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A Subsidization Scheme for Maximizing Social Welfare in Mobile Communications Markets

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Abstract: In contemporary mobile communications markets, various agents or players interact to pursue welfare. Regulatory policies enacted by governments in certain markets aim to maximize social welfare. However, some countries, both least developed and developing, often adopt successful models from developed nations without local market validation. Therefore, network economics serves as a pertinent framework for analyzing such policies. This paper introduces a novel scheme based on constrained optimization problems, where the constraints represent multilevel economic game equilibria within a system model involving three agents: the central planner, the mobile network operator, and the mobile data users. These agents strategically optimize their payoff functions by considering benefit factors and decision variables such as the subsidization factor, pricing, and data consumption. To this end, a three-stage dynamic game is proposed to model the players' interactions, employing the backward induction method to ascertain the subgame perfect equilibrium from the Nash equilibrium. A case study is presented, demonstrating a 31.16% increase in social welfare between scenarios involving no adoption of the subsidization factor and its adoption at the optimal value when the central planner enacts it to other players in the game, even if they do not necessarily attain maximum payoff values. In countries aligning with this proposed model, social welfare is maximized through a subsidization scheme.

Keywords: nonlinear programming; game theory; social welfare; network economics; mobile communication

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1. Introduction

Mobile communication is a general-purpose technology to increase innovation across all industries and sectors [1]. For instance, the fourth generation (4G) of Long-Term Evolution (LTE) mobile telecommunications technology standardizes high-speed wireless communications for mobile phones and data terminals [2]. In that regard, the fifth generation (5G) of wireless network technologies is recognized as the key enabler for realizing the full economic and social benefits of digital transformation [3]. However, its impact is limited by regulatory actions to support profitable long-term investments for all major stakeholders while promoting competition, innovation, sharing of infrastructure, and the emergence of new services [3]. A perfect example of other industries that are significantly growing thanks to the Internet and 5G services are modern transportation services such as bike-sharing, scooter-sharing, etc. These are also all services related to the development of mobility as a service [4–7].

One of the sectors with the greatest rate of growth at the moment is the wireless communications sector, which also serves as the main data source for game-theoretic and other network economics modeling challenges [8]. In this context, game theory has been applied to a wide range of networking problems in order to capture the interaction of players who seek a maximum value for their private utility [9]. According to [10], when the interactions and possible conflicts between the stakeholders have been defined, it is possible to structure a system model as a game by stages, where each of them is a subgame that seeks the perfect equilibrium using the backward induction method. That same method has been used in works applied to different sectors, as is clearly presented in [11].

Different notions of social welfare have been introduced in [12]. Moreover, social welfare has been proposed to evaluate the degree of welfare for the whole society [13]. Therefore, the concept of social welfare can be suitable for dealing with relationships among stakeholders. In that sense, welfare theories are commonly used in the discipline of public economics, which focuses on how government intervention might improve social welfare [14]. Thus, by defining key interactions between the demand side and the supply side, social welfare can be maximized by a social planner supported by subsidy or fiscal incentive strategies [15].

Subsidization schemes in mobile communications are used as strategies for improving the stakeholders' welfare [16]. In particular, they can be proposed both to enhance the demand's data consumption and to bring in additional revenue for the supply [16,17]. Furthermore, they can be modeled using multi-stage approaches [16]. For instance, multilevel or multistage programming problems have been used to model multiple decision-maker problems, and researchers from various fields have recently focused their attention on this class of optimization problems, which use a hierarchical structure to solve a variety of applied issues [18].

Mobile Data Users (MDUs) of telecommunication systems are always interested in receiving a greater amount of mobile data as long as the pricing does not exceed their expectations. Certainly, the data that is consumed should include both the Quality of Service (QoS) and Quality of Experience (QoE). In this sense, some companies provide wireless communications services because business opportunities arise within the framework of Mobile Communications Networks Markets (MCNMs). Those companies are called Mobile Network Operators (MNOs), and they must have a spectrum license granted by the government to offer service to MDUs [1]. This work only considers companies such as the MNO and not other Internet service providers that use optical fiber. Due to regulatory actions and their power, the Central Planner (CP) intervenes as the spectrum's owner, guaranteeing, at least, efficient spectrum management, adequate service quality for MDUs, and social welfare. Consequently, this paper is based on the CP's regulatory power to maximize the social welfare of the system through a novel mechanism that could be included in the regulation for both the MNO and the MDUs.

Research articles and extensive reviews have been conducted to study game theory schemes in mobile communications and wireless networks. For instance, references [19,20] offer important descriptions of the main issues and challenges of game theory and network economics in MCNMs. Subsequently, specific publications are discussed below.

A model to study the best use of spectrum is developed in [16] by improving user welfare while guaranteeing profits to network operators within mobile communication services from the regulator's perspective. The authors proposed a data subsidy scheme where the regulator offers a spectrum price discount to network operators in return for imposing the responsibility of providing a predefined data amount to users free of charge. They also formulate the interactions among the regulator, network operators, and users as a two-stage competition composed of the Cournot and Bertrand models. The impact of spectrum constraints, investment costs, and bandwidth allocation decisions is considered in [21]. In that work, the authors kept in mind the need to evaluate the corresponding social welfare implications both for monopolists and competitive service providers. A model examining two competing heterogeneous mobile telecom carriers and two competing heterogeneous Internet content providers, focusing on their cooperative optimal pricing and subsidizing through sponsored data plans, is discussed in [22]. The authors employed a two-leaders and two-followers Stackelberg game, where the mobile telecom carriers act as leaders and the Internet content providers as followers, strategically and tactically deciding on the proportion of data traffic consumption to subsidize the end users. Additionally, a theoretical framework of an economic model is derived in [23] to study the company's incentives and how they offer customized pricing plans when consumers are at risk of moving to another company. In that sense, a two-stage model is developed in which firms can offer a price discount to their customers who signal the intention to switch to a competitor. In contrast to this paper, all those works rely on the assumption that there is more than one MNO to compete for the pricing offered to users using distinct approaches based on two-stage models.

The business-related economics of mobile data offloading in cellular networks, in which a mobile network operator interacts with both price-setting and price-taking to offload the mobile traffic, is studied in [24]. It proposes an incentive framework for the data offloading market, and it models the interactions in the market as a three-stage Stackelberg game. The reward to content providers in the framework of mobile data subsidization by analyzing the rational behaviors of both content providers and users is analyzed [17]. Furthermore, it investigates the interactions among the stakeholders by formulating a hierarchical Stackelberg game model. In that same sense, a hierarchical Stackelberg game is proposed in [25] to model the interactions among the service provider, the content provider, and mobile users under the sponsored content policy, where the authors studied agents' optimal strategies and how they impact network performance. The interactions among the operator, users, and advertisers by a two-stage Stackelberg game are modeled in [26] considering a general data consumption utility function and a general distribution of user valuation. Nevertheless, the previous works do not focus on optimizing the social welfare of the system through benefit factor schemes for each agent according to their interests.

The pricing strategy for virtual operators to maximize their profits in wireless virtualization communication service markets is analyzed in [27]. In that work, the interactions between end users and virtual and non-virtual operators are modeled as a three-stage dynamic process in which operators make coordinated spectrum leasing decisions in Stage I, pricing decisions in Stage II, and finally, user decisions in Stage III. Another perspective is given by the interactions among the market entities in a two-stage hierarchical game. In particular, a transparent market with a single Internet service provider, a heterogeneous group of content service providers, and several mobile users are modeled in [28]. Those models use the properties and conditions of the game to design a distributed iterative scheme that seeks market equilibrium. A multi-dimensional contract design for addressing the problem of engaging users with proper economic incentives in data-rewarding systems is proposed in [29], in which users are involved with multi-dimensional private information. A novel formulation of the two-sided data trading market, the closed-form solution of the three-stage problem, and the benefits of the data trading market to a small operator are proposed in [30]. Thus, the authors utilized a three-stage Stackelberg game to model the interactions between the mobile operator and the users. In stage I, the operator chooses the operational fee imposed on the sellers to maximize its profit. In Stage II, the users select their operators, and in Stage III, the users of the data trading market make their trading decisions. However, neither of them has explicitly considered a subsidization factor as an important metric for the system.

A method based on clustering algorithms to allow modeling a wireless market at multiple levels of detail is introduced in [31]. It also proposes an algorithm that analyzes the game of providers at various subsets of the strategy space, and it combines the results from these subsets to compute the global Nash equilibrium. The price competition for an environment where there are multiple service levels is modeled in [9]. In that work, providers and brokers offer different QoS to users and maximize their profit by competing in a Bertrand game. Joint pricing and bandwidth optimization of operators with both heterogeneous infrastructure and users are studied in [32]. To this end, two prices are announced for accessing network cells so that users can choose an association that maximizes their utility. A revenue maximization framework with dynamic pricing schemes in the mobile social data market is presented in [33]. A sequential dynamic pricing scheme is also proposed, where the network operator individually offers a certain price to each user for social data access in multiple sequential and repeating periods. Although these works only focus on the setting of two-stage problems, there is the possibility of including a new stage for modeling an agent with the power to make decisions in the context of MCNMs.

