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Application of the ARA Method in Solving Integro-Differential Equations in Two Dimensions

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Abstract: The main purpose of this study is to investigate solutions of some integral equations of different classes using a new scheme. This research introduces and implements the new double ARA transform to solve integral and partial integro-differential equations. We introduce basic theorems and properties of the double ARA transform in two dimensions, and some results related to the double convolution theorem and partial derivatives are presented. In addition, to show the validity of the proposed technique, we introduce and solve some examples using the new approach.

Keywords: ARA transform; double ARA transform; partial integral equations; partial integro-differential equations

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Citation: Saadeh, R. Application of the ARA Method in Solving Integro-Differential Equations in Two Dimensions. *Computation* **2023**, *11*, 4. https://doi.org/10.3390/computation11010004

Academic Editors: Babak Shiri and Zahra Alijani

Received: 22 November 2022
Revised: 16 December 2022
Accepted: 21 December 2022
Published: 29 December 2022

1. Introduction

One of the most effective methods in solving integral equations and partial differential equations (PDEs) is integral transformations. Mathematical models are very important in the fields of science and engineering [1–7], and most of them can be expressed by PDEs or integral equations; as a result, most researchers invest much effort in establishing new methods to seek solutions [8–13]. Herein, we mention one of the most popular methods in these aspects, the integral transform method, and mention that a large number of mathematicians studied integral transforms and improved them.

Laplace transform, Fourier transform, Sumudu transform, Natural transform, ARA transform, formable transform, and others [14–24] are all examples in a series of a large number of popular transforms that are created and implemented to solve different types of partial and integral equations. The main advantage of using these integral transforms in solving equations is that they do not require linearization, discretization or differentiation when using them to solve problems. When applying an integral transform on a PDE, we obtain an ordinary differential equation if we use a single integral transform, or we obtain an algebraic one if we use a double integral transform.

Another aspect in this area are double integral transforms that also have great applications in handling PDEs and integral equations. They show high efficiency and simplicity in solving equations in comparison with other methods. In the present work, we state some famous double transforms in the literature, such as double Laplace transform, double Sumudu transform, double ARA-Sumudu transform, and others [25–43]. Meddahi et al. [44] introduces a general formula for integral transforms, which could be a generalization of double transforms, but even so, we still need to study each double transform alone to determine its efficiency in handling problems.

In 2020, Saadeh et al. [23] introduced an interesting transform known as ARA transform. This transform attracted a lot of attention of researchers because of its ability to generate multi transforms of index n, and it could also simply overcome the challenge of having singular points in differential equations. For all of these merits, it could be applied to solve different types of problems.

Our motivation in this study is to present a new approach in double transforms that is DARAT. We present the basic definition, properties and theorems of the new DARAT. In addition, we compute the values of DARAT to elementary functions. Some results related to partial derivatives and the double convolution theorem are presented and proved; then, we use some of these outcomes to implement solutions of different types of integral equations. The strength of this work appears in creating this double transform for the first time, and using it for solving integral equations. Moreover, the simplicity and applicability of DARA in handling some integral equations are illustrated in the proposed examples.

2. Basic Definitions and Theorems of ARA Transform

In this section, we spotlight some preliminaries of ARA transform [23].

Definition 1. *Assume that $f(x)$ is a continuous function on the interval $(0, \infty)$, then ARA integral transform of order n of $f(x)$ is defined as*

$$\mathcal{G}_n[f(x)](\upsilon) = F(n,\upsilon) = \upsilon \int_0^\infty x^{n-1} e^{-\upsilon x} f(x)dx, \ \upsilon > 0. \tag{1}$$

The inverse ARA transform is provided by

$$\mathcal{G}_{n+1}^{-1}[\mathcal{G}_{n+1}[f(x)]] = \frac{(-1)^{2n}}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\upsilon x} F(\upsilon) d\upsilon = f(x), \tag{2}$$

where

$$F(\upsilon) = \int_0^\infty e^{-\upsilon x} f(x)dx.$$

The ARA transform of first order $\mathcal{G}_1[\mathrm{f}(x)]$ on $[0,\infty)$ that we will focus on in our study is defined as

$$\mathcal{G}_1[f(x)](\upsilon) = F(\upsilon) = \upsilon \int_0^\infty e^{-\upsilon x} f(x)dx, \ \upsilon > 0. \tag{3}$$

For simplicity, let us denote $\mathcal{G}_1[f(x)]$ by $\mathcal{G}[f(x)]$.

Theorem 1. *If $f(x)$ is a continuous function in every finite interval $0 \le x \le \alpha$ and satisfies*

$$\left| x^{n-1} f(x) \right| \le k e^{\alpha x} \tag{4}$$

where k is a positive real number, then the ARA transform exists for all $\upsilon > \alpha$.

