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A Novel Model for Economic Recycle Quantity with Two-Level Piecewise Constant Demand and Shortages

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Abstract: This paper focuses on the production systems that may produce a proportion of recyclable defective products. The developed model is called an Economic Recycle Quantity (ERQ) model with the assumption of a full recovery of defective items. The defective parts are collected during the production-off time and can be used during the next production cycle of the same category. The demand rate of the non-defective items is a two-level piecewise factor—one during the production-run time and another during the production-off time. The developed model aims to optimize the total inventory cost, the order quantity, and the amount of recyclable defective items that represent the ERQ. The mathematical formulations of the model are deduced theoretically. The model was solved analytically, and the optimal results are illustrated. Sensitivity analysis is carried out to investigate the effect of varying system parameters and validate the proposed model. Results of the sensitivity analysis show that the consideration of defective part recycling reduces the total inventory cost where the raw material is reduced. The cost reduction is about 1%; of course, the environmental impact is more appreciated. Furthermore, the managerial implications are described, and the future perspectives are discussed.

Keywords: Economic Recycle Quantity (ERQ); inventory; shortages; production-run time; production-off time

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Citation: Attia, E.-A.; Miah, M.M.; Arif, A.S.; AlArjani, A.; Hasan, M.; Uddin, M.S. A Novel Model for Economic Recycle Quantity with Two-Level Piecewise Constant Demand and Shortages. *Computation* **2024**, *12*, 13. https://doi.org/10.3390/computation12010013

Academic Editor: Pedro G. Lind

Received: 10 October 2023
Revised: 21 December 2023
Accepted: 2 January 2024
Published: 11 January 2024

1. Introduction

For reasons of the turbulent working environment due to rapid market changes, instability of demand, and changes of customer needs, organizations should have inventory management policies. These policies aim to identify precisely which items are to be purchased, with which quantities, and at which times. Adopting robust inventory management policies can reduce the risk of material shortages and the associated consequences. Inventory management has a set of main functions that include forecasting, ordering policy, material handling, material storing, monitoring, and continuous recording of different transactions (adding or withdrawing). The efficient implementation of methods and algorithms to achieve these functions leads to satisfying customer needs and producer demands on time at minimum cost. In inventory management, these methods and algorithms are known as models. There are many inventory models including planning and classification models. Planning models identify the right number of materials/items/products to be ordered/stocked and how often to reorder. The main objectives of planning models are to minimize the cost of keeping supplies in inventory, avoid the cost of item shortages, and achieve satisfactory service levels. Inventory managers determine the ideal inventory order quantity using the well-known Economic Order Quantity (EOQ) or Economic Production

Quantity (EPQ) according to the application. In the beginning, Taft [1] extended the EOQ model introduced by Harris [2] by assuming ideal product quality and faultless manufacturing procedures. Richter [3,4] discussed the EOQ model with variable setup numbers for repair and waste disposal.

In industry, defective items can be produced as production wastages. If a manufacturer produces defective items, the manufacturer may prefer to send them out for rework, rather than simply salvage them at a lower price. Reworked items are generally considered new and sold at full retail price. Items of suboptimal quality can be dealt with either by reformulating them and making them suitable for the primary or secondary market or by disposing of them. Rework and recovery options also play a vital role in dealing with imperfect quality articles. In some cases, the imperfect items can be completely recycled. Both EOQ and EPQ models that consider defective/imperfect products have been found in the literature for years. For example, an EPQ model was created by Salameh and Jaber [5] based on the assumption that defective units can be sold as a single batch following the 100% inspection process. Later, an EPQ model was designed by Hayek and Salameh [6] that considers the repair of a proportion of defective products in a fixed repair time. Defective products can be considered repairable or non-repairable products according to Chiu [7]. Furthermore, Eroglu and Ozdemir [8] along with Wang et al. [9] investigated inventory models with faulty items. Krishnamoorthi [10] adapted Panayappan and Krishnamoorthi's [11] inventory model with product life cycles by including defective products to accommodate shortages.

In many practical situations, the deterministic demand rates for non-defective items may fluctuate from one stage of the manufacturing cycle to the next. Because of their freshness or other considerations, some products may be in higher demand during the production-run time than during the production-off time. Some products may be in higher demand following a manufacturing halt caused by a stock-out panic. Due to different uncertainties, supply shortages are unavoidable in many practical scenarios. Production inventory models for two-level continuous demand patterns have not been created. On the other hand, most models of defective production systems consider reworks, repair, and remanufacture, as well as different production and supply policies. However, the consideration of recycling is missing; only a few attempts were performed.

As a result, there is a research gap in production inventory models relating to raw material recovery by recycling systems with two-level constant demand rates. The model in this paper is inspired by Oh and Hwang's [12] and Bai and Varanasi's [13] inventory models, and it introduces a recycling process of defective items with 100% recovery under two-level piecewise constant demand rates during the production-run time and production-off time under shortages. To the best of our knowledge, this consideration is not covered in the literature yet. This study may help inventory managers handle such inventory problems by considering the recycling process of defective items. The recycling concept can be considered an extension of the EPQ model. This extension could be named the "Economic Recycle Quantity (ERQ) model". In ERQ, the goal is to determine the optimal acceptable amount of damaged products that can be recycled while maintaining a low total inventory cost. The proposed ERQ model considers the different types of inventory costs, including setup, raw material, production, shortage, holding, and recycling costs.

This paper is organized as follows: The next section introduces a literature review, followed by a problem statement section. Section 4 introduces the mathematical model of the problem considered. Section 5 presents the solution procedures of the model. After that, a numerical analysis is presented in Section 6, followed by a sensitivity analysis and discussion in Section 7. Finally, the conclusions and perspectives are discussed.

2. Literature Review

The literature provides many EPQ models for dealing with imperfect production. For example, Schrady [14] was the first to consider the reuse of repairable items. Chiu S. and Chiu Y. [15] considered the rework of repairable defective items in their developed EPQ. King et al. [16] discussed waste reduction through repairing, reconditioning, re-manufacturing, or recycling. An optimal EPQ policy for defective operations and flawed repair was developed by Liao et al. [17]. Hsu et al. [18] compared a set of four strategic actions for the waste reduction of end-of-life waste, namely repair, reconditioning, re-manufacturing, or recycling. These strategies are adopted from the global concern of environmental protection by reusing metals, paper, glass, etc. Liao and Sheu [19] constructed an EPQ model for the production process with stochastic failure considering minimum repairing and defective maintenance. An optimal replenishment policy for deteriorating items was developed by Dye and Hsieh [20] where they emphasized effective investment in preservation technology. Taleizadeh et al. [21] discussed repair failure and limited capacity. Al-Salamah [22,23] determined the decision variables of their EPQ using bee colony heuristics considering imperfect quality and random equipment failure and repair.

In addition, the inventory models can be distinguished by the nature of the demand. The demand could be constant, stock-dependent, time-dependent, or price-dependent demand patterns. Gupta and Vrat [24] first introduced the stock-dependent demand model. Baker and Urban [25] later included stock-dependent demand patterns in a power form, while Pal et al. [26] modified their model to include part deterioration. Datta and Pal [27] presented a deterministic model without shortages that considers decaying objects. Giri and Chaudhuri [28] established probabilistic models of perishable products with rates matched to stock-dependent demand and nonlinear inventory holding costs. Many more studies, such as Sarker et al. [29], examined inventory models of stock-dependent demand. Manna et al. [30] discussed an EPQ model with a production rate-dependent defective rate and advertisement-dependent demand and then developed an algorithm to find the optimal profit of the imperfect production inventory model. Mashud et al. [31] proposed a non-instantaneous inventory model with stock- and price-dependent demand in the presence of partially backlogged shortages. Shaikh et al. [32] established a stock-dependent demand with a price discount facility under partial backlogging. Panda et al. [33] used accumulated credit plans in a two-warehouse inventory model for the deterioration of commodities with price-dependent and stock-dependent demand based on a partial backlog. Of course, the demand is not regular over the planning horizon. Consequently, the real demand model should gather more than one pattern. A piecewise function can be used to join the different natures of the demand along the planning horizon. Datta and Pal [27] considered the demand rate as a piecewise function of the inventory level. Later on, piecewise constant demand was also proposed by Bai and Varanasi [13]. Bhunia and Shaikh [34] studied inventory level-dependent demand rates. Koh et al. [35] proposed a combined EOQ and EPQ model in which demand is projected to come from two sources: recycled and freshly purchased commodities. Oh and Hwang [12] proposed a deterministic inventory model in which a portion of demand is returned and reused for the creation of new products after recycling, with the holding costs of raw materials and usable items determined unequally. Leal Filho W. et al. [36] investigated current textile recycling trends and the associated socioeconomic benefits.