All of the aforementioned works are based on approaches concerning game theory for modeling systems within the context of MCNMs, considering various topics associated with the network economics framework. However, they do not incorporate the utilization of a subsidization factor aimed at maximizing social welfare derived from demand utility and supply cost, both of which are influenced by key benefit factors for each agent within the system model. In particular, this paper introduces a novel three-stage dynamic game featuring constrained optimization problems, where the constraints also manifest as equilibrium equations recognized as multilevel economic games. Consequently, in line with the proposed setting, the social welfare of the system is optimized through a subsidization scheme implemented by the CP for both the MNO and the MDUs, who, in turn, optimize their payoffs. To the best of the authors' knowledge, no other work in the literature proposes a system model or framework similar to the one adopted in this paper. Nevertheless, works concerning social welfare maximization in diverse settings or contexts, distinct from this paper, are documented in the following articles: [15,32,34–40].

The main contributions of this work are summarized as follows:

1. A novel system model is proposed, in which:
   (a) Subsidization, pricing, data consumption, and benefit factors are used to define and describe the interactions of the system model's agents within the declared setting.
   (b) The payoff functions of the agents that intervene in the system model are declared using subsidization schemes based on a three-stage dynamic game. Because the CP decides the subsidization factor to influence the other agents' decisions, the MNO maximizes its profit and obtains a tax rate reduction, while the MDUs maximize their payoff and consume additional data.
   (c) The backward induction method is used to derive a unique Nash equilibrium in the system model stages based on modeling an approach through constrained optimization problems, achieving subgame perfect equilibrium and maximizing the social welfare of the system.
2. The numerical assessments demonstrate that the definition of a subsidization factor maximizes the social welfare of the system. In particular, the maximization of social welfare under the conditions and constraints of the proposed setting establishes that the CP is an agent that does not take sides with any of the other agents but instead seeks to socially influence the welfare of the system using regulation as a mechanism to enact the defined subsidization factor. Then, the numerical evidence shows that the individual outcomes of both the MNO's profit and the MDUs' payoff do not necessarily reach their maximum value after the adoption of the optimal subsidization factor.

The rest of the paper is organized as follows. Section 2 describes the setting of the system model. In Section 3, the problem statement is declared for each of the proposed stages. Section 4 presents the game theory formulation and the existence of the subgame perfect equilibrium in the system model. In Section 5, a case study, results, and discussion are presented, and finally, in Section 6, the conclusions resulting from the development of the system model are given.

2. Setting

This paper considers a three-stage setting in which agents interact with each other. In particular, the higher-stage agent aims to maximize the social welfare of the system. The agents of stages I, II, and III are the CP, the MNO, and the MDUs, respectively. Only one agent per stage is contemplated. However, it is essential to note that MDUs are considered aggregate users and represent a unique agent within the corresponding stage. Next, the general interpretation of the setting, the benefit factors, and the description of the agents are presented.

2.1. General Framework

The system model assumed for this paper is shown in Figure 1. That illustration depicts interactions between agents and their motivations. Based on [8,41], this work considers that an agent is rational if it consistently makes decisions to attain its objectives by maximizing its utility or payoff, taking into account knowledge of the environment. That condition is known to all stakeholders. Self-interest is essentially an implication of rationality. Therefore, maximizing utility is not necessarily the same as maximizing money returns. The above means that utility is not only represented in monetary returns because satisfaction levels can be considered part of welfare.

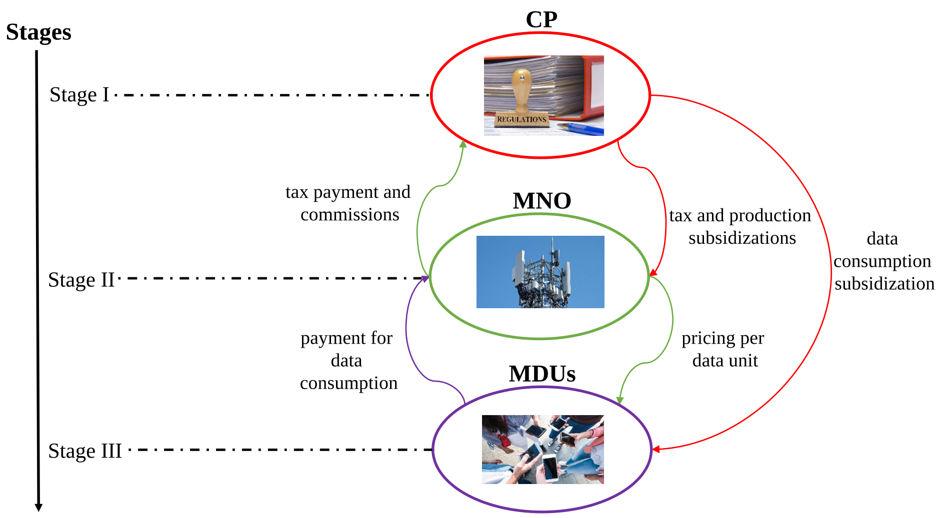


Figure 1. System model of the proposed setting, including both agents and their interactions.

According to Figure 1, the CP gives subsidizations to the other agents to maximize the social welfare of the system. At the same time, the MNO interacts with the CP using the spectrum and paying both taxes due to their earnings and commissions. Moreover, the costs of the additional data consumption by the MDUs and the reduction of the effective tax rate are recognized by the MNO. On the other hand, the interaction between the CP and the MDUs is through the MNO, which gives them additional data free of charge to increase their payoff. The MDUs pay taxes to the CP when paying the MNO for the service provided. Similarly, the MNO also interacts with the MDUs. In that case, the MNO offers data to the MDUs in compliance with the current QoS and QoE regulations, and it transfers them the data subsidization given by the CP. Please note that the MDUs demand the services offered by the MNO to improve their satisfaction levels, which impacts their payoff function.

It is important to mention that the decisions of the agents follow a temporality and sequence established according to the hierarchy between the stages. Figure 2 shows these aspects. Moreover, the order of decision-making is presented as follows:

1. The agent who first makes the decision is the CP since it has all the resources for this purpose. Once the CP makes the decision, it can announce it through the regulation so that the other agents can access this information.
2. The MNO makes the second-order decision. It studies the resolution established with its work team so that, based on the announced subsidization factor and other considerations regarding profit and utility functions, it can design the pricing that it offers to the MDUs.
3. In the end, the MDUs decide whether or not to purchase a data plan based on their knowledge and the pricing that the MNO has offered.

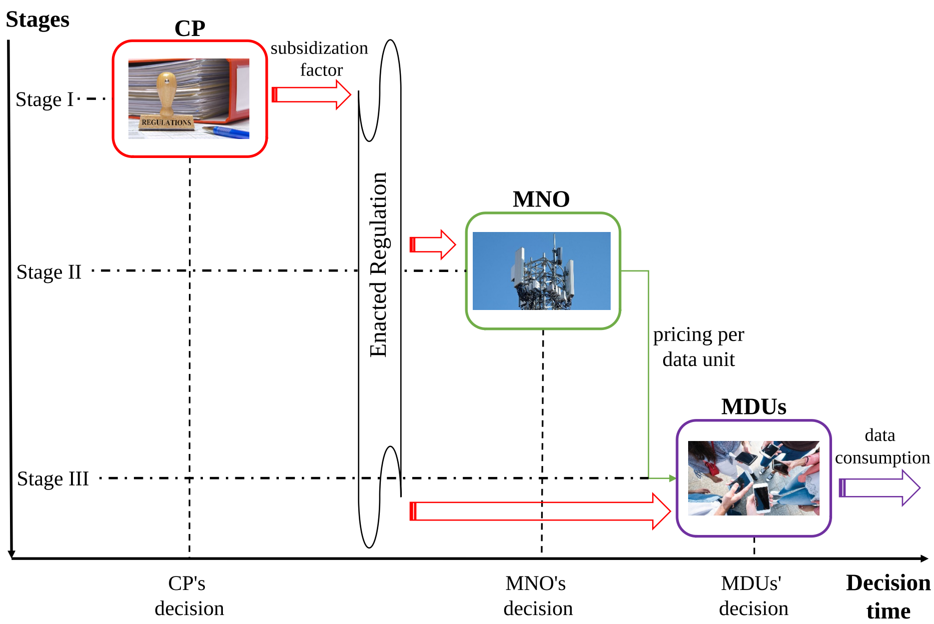


Figure 2. Temporality and sequence of the agent decisions.

In general terms, the agent's decisions are sequential, and they have different motivations according to their interests. Initially, a resolution with the MCNM's guidelines must be socialized to the agents. In that sense, the CP has to perform behavioral studies of other agents and classify them to design the resolution. Then, the MNO analyzes the resolution, adjusts its business plan, and defines the pricing for the MDUs. Finally, the MNO offers the data plan to the MDUs so they can analyze it and make their final decision about the purchase.

In the following subsections, the benefit factors, variables, and parameters utilized in this work are presented. Among these, the benefit factors are proposed to model the interactions between agents. Specifically, the variables α, p, and r denote the subsidization factor, pricing, and data consumption, respectively. These variables are determined within the stages of the proposed system model. Conversely, the parameters $\bar{r}$, σ, and ϕ represent the maximum data that the MDUs are willing to consume, the marginality of the MDUs' utility, and the production factor of the MNO, respectively. The aforementioned parameters are discerned by the CP through econometric analysis and announced to the other agents. Moreover, the payoff functions of the three agents are elaborated upon, incorporating both the subsidization factor and the benefit factors.

