\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2\right)^{r_j}+\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{r_j}\right)-\prod_{\vartheta\in S_n}\left(\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{r_j}\right)},\right.$$
$$\left.\sqrt{1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2\right)^{r_j}+\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{r_j}\right)}\right). \tag{31}$$

Further,

$$\left(\prod_{\vartheta\in S_n}\sum_{j=1}^{n}\left(r_jp_{\vartheta(j)}\right)\right)^{\frac{1}{n!}}=$$
$$\left(\sqrt{\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2\right)^{r_j}+\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{r_j}\right)^{\frac{1}{n!}}-\prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{\frac{r_j}{n!}}},\right.$$
$$\left.\sqrt{1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2\right)^{r_j}+\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{r_j}\right)^{\frac{1}{n!}}}\right). \tag{32}$$

Therefore,

$$\frac{1}{\sum_{j=1}^{n}r_j}\left(\prod_{\vartheta\in S_n}\sum_{j=1}^{n}(r_jp_{\vartheta(j)})\right)^{\frac{1}{n!}}=$$
$$\left(\left(1-\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2\right)^{r_j}+\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{r_j}\right)^{\frac{1}{n!}}+\prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{\frac{r_j}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}r_j}}\right)^{\frac{1}{2}},\right.$$
$$\left.\left(\left(1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2\right)^{r_j}+\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{r_j}\right)^{\frac{1}{n!}}+\prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{\frac{r_j}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}r_j}}-\prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{\frac{r_j}{n!\sum_{j=1}^{n}r_j}}\right)^{\frac{1}{2}}\right) \tag{33}$$

Therefore, Equation (28) is kept.

In the following, we prove the aggregated value is a PFN. For convenience, let

$$\mu = \left(1 - \left(1 - \prod_{\vartheta\in S_n}\left(1 - \prod_{j=1}^{n}\left(1-\mu^2_{\vartheta(j)}\right)^{r_j} + \prod_{j=1}^{n}\left(1-\mu^2_{\vartheta(j)}-v^2_{\vartheta(j)}\right)^{r_j}\right)^{\frac{1}{n!}} + \prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu^2_{\vartheta(j)}-v^2_{\vartheta(j)}\right)^{\frac{r_j}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} r_j}}\right)^{\frac{1}{2}}$$

$$v = \left(\begin{array}{l}\left(1 - \prod_{\vartheta\in S_n}\left(1 - \prod_{j=1}^{n}\left(1-\mu^2_{\vartheta(j)}\right)^{r_j} + \prod_{j=1}^{n}\left(1-\mu^2_{\vartheta(j)}-v^2_{\vartheta(j)}\right)^{r_j}\right)^{\frac{1}{n!}} + \prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu^2_{\vartheta(j)}-v^2_{\vartheta(j)}\right)^{\frac{r_j}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} r_j}} \\ -\prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu^2_{\vartheta(j)}-v^2_{\vartheta(j)}\right)^{\frac{r_j}{n!\sum_{j=1}^{n} r_j}}\end{array}\right)^{\frac{1}{2}}.$$

Evidently,

$$\mu_{\vartheta(j)} \in [0,1], v_{\vartheta(j)} \in [0,1],\ 0 \le \mu^2_{\vartheta(j)} + v^2_{\vartheta(j)} \le 1. \tag{34}$$

Therefore,

$$0 \le \left(1-\mu^2_{\vartheta(j)}\right)^{r_j} \le 1, 0 \le \left(1-v^2_{\vartheta(j)}\right)^{r_j} \le 1, 0 \le \left(1-\mu^2_{\vartheta(j)}-v^2_{\vartheta(j)}\right)^{r_j} \le 1 \tag{35}$$

Further,

$$0 \le \prod_{j=1}^{n}\left(1-\mu^2_{\vartheta(j)}-v^2_{\vartheta(j)}\right)^{r_j} \le 1,\ 0 \le \prod_{j=1}^{n}\left(1-\mu^2_{\vartheta(j)}\right)^{r_j} - \prod_{j=1}^{n}\left(1-\mu^2_{\vartheta(j)}-v^2_{\vartheta(j)}\right)^{r_j} \le 1. \tag{36}$$

Thus,

$$0 \le 1 - \prod_{j=1}^{n}\left(1-\mu^2_{\vartheta(j)}\right)^{r_j} + \prod_{j=1}^{n}\left(1-\mu^2_{\vartheta(j)}-v^2_{\vartheta(j)}\right)^{r_j} \le 1. \tag{37}$$

Further,

$$0 \le \prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu^2_{\vartheta(j)}-v^2_{\vartheta(j)}\right)^{\frac{r_j}{n!}} \le 1, 0 \le \prod_{\vartheta\in S_n}\left(1 - \prod_{j=1}^{n}\left(1-\mu^2_{\vartheta(j)}\right)^{r_j} + \prod_{j=1}^{n}\left(1-\mu^2_{\vartheta(j)}-v^2_{\vartheta(j)}\right)^{r_j}\right)^{\frac{1}{n!}} \le 1. \tag{38}$$