The development of EPQ models considering different aspects of the problem is still an active topic in the literature. Gharaei et al. [37] developed a bi-objective mixed-integer linear programming model to optimize the cost and the profit simultaneously using an EPQ inventory model. Their model aims to determine the optimal number of supply shipments and the quantity of each product shipment. The developed EPQ model considers the idea of defective production. They classified the defective items into two categories: imperfect items that will be sold at a low price and scrapped items that will be salvaged. Ruidas et al. [38] developed an EPQ model considering the concept of repair of imperfect production. The defective items can be repaired if defects are discovered before being sent to the customer, but they will be refunded if they are returned by the customer. The production rate is dependent on the demand rate. However, the demand rate is dependent on the stock level and the selling price. The developed model considers the interval nature of the estimated factors, not the deterministic numbers. The model was then solved by particle swarm optimization (PSO). Another work of Zidan et al. [39] developed an EPQ model in which production batches are classified as complete or incomplete, relying on 100% inspection. The incomplete batches are divided into defective and non-defective. All complete, non-defective lots are stored, resulting in a holding cost, and all defective lots are rejected, resulting in a disposal cost. Backorder costs occur when a customer agrees to pick up the order in the next period and lost sales costs occur when a customer does not accept the order in the next period. Besides these considerations, their developed model considered the setup and backorders. Additionally, Biswas and Schultz [40] considered an EPQ model with two-stage manufacturing processes. The first stage deals with a normal production run that may have imperfect items. The second manufacturing process focuses on the repair processes. Scrapped items can be discovered during the production, inspection, or repair processes. Ganesan and Uthayakumar [41] developed two inventory models dealing with a fixed-length and a variable-length warm-up production period. The considered production cycle is divided into four segments: warm-up, maintenance, standard production, and repair. The defective rate of the warm-up production period is different from that of the production period. Furthermore, they assumed that all the defective items could be 100% repaired. Mokhtari et al. [42] developed an EPQ model considering the imperfect production of items. They assumed that imperfect production could be recovered completely with the repair processes. Their model considers some production constraints e.g., machine capacity constraints, inventory space constraints, budget constraints, and the number of setup constraints.

Recently, Sharma et al. [43] developed an EPQ model considering imperfect production and a deteriorating item that deteriorates with the Weibull distribution of time. The demand is a function of the selling price and the stock level; they also considered the inflation rate. Nobil et al. [44] considered an EPQ model with scrapping and reworking items while modeling a machine warm-up time that is related to its downtime. They extended the model developed by Nobil et al. [45]. The rate of defective parts is associated with the period type of warm-up or normal working. The defective rate of the warm-up period is greater than of the normal working period. A shortage is not allowed in their model. Priyan et al. [46] proposed to consider carbon emissions from logistics, manufacturing processes, and storage activities. For imperfect production, they considered rework processes that could be synchronous or asynchronous with flawless operations. Different costs were also considered, e.g., setup costs, inspection, shortage, and inventory costs. They assumed that the manufacturer had a plan of investment to switch to green energy with a budget constraint. Narang et al. [47] considered the problem of defective items in a multi-stage supply chain with suppliers and manufacturers. The developed model considers carbon emissions. The production process produces defective and perfect items. These defective items can be reworked or scrapped. The product demand is advertising dependent. This advertising is paid for by the manufacturer. No shortage is allowed where the production rate is greater than the customer demand. In addition, Kausar et al. [48] developed an inventory model that considers a defective production system in which the defective items are sold. In their model, they considered the cost of energy usage. Moreover, the demand is not constant, it depends on the level of advertisement and selling price. Gautam et al. [49] presented a two-decade literature review over the period from 2000 up to 2020; the scientific contributions considered an imperfect-quality item for EOQ or EPQ models. In addition, Karim and Nakade [50] reviewed the literature by focusing on the integration of EPQ, carbon emissions, and recycling. They highlighted the research gap in these directions. They also showed a lack of consideration for product recycling, carbon emissions, and stochastic models. In addition, Table 1 shows the lack of consideration of the product-recycling concept, besides the piecewise function of the demand. Responding to such need, the current paper proposes to integrate an EPQ model with the product-recycling concept and piecewise function demand.

Table 1. Comparative study based on the state-of-the-art review.

	Reference	EOQ	EPQ	ERQ	Setup	Backorders	Imperfect	Inspection	Defective Parts				Production Rate	Demand		Other Considerations
									Sold	Recycle	Repairable Items			Constant	Piecewise	
											Scrap	Reworked				
1	Salameh and Jaber [5]	×	×				×	100%	×					×		Yearly demand
2	Hayek and Salameh [6]		×		×	×	×					×	Constant	×		Reworked during off time
4	Chiu S. and Chiu Y. [15]		×		×						×	×	Constant	×		
5	Chan et al. [51]		×		×		×	100%			×	×		×		Sold non-repairable at a lower price
6	Eroglu and Ozdemir [8]	×				×	×	P			×			×		Sold non-repairable at a lower price
7	Wang et al. [9]	×				×	×	P			×			×		Sold non-repairable at a lower price
10	Krishnamoorthi [10]		×		×	×	×		×					×		
11	Dye and Hsieh [20]					×								×		
12	Singh et al. [52]		×		×	P	×	×				×	Constant	×		
13	Tai [53]		×		×	×	×	×			×	×	Constant	×		
14	Pal et al. [54]		×				×	×				×	Constant	×		
15	Sarkar et al. [55]		×		×	×	×	100%				×	Constant	×		Neglected the inspection cost
17	Chiu et al. [56]		×		×		×	100%			×	×				Multi-product and multi-deliveries
20	Priyan and Uthayakumar [57]		×		×		×	100%			×	×	Constant			Dependent demand, multiple shipments
22	Viji and Karthikeyan [58]		×			×								×		
23	Ritha and Priya [59]		×		×		×	P				×	Multi-level	×		Cost: energy, transportation, emission
24	Khanna et al. [60]		×		×		×	×			×	×	Constant			Inspection error, returns, trade credits
25	Manna et al. [30]		×				×	P	×				Constant			Demand (advertisement, deprecation)
26	Al-Salamah [23]		×		×	×	×	100%				×	Constant	×		Rework rate, repair synchronous asynchronous with production
27	Ruidas et al. [38]		×		×	×	×					×	Dependent			Demand (stock level, selling price)
28	Ganesan and Uthayakumar [41]		×		×	×						×	Constant	×		
29	Gharaei et al. [37]		×				×	100%			×		Constant	×		
30	AlArjani et al. [61]		×	×	×	×	×	100%		×			Constant	×	×	Three-level constant demand
31	Mokhtari et al. [42]		×		×							×	Constant	×		Muti-products
32	Biswas and Schultz [40]		×			×		×			×	×	Constant	×		
33	Zidan et al. [39]		×		×	×	×	100%	×				Constant	×		
34	Priyan et al. [46]		×		×	×		×				×	Constant	×		Synchronous or asynchronous rework
35	Sharma et al. [43]		×		×	P			×				Constant			Demand (stock level, selling price)
36	Kausar et al. [48]		×		×			×	×				Constant			Demand (selling price, advertisement)
38	Nobil et al. [44]		×		×						×	×	Multi-level	×		
39	Narang et al. [47]		×		×			×			×	×	Constant		×	Supply chain considering carbon emissions
40	The current manuscript		×	×	×	×	×	100%		×			Constant	×	×	Two-level constant demand

P: Proportion.