2.2. Benefit Factors

According to the above general framework, this paper adopts the term *benefit factors* to declare relevant attributes of the agents based on their behavior. In particular, γ, β, and τ represent associated benefit factors on the additional data for the MDUs and the revenue and tax rate for the MNO, respectively. Those benefit factors are related to the subsidization factor, $\alpha \in [0, 1]$, whose value is optimally decided by the CP to maximize the social welfare of the system. Therefore, α can be interpreted as the percentage of the subsidization that an agent has available to maximize its payoff function. Subsequently, from the α value optimally decided by the CP, the MDUs and the MNO use the benefit factors given by Equations (1)–(3), which are proposed equations by the authors of this work, namely:

$$\gamma = 1 + \alpha \tag{1}$$

$$\beta = \frac{4\alpha}{1 + 4\alpha} \tag{2}$$

$$\tau = \frac{1}{1 + \alpha} \quad (3)$$

The behavior of γ, β, and τ as a function of α are shown in Figure 3. It can be seen that as α increases, γ and β also increase, while τ decreases. According to the next subsections, this makes sense because γ and β impact both the MDUs' utility function and the MNO's revenue function, and they benefit from an increase in the subsidization granted by the CP. The factor τ is also considered a benefit. However, given that it is situated in the MNO's cost function, its objective is to reduce the income tax rate.

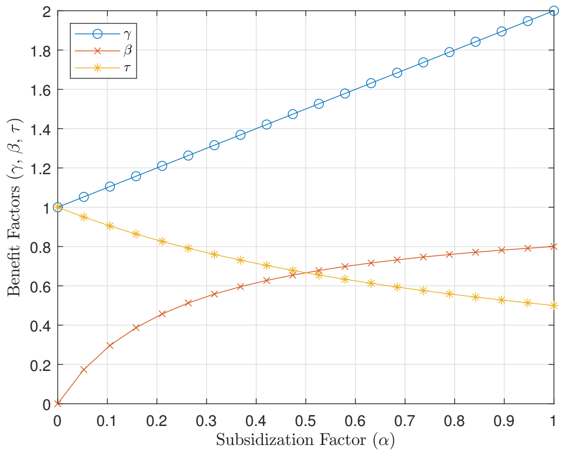


Figure 3. Benefit factors versus subsidization factor in order to verify the behavior of γ, β, and τ as a function of α.

It is important to emphasize the key role of the subsidization factor in determining the final value of the benefit factors. For instance, considering a subsidization factor of 0.30, the following framework can be established for each of the proposed benefit factors:

- Let γ be the factor that adjusts to the data consumption preferences of MDUs, encompassing both contracted and subsidized data. When $\alpha = 0.30$ and $\gamma = 1.30$, it indicates that MDUs are entitled to consume 30.00% more data than their contracted amount with the MNO.
- Let β be the factor corresponding to the MNO's declared cost for producing the data consumed by MDUs in a subsidized manner. Under ideal conditions, the MNO is recognized for up to 80.00% of the declared cost. With $\alpha = 0.30$ and $\beta = 0.55$, it means that the MNO receives recognition for 55.00% of the cost for producing the subsidized data to be consumed by MDUs.
- Let τ be the factor adjusting the income tax payable by the MNO. In the best-case scenario, the MNO covers up to 50.00% of the income tax rate. When $\alpha = 0.30$ and $\tau = 0.77$, it implies that the MNO receives a 23.00% tax discount and pays 77.00% of the total taxes owed.

Based on the aforementioned analysis, it is evident that the subsidization factor plays a crucial role and significantly influences both the benefit factors and the involved agents. Specifically, it reveals how the subsidization factor, represented by γ, β, and τ, facilitates interactions among the three agents. It is noteworthy that when the CP determines the value of α, it directly affects the MNO by recognizing a portion of the cost as revenue, which is generated for producing data to MDUs free of charge. This revenue is contingent upon various factors such as β, pricing, and other pertinent parameters. Additionally, the provision of data free of charge is contingent upon α and the data consumption. Similarly, the MNO is also affected by tax discounts, which are dependent on τ, data consumption, and pricing. Consequently, a network of interactions is established among the agents based on the proposed benefit factors.

It should be noted that Equations (1)–(3) are announced by the CP as part of the regulation. Then, it is assumed that both the MNO and the MDUs know the regulation because it is in the public domain.

2.3. Mobile Data Users Description

The MDUs are considered rational and aggregate. They make decisions and maximize their payoff. In particular, the MDUs are represented based on aggregate demand information, and they consume data from the MNO's wireless network infrastructure. In this regard, by signing a contract to consume mobile data services, the MDUs can gain direct access to the MNO's network. It is considered that the MDUs can terminate the contract unilaterally and without penalty. In Colombia, for example, since 2014, the Communications Regulatory Commission (CRC) has enacted Resolution 4444, which prohibits fixed terms in mobile communications contracts, commonly referred to as permanency clauses [42]. Without loss of generality, it is assumed that the pricing paid by the MDUs to the MNO is exempt from sales tax.

The MDUs' satisfaction level is represented by a utility function. In the literature, there are several functions to represent the utility of the MDUs [32]. For instance, functions such as logarithmic, α-fair, or polynomials. This work adopts a polynomial utility function.

Definition 1. *The MDUs' utility function is a second-order polynomial, defined by Equation (4).*

$$U(r;\gamma,\sigma,\bar{r}) = \begin{cases} -\sigma\gamma^2 r^2 + 2\sigma\gamma^2\bar{r}r & 0 \leq r \leq \bar{r} \\ \sigma\gamma^2\bar{r}^2 & r > \bar{r} \end{cases} \tag{4}$$

where the variable r is the MDUs' data consumption. The parameters $\bar{r}$, σ, and γ represent the maximum data that the MDUs are willing to consume, the marginality utility, and the benefit factor in data, respectively.

Figure 4 shows the behavior of the MDUs' utility function versus data consumption. It is depicted that as the MDUs consume more data, the utility is increased. The maximum level of data consumption is $\bar{r}$. Consuming more of $\bar{r}$ does not give a higher level of satisfaction to the MDUs. If the MDUs consume $\bar{r}$, they are considered satisfied agents, and their utility remains constant at $\sigma\gamma^2\bar{r}^2$. It is also identified that the MDUs are risk-averse according to the concavity of the utility function presented.

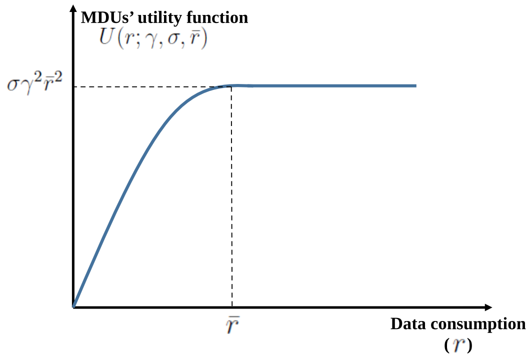


Figure 4. Behavior of the MDUs' utility function for different values of r.

The function $U(r;\gamma,\sigma,\bar{r})$ satisfies the properties proposed in [32,43,44] to be a valid utility function that represents the preferences of MDUs from an economics perspective, namely:

Property 1. *$U(\cdot)$ is zero when the consumption level is zero. Therefore, $U(0;\gamma,\sigma,\bar{r}) = 0$, i.e., there is no utility when no data are consumed, $r = 0$.*

Property 2. *$U(\cdot)$ is a strictly increasing function. The MDUs have elastic requirements on data consumption, and the marginal benefit is non-negative, $0 < \frac{\partial U(\cdot)}{\partial r} < +\infty$, for $0 < r \leq \bar{r}$.*

Property 3. *$U(\cdot)$ is a strictly concave function. This implies that the marginal benefit is non-increasing, $\frac{\partial^2 U(\cdot)}{\partial r^2} < 0$, for $0 \leq r \leq \bar{r}$.*

Property 4. *$r\frac{\partial U(\cdot)}{\partial r}$ is twice differentiable, for $0 \leq r \leq \bar{r}$.*

As a result of the utility function's definition, the following definitions can be presented:

Definition 2. *The MDUs' cost function is established as follows: $C_u = rp$, where r is the data consumption and p is the pricing. This cost is equivalent to a portion of the MNO's revenue.*

Definition 3. *The MDUs' payoff function is stated as follows $G(r;\gamma,\sigma,\bar{r},p) = U(r;\gamma,\sigma,\bar{r}) - C_u$, where the variable r and the other parameters were declared in Equation (4) and Definition 2. The benefit factor γ is given by Equation (1).*

2.4. Mobile Network Operator Description

The MNO is a rational for-profit private firm that seeks to increase its revenue and reduce its costs by maximizing its profit. The MNO leases the spectrum of the CP to effectively carry out its operation, providing data plans to MDUs. In addition, it can hire suitable personnel to make strategic decisions and design optimization models in the context of MCNMs.

The spectrum license acquired by the MNO is for several years to have legal stability in the design and implementation of its business model. Usually, that license is allocated through auction mechanisms. For instance, Colombia has enacted legislation aimed at the modernization of the Information and Communication Technology (ICT) sector. This legislation, formally known as Law 1978 of 25 July 2019, regulates various aspects of the sector. Notably, it stipulates that the spectrum allocated to a MNO remains valid for up to 20 years and is subject to renewal. Renewal considerations are based on factors that prioritize the maximization of social welfare, in accordance with current regulations.

Without loss of generality, it is assumed that the investment costs associated with the wireless network infrastructure required by the MNO for its operation are fixed. Likewise, it is also considered that the MNO has a sufficient network capacity to provide data services to the MDUs by fulfilling the QoS and QoE levels established in the regulation. Notice that the network capacity of the communications system is determined both by the amount of spectrum used and by the spectral efficiency of the technology selected to offer the service. In light of that, the CP guarantees technological autonomy to the MNO, and, in turn, it transfers network neutrality to the MDUs.

