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Modeling and Solving the Joint Replenishment Problem with Cross-Selling Effects Considering One Shared Minor Item

Meng Yi * and Renqian Zhang

School of Economics and Management, Beihang University, Beijing 100191, China; zhangrenqian@buaa.edu.cn
* Correspondence: yimeng@buaa.edu.cn

Abstract: In this paper, we provide a model to handle multiple replenishment cycles and the cross-selling of multiple major items with one minor item, while allowing partial late delivery. The optimization analytic expression of the model is finally obtained by utilizing the convexity of cost function for F and using the first-order conditions in optimization theory. Numerical examples and sensitivity analysis demonstrate the effectiveness of the model and algorithm, which offers a competent solution for practical applications.

Keywords: inventory; joint replenishment problem; cross-selling

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1. Introduction

Demand independence is one of the fundamental assumptions in classical economic order quantity (EOQ) models and joint replenishment problem (JRP) models [1–3]. However, in practice, managing products with independent demand often falls short of achieving the desired level of customer satisfaction [4]. Therefore, studies have long proposed that dependent demand should be considered in inventory management and production planning decisions. Stock-dependent demand [5], and demand for substitute products [6,7] are common non-independent demands in research. In recent years, there has been a significant increase in research on non-independent demands, leading to the development of a broad category of inventory and production planning models and decision-making methods for such demands [8,9].

Cross-selling implies that the demand for or sale of a major item will lead to an additional demand for its minor items, making it another significant factor in causing non-independent demand. Agrawal et al. [10] introduced the concept of "itemset" in the context of cross-selling, which generalized the description of cross-selling effects and expanded its application domain. Some scholars have addressed the issue of product selection in inventory by using association rules to reflect the impact of cross-selling effects [11–13]. Recently, the attention of scholars has focused on cross-selling in omnichannel strategies, cross-selling in recommender systems, and the usage of data mining methods in cross-selling. Liu et al. [14] have focused on omnichannel retailing with different order fulfillment and return options and found that the cross-selling benefit and the offline search cost have a significant impact on the retailer's optimal omnichannel strategy. Yang and Ji [15] discussed the impact of cross-selling on managing consumer returns in omnichannel operations. Ghoshal et al. [16] have studied recommendations and cross-selling pricing strategies when personalizing firms' cross-sell. Mokhtari [17] has addressed an economic order quantity (EOQ) model to determine the joint ordering policy for two products under completion and substitution conditions. Scholars have also used data mining methods to study multiproduct newsvendors with cross-selling and narrow-bracketing behavior [18]. Although these studies have given us many new insights, it is still order strategies that are most closely related to cross-selling.

In the field of inventory management, there have been some advancements in order policies that have considered cross-selling effects [19]. Zhang [20] provided a partial backordering EQO model to handle correlated demands caused by cross-selling in the context of the joint replenishment of multiple minor items. This led to the proposal of a series of JRP models that consider correlated demand caused by cross-selling, and a set of heuristics and exact solution algorithms have been generated around these models [21–23]. However, existing studies have only examined the management problem for one major item and one minor item [24] or the joint replenishment problem (JRP) for one major item and multiple minor item [21]. The many-to-one cross-selling JRP problem still needs to be studied further.

To address the gaps in existing research, this paper sets a more universal application scenario where multiple major products correspond to one minor product for the joint replenishment problem with cross-selling effects. Specifically, in the analysis of correlated demand caused by cross-selling, we divide the relationship of products into major items and minor items. One shared minor item can be applied to multiple major items. For example, a laptop computer may require multiple complementary products such as mice, keyboards, memory sticks, CPUs, and so on, while one of the minor product (e.g., mice, keyboards, memory sticks) can also be used in multiple electronic products such as laptops, desktop computers, and tablets. By proposing the JRP under this scenario and designing corresponding algorithms to solve the optimization problem, we believe that our research can solve the replenishment problem with the partial delayed delivery and related demand determinism caused by many-to-one cross-selling, which extends current relevant research.

The rest of this paper is organized as follows. Section 2 describes the model construction, that is, a joint replenishment problem with cross-selling effects considering one shared minor item. Section 3 addresses the analytical optimization algorithm for our proposed model, including objective function transformation, analytical process, and optimization strategy. Section 4 provides a numerical example for illustration. A detailed sensitivity analysis is performed in Section 5 around the numerical example to observe our model under various parameters. The paper concludes in Section 6.

2. Modeling Analysis

2.1. Model Assumptions

Previous research on deterministic EOQ often limited one major item to be associated with one minor item [20,25]. In our work, this limitation is removed, making the model applicable to a wider range of problems. Assume that the demand for one shared minor item is related to the demand of N major items whose delivery can be partially delayed. Due to cross-selling, the sale of the major item may lead to additional demand for the shared minor item, which can be sold either separately or jointly with the major item. The inventory of the associated minor item will be reduced when the major item is shipped out.

For the other parts of the assumptions, we follow classic practices from the past around deterministic EOQ [24–26]. In other words, even though new work exists that improves the underlying model in various directions to bring it closer to reality, considering the complexity of the solving, we only extend the original classical models, keeping their assumptions intact. This paper mainly assumes that the major item has different cycles and the cycle of the minor item has no multiple relationship with the cycle of major item. At the same time, the following assumptions related to the model are given:

1. Replenishment is instantaneous;
2. Delivery time is zero;
3. The major item can be partially delayed, while the minor item should not be kept in stock;
4. The unit loss of the major item will reduce the demand for the minor item at a constant rate;
5. The demand and cost of each major item are independent of each other.

In real life, the replenishment cycle of various products cannot be exactly the same, especially for large manufacturers. For upstream manufacturers, similar to the retail sector, they need to adjust their production schedules for various products based on demand and source the required raw materials from downstream manufacturers. The same replenishment cycle will lead to excessive transportation pressure and cause problems such as capital turnover, so for different major products the replenishment cycle of each product is not the same. For example, laptop manufacturers produce computers with different functions, including game books, office books, and so on. If the replenishment cycle is the same, it is not only unrealistic in terms of customer demand but it is also unable to have a large inventory to store too many products for replenishment at one time.

2.2. Meaning of Variables in the Model

The meanings of the variables (exogenous variables and decision variables) to be used in this section are shown in Tables 1 and 2 below.