Moreover,

$$0 \le \left(1 - \prod_{\vartheta\in S_n}\left(1 - \prod_{j=1}^{n}\left(1-\mu^2_{\vartheta(j)}\right)^{r_j} + \prod_{j=1}^{n}\left(1-\mu^2_{\vartheta(j)}-v^2_{\vartheta(j)}\right)^{r_j}\right)^{\frac{1}{n!}} + \prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu^2_{\vartheta(j)}-v^2_{\vartheta(j)}\right)^{\frac{r_j}{n!}}\right) \le 1. \tag{39}$$

Therefore,

$$0 \le 1 - \left(1 - \prod_{\vartheta\in S_n}\left(1 - \prod_{j=1}^{n}\left(1-\mu^2_{\vartheta(j)}\right)^{r_j} + \prod_{j=1}^{n}\left(1-\mu^2_{\vartheta(j)}-v^2_{\vartheta(j)}\right)^{r_j}\right)^{\frac{1}{n!}} + \prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu^2_{\vartheta(j)}-v^2_{\vartheta(j)}\right)^{\frac{r_j}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} r_j}} \le 1 \tag{40}$$

i.e., $0 \le \mu \le 1$. Similarly, we can get $0 \le v \le 1$.

Moreover,

$$\mu^2 + v^2 = 1 - \prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu^2_{\vartheta(j)}-v^2_{\vartheta(j)}\right)^{\frac{r_j}{n!\sum_{j=1}^{n} r_j}} \tag{41}$$

As we have proved that

$$0 \le \prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu^2_{\vartheta(j)}-v^2_{\vartheta(j)}\right)^{\frac{r_j}{n!}} \le 1 \tag{42}$$

Therefore,

$$\prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^{2}-v_{\vartheta(j)}^{2}\right)^{\frac{r_j}{n!\sum_{j=1}^{n}r_j}},\ 0\leq 1-\prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^{2}-v_{\vartheta(j)}^{2}\right)^{\frac{r_j}{n!\sum_{j=1}^{n}r_j}}\leq 1. \tag{43}$$

Therefore, $0 \leq \mu^2 + v^2 \leq 1$, which completes the proof. □

Moreover, the PFIDMM has the following properties.

Theorem 5. *(Idempotency) If all* $p_i = (i = 1, 2, \ldots, n)$ *are equal, i.e.,* $p_i = p = (\mu, v)$*, then*

$$PFIDMM^{R} = (p_1, p_2, \ldots, p_n) = p \tag{44}$$

In the following, we investigate some special cases of PFIDMM with respect to R.
Case 1: If $R = (1, 0, \cdots, 0)$, the PFIDMM is reduced to the following

$$PFIDMM^{(1,0,\cdots,0)}(p_1,p_2,\ldots,p_n)=\left(\sqrt{\prod_{j=1}^{n}\left(1-v_i^2\right)^{\frac{1}{n}}-\prod_{j=1}^{n}\left(1-\mu_i^2-v_i^2\right)^{\frac{1}{n}}},\sqrt{1-\prod_{j=1}^{n}\left(1-v_i^2\right)^{\frac{1}{n}}}\right) \tag{45}$$

which is the Pythagorean fuzzy interaction arithmetic averaging operator.
Case 2: If $R = (\lambda, 0, \cdots, 0)$, the PFIDMM is reduced to the following

$$\begin{aligned}PFIDMM^{(\lambda,0,\cdots,0)}(p_1,p_2,\ldots,p_n)=&\left(\sqrt{1-\left(1-\left(1-\prod_{j=1}^{n}\left(1-\mu_i^2\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\mu_i^2-v_i^2\right)^{\lambda}\right)^{\frac{1}{n}}+\prod_{j=1}^{n}\left(1-\mu_i^2-v_i^2\right)^{\frac{\lambda}{n}}\right)^{\frac{1}{\lambda}}},\right.\\&\left.\sqrt{\left(1-\left(1-\prod_{j=1}^{n}\left(1-\mu_i^2\right)^{\lambda}+\prod_{j=1}^{n}\left(1-\mu_i^2-v_i^2\right)^{\lambda}\right)^{\frac{1}{n}}\right)^{\frac{1}{\lambda}}-\prod_{j=1}^{n}\left(1-\mu_i^2-v_i^2\right)^{\frac{1}{n}}}\right),\end{aligned} \tag{46}$$

which is the Pythagorean fuzzy interaction generalized arithmetic averaging operator.
Case 3: If $R = (1, 1, 0, 0, \cdots, 0)$, the PFIDMM is reduced to the following