Because the MNO seeks to maximize its profit as a rational agent, its behavior is represented by Equation (5):

$$\prod(\cdot) = R(\cdot) - C_o(\cdot) - C_{tax}(\cdot) \tag{5}$$

where $\prod(\cdot)$ is the profit function whose decision variable $p \in [0, +\infty)$ denotes the pricing offered to the MDUs. The Right-Hand Side (RHS) of Equation (5) includes the revenue function $R(\cdot)$ and, the cost functions $C_o(\cdot)$ and $C_{tax}(\cdot)$. In particular, the following definitions are established to describe the MNO based on the experience of the authors of this work and are also part of the novel modeling of the article.

Definition 4. *The MNO's revenue function is declared as $R = (rp) + (\alpha rp\phi\beta)$, where r is the MDUs' data consumption, p is the pricing paid by the MDUs, ϕ is the MNO's data production factor expressed as $\phi = 1/k$, and k is the firm's profit factor. The benefit factor β is given by Equation (2).*

The MNO's revenue function has two parts. The first depends on the data consumption paid by the MDUs. The second depends on the data that the MDUs receive as subsidization, which the CP pays on the base price of production (i.e., without profit). It should be noted that the CP does not recognize 100% of the cost declared by the MNO for producing data, but only a benefit of β times the declared value. The preceding is conducted so that the CP's subsidization factor gradually affects the MNO's revenue.

It is worth noting that k and ϕ can be interpreted as follows:

- Let k be the profit factor that represents the MNO's profit over its production costs for selling data to the MDUs. It is a quantity greater than 1. For example, if $k = 1.20$, it then means that the MNO obtains a profit equal to 20% over its production costs.
- Let ϕ be the data production factor that represents the cost factor of producing data to provide to the MDUs. It is the inverse of k.

Definition 5. *The MNO's production cost is established as follows:* $C_o = ((1+\alpha)rp\phi)$*, where the variable* p *and the other parameters were declared in Definition 4.*

The MNO's production cost function includes the cost of providing data to the MDUs for consumption, i.e., data costs both subsidized and unsubsidized. In a microeconomics context, those costs can be considered increasing with the decision variable.

Definition 6. *The MNO's tax cost function includes a benefit of* τ *times the income tax rate, and it is established as follows:* $C_{tax} = (rpI_{tax}\tau)$*, where* I_{tax} *is the firm's income tax rate, the variable* p*, and the other parameters were declared in Definition 4. The benefit factor* τ *is given by Equation (3).*

Definition 7. *As a consequence of Definitions 4–6, it is defined that the MNO's profit function is* $\prod = (rp) + (\alpha rp\phi\beta) - ((1+\alpha)rp\phi) - (rpI_{tax}\tau)$.

2.5. Central Planner Description

The CP is a government entity in a country that traditionally depends on executive power. Since the state owns the spectrum, it must promote efficient mechanisms for its management. For instance, the CP could have spectrum allocation, monitoring, and regulation functions seeking to maximize social welfare. Therefore, this paper proposes that the government, through the CP, grants the other agents a subsidization scheme to be used according to their interests in the context of MCNMs.

In this work, it is established that the social welfare function of the system is derived by summing the demand-side utilities of the agents and subtracting the total supply-side costs incurred by the agents [45]. Without loss of generality, it is assumed that the CP functions as a neutral agent and is not affiliated with either the demand or supply sectors. Furthermore, its operating costs are fixed and do not influence the decision-making processes of the system agents. Similarly, as this paper focuses on a single agent per stage, the utility on the demand side pertains to the MDUs, and the cost on the supply side pertains to the MNO. Consequently, the social welfare function is defined as follows:

Definition 8. *The social welfare function of the system is given by Equation (6).*

$$SW(\cdot) = U(\cdot) - C_o(\cdot) - C_{tax}(\cdot) \tag{6}$$

where $U(\cdot)$ is the MDUs' utility function and, $C_o(\cdot)$ and $C_{tax}(\cdot)$ are the expressions that represent the production and tax costs of the MNO, respectively. The expressions on the RHS of Equation (6) were described in the last subsections. It should be noted that social welfare depends on both variables and parameters of the system agents.

Consequently, the CP has the task of interpreting the rational behavior of both the MNO and the MDUs. Therefore, once the resolution is promulgated, they determine how to model their objective functions so that profit and payoff can be maximized at each stage.

3. Problem Statement

The problem in this paper is to maximize the system's social welfare by the CP using its government power to define a subsidization factor that influences the decision-making of the other agents. Therefore, each of them can use it to maximize their payoffs according to the established constraints within the system model. Consequently, the central question of this paper is: What is the optimal subsidization factor that the CP has to enact so that the social welfare of the system is maximized based on the interaction among agents?

Next, the models of each stage and their hierarchy are presented.

3.1. Central Planer Model in Stage I

The main interest of the CP is to maximize the social welfare of the system, and it has to model a problem that includes both the demand side and the supply side. As a result, according to Definitions 1, 5, 6, and 8, the social welfare function of the system given by Equation (6) can be updated by Equation (7)

$$SW(\alpha; p, r, \gamma, \sigma, \bar{r}, \phi, I_{tax}, \tau) = (-\sigma\gamma^2 r^2 + 2\sigma\gamma^2 \bar{r} r) - ((1+\alpha) r p \phi) - (r p I_{tax} \tau) \tag{7}$$

where Equation (7) is the objective function of the CP's optimization problem, and α is the decision variable, which reflects the system model's subsidization factor.

The constraint given by Equation (8), which represents the variation range of the decision variable α, ensures that the CP's subsidization factor includes both the option of not giving subsidization and giving it progressively up to unity.

$$0 \leq \alpha \leq 1 \tag{8}$$

Finally, in stage I, the optimization problem is given by Equation (9) to find the optimal value of α that maximizes Equation (7) subject to the constraint Equation (8).

$$\begin{aligned} \max_{\alpha} \quad & SW(\alpha; p, r, \gamma, \sigma, \bar{r}, \phi, I_{tax}, \tau) \\ \text{s.t.} \quad & 0 \leq \alpha \leq 1 \end{aligned} \tag{9}$$

3.2. Mobile Network Operator Model in Stage II

The main interest of the MNO is to maximize its profit, and it has to model a problem that includes its revenue and costs. As a result, the profit function of Definition 7 is considered the objective function of the MNO's optimization problem given by Equation (10), whose decision variable is p.

$$\prod(p; r, \alpha, \phi, \beta, I_{tax}, \tau) = (rp) + (\alpha r p \phi \beta) - ((1+\alpha) r p \phi) - (r p I_{tax} \tau) \tag{10}$$

The optimization problem modeled at this stage has a constraint given by Equation (11), which represents the variation range of the decision variable p to guarantee its non-negativity and maximum pricing. Let P_{max} be the maximum pricing for the MDUs, whose value can only be defined based on the interactions and responses of the other agents because the MNO is localized in the intermediate stage of the proposed system model. Consequently, the final value of P_{max} is derived in the next section.

$$0 \leq p \leq P_{max} \tag{11}$$

Finally, in Stage II, the optimization problem is given by Equation (12) to find the optimal value of p that maximizes Equation (10) subject to the constraint Equation (11).

$$\begin{aligned} \max_{p} \quad & \prod(p; r, \alpha, \phi, \beta, I_{tax}, \tau) \\ \text{s.t.} \quad & 0 \leq p \leq P_{max} \end{aligned} \tag{12}$$

3.3. Mobile Data Users Model in Stage III

The main interest of the MDUs is to maximize their payoff, and they have to model a problem that includes their utility and costs. As a result, the payoff function of Definition 3 is considered the objective function of the MDUs' optimization problem given by Equation (13), whose decision variable is r.

$$G(r;\gamma,\sigma,\bar{r},p) = \begin{cases} -\sigma\gamma^2 r^2 + 2\sigma\gamma^2\bar{r}r - rp & 0 \leq r \leq \bar{r} \\ \sigma\gamma^2\bar{r}^2 - rp & r > \bar{r} \end{cases} \tag{13}$$

The optimization problem modeled in this stage has a single constraint that establishes the variation range of the decision variable r and guarantees its non-negativity and maximum data consumption. Consequently, Lemma 1 is presented to support the constraint of the MDUs' maximum data consumption. Its proof is shown in Appendix A.

Lemma 1. *The maximum value that the variable r can take in Equation (13) is also the maximum value that it can take in Equation (4). Therefore, $\bar{r}$ is the maximum data consumption for the MDUs, and the variation range of the decision variable r is given by Equation (14)*

$$0 \leq r \leq \bar{r} \tag{14}$$

The aim of Lemma 1 is to define the maximum data consumption of the MDUs. Therefore, if the MDUs want to consume more data from $\bar{r}$, then they do not obtain a higher level of satisfaction. On the contrary, they reduce their payoff. It should be noted that the minimum data consumption is zero because MDUs can choose not to consume data.

Finally, in Stage III, the optimization problem is given by Equation (15) to find the optimal value of r that maximizes Equation (13) subject to the constraint Equation (14).

$$\begin{aligned} \max_{r} \quad & G(r;\gamma,\sigma,\bar{r},p) \\ \text{s.t.} \quad & 0 \leq r \leq \bar{r} \end{aligned} \tag{15}$$

3.4. Hierarchical Problem by Stages

The problem proposed in this paper is structured in three stages. Then, Equation (16) describes the problems that agents have to solve in their stages jointly and comprehensively. It is a constrained optimization problem in which the CP problem is at the top, and most of the constraints are also, in turn, optimization problems that must be solved by the other agents in each of the stages.

$$\begin{aligned} \max_{\alpha} \quad & SW(\alpha;p^*,r^*,\gamma,\sigma,\bar{r},\phi,I_{tax},\tau) \\ \text{s.t.} \quad & 0 \leq \alpha \leq 1 \\ & p^* = \arg\max_p \prod(p;r^*,\alpha,\phi,\beta,I_{tax},\tau) \\ & \quad \text{s.t.} \quad 0 \leq p \leq P_{max} \\ & r^* = \arg\max_r G(r;\gamma,\sigma,\bar{r},p) \\ & \quad \text{s.t.} \quad 0 \leq r \leq \bar{r} \end{aligned} \tag{16}$$

It is pointed out that Equation (16) is the model based on a three-stage constrained optimization problem, where integrally, each associated stage has a hierarchy level in the proposed mathematical structure. Finally, the solution for Equation (16) is presented in the next section.