3. Problem Statement

In almost all industrial manufacturing, defective products can be found beside perfect production. These defective products can be repaired, reconditioned, or even recycled. A recycling concept to recover defective items in inventory planning is considered in this article, wherein the production systems may produce a proportion of recyclable defective products. These defective parts can be screened and separated along the production processes. Accordingly, the defective parts are collected during the production-off time and can be used during the next production cycle of the same category. Manufacturing systems of bricks can fit this nature of production lines with continuous screening during the different manufacturing operations. The identification of defective materials takes place during stages such as setting, drying, and firing in the brick industry. Before the firing process, the faulty components are recovered. During periods when production is on temporary hiatus, these flawed bricks are recycled and combined with raw materials.

Regarding the demand, products like bricks, mustard oil, paper, plastics, jute, and clothing may have varying constant rates during both production periods and periods of inactivity. In various practical scenarios, the demand patterns for non-defective items may exhibit fluctuations across different stages of the manufacturing cycle. Certain products may experience higher demand during periods of activity than inactivity, probably due to factors like freshness or other considerations. In the aftermath of a stock-out panic that disrupts manufacturing operations, there may be a surge in demand for certain products. In many practical situations, supply shortages are inevitable due to various uncertainties, leading to heightened demand for specific items. The primary objective of this research is to facilitate the recovery of raw materials in production inventory systems through recycling systems that feature two-level constant demand rates. Accordingly, the Economic Recycling Quantity (ERQ) model is mathematically formulated. The ERQ model is formulated relying on the concept of recycling defective parts. These defective parts are taken and completely recycled to be considered as a part of the raw material for the next production period. The resultant inventory model is represented in the next section. The different costs are taken into consideration and include the setup cost, raw material cost, production cost, item shortage cost, holding cost, and recycling cost. To delineate this inventory problem, the following steps were implemented:

- Formulate the mathematical model of the EPQ model.
- Expand the EPQ model to formulate the mathematical model of the ERQ model.
- Solve both models optimally.
- Perform sensitivity analysis of the developed inventory models.

4. Mathematical Formulation

4.1. Model Assumptions

The following points summarize the model assumptions:

1. The production rate is known, constrained, constant, and is greater than the sum of demand and defective rates.
2. A fixed portion of defective items is randomly produced.
3. The demand rate of the good product is a piecewise constant function:

$$\begin{cases} d; \text{during production-run time} \\ cd; \text{during production-off time} \end{cases}$$

4. Production lead time is zero.
5. This is a single-product manufacturing system.
6. Defective products are completely recyclable. In addition, the recycled material can be used in the manufacturing process of the same products during the next production cycle time.
7. The item holding cost for a defective or good product is the same.
8. Different cost parameters are known and fixed.

4.2. Formulation of Total Inventory Cost

On-hand inventory situations are depicted in Figures 1 and 2 below.

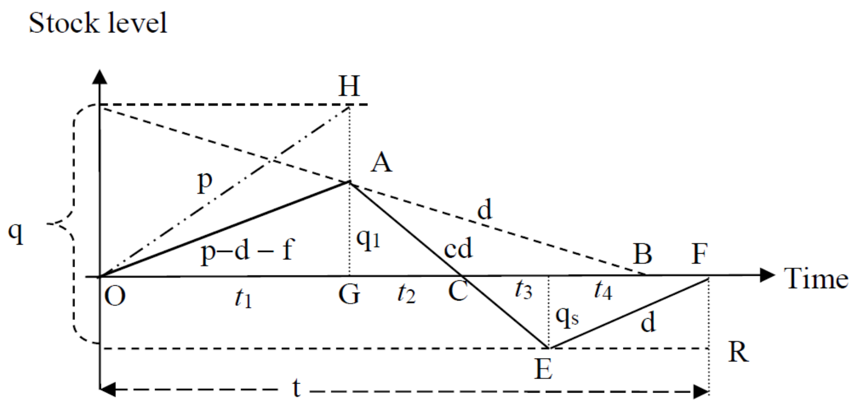

Figure 1. On-hand inventory when c > 1.

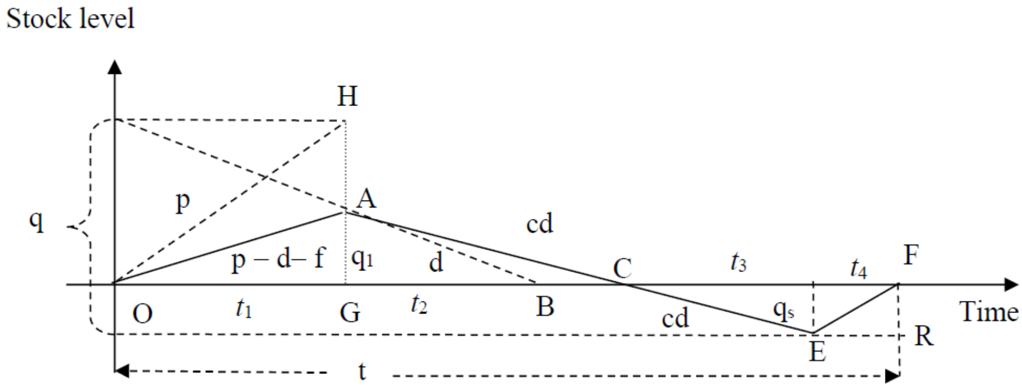

Figure 2. On-hand inventory when 0 < c < 1.

The replenishment rate of non-defective items is 'p − d − f' during the production-run time and the stock starts to be positive through the line OA during time t_1 and replenishment ends at point A with maximum q_1 units of non-defective items. If f = 0 and d = 0, the inventory would be replenished through the line OH.

Therefore,

$$t_1 = \frac{q_1}{p - d - f} \tag{1}$$

During t_2, inventory decreases at the demand rate 'cd' through the line AC and at the end of t_2, the stock level reaches level C with zero on-hand storage. If the demand rate is d during t_2, then the inventory line would be decreased through line AB. Therefore,

$$t_2 = \frac{q_1}{cd} \tag{2}$$

$$\text{Hence } t_1 + t_2 = \frac{\{p - (1 - c)d - f\}q_1}{cd(p - d - f)} \tag{3}$$

The stock starts to be negative with the demand rate 'cd' during time t_3 and shortage reaches q_s at the point E. Thus

$$t_3 = \frac{q_s}{cd} \tag{4}$$

Production starts with the beginning of time t_4 and the shortfall quantity q_s is fulfilled with the rate of 'p − d − f' in addition to the period demand that is satisfied with rate 'd' and stock becomes zero again at the point F at the end of t_4. Therefore,

$$t_4 = \frac{q_s}{p - d - f} \tag{5}$$

$$\text{And } t_3 + t_4 = \frac{\{p - (1-c)d - f\}q_s}{cd(p - d - f)} \tag{6}$$

$$\text{Also } t_1 + t_4 = \frac{w}{f} = \frac{q}{p} \tag{7}$$

Adding (1) and (5) and using (7), we obtain

$$q_1 + q_s = (p - d - f)\frac{w}{f} \tag{8}$$

From similar triangle property,

$$\frac{t_3}{q_s} = \frac{t_2}{q_1} = \frac{t_3 + t_2}{q_s + q_1}$$

$$\frac{t_2}{cdt_2} = \frac{t_3 + t_2}{(p - d - f)\frac{w}{f}}$$

$$t_2 + t_3 = (p - d - f)\frac{w}{cfd} \tag{9}$$

$$\text{Therefore, } t = t1 + t2 + t_3 + t_4 = \frac{\{p - (1-c)d - f\}w}{cfd} \tag{10}$$

$$\text{Then, } \frac{t_3 + t_4}{t} = \frac{f.q_s}{w(p - d - f)} \tag{11}$$

$$\frac{t_1 + t_2}{t} = \frac{f.q_1}{(p - d - f)w} \tag{12}$$

If we decide to recycle defective items, then the inventory model is an ERQ model; otherwise, it is a standard EPQ model. Therefore, two cases arise: Case 1 represents an EPQ model with defective items for a two-level piecewise constant demand that allows shortage. Case 2 represents a new proposed model named the ERQ model with defective items for a two-level piecewise constant demand that allows shortage.