$$\begin{aligned}&PFIDMM^{(1,1,0,0,\cdots,0)}(p_1,p_2,\ldots,p_n)=\\&\left(\left(\left(1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-\mu_i^2\right)\left(1-\mu_j^2\right)+\left(1-\mu_i^2-v_i^2\right)\left(1-\mu_j^2-v_j^2\right)\right)^{\frac{1}{n(n-1)}}+\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(1-\mu_i^2-v_i^2\right)\left(1-\mu_j^2-v_j^2\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right),\right.\\&\left.\left(\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-\mu_i^2\right)\left(1-\mu_j^2\right)+\left(1-\mu_i^2-v_j^2\right)\left(1-\mu_j^2-v_j^2\right)\right)^{\frac{1}{n(n-1)}}+\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(1-\mu_i^2-v_j^2\right)\left(1-\mu_j^2-v_j^2\right)\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right.\right.\\&\left.\left.-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(\left(1-\mu_i^2-v_j^2\right)\left(1-\mu_j^2-v_j^2\right)\right)^{\frac{1}{2n(n-1)}}\right)^{\frac{1}{2}}\right),\end{aligned} \tag{47}$$

which is the Pythagorean fuzzy interaction arithmetic BM operator.
Case 4: If $R = \left(\overbrace{1, 1, \cdots, 1}^{k}, \overbrace{0, 0, \cdots, 0}^{n-k}\right)$, the PFIDMM is reduced to the following

$$PFIDMM^{\left(\overbrace{1,1,\cdots,1}^{k},\overbrace{0,0,\cdots,0}^{n-k}\right)}(p_1,p_2,\ldots,p_n)=$$
$$\left(\left(1-\left(1-\prod_{1\le i_1\prec\cdots\prec i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\mu_{i_j}\right)^2+\prod_{j=1}^{n}\left(1-\mu_{i_j}-v_{i_j}\right)^2\right)^{\frac{1}{C_n^k}}+\prod_{1\le i_1\prec\cdots\prec i_k\le n}\prod_{j=1}^{n}\left(1-\mu_{i_j}-v_{i_j}\right)^{\frac{2}{C_n^k}}\right)^{\frac{1}{k}}\right)^{\frac{1}{2}},\right.$$
$$\left.\left(\begin{array}{l}\left(1-\prod_{1\le i_1\prec\cdots\prec i_k\le n}\left(1-\prod_{j=1}^{k}\left(1-\mu_{i_j}\right)^2+\prod_{j=1}^{n}\left(1-\mu_{i_j}-v_{i_j}\right)^2\right)^{\frac{1}{C_n^k}}+\prod_{1\le i_1\prec\cdots\prec i_k\le n}\prod_{j=1}^{n}\left(1-\mu_{i_j}-v_{i_j}\right)^{\frac{2}{C_n^k}}\right)^{\frac{1}{k}}\\-\prod_{1\le i_1\prec\cdots\prec i_k\le n}\prod_{j=1}^{n}\left(1-\mu_{i_j}-v_{i_j}\right)^{\frac{2}{kC_n^k}}\end{array}\right)^{\frac{1}{2}}\right), \tag{48}$$

which is the Pythagorean fuzzy interaction Maclaurin symmetric mean operator.
Case 5: If $R=(1,1,\cdots,1)$, the PFIDMM is reduced to the following

$$PFIDMM^{(1,1,\cdots,1)}(p_1,p_2,\ldots,p_n)=\left(\sqrt{1-\prod_{i=1}^{n}(1-\mu_i^2)^{\frac{1}{n}}},\sqrt{\prod_{i=1}^{n}(1-\mu_i^2)^{\frac{1}{n}}-\prod_{i=1}^{n}(1-\mu_i^2-v_i^2)^{\frac{1}{n}}}\right) \tag{49}$$

which is the Pythagorean fuzzy interaction arithmetic averaging operator.
Case 6: If $R=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)$, the PFIDMM is reduced to the Pythagorean fuzzy interaction arithmetic averaging operator, which is shown as Equation (49).

4.2. The Pythagorean Fuzzy Interaction Dual Weighted Muirhead Mean Operator

In the following, we introduce the PFIDWMM operator so as to consider the weights vector of the attribute values.

Definition 11 . *Let $p_i=(\mu_i,v_i)(i=1,2,\cdots,n)$ be a collection of PFNs, $w=(w_1,w_2,\cdots,w_n)^T$ be the weight vector of $p_i(i=1,2,\cdots,n)$, which satisfies $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$, and let $R=(r_1,r_2,\ldots,r_n)\in R^n$ be a vector of parameters. If*

$$PFIDWMM^R(p_1,p_2,\ldots,p_n)=\frac{1}{\sum_{j=1}^{n}r_j}\left(\prod_{\vartheta\in S_n}\sum_{j=1}^{n}\left(r_j{p_{\vartheta(j)}}^{nw_{\vartheta(j)}}\right)\right)^{\frac{1}{n!}} \tag{50}$$

then we call $PFIDWMM^R$ the PFIDWMM operator, where $\vartheta(j)=(j=1,2,\ldots,n)$ is any a permutation of $(1,2,\ldots,n)$, and S_n is the collection of all permutations of $(1,2,\ldots,n)$.