Table 1. Variable Definition.

D_i	Demand per unit of time for major item i per unit of time, in pieces/year
A_i	Fixed order cost of major item i, in \$/order
C_{Oi}	Opportunity cost of lost sales of major item i, in \$/unit
C_{hi}	Holding cost per unit of time per unit of major item i, in \$/piece/year
C_{bi}	Cost per backorder of major item i per unit of time, in \$/unit/year
β_i	Backorder rate of major item i
$D_j{}'$	Demand per unit time of associated minor item j per unit time, in piece/year
A'	Fixed order cost of minor item, in \$/order
$C_o{}'$	Opportunity cost of lost sales per unit of minor item per order cycle time, in \$/piece
$C_h{}'$	Holding cost per unit of minor item per unit of time, in \$/piece/year
λ_i	Proportion of sales loss of minor items caused by sales loss of major item i
T	Order cycle
Q_i	Order volume
F_i	Inventory/actual demand

Table 2. Table of interpretation of decision variables.

T_0	Order cycle time of related products
$\boldsymbol{T}$	Order cycle time of major items, i.e., basic order cycle, in years
$\boldsymbol{F}$	Demand satisfaction rate of major items

2.3. Model Description

According to the basic deterministic EOQ model [25], which allows for stock-outs, the ordering cost per unit of time for major item i can be shown by Equation (1).

$$\Gamma_i(T_i, F_i) = \frac{A_i}{T_i} + \frac{C_{hi} D_i T_i F_i^2}{2} + \frac{\beta_i C_{bi} D_i (1-F_i)^2}{2} + C_{oi} D_i (1-\beta_i)(1-F_i) \quad (1)$$

There has been some work introducing cross-selling into the inventory management problem [19], and we use the same approach to construct our model. Similarly, the cost per unit of time function for the minor item can be given by Equation (2).

$$\Gamma(T_0, F_i) = \frac{A'}{T_0} + \sum_{i=1}^{N} \left[\frac{\lambda_i C_h'}{2} D_i T_i F_i^2 + \lambda_i C_o' D_i (1-\beta_i)(1-F_i) + \frac{C_h'(D' - \lambda_i D_i) T_i}{2} \right] \quad (2)$$

The final objective function is to minimize the ordering cost of the items, i.e., the sum of the unit time cost of all major items plus the unit time cost of the minor item, and then the total ordering cost of the manufacturer per unit time can be obtained, as shown in Equation (3).

$$\min \Gamma(\boldsymbol{T}, T_0, \boldsymbol{F}) = \sum_{i=1}^{N} \Gamma_i(T_i, F_i) + \Gamma(T_0, F_i) \quad (3)$$

3. Model Solving

3.1. Objective Function Transformation

The analytical solution method is chosen in this section to derive the results of the model optimization problem established in Section 2. Compared with the computer simulation solution, the analytical solution does not require complex programming algorithms or simulation calculations, and the results obtained by mathematical derivation are more theoretical and more accurate than the numerical simulation results.

First of all, the objective function is written for the above optimization problem as shown in Equation (4), i.e., the expansion form of Equation (3).

$$\begin{aligned}\Gamma(\boldsymbol{T},T_0,\boldsymbol{F}) = \sum_{i=1}^{N}\frac{A_i}{T_i} + \sum_{i}^{N}\left(\frac{C_{hi}+\lambda_i C_h'}{2}\right)D_iT_iF_i^2 + \sum_{i}^{N}\frac{\beta_i C_{bi}D_i(1-F_i)^2}{2} \\ +\sum_{i}^{N}\left(C_{oi}+\lambda_i C_o'\right)D_i(1-\beta_i)(1-F_i) \\ +\sum_{i}^{N}\frac{C_h'(D'-\lambda_i D_i)}{2}T_i + \frac{A'}{T_0}\end{aligned} \tag{4}$$

As assumed in the model in Section 2.1, the demand and cost of each major item are independent of each other, so the total optimization cost and the cost of each major item are equivalent to the cost generated by the major item and the optimization of the addition term can be converted into the optimization solution of each term. By collapsing the total cost with respect to the cost of each major item and the cost due to the major item as $\widetilde{\Gamma}_i(T_i,F_i)$, Equation (4) can be written as:

$$\widetilde{\Gamma}(\boldsymbol{T},T_0,\boldsymbol{F}) = \sum_{i}^{N}\widetilde{\Gamma}_i(T_i,F_i) + \frac{A'}{T_0} \tag{5}$$

$$\widetilde{\Gamma}_i(T_i,F_i) = \frac{A_i}{T_i} + \frac{C_{hi}+\lambda_i C_h'}{2}D_iT_iF_i^2 + \frac{\beta_i C_{bi}D_i}{2}(1-F_i)^2 + (C_{oi}+\lambda_i C_o')D_i(1-\beta_i)(1-F_i) + \frac{C_h'(D'-\lambda_i D_i)}{2}T_i \tag{6}$$

For ease of reading, $\widetilde{\Gamma}_i(T_i,F_i)$ is abbreviated as $\widetilde{\Gamma}_i$ in the derivation and calculation section later in this section. In addition, in order to facilitate the solution and enhance the readable lines of the solution step, the non-decision variables in (6) are organized into the form of simplified coefficients, which are, respectively, $G_{1i}, G_{2i}, G_{3i}, G_{4i}$, as shown in Equations (7)–(10). According to the definition of each variable, it can be seen that they are positive real numbers greater than zero.

$$G_{1i} = \frac{C_{hi}+\lambda_i C_h'}{2}D_i > 0 \tag{7}$$

$$G_{2i} = \frac{\beta_i C_{bi}D_i}{2} > 0 \tag{8}$$

$$G_{3i} = \left(C_{0i}+\lambda_i C_o'\right)D_i(1-\beta_i) > 0 \tag{9}$$

$$G_{4i} = \frac{C_h'(D'-\lambda_i D_i)}{2} > 0 \tag{10}$$

Substituting $G_{1i}, G_{2i}, G_{3i}, G_{4i}$ into Equation (6), a concise representation of $\widetilde{\Gamma}_i$ can be obtained, as shown in Equation (11).

$$\widetilde{\Gamma}_i = \frac{A_i}{T_i} + G_{1i}T_iF_i^2 + G_{2i}(1-F_i)^2 + G_{3i}(1-F_i) + G_{4i}T_i \tag{11}$$

At this time, the total cost, $\widetilde{\Gamma}(\boldsymbol{T}, T_0, \boldsymbol{F})$, is a function only related to $\boldsymbol{T}$, $\boldsymbol{F}$, T_0, and this function can roughly separate $\boldsymbol{T}$ and $\boldsymbol{F}$, so that the two have only one multiplying term, which is a very good optimization function and can bring a significant amount of convenience in the derivation.