4. Best Responses and Subgame Perfect Equilibrium

From this section onward, agents assume the roles of players. This new nomenclature aligns with the terminology of game theory. The subsequent subsections delve into the key aspects of the game, spanning from its formulation to its resolution.

4.1. Formulation and Methodology of the Proposed Game

The proposed approach is a dynamic game where each player can choose their strategy given the information available about the strategies selected by the other players [10]. Because a player observes the strategies of all the other players, in the proposed system model, the game is considered a complete information game, and each stage is a subgame. According to [46], once the CP has made a decision, the other players play a parameterized game. One technique to identify the best strategy is to solve all the subgames, i.e., find the best response at each stage by solving each player's optimization problem. Due to the fact that the proposed model is a multi-stage dynamic game, the concept of subgame perfect equilibrium is used, which means that the players' strategies form a Nash equilibrium in each subgame of the original game, and the backward induction method is used to solve it [10,19,46].

The solution method for Equation (16) is shown in Figure 5, where the backward induction method is proposed.

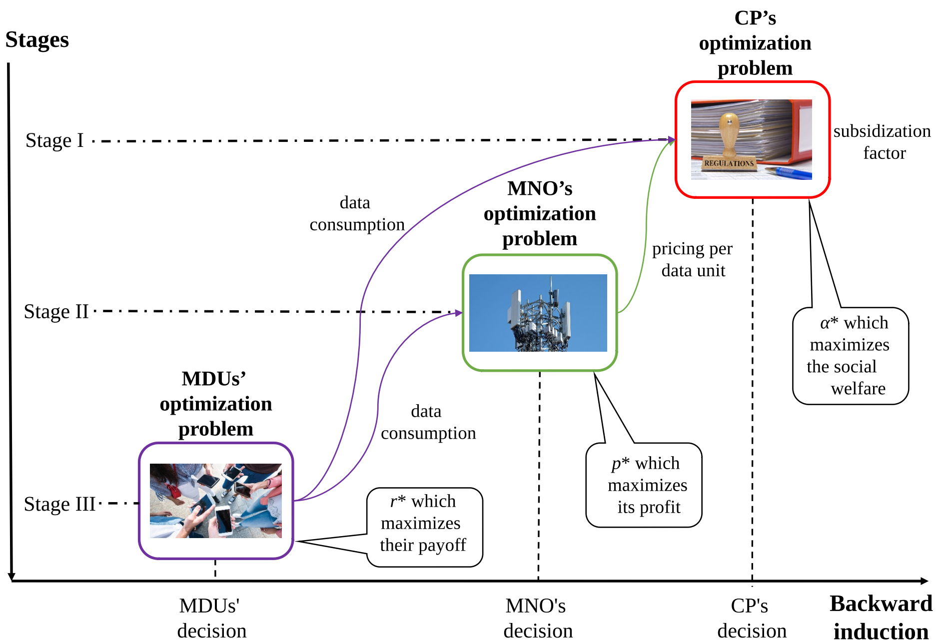


Figure 5. The proposed methodology for solving the game using the backward induction method.

4.2. Subgame in Stage III

In this subgame, the player is represented by the MDUs who maximize their payoff. Using the backward induction method, since the MDUs are at the lower level of the system model, they make their decision first compared to the other players based on the available information.

The solution of the MDUs' optimization problem, given by Equation (15), is presented in Theorem 1. Its proof is shown in Appendix B.

Theorem 1. *The best response in Stage III is unique, and it is represented by the optimal policy given by Equation (17). Since Equation (17) has been found, the final value of P_{max} is given by Equation (18)*

$$r^* = \arg\max_{r} \; G(r;\gamma,\sigma,\bar{r},p) = \bar{r} - \frac{p}{2\sigma\gamma^2} \tag{17}$$

$$0 \leq p \leq 2\sigma\gamma^2\bar{r} \quad \text{then} \quad P_{max} = 2\sigma\gamma^2\bar{r} \tag{18}$$

The result of Theorem 1 allows obtaining the optimal policy r^* of the MDUs and the maximum value of p. In particular, r^* maximizes the MDUs' payoff by adjusting the data consumption, which mainly depends on the pricing established by the MNO. Moreover, Theorem 1 derives the upper bound for the pricing by defining the value of P_{max} for the non-negative decision variable r.

4.3. Subgame in Stage II

In this subgame, the player is the MNO whose goal is to maximize its profit. By applying the backward induction method, as the MNO is situated in the intermediate stage of the system model, it makes its decision based on the information available after the MDUs have decided. Consequently, it is pointed out that the MDUs' optimal policy and P_{max} are already known, and they are now adopted as new constraints of the MNO's optimization problem. Therefore, Equation (12) can be updated, and the new optimization problem of the MNO is given by Equation (19)

$$\begin{array}{ll} \max\limits_{p} & \prod(p;\bar{r}-\frac{p}{2\sigma(1+\alpha)^2},\alpha,\phi,\beta,I_{tax},\tau) \\ \text{s.t.} & 0\leq p\leq 2\sigma(1+\alpha)^2\bar{r} \end{array} \tag{19}$$

The solution of the MNO's optimization problem, given by Equation (19), is presented in Theorem 2. Its proof is shown in Appendix C.

Theorem 2. *The best response in Stage II is represented by the optimal policy given by Equation (20), and it can be guaranteed under conditions given by Equation (21)*

$$p^* = \arg\max_{p} \prod(p;\bar{r}-\frac{p}{2\sigma(1+\alpha)^2},\alpha,\phi,\beta,I_{tax},\tau) = \sigma(1+\alpha)^2\bar{r} \tag{20}$$

$$0<\phi<\frac{1+4\alpha}{1+5\alpha}\left[1-\frac{I_{tax}}{1+\alpha}\right] \tag{21}$$

The result of Theorem 2 allows obtaining the optimal policy p^* of the MNO. However, this optimal policy depends on the fulfillment of the inequality given by Equation (21), which involves values of ϕ to establish the feasible region of the MNO's optimization problem from α and I_{tax}.

4.4. Subgame in Stage I

In this subgame, the player is the CP and maximizes the social welfare of the system. By applying the backward induction method, the CP makes its decision based on the information available after the other players have decided. Consequently, the optimal strategies of the MDUs and the MNO are already known, and they are now adopted as new constraints of the CP's optimization problem. Therefore, Equation (9) can be updated, and the new CP optimization problem is given by Equation (22)

$$\begin{array}{ll} \max\limits_{\alpha} & SW(\alpha;\sigma(1+\alpha)^2\bar{r},\bar{r}-\frac{p}{2\sigma(1+\alpha)^2},\gamma,\sigma,\bar{r},\phi,I_{tax},\tau) \\ \text{s.t.} & 0\leq\alpha\leq 1 \end{array} \tag{22}$$

The solution of the CP's optimization problem, given by Equation (22), is presented in Theorem 3. Its proof is shown in Appendix D.

Theorem 3. *The best response in Stage I is represented by the optimal policy given by Equation (23) and can be guaranteed under conditions given by Equation (24)*

$$\begin{gathered} \alpha^* = \arg\max_{\alpha} SW(\alpha;\sigma(1+\alpha)^2\bar{r},\bar{r}-\frac{p}{2\sigma(1+\alpha)^2},\gamma,\sigma,\bar{r},\phi,I_{tax},\tau) \\ \alpha^* = \frac{\sqrt{1-\frac{4\phi I_{tax}}{3}}-2\phi+1}{2\phi} \end{gathered} \tag{23}$$

$$\frac{1-2\phi}{2\phi} < \alpha \leq 1 \tag{24}$$

The result of Theorem 3 allows obtaining the optimal policy α^* of the CP. Nevertheless, this optimal policy depends on the fulfillment of the inequality given by Equation (24), which involves values of α to establish the feasible region of the CP's optimization problem from ϕ.