Now, we present some basic properties of ARA transform of order n, and for readers who are interested in more details, they can see [23]. Let $F(n,\upsilon) = \mathcal{G}_n[f(x)]$ and $G(n,\upsilon) = \mathcal{G}_n[g(x)]$ and $a, b \in \mathbb{R}$. Then,

- $\mathcal{G}_n[a\, f(x) + b\, g(x)] = a\, \mathcal{G}_n[f(x)] + b\, \mathcal{G}_n[g(x)]$;
- ${\mathcal{G}_n}^{-1}[a\, F(n,\upsilon) + b\, G(n,\upsilon)] = a\, {\mathcal{G}_n}^{-1}[F(n,\upsilon)] + b\, {\mathcal{G}_n}^{-1}[G(n,\upsilon)]$;
- $\mathcal{G}_n[x^\alpha] = \frac{\Gamma(\alpha+n)}{\upsilon^{\alpha+n-1}}$, $\alpha > 0$;
- $\mathcal{G}_n[e^{ax}] = \frac{\upsilon\Gamma(n)}{(\upsilon-a)^n}$, $a \in \mathbb{R}$;
- $\mathcal{G}_n\left[f^{(n)}(x)\right] = (-1)^{n-1} \upsilon \frac{d^{n-1}}{d\upsilon^{n-1}} \left(\upsilon^{n-1} \mathcal{G}_1[f(x)] - \sum_{k=1}^{n} \upsilon^{n-k} f^{(k-1)}(0) \right)$;
- $\mathcal{G}_n[\, x^m f(x)] = Q(n+m,\upsilon)$.

In the following table (Table 1), we state some basic properties of ARA transform, where $f(x)$ and $g(x)$ are two continuous functions and $a, b \in \mathbb{R}$.

Table 1. Properties of ARA transform.

Function	ARA Transform
$a\,f(x)+b\,g(x)$	$a\,\mathcal{G}[f(x)]+b\,\mathcal{G}[g(x)]$
x^{α}	$\frac{\Gamma(\alpha+1)}{\upsilon^{\alpha}},\ \alpha>0$
e^{ax}	$\frac{\upsilon}{\upsilon-a}$
$f'(x)$	$\upsilon\mathcal{G}[f(x)]-\upsilon f(0)$
$f^{(n)}(x)$	$\upsilon^{n+1}\mathcal{G}[f(x)]-\sum\limits_{k=1}^{n}\upsilon^{n-k+1}f^{(k-1)}(0)$

3. Double ARA Transform (DARAT)

In this section, a new double integral transform, DARAT, is introduced that combines two ARA transforms of order one. We present fundamental properties and theorems of the new transformation.

Definition 2. *Assume that $\psi(x,t)$ is a continuous function of the variables x and t, where $x>0$ and $t>0$, then the two-dimensional ARA transform denoted by DARAT of $\psi(x,t)$ is defined as*

$$\mathcal{G}_x\mathcal{G}_t[\psi(x,t)]=\Psi(\upsilon,s)=\upsilon s\int\limits_0^{\infty}\int\limits_0^{\infty}e^{-\upsilon x-st}\psi(x,t)dxdt,\ \upsilon,s>0. \tag{5}$$

provided the double integral exists.

Obviously, DARAT is a linear integral transform:

$$\mathcal{G}_x\mathcal{G}_t[a\,\psi(x,t)+b\,\phi(x,t)]=a\,\mathcal{G}_x\mathcal{G}_t[\,\psi(x,t)]+b\,\mathcal{G}_x\mathcal{G}_t[\,\phi(x,t)], \tag{6}$$

where a and b are constants.

The inverse of the DARAT is provided by

$$\mathcal{G}_x^{-1}\mathcal{G}_x^{-1}[\Psi(\upsilon,s)]=\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\frac{e^{\upsilon x}}{\upsilon}d\upsilon\frac{1}{2\pi i}\int\limits_{r-i\infty}^{r+i\infty}\frac{e^{st}}{s}\Psi(\upsilon,s)ds=\psi(x,t). \tag{7}$$

Property 1. Let $\psi(x,t)=\psi_1(x)\psi_2(t),\ x>0,t>0$. Then,

$$\mathcal{G}_x\mathcal{G}_t[\psi(x,t)]=\mathcal{G}_x[\psi_1(x)]\mathcal{G}_t[\psi_2(t)]. \tag{8}$$

Proof.