Theorem 6. *Let $p_i=(\mu_i,v_i)(i=1,2,\cdots,n)$ be a collection of PFNs, we can see that the aggregation result from by the PFIDWMM is still a PFN, it can be obtained as follows:*

$$PFIDWMM^R(p_1,p_2,\ldots,p_n)=$$
$$\left(\left(1-\left(\begin{array}{l}1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-v_{\vartheta(j)}^2\right)^{nw_{\vartheta(j)}}+\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{nw_{\vartheta(j)}}\right)^{r_j}+\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{nw_{\vartheta(j)}r_j}\right)^{\frac{1}{n!}}\\+\prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{\frac{nw_{\vartheta(j)}r_j}{n!}}\end{array}\right)^{\frac{1}{\sum_{j=1}^{n}r_j}}\right)^{\frac{1}{2}},\right.$$
$$\left.\left(\begin{array}{l}\left(\begin{array}{l}1-\prod_{\vartheta\in S_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-v_{\vartheta(j)}^2\right)^{nw_{\vartheta(j)}}+\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{nw_{\vartheta(j)}}\right)^{r_j}+\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{nw_{\vartheta(j)}r_j}\right)^{\frac{1}{n!}}\\+\prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{\frac{nw_{\vartheta(j)}r_j}{n!}}\end{array}\right)^{\frac{1}{\sum_{j=1}^{n}r_j}}\\-\prod_{\vartheta\in S_n}\prod_{j=1}^{n}\left(1-\mu_{\vartheta(j)}^2-v_{\vartheta(j)}^2\right)^{\frac{nw_{\vartheta(j)}r_j}{n!\sum_{j=1}^{n}r_j}}\end{array}\right)^{\frac{1}{2}}\right). \tag{51}$$

Proof. Because $p_{\vartheta(j)}^{nw_{\vartheta(j)}} = \left(\sqrt{\left(1 - v_{\vartheta(j)}^2\right)^{nw_{\vartheta(j)}} - \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{nw_{\vartheta(j)}}}, \sqrt{1 - \left(1 - v_{\vartheta(j)}^2\right)^{nw_{\vartheta(j)}}} \right)$, we can replace $\mu_{\vartheta(j)}$ in Equation (28) with $\sqrt{(1 - v^2)^{nw_{\vartheta(j)}} - (1 - \mu^2 - v^2)^{nw_{\vartheta(j)}}}$, and $v_{\vartheta(j)}$ in Equation (28) with $\sqrt{1 - (1 - v^2)^{nw_{\vartheta(j)}}}$, then we can get Equation (51).

Because $p_{\vartheta(j)}$ is a PFN, $p_{\vartheta(j)}^{nw_{\vartheta(j)}}$ is also a PFN. By Equation (28), we have $PFIDWMM^R(p_1, p_2, \ldots, p_n)$ is a PFN.

Just the same as the PFIDMM operator, the PFIDWMM operator still does not have the monotonicity and the boundedness. □