4.5. Subgame Perfect Equilibrium

In the conclusion of the game, the intersection of the best responses found in each stage of the system model produces the subgame perfect equilibrium, which is a Nash equilibrium. Then, the optimal policies obtained in each stage and their intersections are presented in Corollary 1 as follows:

Corollary 1. *From the results obtained in Theorem 1–Theorem 3, then the intersections of the optimal policies α^*, p^*, and r^* are given by Equation (25), Equation (26), and Equation (27), respectively.*

$$\alpha^* = \frac{\sqrt{1-\frac{4\phi I_{tax}}{3}} - 2\phi + 1}{2\phi} \tag{25}$$

$$p^* = \sigma\left[1 + \left(\frac{\sqrt{1-\frac{4\phi I_{tax}}{3}} - 2\phi + 1}{2\phi}\right)\right]^2 \bar{r} \tag{26}$$

$$r^* = \frac{\bar{r}}{2} \tag{27}$$

The results of Corollary 1 establish the subgame perfect equilibrium of the whole game. The above is also used to calculate the payoff functions of the players. For this purpose, Corollary 2 is defined as follows:

Corollary 2. *From the results of Corollary 1, it is possible to substitute the optimal policy of each player directly in the objective functions declared in the stages of the proposed system model. Then, the social welfare of the system, the MNO's profit, and the MDUs' payoff are given by Equation (28), Equation (29), and Equation (30), respectively.*

$$SW = \sigma\bar{r}^2\left[\frac{1}{4}(3-2\phi-2I_{tax}) + \frac{3}{2}(1-\phi-\frac{1}{3}I_{tax})\alpha^* + \frac{3}{4}(1-2\phi)(\alpha^*)^2 - \frac{1}{2}\phi(\alpha^*)^3\right] \tag{28}$$

$$\prod = (r^*p^*) + \left[\frac{4(\alpha^*)^2 r^* p^* \phi}{(1+4\alpha^*)}\right] - ((1+\alpha^*)r^*p^*\phi) - \left[\frac{r^*p^*I_{tax}}{(1+\alpha^*)}\right] \tag{29}$$

$$G = -\sigma(1+(\alpha^*))^2(r^*)^2 + 2\sigma(1+(\alpha^*))^2\bar{r}r^* - r^*p^* \tag{30}$$

The results of Corollary 2 establish the final and optimal values of the player-payoff functions according to their selfish interests and the best responses chosen for the game modeled in this work.

5. Case Study, Results, and Discussion

In this section, a case study is jointly proposed, along with its numerical results and discussion. Please note that the CP has access to several data sources with the information required to design and announce the regulation to the other players. It is important to highlight that the regulation governs the entire proposed system model. In that regard, the present case study provides values for some of the parameters based on actual information associated with the MCNMs.

5.1. Case Study Description

This case study is initiated by setting the value of the income tax rate. The corporate tax is charged to for-profit firms and changes from country to country. In this way, the value of the income tax rate is chosen from that established in the United States of America according to [47]. In particular, for the MNO, a corporate income tax rate of 21.00% has been adopted so that $I_{tax} = 0.21$.

Because the CP decides first, the optimal value of α must be found first. Based on Equations (21), (24) and (25); Figure 6 shows the behavior of α as a function of ϕ, and the two constraints associated with the feasible region of ϕ. In Equation (25) is observed that when ϕ increases, α decreases. A suitable value to choose ϕ from the profit factor is to set $k = 1.50$, which is reasonable as MNO's profit factor. In particular, a company that offers its products or services in the market at an increased rate of 50% over its costs could obtain a competitive profit margin. Therefore, given the value of k and found $\phi = 0.67$, the subsidization factor value for the system model is calculated from Equation (25), obtaining $\alpha = 0.43$. Likewise, through Figure 6, it is confirmed that the α value is part of the established feasible region. Moreover, the values of γ, β, and τ are found from Equation (1), Equation (2), and Equation (3), namely 1.43, 0.63, 0.70, respectively.

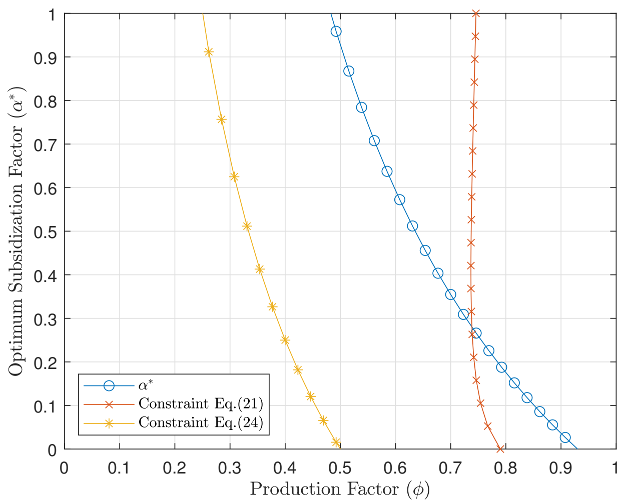


Figure 6. Optimum subsidization factor versus production factor, in order to verify both the behavior of α as a function of ϕ and the α's feasible region.

The description of the case study is followed by selecting the parameters $\bar{r}$ and σ. For instance, In Colombia, the national added value of the ICT industry represents 3.70% of the Gross Domestic Product (GDP). This data, sourced from the National Administrative Department of Statistics (DANE), pertains to the year 2019. Furthermore, the MCNM is concentrated in three MNOs, with a respective share per MNO of 46.93%, 24.26%, and 17.56%, totaling 88.75%. This data, which is accurate as of December 2019, has been directly sourced from the Ministry of Information and Communication Technologies (MinTIC). In addition, the remainder of the market is held by other MNOs with shares of less than 4.00%, including the Mobile Virtual Network Operators (MVNOs). In consequence, the data inputs to formulate this case study, such as data consumption and pricing, are based on the data plans offered by those three MNOs in the last quarter of 2020, as is presented in Table 1. The last input for this case study is adopted as the average of the data from those three operators and is presented in the final row of Table 1. Because those data consumption plans offer 8.00 GB, it is assumed that the MDUs would be satisfied with a 16.00 GB data plan, then $\bar{r}$ is set as 16.00 GB. Moreover, σ is established using Equation (18) with $\gamma = 1.00$ and making the pricing equal to 16.29 USD. Therefore, the marginality of the MDUs' utility is calculated as $\sigma = 0.51$ USD/GB.

Table 1. Data consumption plans that are offered by the MNOs to the MDUs.

MNO	Data Consumption Plan	Pricing Plan
Operator 1	8.00 GB	16.47 USD
Operator 2	8.00 GB	15.94 USD
Operator 3	8.00 GB	16.45 USD
Operator for the case study	8.00 GB	16.29 USD

5.2. Results and Discussion

Based on the stated case study, it is now possible to compare and discuss various results related to the proposed system model. The social welfare of the system is calculated from Equation (28) and is shown in Figure 7 as a function of the subsidization factor. This illustration depicts the social welfare's behavior for four values of the parameter k, including the one proposed in the case study. If the CP does not establish a subsidization factor, then it would lose the possibility of maximizing the social welfare of the system, as well as if it is established very high. As a result, Figure 7 enables the CP to determine which value of the subsidization factor should be used to optimize the system's social welfare. Once the variable α is defined, the regulation is announced so that the MNO and the MDUs can learn about it and analyze it in light of their respective interests. In compliance with the proposed case study, the optimal value of the subsidization factor for the system model is $\alpha^* = 0.43$ according to the curve for $k = 1.50$ shown in Figure 7.

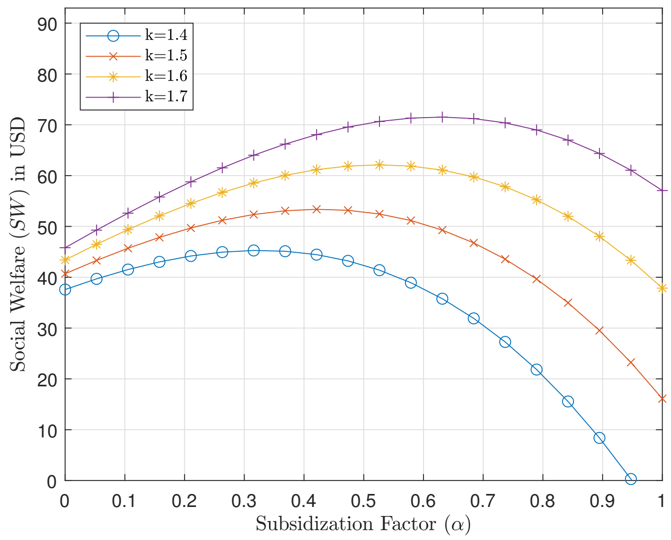


Figure 7. Behavior of the system's social welfare function for different values of k.

It is now the turn of the MNO, which seeks to maximize its profit. The variable α and the other parameters of the system model are already known by the regulation. In consequence, the values of β and τ can already be calculated. For example, Figure 8 shows the behavior of the MNO's profit function concerning the pricing from Equation (29). If the MNO wants to maximize its profit, then it must assign the optimal value of $p = 16.69$ USD. However, the MNO can analyze its profit function in more detail, for example, from the information given in the regulation, and assume α and k as parameters in the curves presented in Figure 9a,b, respectively. It is important to point out that the MNO's profit for different values of α and k is changed. In that sense, the MNO could obtain a greater profit by increasing the value of α. Nevertheless, this would also increase the pricing and would not be attractive to the MDUs. In the same sense, it can be analyzed that by increasing k or decreasing ϕ, the MNO obtains a greater benefit given the optimal value of α even without increasing the pricing offered to the MDUs.