$$\begin{aligned}\mathcal{G}_x\mathcal{G}_t[\psi(x,t)]= &\quad \mathcal{G}_x\mathcal{G}_t[\psi_1(x)\psi_2(t)]=\upsilon s\int\limits_0^{\infty}\int\limits_0^{\infty}e^{-\upsilon x-st}\psi_1(x)\psi_2(t)dxdt\\ &=\upsilon\int\limits_0^{\infty}\psi_1(x)e^{-\upsilon x}dx\ s\int\limits_0^{\infty}\psi_2(t)e^{-st}dt=\mathcal{G}_x[\psi_1(x)]\mathcal{G}_t[\psi_2(t)].\end{aligned}$$

□

Property 2. DARAT of basic functions

i. Let $\psi(x,t)=1,x>0,t>0$. Then,

$$\mathcal{G}_x\mathcal{G}_t[1]=\upsilon s\int\limits_0^{\infty}\int\limits_0^{\infty}e^{-\upsilon x-st}dxdt=\upsilon\int\limits_0^{\infty}e^{-\upsilon x}dx\ s\int\limits_0^{\infty}e^{-st}dt=1,$$

where $Re(\mathcal{s}) > 0$.

ii. Let $\psi(x,t) = x^a t^b,\ x > 0,\ t > 0$ and a, b are constants. Then,

$$\mathcal{G}_x\mathcal{G}_t\left[x^a t^b\right] = v\mathcal{s}\int_0^\infty\int_0^\infty e^{-vx-\mathcal{s}t}x^a t^b dxdt = v\int_0^\infty e^{-vx}x^a dx\ \mathcal{s}\int_0^\infty e^{-\mathcal{s}t}t^b dt = \frac{\Gamma(a+1)\,\Gamma(b+1)}{v^{a+1}\mathcal{s}^b},$$

where $Re(a) > -1$ and $Re(b) > -1$.

iii. Let $\psi(x,t) = e^{ax+bt}$, $x > 0, t > 0$ and a, b are constants. Then,

$$\mathcal{G}_x\mathcal{G}_t\left[e^{ax+bt}\right] = v\mathcal{s}\int_0^\infty\int_0^\infty e^{-vx-\mathcal{s}t}e^{ax+bt}dxdt = v\int_0^\infty e^{-vx}e^{ax}dx\ \mathcal{s}\int_0^\infty e^{-\mathcal{s}t}e^{bt}dt = \frac{v\mathcal{s}}{(v-a)(\mathcal{s}-b)}.$$

Similarly,

$$\mathcal{G}_x\mathcal{G}_t\left[e^{i(ax+bt)}\right] = \frac{v\mathcal{s}}{(v-ia)(\mathcal{s}-ib)} = \frac{v\mathcal{s}(\mathcal{s}v-ab)+iv\mathcal{s}(vb+\mathcal{s}a)}{(v^2+a^2)(\mathcal{s}^2+b^2)}.$$

Consequently,

$$\mathcal{G}_x\mathcal{G}_t[\sin(ax+bt)] = \frac{v\mathcal{s}(vb+\mathcal{s}a)}{(v^2+a)(\mathcal{s}^2+b^2)},$$

$$\mathcal{G}_x\mathcal{G}_t[\cos(ax+bt)] = \frac{v\mathcal{s}(\mathcal{s}v-ab)}{(v^2+a^2)(\mathcal{s}^2+b^2)}.$$

iv. Let $\psi(x,t) = \sinh(ax+bt)$ or $\psi(x,t) = \cosh(ax+bt)$. Recall that

$$\mathcal{G}_x\mathcal{G}_t[\sinh(ax+bt)] = \frac{v\mathcal{s}(vb+\mathcal{s}a)}{(v^2-a^2)(\mathcal{s}^2-b^2)},$$

$$\mathcal{G}_x\mathcal{G}_t[\cosh(ax+bt)] = \frac{v\mathcal{s}(\mathcal{s}v+ab)}{(v^2-a^2)(\mathcal{s}^2-b^2)}.$$

v. Let $\psi(x,t) = J_0\left(c\sqrt{xt}\right)$, where J_0 is the zero Bessel function. Then,

$$\begin{aligned}\mathcal{G}_x\mathcal{G}_t\left[J_0\left(c\sqrt{xt}\right)\right] &= v\mathcal{s}\int_0^\infty\int_0^\infty e^{-vx-\mathcal{s}t}J_0\left(c\sqrt{xt}\right)dxdt \\ &= v\int_0^\infty e^{-vx}J_0\left(c\sqrt{xt}\right)dx\cdot\mathcal{s}\int_0^\infty e^{-\mathcal{s}t}dt \\ &= v\mathcal{s}\int_0^\infty e^{-\frac{c^2}{4\mathcal{s}}t}e^{-\mathcal{s}t}dt = \frac{4v\mathcal{s}}{4v\mathcal{s}+c^2}.\end{aligned}$$

If a function $\psi(x,t)$ satisfies the following condition

$$|\psi(x,t)| \le ke^{ax+bt},$$

where a, b and k are positive constants, then we say that $\psi(x,t)$ is a function of exponential orders a and b.