4.2.1. Inventory Cost for Case 1

The holding cost of defective items is ignored in this case. Thus,

$$\text{Total inventory} = \frac{1}{2}q_1 \times t_1 + \frac{1}{2}q_1 \times t_2 = \frac{1}{2}q_1(t_1 + t_2)$$

$$\text{And the average inventory} = \frac{1}{2}q_1\left(\frac{t_1 + t_2}{t}\right) = \frac{fq_1{}^2}{2(p - d - f)w} = \frac{\{(p - d - f)W - f.q_s\}^2}{2f(p - d - f)w}$$

The following equations can represent the different terms of the inventory cost function per unit time:

(1) Average setup cost = $\frac{1}{t}O = \frac{cfdO}{\{p-(1-c)d-f\}w}$

(2) Average production cost = $\frac{1}{t}qK = \frac{cpdK}{\{p-(1-c)d-f\}}$

(3) Average shortage cost = $\frac{q_s}{2}\frac{t_3+t_4}{t}S = \frac{S.f.q_s{}^2}{2w(p-d-f)}$

(4) Raw material cost = $\frac{1}{t}qR = \frac{cpdR}{\{p-(1-c)d-f\}}$

(5) Holding cost = $\frac{\{(p-d-f)w-f.q_s\}^2}{2f(p-d-f)w}H$

Therefore,

$$TC_1 = \frac{cfdO}{\{p - (1-c)d - f\}w} + \frac{cpdR}{\{p - (1-c)d - f\}} + \frac{cpdK}{\{p - (1-c)d - f\}} + \frac{fq_s{}^2S}{2w(p - d - f)} + \frac{\{(p - d - f)w - fq_s\}^2H}{2f(p - d - f)w} \tag{13}$$

Without considering the recycling concept, Equation (13) represents the total inventory cost based on EPQ. TC_1 is a function of 'w' and q_s. The main objective is to minimize both the number of defective units and the shortage that leads to minimum inventory cost.

4.2.2. Inventory cost for Case 2

During the production run, a defined proportion of the defective items that can be recycled are received. During the production-off period, the recycling procedure is carried out. As a result, the recycled raw materials are returned to the production cycle's manufacturing process. The additional supplies needed are obtained from suppliers. The process is depicted in Figure 3.

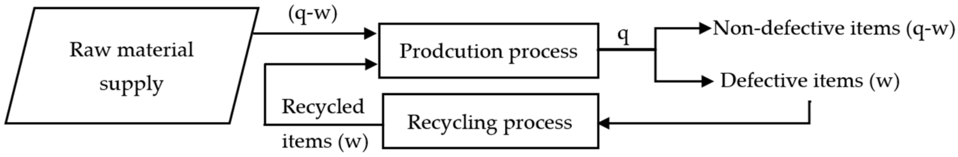

Figure 3. Production system with recycling.

In this case, the cycle starts with the shortage amount q_s in the inventory which is served at the rate '$p - d - f$' over the time t_4, alongside the current demand, which is also fulfilled at the rate 'd'. In the production-run time $t_1 + t_4$, w units of defective items are produced and they are stored properly to be recycled during the production-off time t_2 + t_3.

The holding cost for defective items is included during $t_1 + t_4$ time before sending them for the recycling process as perfect items. Therefore,

$$\begin{aligned}\text{Total Inventory} &= \tfrac{1}{2}q_1 \times t_1 + \tfrac{1}{2}q_1 \times t_2 + \tfrac{1}{2}f(t_1 + t_4) \times (t_1 + t_4)\\ &= \tfrac{1}{2}q_1(t_1 + t_2) + \tfrac{1}{2}f(t_1 + t_4)^2\end{aligned}$$

$$\begin{aligned}\text{Average Inventory} &= \tfrac{1}{2}q_1\left(\tfrac{t_1+t_2}{t}\right) + \tfrac{1}{2}f\tfrac{(t_1+t_4)^2}{t}\\ &= \tfrac{1}{2}q_1\tfrac{fQ_d}{(p-d-f)w} + \tfrac{cdw}{2\{p-(1-c)d-f\}} = \tfrac{fq_1{}^2}{2(p-d-f)w} + \tfrac{cdw}{2\{p-(1-c)d-f\}}\\ &= \tfrac{f\left\{(p-d-f)\frac{w}{d}-q_s\right\}^2}{2(p-d-f)w} + \tfrac{cdw}{2\{p-(1-c)d-f\}}\end{aligned}$$

A total of w units are recycled during the production-off time. In each cycle, the inventory managers procure the required raw materials for $q - w$ units after the first cycle, then the process repeats. The following formula represent the different components of the inventory total cost per unit time:

(1) Average setup cost = $\frac{1}{t}O = \frac{cfdO}{\{p-(1-c)d-f\}w}$
(2) Average production cost = $\frac{1}{t}qK = \frac{cpdK}{\{p-(1-c)d-f\}}$
(3) Average shortage cost = $\frac{q_s}{2}\frac{t_3+t_4}{t}S = \frac{fq_s{}^2}{2w(P-D-d)}S$
(4) Raw material cost = $\frac{1}{t}(q-w)R = \frac{c(p-f)dR}{\{p-(1-c)d-f\}}$
(5) Holding Cost = $\left[\frac{f\left\{(p-d-f)\frac{w}{d}-q_s\right\}^2}{2(p-d-f)w} + \frac{cdw}{2\{p-(1-c)d-f\}}\right]H$
(6) Average recycling cost = $\frac{1}{t}wr = \frac{cfdr}{p-(1-c)d-f}$

$$\text{Then, } TC_2 = \frac{cfdO}{\{p-(1-c)d-f\}w} + \frac{cpdK}{\{p-(1-c)d-f\}} + \left[\frac{\{(p-d-f)w-dq_s\}^2}{2f(p-d-f)w} + \frac{cdw}{2\{p-(1-c)d-f\}}\right]H + \frac{fq_s{}^2S}{2w(p-d-f)} + \frac{c(p-f)dR}{\{p-(1-c)d-f\}} + \frac{cfdr}{p-(1-c)d-f} \quad (14)$$

Considering the recycling of the defective production, Equation (14) represents the total inventory cost based on ERQ. TC_2 is a function of 'w' and q_s to minimize the total cost of the inventory. The optimal quantity of the defective units (ERQ) and the optimum shortage level can be identified along with the corresponding optimal production lot size.

5. Optimal Solution

The total cost functions TC_1 and TC_2 are both functions of 'w' and 'q_s', the convexity of them will be justified by a Hessian matrix, and a unique optimal solution is obtained. Special cases of the optimal values of both models are also discussed.

5.1. Optimal Solution of Case 1

Theorem 1 proves the convexity of TC_1 and the uniqueness of the optimal solution.