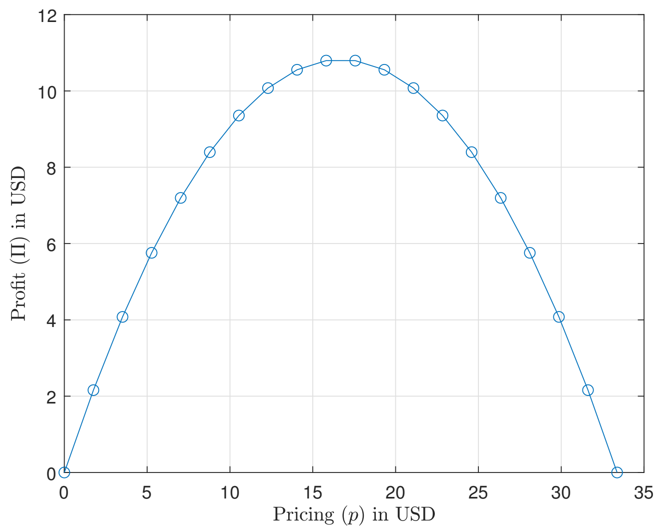
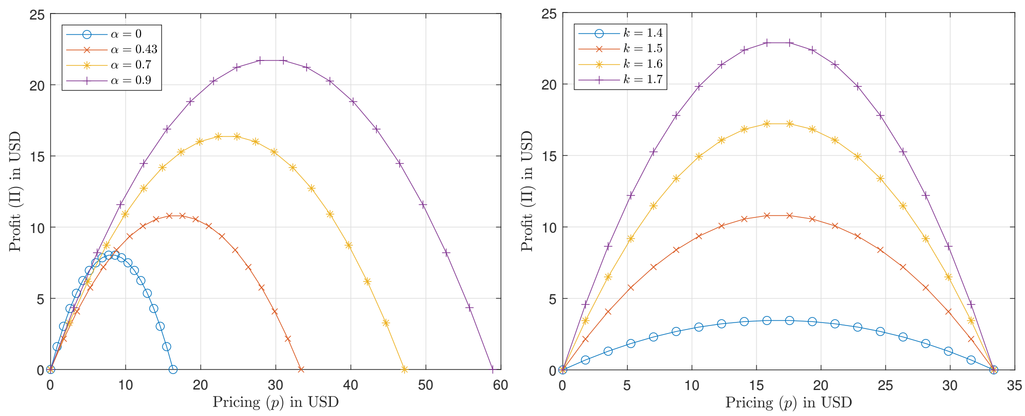


Figure 8. Behavior of the MNO's profit function for optimal α.

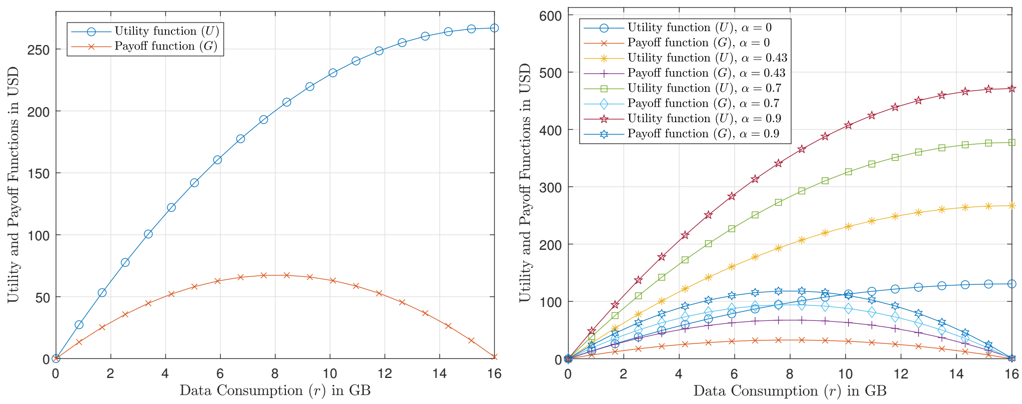


(**a**) (**b**)

Figure 9. Behavior of the MNO's profit function for (**a**) different values of α and (**b**) different values of k.

Now, it is the MDUs' turn to maximize their payoff function due to the subsidization factor announced by the CP, the pricing offered by the MNO, and the other information included in the regulation. In consequence, the behaviors of MDUs' utility and payoff functions from Equation (4) and Equation (30), respectively, is shown in Figure 10a. It can be seen that, based on the pricing offered by the MNO, the optimal value of data consumption is 8.00 GB. However, the MDUs obtain a higher level of satisfaction when purchasing the data plan because of the benefit of consuming more data for the same pricing. Therefore, if the MDUs pay for the data plan at 16.69 USD, then the amount of data that can be consumed is the contracted 8.00 GB plus the benefit $\gamma = 1.43$, namely 11.44 GB, which is a higher level of satisfaction. Moreover, it can be observed in Figure 10a that the MDUs' utility and payoff are reduced when the pricing is increased by the MNO. In that context, the curves depicted in Figure 10b show the behavior of the MDUs' interests as the value of the subsidization factor is changed. Therefore, considering constant pricing, the MDUs can increase their utility and payoff when the subsidization factor is increased.

Following both the results of the state-of-the-art review presented in Section 1 and the findings presented in this section, a comparative analysis of the scientific literature most relevant to this paper is provided in Table 2, which highlights the strengths of this work and its alignment with the reviewed state-of-the-art. In particular, Table 2 provides a qualitative assessment of key aspects, including the game theory or network economics approach employed, the selection of agents or players, the mathematical framework utilized, the optimization of decision variables, the maximization of social welfare through subsidization, the reliance on numerical evaluations derived from case studies, and the specific application sector addressed in the work. To the best of the authors' knowledge, no other work in the literature proposes a system model or setting similar to the one adopted in this paper. Consequently, the quantitative results of this study have not been directly compared with those of other works.



(**a**) (**b**)

Figure 10. Behavior of the MDUs' utility and payoff functions for (**a**) optimal α and (**b**) different values of α.

The case study presented with $k = 1.50$ provides the values of the payoff functions for six different α values, which are presented and consolidated in Table 3. Clearly, this work focuses on maximizing the social welfare of the system rather than maximizing the particular or individual interests of the MNO and the MDUs. Consequently, it can be inferred from Table 3 that the maximum subsidization factor of 1 implies the maximum individual values of the payoff functions for these players but not the maximum value of the social welfare to be ensured by the CP. On the other hand, when the CP enacts the subsidization factor to the optimal value of 0.43, the payoff functions of the other players are adjusted to achieve the maximum value of the social welfare of the system, and they do not reach their maximum values under the conditions and constraints of the proposed setting. Therefore, it is not about the CP taking sides with either the demand or the supply players. The important aspect is that the CP fulfills its role as a neutral player that guarantees the maximization of the social welfare of the system, which effectively increases to the value of 53.37 USD. Moreover, when the CP does not enact a subsidization factor, i.e., $\alpha = 0$, the players' payoff functions reach a value below their optimum, demonstrating the importance of adopting a subsidization scheme.

It should be noted that when the CP increases or decreases the enacted subsidization factor, the payoff functions of the other players are impacted through different behaviors shown in Table 3. For instance, it might be assumed that the players are interested in increasing the subsidization factor to obtain higher values in their payoff functions. However, even for the MNO, this is not advisable since a very high subsidization factor implies increasing the pricing offered to the MDUs for consuming data, therefore altering the demand behavior for the offered service. Similarly, the MDUs are not very interested in having excess data to consume due to the high pricing, as they have a limit on their satisfaction level upon consuming data.

Table 2. Comparison of the different works related to this paper.

Work	Network Economics or Game Theory Approach	Agents or Players	Mathematical Framework	Optimized Decision Variables	Social Welfare Maximization Based on a Subsidization	Numerical Assessments Based on a Case Study	Application Sector
[32]	Network economics and a sequential game	The mobile and fixed users, and the services providers	A two-stage sequential optimization procedure	The pricing and the bandwidth allocation	–	X	Heterogeneous wireless networks
[38]	Stackelberg game-based evolutionary game	The generators and the energy users	A bi-level optimization problem	The electricity price, the generation power, and the demand power	–	X	Smart grids
[39]	Four basic game-theoretical model types	The government, the module supplier, and the photovoltaic system assembler	A two-stage optimization problem	The price of a module, the price of a photovoltaic system, and the government's subsidy factor	X	–	Photovoltaic industry
[40]	Network economics	The users and the service provider	An optimization problem	The bandwidth fraction	–	–	Heterogeneous cellular networks
This paper	Network economics and a dynamic game	The central planner, the mobile network operator, and the mobile data users	A three-stage constrained optimization problem	The subsidization factor, the pricing, and the data consumption	X	X	Mobile communication markets

Table 3. The values of the players' payoff functions for different subsidization factors.

Player's Payoff Function	$\alpha = 0$	$\alpha = 0.3$	α^* = 0.43	$\alpha = 0.6$	$\alpha = 0.86$	$\alpha = 1$
The CP's SW	40.69 USD	52.05 USD	53.37 USD	50.48 USD	33.22 USD	16.10 USD
The MNO's $\prod$	8.05 USD	8.92 USD	10.82 USD	14.11 USD	20.61 USD	24.81 USD
The MDUs' G	32.64 USD	55.16 USD	66.72 USD	83.55 USD	112.92 USD	130.56 USD

* Optimal value.

This case study has presented several aspects of network economics in mobile communications markets using a specific game theory framework. In particular, in numerical terms, the results shown in Table 4 confirm that when the subsidization factor of 0.43 is enacted and used by the players, compared to a scheme without subsidization, the social welfare of the system is maximized at 31.16%, and both the MNO and the MDUs optimize their payoff functions at 34.41% and 104.41%, respectively. According to the above, it is important to note that the subsidization factor is practically critical in practical terms when designing regulations for MCNMs, and its definition allows the social welfare maximization of the system by strategically playing from the intersection of each player's best responses within the game.

Table 4. Comparison of players' payoff functions with and without the optimal subsidization factor and their percentage increase.

Player's Payoff Function	$\alpha = 0$	α^* = 0.43	Percentage Increase
The CP's SW	40.69 USD	53.37 USD	31.16%
The MNO's $\prod$	8.05 USD	10.82 USD	34.41%
The MDUs' G	32.64 USD	66.72 USD	104.41%

* Optimal value.