Theorem 2. *Assume that the function $\psi(x,t)$ is continuous on the region $[0,X)\times[0,T)$ and is of exponential orders a and b. Then, $\mathcal{G}_x\mathcal{G}_t[\psi(x,t)]$ exists for v and $\mathcal{s}$, provided $Re(v) > a$ and $Re(\mathcal{s}) > b$.*

Proof. From the definition of DARAT, one can conclude

$$|\Psi(v,\mathcal{s})| = \left|v\mathcal{s}\int_0^\infty\int_0^\infty e^{-vx-\mathcal{s}t}\psi(x,t)dxdt\right| \le vt\int_0^\infty\int_0^\infty e^{-vx-\mathcal{s}t}|\psi(x,t)|dxdt \le kv\int_0^\infty e^{-(v-a)x}dx\cdot\mathcal{s}\int_0^\infty e^{-(\mathcal{s}-b)t}dt = \frac{kv\mathcal{s}}{(v-a)(\mathcal{s}-b)},$$

where $Re(v) > a$ and $Re(\mathcal{s}) > b$. □

Thus, $\mathcal{G}_x\mathcal{G}_t[\psi(x,t)]$ exists for υ and $\jmath$, provided $Re(\upsilon) > a$ and $Re(\jmath) > b$.

Theorem 3. *(Convolution Theorem). Let $\mathcal{G}_x\mathcal{G}_t[\psi(x,t)]$ and $\mathcal{G}_x\mathcal{G}_t[\phi(x,t)]$ be exist, and $\mathcal{G}_x\mathcal{G}_t[\psi(x,t)] = \Psi(\upsilon,\jmath)$, $\mathcal{G}_x\mathcal{G}_t[\phi(x,t)] = \Phi(\upsilon,\jmath)$. Then,*

$$\mathcal{G}_x\mathcal{G}_t[\phi(x,t) * *\phi(x,t)] = \frac{1}{\upsilon\jmath}\Psi(\upsilon,\jmath)\Phi(\upsilon,\jmath), \tag{9}$$

where

$$\psi(x,t) * *\phi(x,t) = \int_0^x \int_0^t \phi(x-\rho, t-\tau)\phi(\rho,\tau)d\rho d\tau,$$

and the symbol $**$ denotes the double convolution with respect to x and t.

Proof. The definition of DARAT implies

$$\begin{aligned} &\mathcal{G}_x\mathcal{G}_t[\psi(x,t) * *\phi(x,t)] \\ &\quad = \upsilon\jmath \int_0^\infty \int_0^\infty e^{-\upsilon\jmath-\jmath t}(\psi(x,t) * *\phi(x,t))dxdt \\ &\quad = \upsilon\jmath \int_0^\infty \int_0^\infty e^{-\upsilon x-\jmath t}\left(\int_0^x \int_0^t \psi(x-\rho,t-\tau)\phi(\rho,\tau)d\rho d\tau\right)dxdt. \end{aligned} \tag{10}$$

□

Using the Heaviside unit step function, Equation (10) can be written as

$$\begin{aligned} \mathcal{G}_x\mathcal{G}_t[u * *w(x,t)] &= \\ &= \upsilon\jmath \int_0^\infty \int_0^\infty e^{-\upsilon x-\jmath t}\left(\int_0^\infty \int_0^\infty \psi(x-\rho,t-\tau)H(x-\rho,t-\tau)\phi(\rho,\tau)d\rho d\tau\right)dxdt \\ &= \int_0^\infty \int_0^\infty \phi(\rho,\tau)d\rho d\tau\left(\upsilon\jmath \int_0^\infty \int_0^\infty e^{-\upsilon(x+\rho)-\jmath(t+\tau)}\psi(x-\rho,t-\tau)H(x-\rho,t-\tau)\right)dxdt \\ &= \Psi(\upsilon,\jmath)\int_0^\infty \int_0^\infty e^{-\upsilon\rho-\jmath\tau}\phi(\rho,\tau)d\rho d\tau = \frac{1}{\upsilon\jmath}\Psi(\upsilon,\jmath)\Phi(\upsilon,\jmath) \end{aligned}$$

Theorem 4. *If $\psi(x,t)$ is a continuous function and $\mathcal{G}_x\mathcal{G}_t[\psi(x,t)] = \Psi(\upsilon,\jmath)$, then we obtain the following:*

(a) $\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial\psi(x,t)}{\partial t}\right] = \jmath\Psi(\upsilon,\jmath) - \jmath\mathcal{G}_x[\psi(x,0)]$.