3.2. Model Solving Procedure

Having simplified this equation, we can find the analytic solution of this function by taking the derivative of $\widetilde{\Gamma}_i$. According to the following derivation result, Equations (12) and (13), it can be seen that this is a binary quadratic equation.

$$\frac{d\Gamma_i}{dT_i} = -A_i T_i^{-2} + G_{1i} {F_i}^2 + G_{4i} \tag{12}$$

$$\frac{d\Gamma_i}{dF_i} = 2G_{1i} T_i F_i - 2G_{2i} + 2G_{2i} F_i - G_{3i} \tag{13}$$

According to the first-order condition, the following optimization solution steps can be made. First, set Equation (12) to 0, i.e., $-A_i T_i^{-2} + G_{1i} {F_i}^2 + G_{4i} = 0$, and the expression of ${T_i}^*$ can be calculated, as shown in Equation (14).

$${T_i}^* = \sqrt{\frac{A_i}{G_{1i} {F_i}^2 + G_{4i}}} \tag{14}$$

Similarly, if Equation (14) is set to 0, i.e., $2G_{1i} T_i F_i - 2G_{2i} + 2G_{2i} F_i - G_{3i} = 0$, the expression of optimal ${F_i}^*$ regarding T_i can be calculated, as shown in Equation (15).

$${F_i}^* = \frac{2G_{2i} + G_{3i}}{2G_{1i} T_i + 2G_{2i}} \tag{15}$$

Since ${T_i}^*$ is included in the expression of ${F_i}^2$, the expression of ${F_i}^*$ (the square of Equation (15)) can be substituted into Equation (14) to facilitate the solution of the expression of F_i without T_i^*. The square of Equation (15) is shown in Equation (16).

$$\frac{4{G_{2i}}^2 + 4G_{2i} G_{3i} + {G_{3i}}^2}{4G_{1i} {T_i}^2 + 8G_{1i} G_{2i} T_i + 4{G_{2i}}^2} = {F_i}^2 \tag{16}$$

Substituting the result of Equation (16) into Equation (14), an equation containing only one decision variable, T_i, can be obtained, as shown in Equation (17).

$$\frac{G_{1i} (2G_{2i} + G_{3i})^2}{(2G_{1i} T_i + 2G_{2i})^2} = \frac{A_i - G_{4i} {T_i}^2}{T_i^2} \tag{17}$$

It is easy to see that Equation (17) is a function only related to $\boldsymbol{T}$, so it is inevitable that an analytical solution of $\boldsymbol{T}$ is obtained. The form of Equation (17) is rewritten into a polynomial form, as shown in Equation (18).

$$\begin{aligned} &4G_{1i} {G_{2i}}^2 {T_i}^2 + 4G_{1i} G_{2i} G_{3i} {T_i}^2 + {G_{3i}}^2 {T_i}^2 \\ &\quad = 4A_i G_{1i} {T_i}^2 + 8A_i G_{1i} G_{2i} T_i + 4A_i {G_{2i}}^2 - 4G_{1i} G_{4i} {T_i}^4 \\ &\quad -8G_{1i} G_{2i} G_{4i} {T_i}^3 - 4G_{2i}^2 G_{4i} {T_i}^2 \end{aligned} \tag{18}$$

After combining similar terms, the simplified form of Equation (18) can be obtained as Equation (19).

$$\begin{aligned} &4G_{1i} G_{4i} {T_i}^4 + 8G_{1i} G_{2i} G_{4i} {T_i}^3 + [G_{1i} (2G_{2i} + G_{3i})^2 - 4A_i G_{1i} + 4G_{2i}^2 G_{4i}] {T_i}^2 \\ &\quad -8A_i G_{1i} G_{2i} T_i + 4A_i G_{2i}^2 = 0 \end{aligned} \tag{19}$$

According as Equation (19), this is a quartic equation of $\boldsymbol{T}$, and the optimal solution of T^* is the solution of this quartic equation.

3.3. The Optimization Strategy of the Model

In order to facilitate the representation and calculation in the process of model optimization and enhance the simplicity of the operation process, Equation (6) was re-simplified and written in the following form as Equation (20),

$$\widetilde{\Gamma_i}(T_i, F_i) = \frac{A_i}{T_i} + T_i u(F_i) + v(F_i), \tag{20}$$

where $u(F_i)$ and $v(F_i)$ in Equation (20) are two functions of F_i, as shown in Equations (21) and (22).

$$u(F_i) = G_{1i}F_i^2 + G_{4i} \tag{21}$$

$$v(F_i) = G_{2i}(1 - F_i)^2 + G_{3i}(1 - F_i) \tag{22}$$

Therefore, Equation (14) can be written as Equation (23) in the following form:

$$T_i^*(F_i) = \sqrt{\frac{A_i}{u(F_i)}} \tag{23}$$

Substituting Equation (23) into Equation (20), the form of Equation (24) can be obtained as follows:

$$\begin{aligned}
\overline{\Gamma_i}(F_i) &= \frac{A_i}{\sqrt{\frac{A_i}{u(F_i)}}} + \sqrt{\frac{A_i}{u(F_i)}} \cdot u(F_i) + v(F_i) \\
&= \frac{A_i[u(F_i)]^{\frac{1}{2}}}{A_i^{\frac{1}{2}}} + \frac{A_i^{\frac{1}{2}}}{[u(F_i)]^{\frac{1}{2}}} \cdot u(F_i) + v(F_i) \\
&= A_i^{\frac{1}{2}}[u(F_i)]^{\frac{1}{2}} + A_i^{\frac{1}{2}}[u(F_i)]^{\frac{1}{2}} + v(F_i) \\
&= 2\sqrt{A_i u(F_i)} + v(F_i)
\end{aligned} \tag{24}$$

It is easy to show that the function $\overline{\Gamma_i}(F_i)$ is continuous on an interval of $F \in [0,\ 1]$, so the optimal solution can be obtained by taking the derivative.