Theorem 1. *TC_1 is a strictly convex function of w and q_s simultaneously, as shown by Equation (13) and there is only one optimal solution of w^* and ${q_s}^*$.*

Proof. The first order partial derivatives of TC_1 are as follows:

$$\frac{\partial(TC_1)}{\partial w} = -\frac{cfdO}{\{p-(1-c)d-f\}w^2} - \frac{f{q_s}^2}{2(p-d-f)w^2}(S+H) + \frac{(p-d-f)H}{2f} \tag{15}$$

$$\frac{\partial(TC_1)}{\partial q_s} = \frac{fq_s}{w(p-d-f)}S - H + \frac{fq_s}{(p-d-f)w}H \tag{16}$$

Moreover, the second order partial derivatives of TC_1 are as follows:

$$\frac{\partial^2(TC_1)}{\partial w^2} = \frac{2cfdO}{\{p-(1-c)d-f\}w^3} + \frac{f{q_s}^2}{(p-d-f)w^3}(S+H) \tag{17}$$

$$\frac{\partial^2(TC_1)}{\partial {q_s}^2} = \frac{f}{w(p-d-f)}(S+H) \tag{18}$$

$$\frac{\partial^2(TC_1)}{\partial w \partial q_s} = -\frac{fq_s}{(p-d-f)w^3}(S+H) \tag{19}$$

$$\frac{\partial^2(TC_1)}{\partial q_s \partial w} = -\frac{fq_s}{(p-d-f)w^3}(S+H) \tag{20}$$

The Hessian matrix of TC_1 is

$$H_{ij} = \begin{bmatrix} \frac{\partial^2(TC_1)}{\partial w^2} & \frac{\partial^2(TC_1)}{\partial w \partial q_s} \\ \frac{\partial^2(TC_1)}{\partial q_s \partial w} & \frac{\partial^2(TC_1)}{\partial {q_s}^2} \end{bmatrix}$$

Thus, the first principal minor is

$$|H_{11}| = \frac{\partial^2(TC_1)}{\partial w^2} = \frac{2cfdO}{\{p-(1-c)d-f\}w^3} + \frac{f{q_s}^2}{(p-d-f)w^3}(S+H)$$

It is assumed that $p-d-f>0$ and so $p-(1-c)d-f>0$. Therefore, $|H_{11}|>0$. In addition, the second principal minor is

$$\begin{aligned} |H_{22}| &= \frac{\partial^2(TC_1)}{\partial w^2}\frac{\partial^2(TC_1)}{\partial {q_s}^2} - \frac{\partial^2(TC_1)}{\partial q_s \partial w}\frac{\partial^2(TC_1)}{\partial w \partial q_s} \\ &= \frac{f}{(p-d-f)}\frac{2cfdO}{\{p-(1-c)d-f\}w^4}(S+H) > 0 \end{aligned}$$

The first and second principal minors of the Hessian matrix for TC_1 are positive and the Hessian matrix is positive definite. Hence, TC_1 is a non-negative, differentiable, and strictly convex function with respect to w and q_s concurrently. Therefore, the objective function TC_1 has the global minimum value $w^*, {q_s}^*$. □

Solving $\frac{\partial(TC_1)}{\partial q_s} = 0$ and $\frac{\partial(TC_1)}{\partial w} = 0$, the following results are obtained,

$$\frac{q_s}{w} = \frac{(p-d-f)}{f}\left(\frac{H}{S+H}\right) \tag{21}$$

And

$$\frac{cfd}{\{p-(1-c)d-f\}w^2}O + \frac{f}{2(p-d-f)}(S+H)\left(\frac{q_s}{w}\right)^2 = \frac{(p-d-f)}{2f}H \tag{22}$$

The optimal solution of TC_1 is obtained as

$$w* = f\sqrt{2dO}\sqrt{\frac{c}{(p-d-f)\{p-(1-c)d-f\}}}\sqrt{\frac{S+H}{SH}} \tag{23}$$

$$q_s{}^* = (p-d-f)\sqrt{\frac{2cdO}{(p-d-f)\{p-(1-c)d-f\}}}\sqrt{\frac{H}{(S+H)S}} \tag{24}$$

$$q* = p\sqrt{2dO}\sqrt{\frac{c}{(p-d-f)\{p-(1-c)d-f\}}}\sqrt{\frac{S+H}{SH}} \tag{25}$$

$$q_1{}^* = (p-d-f)\sqrt{2dO}\cdot\sqrt{\frac{c}{(p-d-f)\{p-(1-c)d-f\}}}\sqrt{\frac{S}{(S+H)H}} \tag{26}$$

$$t^* = \sqrt{\frac{2O}{d}}\sqrt{\frac{\{p-(1-c)d-f\}}{c(p-d-f)}}\sqrt{\frac{S+H}{SH}} \tag{27}$$

Special Cases

(i) If f = 0, then w* = 0.

(ii) If c = 1 and f = 0, then

$$q^* = \sqrt{\frac{2pdO}{(p-d)H}}\cdot\sqrt{\frac{(S+H)}{S}}$$

$$\text{and } t^* = \sqrt{\frac{2O}{dH}}\cdot\sqrt{\frac{(S+H)}{S}}$$

These are the optimal results of the standard EPQ model and hence the model complies with the standard EPQ model.

(iii) If c = 1,

$$w^* = f\sqrt{2dO}\sqrt{\frac{1}{(p-d-f)(p-f)}}\cdot\sqrt{\frac{S+H}{SH}}$$

$$q^* = p\sqrt{2dO}\sqrt{\frac{1}{(p-d-f)(p-f)}}\cdot\sqrt{\frac{S+H}{SH}}$$

$$t^* = (p-f)\sqrt{\frac{2O}{d}}\sqrt{\frac{1}{(p-d-f)(p-f)S}}\sqrt{\frac{S+H}{H}}$$

These numbers indicate the EPQ model's optimal outcomes for constant demand with two levels of demand and shortages.

5.2. Optimal Solution of Case 2

Theorem 2 proves the convexity of TC_2 and hence the uniqueness of the optimal solution.

Theorem 2. *TC_2 is strictly convex with respect to w and Q_s and there exists only one optimal solution of w^* and $q_s{}^*$.*

Proof. The relation between TC_2 and TC_1 is given by

$$TC_2 = TC_1 - \frac{cd}{\{p-(1-c)d-f\}}\left[f(R-r) - \frac{w}{2}H\right] \tag{28}$$

Therefore, the following equality holds

$$\frac{\partial^2(TC_2)}{\partial w^2} = \frac{\partial^2(TC_1)}{\partial w^2}$$

$$\frac{\partial^2(TC_2)}{\partial {q_s}^2} = \frac{\partial^2(TC_1)}{\partial {q_s}^2}$$

$$\frac{\partial^2(TC_2)}{\partial w \partial q_s} = \frac{\partial^2(TC_1)}{\partial w \partial q_s}$$

$$\frac{\partial^2(TC_2)}{\partial q_s \partial w} = \frac{\partial^2(TC_1)}{\partial q_s \partial w}$$

The Hessian matrix of TC_2:

$$H_{ij} = \begin{bmatrix} \frac{\partial^2(TC_2)}{\partial w^2} & \frac{\partial^2(TC_2)}{\partial w \partial q_s} \\ \frac{\partial^2(TC_2)}{\partial q_s \partial w} & \frac{\partial^2(TC_2)}{\partial {q_s}^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2(TC_1)}{\partial w^2} & \frac{\partial^2(TC_1)}{\partial w \partial q_s} \\ \frac{\partial^2(TC_1)}{\partial q_s \partial w} & \frac{\partial^2(TC_1)}{\partial {q_s}^2} \end{bmatrix}$$