(b) $\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial\psi(x,t)}{\partial x}\right] = \upsilon\Psi(\upsilon,\jmath) - \upsilon\mathcal{G}_t[\psi(0,t)]$.

(c) $\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial^2\psi(x,t)}{\partial t^2}\right] = \jmath^2\Psi(\upsilon,\jmath) - \jmath^2\mathcal{G}_x[\psi(x,0)] - \jmath\mathcal{G}_x\left[\frac{\partial\psi(x,0)}{\partial t}\right]$.

(d) $\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial^2\psi(x,t)}{\partial \upsilon^2}\right] = \upsilon^2\Psi(\upsilon,\jmath) - \upsilon^2\mathcal{G}_t[\psi(0,t)] - \upsilon\mathcal{G}_t\left[\frac{\partial\psi(0,t)}{\partial x}\right]$.

(e) $\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial^2\psi(x,t)}{\partial x\partial t}\right] = \upsilon\jmath\Psi(\upsilon,\jmath) - \upsilon\jmath\mathcal{G}_x[\psi(x,0)] - \upsilon\jmath\mathcal{G}_t[\psi(0,t)] + \upsilon\jmath\ \psi(0,0)$.

Proof.

(a)
$$\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial\psi(x,t)}{\partial t}\right] = \upsilon\varsigma\int_0^\infty\int_0^\infty e^{-\upsilon x-\varsigma t}\frac{\partial\psi(x,t)}{\partial t}dxdt$$
$$= \upsilon\int_0^\infty e^{-\upsilon x}dx\cdot\varsigma\int_0^\infty e^{-\varsigma t}\frac{\partial\psi(x,t)}{\partial t}dt$$
$$= \upsilon\int_0^\infty e^{-\upsilon t}(\varsigma\,\mathcal{G}_t[\psi(x,t)] - \varsigma\,\psi(x,0))dx = \varsigma\Psi(\upsilon,\varsigma) - \varsigma\mathcal{G}_x[\psi(x,0)].$$

(b)
$$\mathcal{G}_x\mathcal{G}_t\left[\frac{\partial\psi(x,t)}{\partial x}\right] = \upsilon\varsigma\int_0^\infty\int_0^\infty e^{-\upsilon x-\varsigma t}\frac{\partial\psi(x,t)}{\partial x}dxdt$$
$$= \varsigma\int_0^\infty e^{-\varsigma t}dt\;\upsilon\int_0^\infty e^{-\upsilon x}\frac{\partial\psi(x,t)}{\partial x}dx.$$
$$= \varsigma\int_0^\infty e^{-\varsigma t}(\upsilon\,\mathcal{G}_x[\psi(x,t)] - \upsilon\,\psi(0,t))dt = \upsilon\Psi(\upsilon,\varsigma) - \upsilon\mathcal{G}_t[\psi(0,t)].$$

The proof of parts (c), (d) and (e) can be obtained by similar arguments.

□

The previous results of DARAT are summarized in the following table, Table 2.

Table 2. DARAT for some elementary functions.

$\psi(x,t)$	$\mathcal{G}_x\mathcal{G}_t[\psi(x,t)] = \Psi(\upsilon,\varsigma)$
1	1
$x^a t^b$,	$\frac{\Gamma(a+1)\ \Gamma(b+1)}{\upsilon^a\varsigma^b}$
e^{ax+bt}	$\frac{\upsilon\varsigma}{(\upsilon-a)(\varsigma-b)}$
$e^{i\ (ax+bt)}$	$\frac{\upsilon\varsigma}{(\upsilon-ia)(\varsigma-ib)}$
$\sin(ax+bt)$	$\frac{\upsilon\varsigma(\upsilon b+\varsigma a)}{(\upsilon^2+a^2)(\varsigma^2+b^2)}$
$\cos(ax+bt)$	$\frac{\upsilon\varsigma(\varsigma\upsilon-ab)}{(\upsilon^2+a^2)(\varsigma^2+b^2)}$
$\sinh(ax+bt)$	$\frac{\upsilon t(\upsilon b+\varsigma a)}{(\upsilon^2-a^2)(\varsigma^2-b^2)}$
$\cosh(ax+bt)$	$\frac{\upsilon\varsigma(\varsigma\upsilon+ab)}{(\upsilon^2-a^2)(\varsigma^2-b^2)}$
$J_0\left(c\sqrt{xt}\right)$, J_0 is the zero order Bessel function	$\frac{4\upsilon\varsigma}{4\upsilon\varsigma+c^2}$
$e^{ax+bt}\psi(x,t)$	$\frac{\upsilon}{\upsilon-b}\frac{\varsigma}{\varsigma-b}\Psi(\upsilon-a,\varsigma-b)$
$(\psi{*}{*}\phi)(x,t)$	$\frac{1}{\upsilon\varsigma}\Psi(\upsilon,\varsigma)\Phi(\upsilon,\varsigma)$
$\psi_1(x)$	$\mathcal{G}_x[\psi_1(x)] = \Psi_1(\upsilon)$
$\psi_2(t)$	$\mathcal{G}_t[\psi_2(t)] = \Psi_2(\varsigma)$
$\psi_1(x)\psi_2(t)$	$\Psi_1(\upsilon)\Psi_2(\varsigma)$