Firstly, the first and second derivatives of functions $u(F_i)$ and $v(F_i)$ are given, as shown in Equations (25)–(28).

$$u'(F_i) = 2G_{1i}F_i \tag{25}$$

$$u''(F_i) = 2G_{1i} \tag{26}$$

$$v'(F_i) = 2G_{2i}F_i - 2G_{2i} - G_{3i} \tag{27}$$

$$v''(F_i) = 2G_{2i} \tag{28}$$

Then, the first and second derivatives of $\overline{\Gamma_i}(F_i)$ can be calculated, as shown in Equations (29) and (30).

$$\frac{\partial \overline{\Gamma_i}}{\partial F_i} = \sqrt{A_i}\frac{u'(F_i)}{\sqrt{u(F_i)}} + v'(F_i) = \sqrt{A_i}\frac{2G_{1i}F_i}{\sqrt{u(F_i)}} + 2G_{2i}F_i - 2G_{2i} - G_{3i} \tag{29}$$

$$\frac{\partial^2 \overline{\Gamma_i}}{\partial^2 F_i} = \frac{\sqrt{A_i}[2u''(F_i)u(F_i) - (u'(F_i))^2]}{2[u(F_i)]^{\frac{3}{2}}} + v''(F_i) = \frac{\sqrt{A_i}(8G_{1i}^2 F_i - 4G_{1i}^2 F_i^2)}{2[u(F_i)]^{\frac{3}{2}}} + 2G_{2i} \tag{30}$$

According to Equations (7)–(9) and (21) and the definition of satisfying rate F_i, the following conditions can be easily obtained:

$$G_{1i} > 0,\ 0 \le F_i \le 1,\ 2G_{2i} > 0,\ u(F_i) > 0$$

Therefore, it is easy to draw the inequality, as in Equation (31).

$$8G_{1i}^2 F_i - 4G_{1i}^2 F_i^2 > 0 \tag{31}$$

As the inequality relation of Equation (31), it is easy to deduce $\frac{\partial^2 \overline{\Gamma_i}}{\partial^2 F_i} > 0$. According to the second-order condition of convex function, it is obvious that $\overline{\Gamma}_i(F_i)$ is convex on the domain of F_i.

Since $F_i \in [0,\ 1]$, three cases are discussed below.

Case 1: $F_i = 0$

As shown in Figure 1, when $F_i = 0$, $\frac{\partial \overline{\Gamma_i}}{\partial F_i} = -2G_{2i} - G_{3i} \le 0$ can be calculated. It follows from the first-order property that $\overline{\Gamma_i}(F_i)$ is monotonically decreasing near $F_i = 0$, which means that the optimal solution can never be taken at $\boldsymbol{F} = 0$.

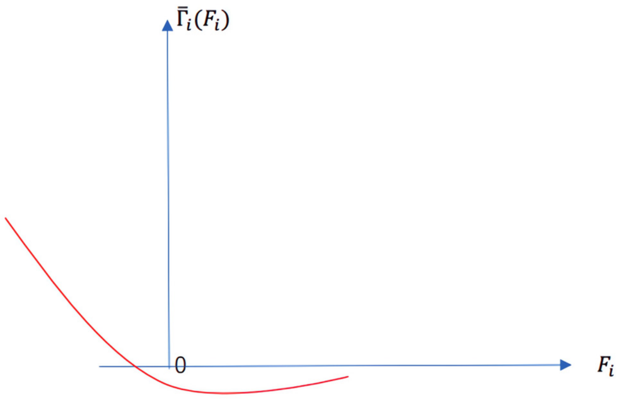

Figure 1. The situation of $\overline{\Gamma}_i(F_i)$ when $F_i = 0$.

Therefore, the following conclusion can be drawn: the objective function cannot obtain the optimal value when $F_i = 0$.

Case 2: $F_i = 1$

If $F_i = 1$, Equation (32) can be calculated.

$$\left.\frac{\partial \overline{\Gamma_i}}{\partial F_i}\right|_{F_i = 1} = \frac{\sqrt{A_i} \cdot 2G_{1i}}{\sqrt{G_{1i} + G_{4i}}} + 2G_{2i} - 2G_{2i} - G_{3i} \tag{32}$$

Since Equation (32) depends on parameter values, its positive and negative values are difficult to determine, so it needs to be discussed by case.

As shown in Figure 2, similar to the case of a major item with one minor item, the optimal solution $F_i{}^* \in (0, 1)$ when Equation (32) > 0, i.e., the optimal solution, $T_i{}^*$, is the positive real root of the quartic equation of one variable, Equation (19), and then the optimal solution, $F_i{}^*$, is obtained by Equation (15).

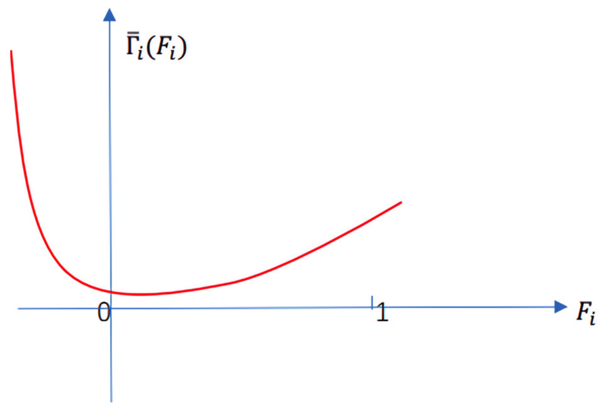

Figure 2. The situation of $\overline{\Gamma}_i(F_i)$ when $F_i = 1$.