The first principal minor: $|H_{11}| = \frac{\partial^2(TC_2)}{\partial w^2} = \frac{\partial^2(TC_1)}{\partial w^2} > 0$.
The second principal minor:

$$|H_{22}| = \frac{\partial^2(TC_2)}{\partial w^2}\frac{\partial^2(TC_2)}{\partial {q_s}^2} - \frac{\partial^2(TC_2)}{\partial q_s \partial w}\frac{\partial^2(TC_2)}{\partial w \partial q_s} = \frac{\partial^2(TC_1)}{\partial w^2}\frac{\partial^2(TC_1)}{\partial {q_s}^2} - \frac{\partial^2(TC_1)}{\partial q_s \partial w}\frac{\partial^2(TC_1)}{\partial w \partial q_s} > 0$$

Hence, the Hessian matrix is positive definite and TC_2 is non-negative, differentiable, and strictly convex with respect to w and q_s simultaneously and there exists a unique optimal solution w^* and ${q_s}^*$. □

The necessary conditions of the minimization of TC_2 are $\frac{\partial(TC_2)}{\partial q_s} = 0$ and $\frac{\partial(TC_2)}{\partial W} = 0$ and we obtain

$$\frac{q_s}{w} = \frac{(p-d-f)}{f}\left(\frac{H}{S+H}\right) \tag{29}$$

And

$$\frac{cfdO}{\{p-(1-c)d-f\}w^2} + \frac{f(S+H)}{2(p-d-f)}\left(\frac{q_s}{w}\right)^2 = \frac{cdH}{2\{p-(1-c)d-f\}} + \frac{(p-d-f)H}{2f} \tag{30}$$

And the unique optimal solution of TC_2 is obtained as:

$$w* = f\sqrt{\frac{2cdO}{(p-d-f)\{p-(1-c)d-f\}S + cfd(S+H)}}\sqrt{\frac{S+H}{H}} \tag{31}$$

$${q_s}^* = (p-d-f)\sqrt{\frac{2cdO}{(p-d-f)\{p-(1-c)d-f\}S + cfd(S+H)}}\sqrt{\frac{H}{S+H}} \tag{32}$$

$$q* = p\sqrt{\frac{2cdO}{(p-d-f)\{p-(1-c)d-f\}S + cfd(S+H)}}\sqrt{\frac{S+H}{H}} \tag{33}$$

$${q_1}^* = (p-d-f)\sqrt{\frac{2cdO}{(p-d-f)\{p-(1-c)d-f\}S + cfd(S+H)}}\frac{S}{\sqrt{(S+H)H}} \tag{34}$$

$$t^* = \sqrt{\frac{2O}{cd}} \frac{\{p-(1-c)d-f\}}{\sqrt{(p-d-f)\{p-(1-c)d-f\}S+cfd(S+H)}} \sqrt{\frac{S+H}{H}} \tag{35}$$

Special Cases

If f = 0, then w* = 0.

That is, if the deficient item production rate is zero, then there is no Economic Recycle Quantity to be recycled.

5.3. Comparative Study with the Existing Methods and the Proposed Methods

(i) **Convergence of Classical EPQ Model:**

If c = 1 and f = 0, then

$$q^* = \sqrt{\frac{2pdO}{(p-d)H}} \cdot \sqrt{\frac{(S+H)}{S}}$$

$$t^* = \sqrt{\frac{2O}{dH}} \cdot \sqrt{\frac{(S+H)}{S}}$$

These are optimal results in the standard EPQ model with constant demand and shortages. Therefore, the proposed model can be converged to the classical EPQ model by considering particular cases.

(ii) **Convergence of ERQ model for constant demand:**

If c = 1, then

$$w* = f\sqrt{2dO}\sqrt{\frac{1}{(p-d-f)(p-f)S+fd(S+H)}} \cdot \sqrt{\frac{S+H}{H}}$$

$$q* = p\sqrt{2dO}\sqrt{\frac{1}{(p-d-f)(p-f)S+fd(S+H)}} \cdot \sqrt{\frac{S+H}{H}}$$

$$t^* = (p-f)\sqrt{\frac{2O}{d}}\sqrt{\frac{1}{(p-d-f)(p-f)S+fd(S+H)}}\sqrt{\frac{S+H}{H}}$$

These formulas represent the optimal results of the ERQ model with defective units and shortages for constant demand.

6. Numerical Analysis and Convexity Graphs

Some items, such as bricks, mustard oil, paper, jute, and clothing, have differing constant rate demands throughout production-run periods and production-off periods. In such circumstances, the model is examined using a two-level piecewise constant demand rate that varies from production-run time to production-off time. In the brick industry, the screening of defective materials occurs during the setting, drying, and fire processes. Furthermore, the defective components are recovered before the firing stage. During the production-off periods, the inadequate bricks are recycled and aggregated with raw material. Furthermore, market demand for bricks is independent of production period features, i.e., it may increase or decrease over the production-run or production-off periods. Numerical examples are discussed below considering the following data set.

Data set #1: p = 5000 units, d = 4500 units, f = 100 units, O = $1000, K = $50, H = $10, R = $50, r = $5, S = $3, c = 0.8.

6.1. Case 1: Without Recycling

After solving the model using the above data set, one can obtain the following results.
w^* = 139 units, q^* = 6982 units, $q_s{}^*$ = 429 units, $q_1{}^*$ = 128 units and t^* = 1.55158 time units.
$t_1{}^*$ =0.3222, $t_2{}^*$ =0.0358, $t_3{}^*$ =0.11935, $t_4{}^*$ =1.07417 time units.
Average setup cost = $644, average production cost = $225,000,
average shortage cost = $495, average raw material cost = $225,000,
average holding cost = $148 and TC_1 = $451,289.

Cycle Time Verification:

$$t_1{}^*+t_2{}^*+t_3{}^*+t_4{}^* = 0.3222 + 0.0358 + 0.11935 + 1.07417 = 1.55152 = t^*$$

Convexity Graphs

In the range 125 < w < 150 and 400 < q_s < 450, Figure 4 depicts the convexity of TC_1 with respect to w and q_s simultaneously.

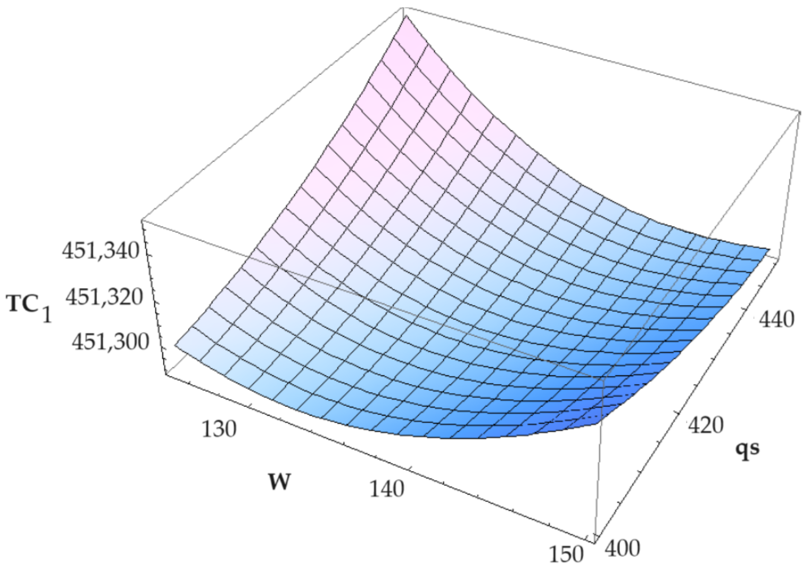

Figure 4. w–q_s–TC_1 graph.

Fixing q_s = 414 and taking 125 < w < 150, Figure 5 depicts the convexity of TC_1 with respect to 'w'.