4. Applications of DARAT in Solving Integral Equations

In this section, DARAT is implemented to solve some classes of integral equations, Volterra integral equations and Volterra partial integro-differential equations.

(I) Integral equations of two variables.

We possess the following Volterra integral equation:

$$\psi(x,t) = \omega(x,t) + a\int_0^x\int_0^t \psi(x-\delta,t-\varepsilon)\phi(\delta,\varepsilon)d\delta d\varepsilon, \tag{11}$$

in which $\psi(x,t)$ is the target function, a is a constant, $\omega(x,t)$ and $\phi(x,t)$ are two known functions.

Then, running DARAT on Equation (11), to get

$$\mathcal{G}_x\mathcal{G}_t[\psi(x,t)] = \mathcal{G}_x\mathcal{G}_t[\omega(x,t)] + \mathcal{G}_x\mathcal{G}_t\left[a\int_0^x\int_0^t \psi(x-\delta,t-\varepsilon)\phi(\delta,\varepsilon)d\delta d\varepsilon\right], \tag{12}$$

where $\mathcal{G}_x\mathcal{G}_t[\omega(x,t)] = W(v,\jmath)$. Using the linearity property (7) and Theorem 4, Equation (11) becomes

$$\Psi(v,\jmath) = W(v,\jmath) + a\frac{1}{v\jmath}\Psi(v,\jmath)\Phi(v,\jmath), \tag{13}$$

where $\Psi(v,\jmath) = \mathcal{G}_x\mathcal{G}_t[\psi(x,t)]$, $W(v,\jmath) = \mathcal{G}_x\mathcal{G}_t[\omega(x,t)]$ and $\Phi(v,\jmath) = \mathcal{G}_x\mathcal{G}_t[\phi(x,t)]$.

Consequently,

$$\Psi(v,\jmath) = \frac{v\jmath\, W(v,\jmath)}{v\jmath - a\,\Phi(v,\jmath)}. \tag{14}$$

Applying the inverse transform $\mathcal{G}_x^{-1}\mathcal{G}_t^{-1}$ to (14), we obtain exact value of $\psi(x,t)$ in (11),

$$\psi(x,t) = \mathcal{G}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{xt\, W(v,\jmath)}{v\jmath - a\,\Phi(v,\jmath)}\right]. \tag{15}$$

We use the result in Equation (15) to solve the following examples.

Example 1. Consider the following integral equation

$$\psi(x,t) = b - a\int_0^x\int_0^t \psi(\delta,\varepsilon)d\delta d\varepsilon, \tag{16}$$

where a and b are constants.

Solution. Taking DARAT to Equation (16) and using the linearity property and convolution theorem, we obtain

$$\Psi(v,\jmath) = b - \frac{a}{\jmath v}\Psi(v,\jmath). \tag{17}$$

As a result,

$$\Psi(v,\jmath) = \frac{b\, v\jmath}{\jmath v + a}. \tag{18}$$

Applying the inverse transform $\mathcal{G}_x^{-1}\mathcal{G}_t^{-1}$ to Equation (18), we obtain the exact solution $\psi(x,t)$ of (16) in the original space

$$\psi(x,t) = \mathcal{G}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{b\, v\jmath}{\jmath v + a}\right] = b\, J_0\left(2\sqrt{axt}\right).$$

Example 2. Consider the following integral equation:

$$b^2 t = \int_0^x\int_0^t \psi(x-\delta, t-\varepsilon)\psi(\delta,\varepsilon)d\delta d\varepsilon, \tag{19}$$

where b is a constant.