In this case, we can take the derivative of $\overline{\Gamma_i}$, Equation (33).

$$\left.\frac{\partial \overline{\Gamma_i}}{\partial F_i}\right|_{F_1} = \frac{\sqrt{A_i}}{\sqrt{G_{1i} + G_{4i}}} - G_{3i} > 0 \tag{33}$$

By simplifying Equations (5) and (33), we can obtain:

$$\frac{\sqrt{A_i}}{\sqrt{G_{1i} + G_{4i}}} - G_{3i} > 0$$

Substituting the original value of G_{1i}, G_{3i}, G_{4i} into Equation (34), a critical value of the out-of-stock rate can be obtained as follows:

$$\begin{aligned} \frac{\sqrt{A_i}}{\sqrt{G_{1i} + G_{4i}}} - (C_{oi} + \lambda_i C_o{}')D_i + (C_{oi} + \lambda_i C_o{}')D_i\beta_i > 0 \\ (C_{oi} + \lambda_i C_o{}')D_i\beta_i > (C_{oi} + \lambda_i C_o{}')D_i - \frac{\sqrt{A_i}}{\sqrt{G_{1i} + G_{4i}}} \\ \beta_i > 1 - \frac{\sqrt{A_i}}{(C_{oi} + \lambda_i C_o{}')D_i\sqrt{G_{1i} + G_{4i}}} \\ \beta_i > 1 - \frac{1}{(C_{oi} + \lambda_i C_o{}')D_i} \cdot \sqrt{\frac{A_i}{G_{1i} + G_{4i}}} \end{aligned} \tag{34}$$

Since the optimal solution to T is $T^* = \sqrt{\frac{A_i}{G_{1i}F_i^2 + G_{4i}}}$, Equation (34) can be rewritten as Equation (35), as follows:

$$\beta_i > \beta^*_{C_i} = 1 - \frac{1}{C_{oi} + \lambda_i C_o{}'} T^*, \tag{35}$$

where a $\beta^*_{C_i}$ is determined by Equation (35); it is the lower threshold of the out-of-stock rate.

Therefore, the process of the optimal strategy is as follows:

1. For each major item, a $\beta^*_{C_i}$ is determined to judge the relationship between the shortage rate and the critical value.
2. When the shortage rate is far less than the critical value, the optimal solution of F_i^* is 0 or 1 and the corresponding value of T_i^* is calculated according to Equation (14), then the value of the objective function is calculated. A group of F_i^* and T_i^* that makes the objective function smaller is selected as the optimal solution.
3. When the shortage rate is greater than the critical value, the optimal T_i^* is a positive real root of the quadratic equation in Equation (19) and the corresponding F_i^* is calculated according to Equation (15).

4. The strategy of one major item and one minor item is used to determine the T_i^*, F_i^* of each major item.
5. T_0 is determined by assuming that T_0 is the largest period of all T_i.
6. All the cost items are totaled to find the final total cost.

4. Numerical Computations

A numerical example is used to validate the algorithm in this section, which includes different cases where the shortage rate is greater than or less than the critical value. To ensure that this numerical example is appropriate for the context of our study, i.e., the demand for one minor item is related to the demand for N major items, we introduce a scenario that is close to reality: Consider the minor item as one type of mouse of a certain brand, and the major items as three different models of laptop computers. Product 1 represents the laptop with a long life cycle and a general market demand. Product 2 represents the emerging laptop with a short life cycle and high market interest. Product 3 represents the laptop with a long life cycle and relatively low market demand for a specific group of people. According to practical experience, one kind of mouse is generally applicable to many models of laptops, and the demand for the mouse tends to increase after the sale of laptops. Different models of laptops are applicable to different groups of people, and there is usually a 10-fold or more difference in the demands. Specific parameters are shown in Tables 3–5:

Major items:

Table 3. Major item Dataset.

Product	Requirement D (Pieces/Year)	Fixed Ordering Cost A ($/Time)	Cost of Sales Losses C_o ($/Piece)	Holding Cost C_h ($/Piece/Year)	Delivery Delaying Cost C_b ($/per/Year)	Allowable Delivery Delaying Rate β
$i = 1$	400	200	5	2	10	0.7
$i = 2$	800	600	15	30	8	0.2
$i = 3$	10	400	2	1	5	0.96

Minor item:

Table 4. Minor item Dataset.

Product	Requirement D (Pieces/Year)	Fixed Ordering Cost A ($/Time)	Cost of Sales Losses C_o ($/Piece)	Holding Cost C_h ($/Piece/Year)
Mouse	400	200	5	1

The proportion of related product sales loss caused by the shortage of a major item:

Table 5. The proportion of related product sales loss caused by the shortage of major item ***i***.

	***i* = 1**	***i* = 2**	***i* = 3**
λ_i	0.1	0.5	0.3

The critical value of β of each major item can be calculated as Table 6:

Table 6. The critical value of β of each major item.

	***i* = 1**	***i* = 2**	***i* = 3**
β_i	0.9995	0.9998	0.9595

According to the algorithm, the optimal ***T*** and ***F*** of each major item can be calculated as Table 7:

Table 7. The optimal ***T*** and ***F*** of each major item.

Optimal Solution	Major Item $i = 1$	Major Item $i = 2$	Major Item $i = 3$
T_i	1.0541	2.4759	1.4173
F_i	1	1	0.0541

The higher the cycle T_0 of the minor item is, the lower the total cost is. Set $T_0 = \max T_i = 2.4759$. Substituting all the T_i, F_j, T_0 obtained into the original objective function, the total cost of the optimal solution of this example can be obtained as $\min \Gamma_i(T_i, F_j, T_0) = 1370.83$.

5. Model Sensitivity Analysis

5.1. Sensitivity Analysis of Model Optimal Cost to Ordering Cost

The numerical example in the numerical calculation experiment in the previous section will continued to be used in this section. The ordering cost A of the minor item and the ordering cost A_i of the major item i are respectively generated, and 10 different examples are calculated according to the range [100, 10,000] to obtain the optimal solution by keeping other parameters in the numerical example unchanged, as shown in Figures 3 and 4.