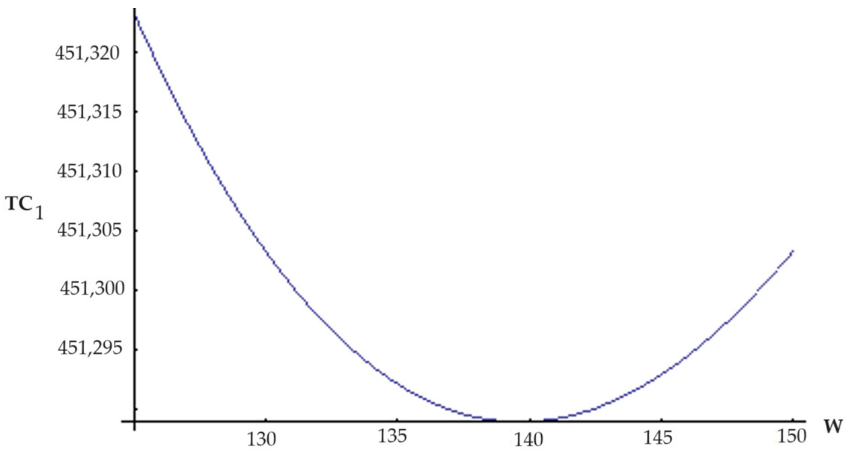

Figure 5. w–TC_1 graph.

Fixing w = 136 and taking 415 < q_s < 445, the following Figure 6 indicates the convex nature of TC_1 with respect to q_s and the minimum total cost will be at q_s = 430.

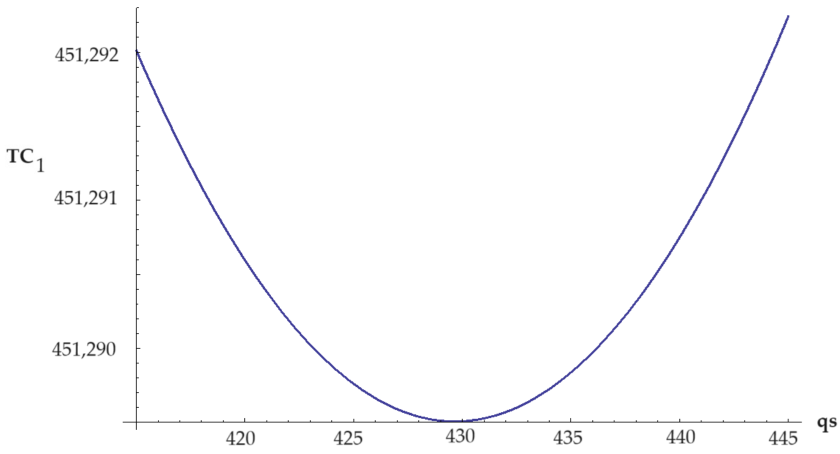

Figure 6. q_s–TC_1 graph.

6.2. Case 2: Considering Recycling

After solving the model using the above data set, one can obtain the following results.
w^* = 99.37 units, q^* = 4968.25 units, $q_s{}^*$ = 305.7 units, $q_1{}^*$ = 91.72 units
and t^* = 1.10406 time units.
$t_1{}^* = 0.2293$, $t_2{}^* = 0.02547$, $t_3{}^* = 0.08492$, and $t_4{}^* = 0.7643$ time units.
Average setup cost = $905.75, average production cost = $225,000,
average shortage cost = $352.77, recycle cost = $450,
average raw material cost = $220,500, holding cost = $552.97, and TC_2 = $447,762

Cycle Time Verification:

$$t_1{}^* + t_2{}^* + t_3{}^* + t_4{}^* = 0.2293 + 0.02547 + 0.08492 + 0.7643 = 1.104 \approx t^*$$

Convexity Graphs

Over the interval 125 < w < 1550 and 400 < q_s < 550, Figure 7 indicates the convex nature of TC_2 with respect to w and q_s simultaneously.

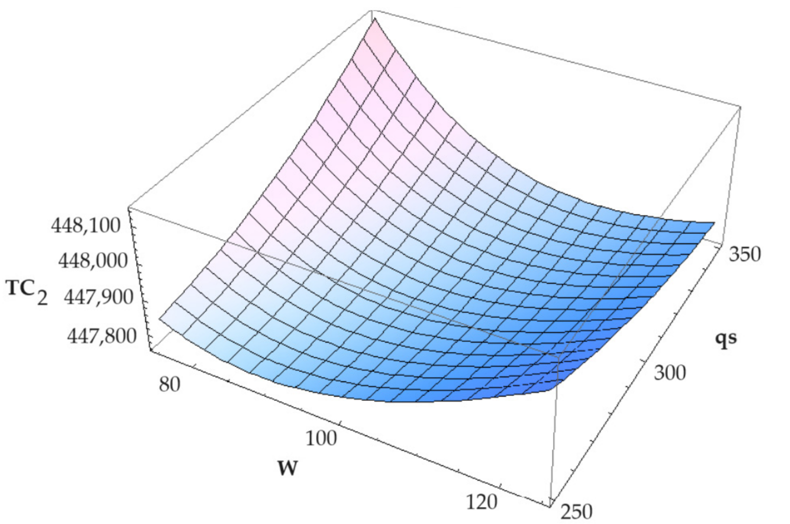

Figure 7. w–qs–TC_2 graph.

Figure 8 depicts the convexity of TC_2 with respect to 'w' while q_s = 305 and 85 < w < 115.

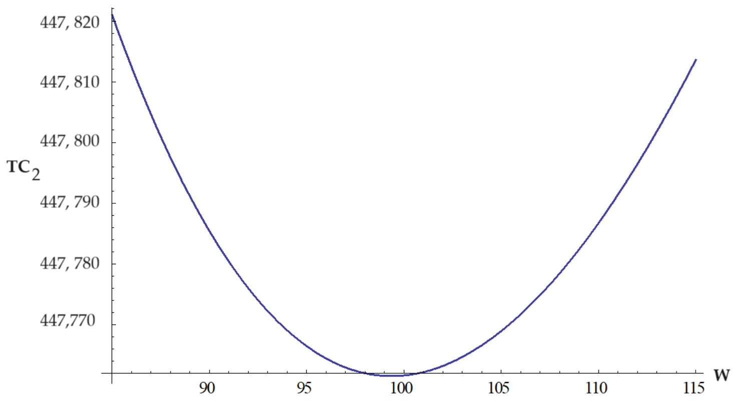

Figure 8. w versus TC_2.

Figure 9 depicts the convexity of TC_2 with respect to q_s while $400 < q_s < 550$ and $w = 99$.

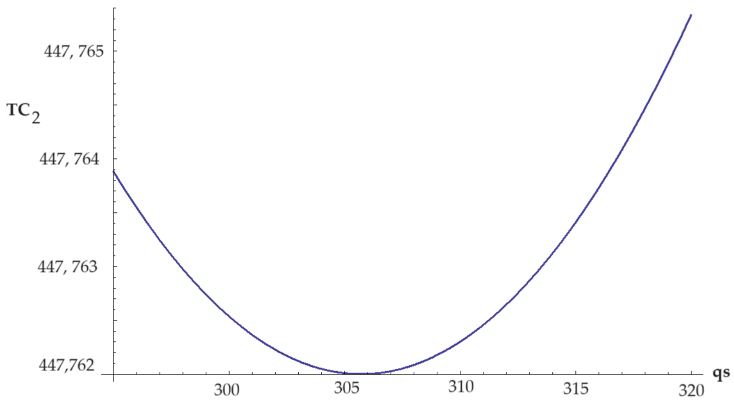

Figure 9. q_s versus TC_2.

7. Analysis and Discussion

7.1. Sensitivity Analysis

Various system parameters have sensitivity to total inventory cost. While the sensitivity of some parameters is observed, values of other parameters are considered static, i.e., the same values as represented in data set #1. As listed in Table 2, the optimal solution was obtained at the different levels of "c" with the associated observation.