Theorem 7. *The PFIDMM operator is a special case of the PFIDWMM operator.*

Proof. When $w = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)$

$$
\begin{aligned}
&PFIDWMM^R(p_1, p_2, \ldots, p_n) = \\
&\left(\left(1 - \left(\begin{array}{l} 1 - \prod\limits_{\vartheta \in S_n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - \left(1 - v_{\vartheta(j)}^2\right)^{nw_{\vartheta(j)}} + \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{nw_{\vartheta(j)}} \right)^{r_j} + \prod\limits_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{nw_{\vartheta(j)} r_j} \right)^{\frac{1}{n!}} \\ + \prod\limits_{\vartheta \in S_n} \prod\limits_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{\frac{nw_{\vartheta(j)} r_j}{n!}} \end{array} \right)^{\frac{1}{\sum_{j=1}^{n} r_j}} \right)^{\frac{1}{2}}, \right. \\
&\left. \left(\begin{array}{l} \left(\begin{array}{l} 1 - \prod\limits_{\vartheta \in S_n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - \left(1 - v_{\vartheta(j)}^2\right)^{nw_{\vartheta(j)}} + \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{nw_{\vartheta(j)}} \right)^{r_j} + \prod\limits_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{nw_{\vartheta(j)} r_j} \right)^{\frac{1}{n!}} \\ + \prod\limits_{\vartheta \in S_n} \prod\limits_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{\frac{nw_{\vartheta(j)} r_j}{n!}} \end{array} \right)^{\frac{1}{\sum_{j=1}^{n} r_j}} \\ - \prod\limits_{\vartheta \in S_n} \prod\limits_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{\frac{nw_{\vartheta(j)} r_j}{n! \sum_{j=1}^{n} r_j}} \end{array} \right)^{\frac{1}{2}} \right) \\
&= \left(\left(1 - \left(1 - \prod\limits_{\vartheta \in S_n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - \left(1 - v_{\vartheta(j)}^2\right) + \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right) \right)^{r_j} + \prod\limits_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{r_j} \right)^{\frac{1}{n!}} + \prod\limits_{\vartheta \in S_n} \prod\limits_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{\frac{r_j}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} r_j}} \right)^{\frac{1}{2}}, \right. \\
&\left. \left(\begin{array}{l} \left(1 - \prod\limits_{\vartheta \in S_n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - \left(1 - v_{\vartheta(j)}^2\right) + \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right) \right)^{r_j} + \prod\limits_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{r_j} \right)^{\frac{1}{n!}} + \prod\limits_{\vartheta \in S_n} \prod\limits_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{\frac{r_j}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} r_j}} \\ - \prod\limits_{\vartheta \in S_n} \prod\limits_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{\frac{r_j}{n! \sum_{j=1}^{n} r_j}} \end{array} \right)^{\frac{1}{2}} \right) \\
&= \left(\left(1 - \left(1 - \prod\limits_{\vartheta \in S_n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2\right)^{r_j} + \prod\limits_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{r_j} \right)^{\frac{1}{n!}} + \prod\limits_{\vartheta \in S_n} \prod\limits_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{\frac{r_j}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} r_j}} \right)^{\frac{1}{2}}, \right. \\
&\left. \left(\begin{array}{l} \left(1 - \prod\limits_{\vartheta \in S_n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{r_j} + \prod\limits_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{r_j} \right)^{\frac{1}{n!}} + \prod\limits_{\vartheta \in S_n} \prod\limits_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{\frac{r_j}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} r_j}} \\ - \prod\limits_{\vartheta \in S_n} \prod\limits_{j=1}^{n} \left(1 - \mu_{\vartheta(j)}^2 - v_{\vartheta(j)}^2\right)^{\frac{r_j}{n! \sum_{j=1}^{n} r_j}} \end{array} \right)^{\frac{1}{2}} \right) \\
&= PFIDMM^R(p_1, p_2, \ldots, p_n).
\end{aligned}
$$

5. A Novel Approach to MAGDM with Pythagorean Fuzzy Information

Based on the proposed operators, this section provides a novel approach to MAGDM problems in which attribute values take the form of PFNs and the weights of attributes take the form of crisp numbers. The description of a typical MAGDM problem with Pythagorean fuzzy information is shown as follows. Let $X = \{x_1, x_2, \cdots, x_m\}$ be a set of alternatives and $G = \{G_1, G_2, \cdots, G_n\}$ be a set of attributes with the weights vector being $w = (w_1, w_2, \cdots, w_n)^T$, satisfying $w_i \in [0, 1]$ and

$\sum_{i=1}^{n} w_i = 1$. For attribute $G_j (j = 1, 2, \cdots, n)$ of alternative $x_i (i = 1, 2, \ldots, m)$, a PFN $p_{ij} = (\mu_{ij}, v_{ij})$ $(i = 1, 2, \cdots, m; j = 1, 2, \cdots, n)$ is utilized to represent decision makers' preference information, in which μ_{ij} denotes that degree that alternative x_i satisfies the criteria G_j and v_{ij} represents the degree that alternative x_i dissatisfies the criteria G_j. Therefore, we can get a Pythagorean fuzzy decision matrix finally, which and be denoted by $P = (p_{ij})_{m \times n}$. In the followings, we introduce an algorithm to solve this problem based on the proposed operators.

Step 1. Standardized the original decision matrix. In real decision-making problems, there exists two kinds of attributes: benefit attributes and cost attributes. Therefore, the original decision matrix should be normalized by

$$p_{ij} = \begin{cases} (\mu_{ij}, v_{ij}) & G_j \in I_1 \\ (v_{ij}, \mu_{ij}) & G_j \in I_2 \end{cases} \tag{52}$$

where I_1 represents benefit attributes and I_2 represents cost attributes.

Step 2. For alternative $x_i (i = 1, 2, \ldots, m)$, utilize the PFIWMM operator

$$p_i = PFIWMM^R (p_{i1}, p_{i2}, \cdots, p_{in}) \tag{53}$$

or the PFIDWMM operator

$$p_i = PFIDWMM^R (p_{i1}, p_{i2}, \cdots, p_{in}) \tag{54}$$

to aggregate all the attributes values, so that a series of comprehensive preference value can be obtained.

Step 3. Rank the overall values $p_i (i = 1, 2, \ldots, m)$ based on their scores according to Definition 3.

Step 4. Rank the corresponding alternatives according to the rank of overall values and select the best alternative.