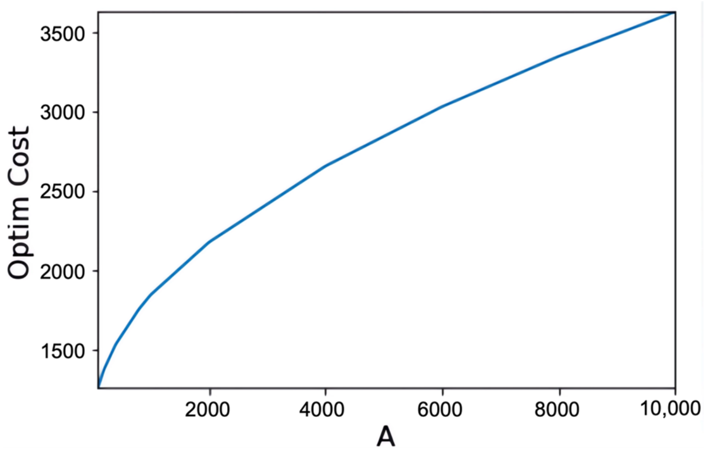

Figure 3. Optimal total cost curve varying with ordering cost of minor items.

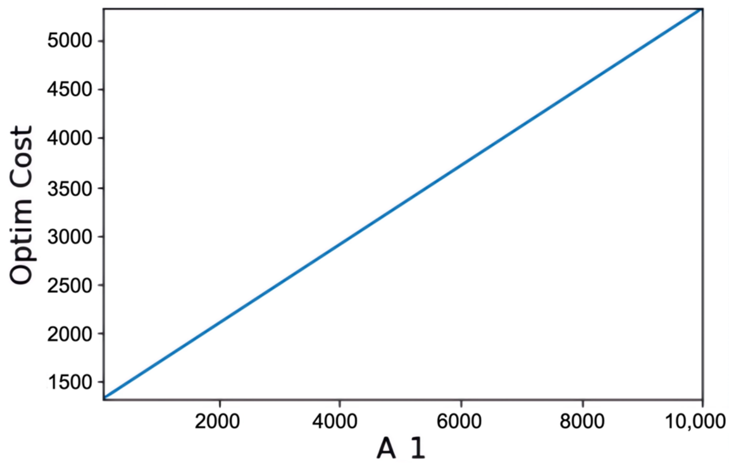

Figure 4. Optimal total cost curve varying with ordering cost of major items.

As can be seen from Figures 3 and 4, the optimal result calculated by the model is greatly influenced by it (reflected in the high rate of change in the figure) when the ordering cost of minor item A is within the interval of [100, 2000]. At the same time, the optimal result calculated by the model is influenced very little by ordering cost, A_i, of the major item i, which is close to linear influence.

5.2. Sensitivity Analysis of Model Optimal Cost to Demand Rate

By keeping other parameters in the numerical example unchanged, the demand rate, D, of the minor item and the ordering cost, D_i, of major item i are respectively calculated for 10 different examples in accordance with the ranges [400, 2200] and [200, 2000] to obtain the optimal solution, as shown in Figures 5 and 6.

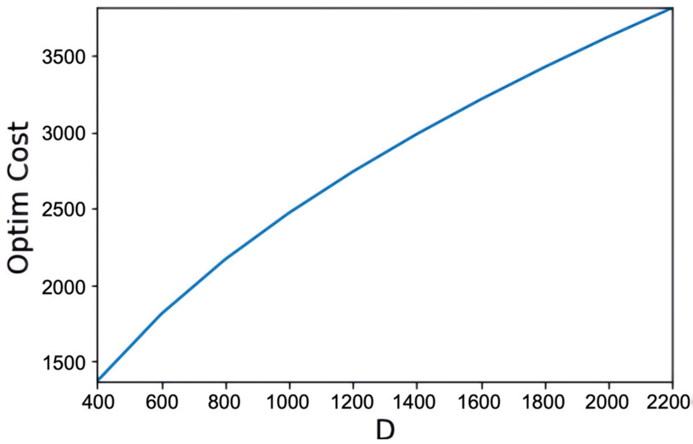

Figure 5. Optimal total cost curve varying with the demand rate of minor item.

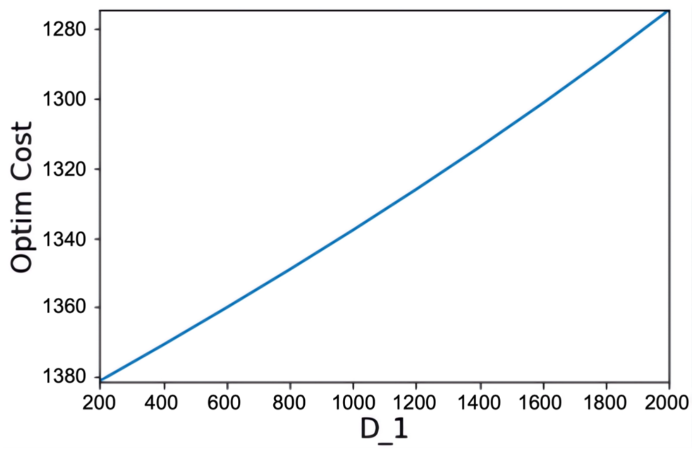

Figure 6. Optimal total cost curve varying with the ordering cost of major item.

As can be seen from the two figures above, when the demand rate, D, of the minor item is within the range of [400, 800], the optimal result obtained by the model calculation is greatly affected by it. At the same time, the optimal result calculated by the model is influenced very little by the ordering cost, D_i, of major item i, which is close to linear influence.

5.3. Sensitivity Analysis of Model Optimal Cost to Order Backlog Cost

By keeping other parameters in the numerical example unchanged, 10 different examples of order backlog cost, C_{bi}, of major item i are generated in accordance with the range [1, 30] and the optimal solution is obtained, as shown in Figure 7.

As can be seen from the figure above, the optimal cost calculated by the model proposed in this section will not be affected by the order backlog cost, C_{bi}, of major item i.

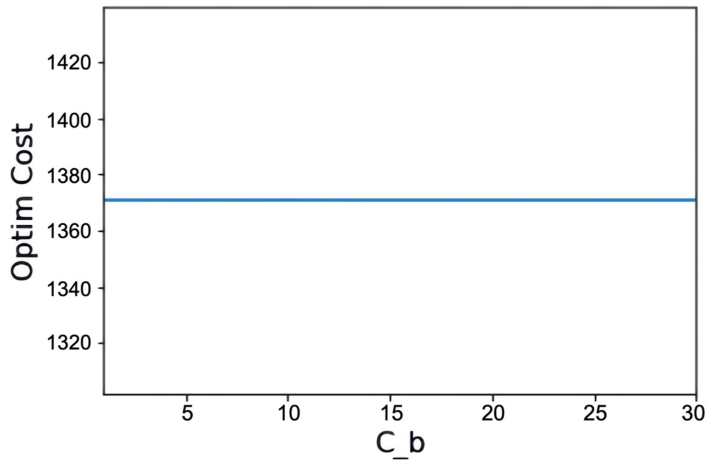

Figure 7. Optimal total cost curve varying with the ordering cost of major item.