Table 2. Sensitivity of 'c' (using data set #1 with different values of c).

c:	**0.5**	**0.8**	**1**	**1.50**	**2**	**Observations**
w*	97.88	99.36	99.87	100.56	100.91	Increase
q*	4894	4968	4993	5028	5045	Increase
q_s*	301	305	307	309	310	Increase
q_1*	90.35	91.72	92.18	92.82	93.14	Increase
t_1*	0.22589	0.2293	0.23047	0.23206	0.2328	Increase
t_2*	0.04106	0.02547	0.02048	0.01375	0.01035	Decrease
t_3*	0.13387	0.084992	0.06828	0.04583	0.0345	Decrease
t_4*	0.75299	0.7643	0.7682	0.7735	0.7762	Increase
t*	1.15292	1.10406	1.0875	1.06519	1.05395	Decrease
Setup Cost	867	905	919	938	949	Increase

Table 2. *Cont.*

c:	0.5	0.8	1	1.50	2	Observations
Raw Material Cost	208,019	220,500	225,000	231,294	234,574	Increase
Production Cost	212,264	225,000	229,592	236,014	239,362	Increase
Holding Cost	519	552	565	581	590	Increase
Shortage Cost	347	352	355	357	358	Increase
Recycle Cost	424	450	459	472	478	Increase
TC_2	**422,442**	**447,762**	**456,890**	**469,657**	**476,313**	**Increase**

Cost–Benefit Analysis

In the following Tables 3–7, the cost saving of Case 2 is observed compared with that of Case 1 considering different effects. The average cost saving and its percentage are computed by:

$$\text{Average Cost Saving} = TC_1 - TC_2 = \frac{\text{cd}}{\{\text{p} - (1-\text{c})\text{d} - \text{f}\}}\left[\text{f}(\text{C}_\text{R} - \text{C}_\text{r}) - \frac{\text{w}}{2}\text{C}_\text{h}\right] \quad (36)$$

$$\text{Percentage of Cost Saving} = \left(\frac{\text{Average Cost saving}}{TC_1} \times 100\right)\% \quad (37)$$

Table 3. Effect of defective rate on cost saving (using data set #1, c = 1.5 while changing f).

f:	100	150	200	250	300
Percentage of Cost Saving	0.8%	1.2%	1.6%	2%	2.5%
Observation	% Cost saving increases if ‘f’ increases.				

Table 4. Effect of recycle cost on cost saving (using data set #1, c = 1.5 while changing r).

r:	5	10	15	20	25
Percentage of Cost Saving	0.8%	0.7%	0.6%	0.5%	0.4%
Observation	% Cost saving decreases while recycle cost increases.				

Table 5. Effect of holding cost on cost saving (using data set #1, c = 1.5 while changing H).

H:	10	20	30	40	50
Percentage of Cost Saving	0.79%	0.73%	0.69%	0.65%	0.61%
Observation	% Cost saving decreases while holding cost increases.				

Table 6. Cost saving with respect to ‘c’ (using data set #1, varying c).

c:	0.5	1	1.5	2	2.5
Percentage of Cost Saving	0.79962%	0.79773%	0.79707%	0.79673%	0.79653%
Observation	% Cost saving has no effect while ‘*c*’ increases.				

Table 7. Effect of raw material cost on cost saving (using data set #1, c = 1.5 while changing R).

***R*:**	**50**	**55**	**60**	**65**	**70**
Percentage of Cost Saving	0.797%	0.854%	0.906%	0.954%	0.997%
Observation	% Cost saving increases while raw material cost increases.				

7.2. Managerial Insights

- The proposed model aims to identify the ERQ, optimal lot size, cycle duration, total inventory cost, maximum on-hand stock, and maximum backorders for inventory management across diverse commodities.
- Inventory managers can gain valuable managerial insights through sensitivity analysis and cost–benefit analysis. Specifically, it is crucial for inventory managers to carefully evaluate the ERQ when considering the recycling of defective products. Striking the right balance in recycling the appropriate number of defective items is vital to effectively minimize overall costs. Recycling a greater or lesser number of defective units rather than ERQ raises inventory costs. As a result, before recycling, managers should compute the ERQ of defective products and the related EPQ. In the event of a significant increase in raw material costs, recycling becomes advantageous for the company.
- Total inventory cost is more sensitive to increased demand during shortages than to increased demand during production-run time.
- The cost–benefit ratio diminishes at a faster rate with rising recycling costs compared to the impact of increased holding costs.
- During the off-season, when demand experiences an upsurge, all costs tend to rise, while the net cost–benefit remains the same.
- If demand increases during the production-off period, stock will be depleted quickly, and both the cycle time and the production-off time will be lowered.

8. Conclusions

In this work, inventory models for imperfect manufacturing systems with and without recycling are compared. The inventory model for recycling is examined, with the cost of recycling taken into account. The cases' total inventory cost functions are strictly convex, and the optimal solution is distinct. The proposed model corresponds to the standard EPQ model. The proposed model's findings will assist manufacturers in determining the EPQ size and ERQ of a defective quantity, which will aid in recycling decisions. The maximum amount of on-hand stock and shortages will assist them in managing storage facilities and developing marketing policy. Recycling raises inventory holding costs while also introducing recycling costs. However, in the event of ambiguity or obligation, it can be a trustworthy and alternate source of raw material. Recycling damaged things minimizes waste and is environmentally friendly. The proposed inventory model makes basic inventory theories more accessible to young pupils. The ERQ model can be used in the textile, glass, paper, jute, and brick industries. The developed ERQ model can be considered an expansion of the EPQ model. The model was solved optimally, and a sensitivity analysis was performed. The results show the impact of considering recycling of defective products on the reduction of the inventory total cost. Accordingly, the managerial implications were discussed. The study is limited to the deterministic model. As perspectives of this model, different considerations can be considered, e.g., the stochastic nature of the different factors. Many parameters are stochastic in nature; the proportion of defective items is a stochastic parameter, and the demand parameter is stochastic also. Moreover, a fuzzy economic production quantity can be developed based on the fuzzy concept.

Author Contributions: Methodology: E.-A.A., M.M.M., A.S.A., A.A., M.S.U. and M.H. Software, E.-A.A., A.A., M.S.U. and M.H.; Validation, E.-A.A., M.M.M., A.S.A., A.A., M.S.U. and M.H. All authors have read and agreed to the published version of the manuscript.

Funding: The authors extend their appreciation to the Deputyship for Research and Innovation, Ministry of Education in Saudi Arabia for funding this research work through the project number (IF2/PSAU/2022/01/22491).

Data Availability Statement: All data derived from the study have been presented in the paper. However, further inquiries could be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

p	Production rate per unit time.
d	Demand rate of the non-defective items during a specified period.
f	Deficient item production rate.
t_1	Production-run time when the inventory has positive stock.
t_2	Production-off time when the inventory has positive stock.
t_3	Production-off time when the inventory has negative stock.
t_4	Production-run time when the inventory has negative stock.
O	Setup cost per cycle.
H	Item inventory holding cost.
R	Item raw material cost.
r	Item recycling cost.
S	Item shortage cost.
K	Item production cost.
TC_1	Total inventory cost (Case 1).
TC_2	Total inventory cost (Case 2).
c	Ratio of demand rates of production-off time and production-run time.
Case 1	EPQ model with defective items for two-level piecewise constant demand that allows shortage.
Case 2	ERQ model with defective items for two-level piecewise constant demand that allows shortage.
Decision Variables:	
w	Number of deficient items per cycle.
q	Lot size.
q_s	Maximum shortage.
t	Production cycle time when $t = t_1 + t_2 + t_3 + t_4$.
q_1	Maximum on-hand stock of non-defective items.

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