Solution. Applying DARAT to Equation (19) and using the convolution theorem on (19), we obtain

$$\frac{b^2}{\jmath} = \frac{1}{v\jmath}\Psi^2(v,\jmath). \tag{20}$$

Thus, we have

$$\Psi(v,\jmath) = b\sqrt{v}. \tag{21}$$

Applying the inverse transform $\mathcal{G}_x^{-1}\mathcal{G}_t^{-1}$ to Equation (21), we obtain the solution $\psi(x,t)$ of Equation (21) as follows:

$$\psi(x,t) = \mathcal{G}_x^{-1}\mathcal{G}_t^{-1}\left[b\sqrt{v}\right] = \frac{b}{\sqrt{\pi}}\frac{1}{\sqrt{x}}\,. \tag{22}$$

Example 3. Consider the following integral equation:

$$\int_0^x \int_0^t e^{\delta-\varepsilon}\psi(x-\delta,t-\varepsilon)d\delta d\varepsilon = xe^{x-t} - xe^x. \tag{23}$$

Solution. Applying DARAT to (23) and using the convolution theorem, we obtain

$$\frac{1}{\upsilon s}\frac{\upsilon s\,\Psi(\upsilon,s)}{(\upsilon-1)(1+s)} = \frac{\upsilon s}{(\upsilon-1)^2(s+1)} - \frac{\upsilon}{(\upsilon-1)^2}. \tag{24}$$

Thus,

$$\Psi(\upsilon,s) = \frac{-\upsilon}{(\upsilon-1)}. \tag{25}$$

Simplifying and taking the inverse transform $\mathcal{G}_x^{-1}\mathcal{G}_t^{-1}$ for Equation (25), we obtain the solution of (23) as follows:

$$\psi(x,t) = \mathcal{G}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{-\upsilon}{(\upsilon-1)}\right] = -e^x. \tag{26}$$

(II) First order partial integro-differential equations.

We possess the following Volterra partial integro-differential equation:

$$\frac{\partial\psi(x,t)}{\partial x} + \frac{\partial\psi(x,t)}{\partial t} = \omega(x,t) + a\int_0^x\int_0^t \psi(x-\delta,t-\varepsilon)\phi(\delta,\varepsilon)d\delta d\varepsilon \tag{27}$$

subject to the conditions

$$\psi(x,0) = f(x),\ \psi(0,t) = g(t), \tag{28}$$

where $\psi(x,t)$ is the unknown function, a is a constant, $\omega(x,t)$ and $\phi(x,t)$ are two known functions.

Applying DARAT to both sides of (27), we obtain

$$t\Psi(t,s) - \upsilon\mathcal{G}_t[\psi(0,t)] + s\Psi(\upsilon,s) - s\mathcal{G}_x[\psi(x,0)] = W(\upsilon,s) + a\frac{1}{\upsilon s}\Psi(\upsilon,s)\Phi(\upsilon,s).$$

Substituting the values of the transformed condition (28),

$$\Psi(\upsilon,s) = \frac{\upsilon s W(\upsilon,s) + \upsilon^2 s G(s) + \upsilon s^2 F(\upsilon)}{\upsilon^2 s + \upsilon s^2 - a\Phi(\upsilon,s)}, \tag{29}$$

where $F(\upsilon) = \mathcal{G}_x[\psi(x,0)]$ and $G(s) = \mathcal{G}_t[\psi(0,t)]$.

Applying the inverse transform $\mathcal{G}_x^{-1}\mathcal{G}_t^{-1}$ to (29), we obtain the solution of (27) as follows:

$$\psi(x,t) = \mathcal{G}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{\upsilon s W(\upsilon,s) + \upsilon^2 s G(s) + \upsilon s^2 F(s)}{\upsilon^2 s + \upsilon s^2 - a\Phi(\upsilon,s)}\right]. \tag{30}$$

We illustrate the above technique by the following example.

Example 4. Consider the following partial integro-differential equation:

$$\frac{\partial\psi(x,t)}{\partial x} + \frac{\partial\psi(x,t)}{\partial t} = -1 + e^x + e^t + e^{x+t} + \int_0^x\int_0^t \psi(x-\delta,t-\varepsilon)d\delta d\varepsilon \tag{31}$$

with the conditions

$$\psi(x,0) = e^x,\ \psi(0,t) = e^t. \tag{32}$$

Solution. Computing the following transforms of the conditions (32) and the source function $\omega(x,t)$ as

$$\begin{cases} P(v) = \mathcal{G}_x[e^x] = \frac{v}{v-1}, \\ Q(s) = \mathcal{G}_t\left[e^t\right] = \frac{s}{s-1}, \\ W(v,s) = \mathcal{G}_x\mathcal{G}_t\left[-1+e^x+e^t+e^{x+t}\right] = \frac{(2sv-1)}{(s-1)(v-1)}. \end{cases} \tag{33}$$

Substituting the values (33) into (30) and after simple computations, we obtain the solution of (31):

$$\psi(x,t) = \mathcal{G}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{vs}{(-1+s)(-1+v)}\right] = e^{x+t}. \tag{34}$$

(III) Second order partial integro-differential equations.