5.4. Sensitivity Analysis of Model Optimal Cost to Inventory Carrying Cost

By keeping other parameters in the numerical examples unchanged, 10 different examples are respectively generated for the inventory holding cost, C_h, of the minor item and the inventory holding cost, C_{hi}, of major item i in accordance with the range [1, 10] to obtain the optimal solution, as shown in Figures 8 and 9.

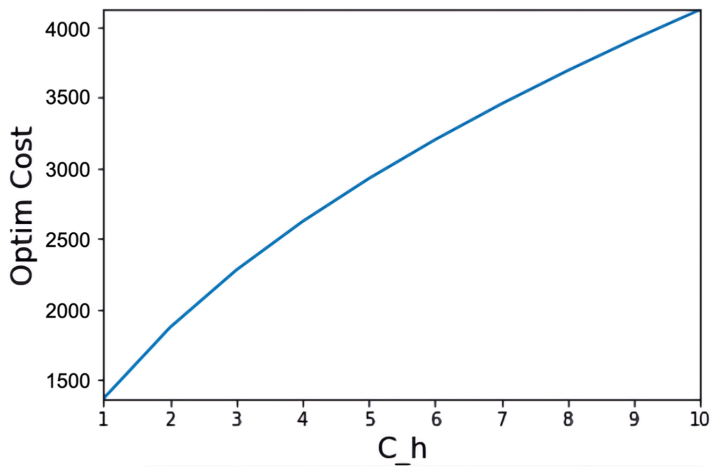

Figure 8. Optimal total cost curve varying with the ordering cost of major item.

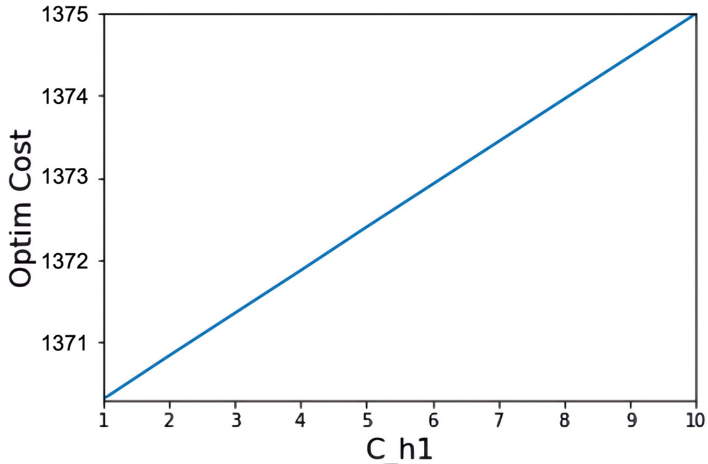

Figure 9. Optimal total cost curve changing with inventory carrying cost of the major item.

As can be seen from the two curves above, the optimal result obtained by this model is influenced by the inventory holding cost, C_h, and the inventory holding cost, C_{hi}, of major item i, which is very small and close to linear.

5.5. Sensitivity Analysis of Model Optimal Cost to Opportunity Cost

By keeping other parameters in the numerical examples unchanged, 10 different examples are respectively generated for the inventory holding cost, C_h, of related products and the inventory holding cost, C_{hi}, of major item i in accordance with the range [1, 10] to obtain the optimal solution, as shown in Figures 10 and 11.

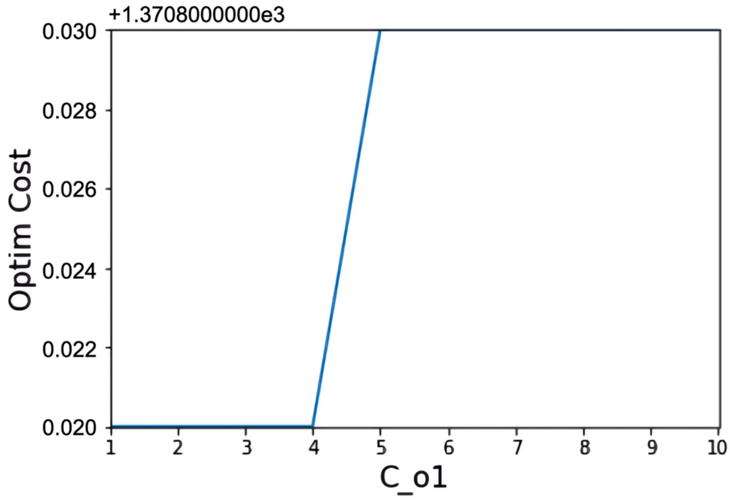

Figure 10. Optimal total cost curve changing with the opportunity cost of minor item.

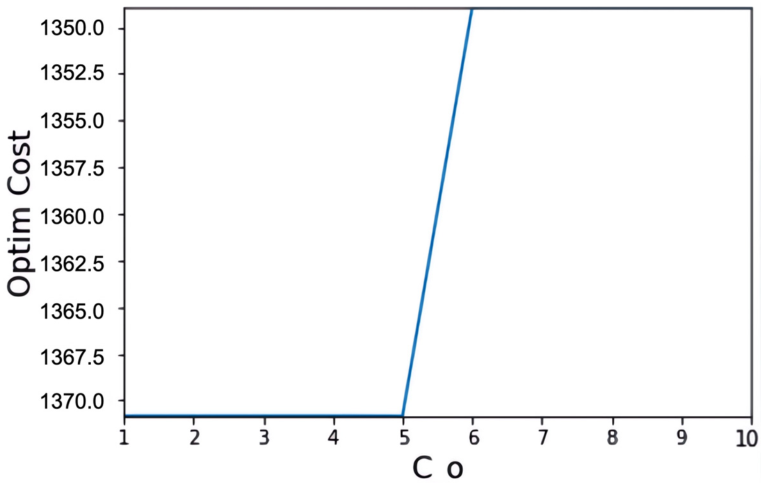

Figure 11. Optimal total cost curve changing with inventory carrying cost of major item.