6. Numerical Example

In the following, we provide a numerical example that is adopted from [21] to illustrate the application of the proposed method. In order to know the best airline in Taiwan, the civil aviation administration of Taiwan (CAAT) organizes several experts to form a committee to assess the four major domestic airlines. The four airlines are the UNI Air (x_1), Transasia (x_2), Mandarin (x_3), and Daily Air (x_4).The alternatives are assessed from four attributes: (1) the booking and ticketing service (G_1); (2) the check-in and boarding process (G_2); (3) the cabin service (G_3); (4) the responsiveness (G_4). Weight vector of the attributes is $w = (0.15, 0.25, 0.35, 0.25)^T$. Experts are required to utilize a PFN $p_{ij} = (\mu_{ij}, v_{ij})$ to express their assessments for attributes $G_j (j = 1, 2, 3, 4)$ of airline $x_i (i = 1, 2, 3, 4)$, and a Pythagorean fuzzy decision matrix $P = (p_{ij})_{4 \times 4} (i, j = 1, 2, 3, 4)$ is shown in Table 1. In the following, we will solve this problem based on the proposed method.

Table 1. The Pythagorean fuzzy decision matrix.

	G_1	G_2	G_3	G_4
x_1	(0.9, 0.3)	(0.7, 0.6)	(0.5, 0.8)	(0.6, 0.3)
x_2	(0.4, 0.7)	(0.9, 0.2)	(0.8, 0.1)	(0.5, 0.3)
x_3	(0.8, 0.4)	(0.7, 0.5)	(0.6, 0.2)	(0.7, 0.4)
x_4	(0.7, 0.2)	(0.8, 0.2)	(0.8, 0.4)	(0.6, 0.6)

6.1. The Decision-Making Process

Step 1. As all of the attribute values are the same type, the original decision matrix does not need to be standardized.

Step 2. For each alternative, utilize Equation (53) to aggregate the assessments. Here, we assume $R = (1, 1, 1, 1)$. Therefore, we can obtain

$$p_1 = (0.3895, 0.2816)\ p_2 = (0.3526, 0.1679)\ p_3 = (0.3415, 0.1979)\ p_4 = (0.3588, 0.1878).$$

Step 3. Based on Definition 3, we can calculate the score function $S(p_i)(i = 1, 2, 3, 4)$ as follows

$$s(p_1) = 0.0724\ s(p_2) = 0.0961\ s(p_3) = 0.0775\ s(p_4) = 0.0935.$$

Therefore, the ranking order of the overall values is $p_2 > p_4 > p_3 > p_1$.

Step 4. According to the ranking order of the overall values, we can get the ranking order of the corresponding alternatives. That is $x_2 \succ x_4 \succ x_3 \succ x_1$. Therefore, x_2 is the best alternative, which means Transasia is the best airline of Taiwan.

In [14], the ranking results by using the Pythagorean fuzzy weighted averaging (PFWA) operator, the symmetric Pythagorean fuzzy weighted averaging (SPFWA) operator and the symmetric Pythagorean fuzzy weighted geometric (SPFWG) operator are also $x_2 \succ x_4 \succ x_3 \succ x_1$, which proves the validity of the proposed method.

In step 2, if we utilize the PFIWDMM operator to aggregate the decision makers' preference information, we can obtain

$$p_1 = (0.3622, 0.3159)\ p_2 = (0.3295, 0.2096)\ p_3 = (0.3439, 0.1938)\ p_4 = (0.3524, 0.1996).$$

Therefore, the scores of the overall values are

$$s(p_1) = 0.0314\ s(p_2) = 0.0647\ s(p_3) = 0.0807\ s(p_4) = 0.0843$$

Thus, the ranking order of the alternatives is $x_4 \succ x_3 \succ x_2 \succ x_1$. In Ref [11], the ranking result by utilizing the Pythagorean fuzzy weighted geometric is also $x_4 \succ x_3 \succ x_2 \succ x_1$, which also illustrate the validity of the proposed approach.

6.2. Further Discussion

The prominent advantage of the proposed aggregation operators is that the interrelationship among all PFNs can be taken into consideration. Moreover, it has a parameter vector that leads to flexible aggregation operators. To show the validity and superiorities of the proposed operators, we conduct a comparative analysis. We solve the same problem by some existing MAGDM approaches including the SPFWA and the SPFWG operators in [22], the Pythagorean fuzzy ordered weighted averaging weighted averaging distance (PFOWAWAD) operator in [22], the Pythagorean fuzzy point (PFP) operator and generalized Pythagorean fuzzy point ordered weighted averaging (GPFPOWA) in [23], the Pythagorean fuzzy Einstein ordered weighted averaging (PFEOWA) operator in [24], the Pythagorean fuzzy Einstein ordered weighted geometric (PFEOWG) operator in [25,26], the Pythagorean fuzzy weighted Bonferroni mean (PFWBM) operator in [30], the Pythagorean fuzzy weighted geometric Bonferroni mean (PFWGBM) operator in [31], the generalized Pythagorean fuzzy weighted Bonferroni mean (GPFWBM) operator and generalized Pythagorean fuzzy Bonferroni geometric mean (GPFBGM) operator in [32], the dual generalized Pythagorean fuzzy weighted Bonferroni mean (DGPFWBM) operator and dual generalized Pythagorean fuzzy weighted Bonferroni geometric mean (DGPFWBGM) operator in [32], the Pythagorean fuzzy weighted Maclaurin symmetric mean (PFWMSM) operator in [33], the generalized Pythagorean fuzzy weighted Maclaurin symmetric mean (GPFWMSM) operator in [34], the Pythagorean fuzzy interaction ordered weighted averaging (PFIOWA) operator and the Pythagorean fuzzy interaction ordered weighted geometric (PFIOWG) operator in [38], the Pythagorean fuzzy weighted Muirhead mean (PFWMM) operator, and Pythagorean fuzzy weighted dual Muirhead mean (PFWDMM) operator [39]. Details can be found in Table 2.