Consider the following partial integro-differential equation:

$$\frac{\partial^2\psi(x,t)}{\partial t^2} - \frac{\partial^2\psi(x,t)}{\partial x^2} + \psi(x,t) + \int_0^x\int_0^t \phi(x-\delta,t-\varepsilon)\psi(\delta,\varepsilon)d\delta d\varepsilon = \omega(x,t) \tag{35}$$

with the conditions

$$\begin{aligned} \psi(x,0) = f_0(x),\ \frac{\partial\psi(x,0)}{\partial t} = f_1(x), \\ \psi(0,t) = g_0(t),\ \frac{\partial\psi(0,t)}{\partial x} = g_1(t). \end{aligned} \tag{36}$$

Applying the DARAT on both sides of (36), we obtain

$$\begin{aligned} s^2\Psi(v,s) - s^2\mathcal{G}_x[\psi(x,0)] - s\mathcal{G}_x\left[\tfrac{\partial\psi(x,0)}{\partial t}\right] - \left(v^2\Psi(v,s) - v^2\mathcal{G}_t[\psi(0,t)] - v\,\mathcal{G}_t\left[\tfrac{\partial\psi(0,t)}{\partial x}\right]\right) + \Psi(v,s) \\ +\tfrac{1}{vs}\Psi(v,s)\Phi(v,s) = W(v,s). \end{aligned}$$

After simple calculations, one can obtain

$$\Psi(v,s) = \frac{s^3F_0(v) + s^2F_1(v) - vsG_0(s) - sG_1(s) + sW(v,s)}{s^3 - v^2s + s + \Phi(v,s)}, \tag{37}$$

where $F_0(v) = \mathcal{G}_x[\psi(x,0)]$, $F_1(v) = \mathcal{G}_x\left[\frac{\partial\psi(x,0)}{\partial t}\right]$, $G_0(s) = \mathcal{G}_t[\psi(0,t)]$ and $G_1(s) = \mathcal{G}_t\left[\frac{\partial\psi(0,t)}{\partial x}\right]$.

Applying the inverse transform $\mathcal{G}_x^{-1}\mathcal{G}_t^{-1}$ to (37), we obtain the exact solution of (35) as follows:

$$\psi(x,t) = \mathcal{G}_x^{-1}\mathcal{G}_t^{-1}\left[\frac{vs^3F_0(v) + vs^2F_1(v) - sv^3G_0(s) - v^2sG_1(s) + vsW(v,s)}{vs^3 - v^3s + vs + \Phi(v,s)}\right]. \tag{38}$$

We illustrate the above technique by the following example.

Example 5. Consider the following partial integro-differential equation:

$$\frac{\partial^2\psi(x,t)}{\partial t^2} - \frac{\partial^2\psi(x,t)}{\partial x^2} + \psi(x,t) + \int_0^x\int_0^t e^{x-\delta+t-\varepsilon}\psi(\delta,\varepsilon)d\delta d\varepsilon\ = e^{x+t} + xte^{x+t} \tag{39}$$

with the conditions

$$\begin{aligned} \psi(x,0) = e^x,\ \psi_t(x,0) = e^x, \\ \psi(0,t) = e^t,\ \psi_t(0,t) = e^t. \end{aligned} \tag{40}$$

Computing the following transformed values of the conditions (40) and the functions $\Phi(\upsilon, \jmath)$ and $\omega(x, t)$ provides

$$\begin{cases} F_0(\upsilon) = F_1(\upsilon) = \frac{\upsilon}{\upsilon - 1}, \\ G_0(\jmath) = G_1(\jmath) = \frac{\jmath}{\jmath - 1}, \\ \Phi(\upsilon, \jmath) = \frac{\upsilon \jmath}{(-1+\jmath)(-1+\upsilon)}, \\ W(\upsilon, \jmath) = \frac{\jmath \upsilon (2 + \jmath(-1+\upsilon) - \upsilon)}{(-1+\jmath)^2(-1+\upsilon)^2}. \end{cases} \tag{41}$$

Substituting the values in (41) into (38) and simplifying, one can obtain the solution of Equation (39) as follows:

$$\psi(x, t) = \mathcal{G}_x^{-1} \mathcal{G}_t^{-1} \left[\frac{\upsilon \jmath}{(-1+\jmath)(-1+\upsilon)} \right] = e^{x+t}. \tag{42}$$

5. Conclusions

In this study, DARAT approach is presented to solve integral differential equations. Theorems and fundamental properties of the new double transformation are introduced. We discuss two kinds of integral equations: partial integral equations and partial integro-differential equations of first and second orders. Some illustrative examples are discussed to show the validity and efficiency of DARAT in solving the proposed equations. In the future, we will solve nonlinear partial integro-differential equations and fractional partial differential equations.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declare no conflict of interests.

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