As can be seen from the two figures above, the optimal result calculated by this model is influenced very little by the opportunity cost of the minor item, C_o, and the inventory carrying cost, C_{oi}, of major item i and only fluctuates within a very small range.

5.6. Sensitivity Analysis of Model Optimal Cost to Order Backlog Rate

By keeping other parameters in the numerical examples unchanged, 10 different examples are respectively generated for the order backlog rate β_i of major item i in accordance with the range [0.1, 0.95] to obtain the optimal solution, as shown in Figure 12.

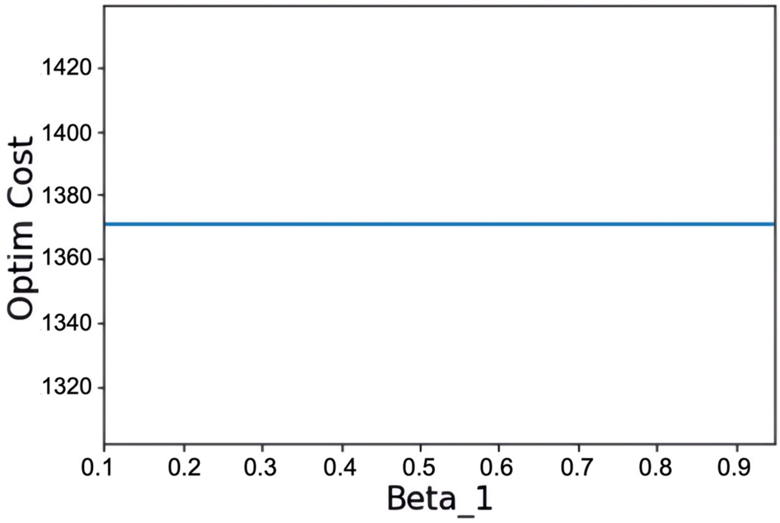

Figure 12. Optimal total cost curve with the change of order backlog rate β_i of major item i.

As can be seen from the two figures above, the optimal result calculated by this model is influenced very little by the opportunity cost of the minor item, C_o, and the inventory carrying cost, C_{oi}, of major item i and only fluctuates within a very small range. The optimal cost calculated by the model presented in this section will not be affected by the order backlog rate, β_i, of major item i. Through the analysis of the solution steps, it can be seen that once the β_i exceeds the critical value, it will not affect the solution process, and then will not affect the optimal solution obtained by the model calculation.

5.7. Sensitivity Analysis of Model Optimal Cost to Sales Loss Rate

By keeping other parameters in the numerical examples unchanged, 10 different examples are respectively generated for the loss ratio, λ_i, of major item i to minor item sales in accordance with the range [0.1, 0.95] to obtain the optimal solution, as shown in Figure 13.

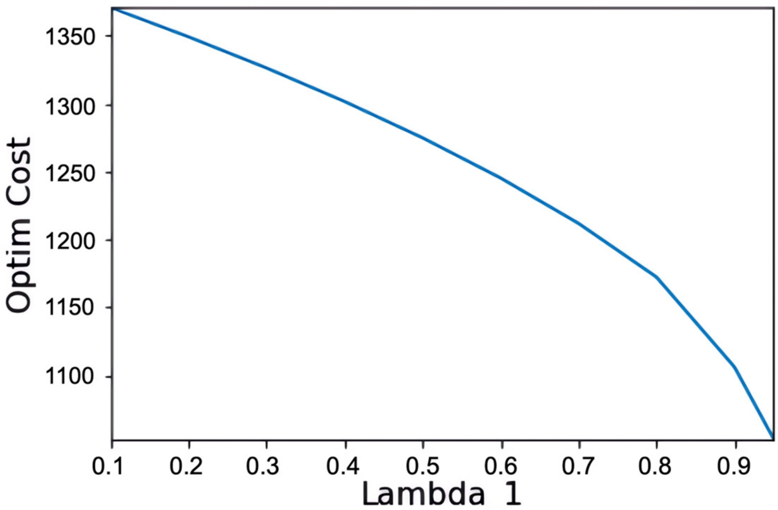

Figure 13. Optimal total cost curve changing with the loss rate of sales of major item to minor item.

As can be seen from the figure above, the optimal cost calculated by the model proposed in this section has a great influence on the optimal cost in the interval $\lambda_i \in [0.8, 0.95]$, and it is close to linear influence in other intervals.

6. Conclusions

In this paper, we consider solving the many-to-one cross-selling JRP problem, which leads to the certainty of partial late delivery and related demand. We extend the proposed similar model to make it capable of handling multiple replenishment cycles and the cross-selling of multiple major items with one minor item, allowing partial late delivery. In this paper, various possibilities are discussed under the condition of the demand satisfaction rate of major items' F. The optimization analytic expression of the model is finally obtained by utilizing the convexity of cost function for F and using the first-order conditions in optimization theory.

The proposed method has been validated in numerical examples of the model, and the acquisition of accurate solutions provides strong evidence for the effectiveness of the proposed approach. The model and approach offer a competent solution for practical applications.

Additionally, if the replenishment cycle of major items is assumed to be consistent in real life, the decision space will often be reduced. Therefore, the relaxation of the conditions of an inconsistent replenishment cycle can also effectively increase the decision space and give the model proposed in this chapter a certain application prospect.

We also considered a widespread scenario in reality, that is, multiple major items versus one shared minor item under the cross-selling effect. From the perspective of practice, our paper may be able to provide some value to enterprises under appropriate conditions. Based on the numerical experiments in Section 5, we can see that some metrics of the minor items can significantly affect total ordering cost when compared to the major items. Specifically, for the ordering cost, demand rate, and inventory carrying cost, under certain conditions (being in a certain interval, controlling for other metrics) total ordering cost is strongly influenced by the minor items and weakly influenced by major items. The insight from this is that if an enterprise wants to reduce its total costs in a more efficient way, under similar conditions, it should focus more on adopting a means of controlling the indicators for minor items. To avoid sudden losses, the enterprise should prepare in advance to prevent large fluctuations caused by minor item suppliers.

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