The approaches in [11–26] are based on a simple weighted averaging operator. The weaknesses of these approaches are (1) they assume that all the input arguments are independent, which is somewhat inconsistent with reality; (2) they cannot consider the interrelationship among input arguments; (3) they cannot capture the interrelationship between membership degree and non-membership degrees. However, on the contrary, the method in the present paper can capture the interrelationship among input arguments. In addition, it provides a feasible aggregation process as it has a parameter vector R. Quite a few existing aggregation operators are special cases of the proposed operators. Moreover, the method is based on the interaction operations for the PFNs. Thus, the proposed method can consider the relationship among membership and non-membership degrees. In other words, the proposed method can effectively handle situations in which a membership degree or a non-membership degree is zero. Thus, the proposed method is more powerful and flexible than the methods in [21–26].

Table 2. Comparison of different aggregation operators.

Approaches	Whether Captures Interrelationship of Two Attributes	Whether Captures Interrelationship of Multiple Attributes	Whether Captures Interrelationship of All Attributes	Whether Captures Relationship of Membership and Non-Membership Degrees	Whether Makes the Method Flexible by the Parameter Vector
SPFWA [21]	No	No	No	No	No
SPFWG [21]	No	No	No	No	No
PFOWAWAD [22]	No	No	No	No	No
PFP [23]	No	No	No	No	No
GPFPOWA [23]	No	No	No	No	No
PFEOWA [24]	No	No	No	No	No
PFEOWG [25,26]	No	No	No	No	No
PFWBM [30]	Yes	No	No	No	No
PFWGBM [31]	Yes	No	No	No	No
GPFWBM [32]	Yes	No	No	No	No
GPFWBGM [32]	Yes	No	No	No	No
DGPFWBM [32]	Yes	Yes	Yes	No	Yes
DGPFWBGM [32]	Yes	Yes	Yes	No	Yes
PFWMSM [33]	Yes	Yes	No	No	No
GPFWMSM [34]	Yes	Yes	No	No	No
PFIOWA [38]	No	No	No	Yes	No
PFIOWG [38]	No	No	No	Yes	No
PFWMM [39]	Yes	Yes	Yes	No	Yes
PFWDMM [39]	Yes	Yes	Yes	No	Yes
PFIWMM	Yes	Yes	Yes	Yes	Yes
PFIWDMM	Yes	Yes	Yes	Yes	Yes

Approaches in [30,31] are based on BM, so that they consider the interrelationships between arguments. However, the main flaw is that they can only capture the interrelationship between any two arguments. Approaches based on GPFWBM and GPFWBGM operators are better than approaches in [32], as the former approaches can capture the interrelationship between any three approaches. Approaches in [33,34] can consider the interrelationship among multiple arguments; however, all the methods [30–34] fail to reflect the interrelationship among all input arguments. Additionally, these methods do not consider the interrelationship among membership degree and non-membership degree. The proposed method in this paper not only captures the interrelationship between all input arguments but also takes the relationship between membership and non-membership degrees.

The approaches in [32] based on the DGPFWBM and GPFWBGM operators are much better than the methods in [30–34], as they can consider the interrelationship among all arguments. Additionally, they have vectors of the parameters, leading to a flexible and feasible aggregation process. However, the main drawback of these operators is that they do not consider the relationship between membership degree and non-membership degree. The proposed method in this paper takes the interrelationships of all arguments into consideration and simultaneously considers the relationship between membership and non-membership degrees. Thus, our method in this paper is more powerful than the method based on the DGPFWBM or GPFWBGM operators.

Compared with the approach based on the PFIOWA and PFIOWG operators, the merit of the proposed approach is that it can reflect the membership and non-membership degrees, as it is based on the interaction operations for PFNs. However, it cannot reflect of the interrelationship among PFNs. Moreover, it is not as flexible as the proposed method. In addition, the Pythagorean fuzzy Muirhead mean operators in [39] are based on basic operational laws, so that the relationship among membership and non-membership degrees is overlooked. In other words, the operators in [39] do not work for the situations in which one membership or non-membership degree is equal to one.

All in all, the proposed method in this paper can reflect the interrelationships among all input arguments. In addition, it works for situations in which a membership degree or a non-membership degree is zero, leading to less information loss and consequently making decision-making results more reasonable. Therefore, the proposed method is more powerful and flexible than others.

It is noted that there exists a vector of parameter R in the proposed method. The parameter vector R plays a significant role in the final ranking results. Some existing Pythagorean fuzzy aggregation operators are special cases of the proposed operators. By assigning different parameter vectors in the proposed operators, different overall values as well as the final ranking results can be obtained. Thus, in the following, we investigate the influence of the vector of parameters R on the score functions and the ranking results. We assign different values to R in the PFIWMM and PFIWDMM operators, and the score function and ranking orders are presented in Tables 3 and 4.

Table 3. Ranking results by utilizing the different parameter vector R in the Pythagorean fuzzy interaction weighted Muirhead mean (PFIWMM) operator.

Parameter Vector R	The Scores of $s(p_i)(i = 1,2,3,4)$	Ranking Results
$R = (1,0,0,0)$	$s(p_1) = 0.0874$ $s(p_2) = 0.2680$ $s(p_3) = 0.1966$ $s(p_4) = 0.2440$	$x_2 \succ x_4 \succ x_3 \succ x_1$
$R = (1,1,0,0)$	$s(p_1) = 0.1040$ $s(p_2) = 0.1706$ $s(p_3) = 0.1337$ $s(p_4) = 0.1625$	$x_2 \succ x_4 \succ x_3 \succ x_1$
$R = (1,1,1,0)$	$s(p_1) = 0.0868$ $s(p_2) = 0.1232$ $s(p_3) = 0.0984$ $s(p_4) = 0.1190$	$x_2 \succ x_4 \succ x_3 \succ x_1$
$R = (1,1,1,1)$	$s(p_1) = 0.0724$ $s(p_2) = 0.0961$ $s(p_3) = 0.0775$ $s(p_4) = 0.0935$	$x_2 \succ x_4 \succ x_3 \succ x_1$
$R = (2,0,0,0)$	$s(p_1) = 0.1423$ $s(p_2) = 0.2989$ $s(p_3) = 0.1888$ $s(p_4) = 0.2646$	$x_2 \succ x_4 \succ x_3 \succ x_1$

As we can see in Table 3, by assigning different vector R to the PFIWMM operator, different scores of the overall assessments can be obtained. However, the ranking results are always the same. In addition, the more interrelationships between PFNs are taken into consideration, the smaller the value of score functions will become. Similarly, as we can see in Table 4, different scores of the overall assessments are obtained with different parameter vector R in the PFIWDMM operator. Similar to the PFIWMM operator, the more interrelationships among attributes are taken into account, the smaller the scores of the overall assessments. However, no matter what the parameter vector is, the ranking result is always the same. Therefore, the parameter vector can be viewed as the decision makers' risk preference.

Table 4. Ranking results by utilizing the different parameter vector R in the PFIWDMM operator.

Parameter Vector R	The Scores of $s(p_i)(i=1,2,3,4)$	Ranking Results
$R=(1,0,0,0)$	$s(p_1)=0.2103$ $s(p_2)=0.2530$ $s(p_3)=0.2990$ $s(p_4)=0.3116$	$x_4 \succ x_3 \succ x_2 \succ x_1$
$R=(1,1,0,0)$	$s(p_1)=0.0818$ $s(p_2)=0.1290$ $s(p_3)=0.1577$ $s(p_4)=0.1647$	$x_4 \succ x_3 \succ x_2 \succ x_1$
$R=(1,1,1,0)$	$s(p_1)=0.0465$ $s(p_2)=0.0862$ $s(p_3)=0.1068$ $s(p_4)=0.1116$	$x_4 \succ x_3 \succ x_2 \succ x_1$
$R=(1,1,1,1)$	$s(p_1)=0.0314$ $s(p_2)=0.0647$ $s(p_3)=0.0807$ $s(p_4)=0.0843$	$x_4 \succ x_3 \succ x_2 \succ x_1$
$R=(2,0,0,0)$	$s(p_1)=0.0807$ $s(p_2)=0.1966$ $s(p_3)=0.3076$ $s(p_4)=0.3078$	$x_4 \succ x_3 \succ x_2 \succ x_1$

7. Conclusions

In the field of aggregation operators, more and more operators have been proposed. However, some operators do not take the correlations among attributes into consideration, which cannot satisfy the needs of real decision-making problems. The MM operator can consider the interaction relationships among any number of attributes with a parameter R. In this paper, we extend the MM operator to PFNs and propose some new Pythagorean fuzzy operators, including the PFIMM, PFIWMM, and PFIDWMM operators. These operators can reflect the correlations among all Pythagorean fuzzy elements. Further, we propose a novel approach to MAGDM by using these operators. Moreover, in order to show the application of the proposed method in this paper, we provide a numerical example and the advantages of the new operator are more obvious by comparing the new operator with the existing ones. Finally, we give the parameter vector R some different values to discuss the advantages of the new approach on the ranking results of the numerical example. In further works, we will apply the proposed method in more practical decision-making problems, such as low carbon supplier selection, hospital-based post-acute care, risk management, medical diagnosis, and resource evaluation, etc. In addition, we will investigate more aggregation operators for fusing Pythagorean fuzzy information.

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