Finally, concerning the improvements and advantages of implementing this case study, it would enable the governments of the countries, in their role as regulators, to adequately design incentives in a mobile communications market to achieve the social welfare of the system. Furthermore, it is possible to state that countries, both least developed and developing, can utilize this approach to formulate public policies within their governments through subsidy schemes aimed at maximizing social welfare based on their mobile communications markets. Therefore, they would not necessarily need to adopt models from developed countries but could instead validate their models using the methodology implemented in this case study, updating the input data concerning their local markets.

6. Conclusions

In this paper, a novel setting was proposed for analyzing and resolving system models within the context of MCNMs. The system comprises three players who seek to maximize their payoffs through a three-stage dynamic game designed for decision-making. The attainment of the Nash equilibrium in this game involved modeling player interactions and determining the subgame perfect equilibrium through the intersection of the best responses at each stage, employing the backward induction method. Then, the maximization of social welfare was achieved through a subsidization factor defined by the CP, confirming the resolution of the stated problem.

The decision regarding the optimal value of the subsidization factor, set at 0.43 by the CP, was formalized through a resolution within the framework of MCNMs to disseminate this decision among the other involved players. Subsequently, the MNO conducted an analysis and made strategic determinations to establish the data pricing consumption for the MDUs at 16.69 USD. Similarly, the MDUs conducted their own analysis, drawing upon their expertise, to attain a data consumption level of 11.44 GB. Moreover, the outcomes were subjected to evaluation and discussion in a case study that examined various scenarios corresponding to different values of the subsidization factor. Notably, the comparison between scenarios involving no adoption of the subsidization factor and its adoption at the optimal value revealed significant increases in the social welfare of the system at 31.16%, the profit margin for the MNO at 34.41%, and the payoff for the MDUs at 104.41%.

The numerical assessments indicate that the subsidization factor maximized the social welfare of the system. Specifically, the maximization of social welfare within the conditions and constraints of the proposed setting highlights that the CP did not align with any particular player but, rather, influenced the system's social welfare through regulatory mechanisms to implement the subsidization factor. Consequently, it is noted as a limitation that the individual outcomes for both the MNO's profit and the MDUs' payoff did not reach their maximum value following the adoption of the optimal subsidization factor.

In future work, the system model will be modified such that both the information is unknown to the players and the player in the higher stage can impose constraints on the other players, therefore modeling a multi-stage dynamic game with incomplete information. Consequently, the new work will consider a system model based on Equilibrium Problems with Equilibrium Constraints (EPEC) and will employ a combination of Bayesian Nash equilibrium and subgame perfect equilibrium to solve the game.

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Abbreviations

The following abbreviations are used in this manuscript:

4G	4th Generation Mobile Networks
5G	5th Generation Mobile Networks
CP	Central Planner
CRC	Agency of Communications Regulation
EPEC	Equilibrium Problems with Equilibrium Constraints
GB	Giga Bytes
GDP	Gross Domestic Product
ICT	Information and Communication Technology
LTE	Long-Term Evolution
MCNMs	Mobile Communications Network Markets
MDUs	Mobile Data Users
MNO	Mobile Network Operator
MinTIC	Ministry of Information and Communication Technologies
MVNOs	Mobile Virtual Network Operators
DANE	National Administrative Department of Statistics
QoE	Quality of Experience
QoS	Quality of Service
RHS	Right-Hand Side
USD	United State Dollar

r	Data consumption in GB
p	Pricing in USD
P_{max}	Maximum pricing in USD
α	Subsidization factor (No units)
σ	Marginality of the MDUs' utility in USD/GB
$\bar{r}$	Maximum data consumption of the MDUs in GB
γ	Data benefit (No units)
β	Cost benefit (No units)
τ	Tax benefit (No units)
ϕ	Production factor (No units)
I_{tax}	Income tax rate (No units)
k	Profit factor (No units)
$U(\cdot)$	MDUs' Utility in USD
$C_u(\cdot)$	MDUs' Cost in USD
$G(\cdot)$	MDUs' Payoff in USD
$\prod(\cdot)$	MNO's Profit in USD
$R(\cdot)$	MNO's Revenue in USD
$C_o(\cdot)$	Production cost of the MNO in USD
$C_{tax}(\cdot)$	Taxes cost of the MNO in USD
$SW(\cdot)$	Social Welfare of the system in USD

Appendix A. Proof of the Lemma 1

Proof. It is convenient to start with the analysis of the utility function. Therefore, the first-order condition is used to find the maximum value of the variable r from Equation (4). Then

$$U(r;\gamma,\sigma,\bar{r}) = \begin{cases} -\sigma\gamma^2 r^2 + 2\sigma\gamma^2\bar{r}r & 0 \le r \le \bar{r} \\ \sigma\gamma^2\bar{r}^2 & r > \bar{r} \end{cases}$$

The interval of interest is $0 \le r \le \bar{r}$, because the other it does not include the variable r. Therefore, $U(r;\gamma,\sigma,\bar{r})_{0\le r\le\bar{r}} = -\sigma\gamma^2 r^2 + 2\sigma\gamma^2\bar{r}r$, and its optimal policy is $r^* = \arg\max_r U(r;\gamma,\sigma,\bar{r})_{0\le r\le\bar{r}}$. Then through the first-order condition yields to $\frac{\partial U(r;\gamma,\sigma,\bar{r})_{0\le r\le\bar{r}}}{\partial r} = 0$. Thus,

$$\frac{\partial U(r;\gamma,\sigma,\bar{r})_{0\le r\le\bar{r}}}{\partial r} = -2\sigma\gamma^2 r + 2\sigma\gamma^2\bar{r} = 0$$

Therefore, setting $r = r^*$ and solving for $r^* = \bar{r}$, then $r^* = \arg\max_r \ U(r;\gamma,\sigma,\bar{r})_{0\le r\le\bar{r}} = \bar{r}$. Now, given Equation (13)

$$G(r;\gamma,\sigma,\bar{r},p) = \begin{cases} -\sigma\gamma^2 r^2 + 2\sigma\gamma^2\bar{r}r - rp & 0 \le r \le \bar{r} \\ \sigma\gamma^2\bar{r}^2 - rp & r > \bar{r} \end{cases}$$

The interval, $r > \bar{r}$, represents a typical decreasing behavior. Therefore, if that interval is considered, there is no maximum value of the payoff function. Then, it must be proved that within the interval, $0 \le r \le \bar{r}$, the maximum value of the function $G(r;\gamma,\sigma,\bar{r},p)$ can always be found. In that sense, setting $\hat{A} = -\sigma\gamma^2 r^2 + 2\sigma\gamma^2\bar{r}r$, and $\hat{B} = rp$. Then $G(r;\gamma,\sigma,\bar{r},p) = \hat{A} - \hat{B}$.

$\hat{A}$ is the same function of Equation (4) in the interval $0 \le r \le \bar{r}$ whose optimal policy was found as $\bar{r}$ and substituting it in Equation (4), it is obtained as the maximum value $\sigma\gamma^2\bar{r}^2$. Then if now $r = \hat{a}\bar{r}$, where $\hat{a} \in \mathbb{R}^+ > 1$ and substituting it in $G(r;\gamma,\sigma,\bar{r},p)$

$$\hat{A} = -\sigma\gamma^2\hat{a}^2\bar{r}^2 + 2\sigma\gamma^2\hat{a}\bar{r}^2 = \sigma\gamma^2\hat{a}\bar{r}^2[-\hat{a}+2]$$

Then the analysis of $\hat{A}$ is as follows:

- If $1 < \hat{a} < 2$, then $\hat{A}$ is a positive value and less that $\sigma\gamma^2\bar{r}^2$.
- If $\hat{a} = 2$, then $\hat{A} = 0$ and less that $\sigma\gamma^2\bar{r}^2$.

- If $\hat{a} > 2$, then $\hat{A}$ is a negative value and less that $\sigma\gamma^2\bar{r}^2$.

Therefore, if the variable r is increased $\hat{a}\bar{r}$ times, then the maximum value of $\hat{A}$ is less than $\sigma\gamma^2\bar{r}^2$.

Now, as $\hat{B} = \hat{a}\bar{r}p$ the analysis of $\hat{B}$ is as follows:

- Since $p \geq 0$, then $\hat{B}$ is positive $\forall\ \hat{a}$ and $\forall\ p$.

Finally, the behavior of Equation (13) depends on both $\hat{A}$ and $\hat{B}$. Its maximum value is always within the interval of the variable r given by Equation (A1)

$$0 \leq r \leq \bar{r} \tag{A1}$$

□

Appendix B. Proof of the Theorem 1

Proof. Next, given Equation (15), the five steps to prove Theorem 1 are presented, namely:

1. It can be verified that the objective function of Equation (15) is concave through:

$$\frac{\partial^2 G(r;\gamma,\sigma,\bar{r},p)}{\partial r^2} < 0$$

Then given that

$$G(r;\gamma,\sigma,\bar{r},p) = -\sigma\gamma^2 r^2 + 2\sigma\gamma^2\bar{r}r - rp$$
$$\frac{\partial G(r;\gamma,\sigma,\bar{r},p)}{\partial r} = -2\sigma\gamma^2 r + 2\sigma\gamma^2\bar{r} - p$$
$$\frac{\partial^2 G(r;\gamma,\sigma,\bar{r},p)}{\partial r^2} = -4\sigma\gamma^2$$

As σ and γ are always positive values, then it is true that $\frac{\partial^2 G(r;\gamma,\sigma,\bar{r},p)}{\partial r^2} < 0$. Therefore, $G(r;\gamma,\sigma,\bar{r},p)$ is a strictly concave function.
2. Based on the result obtained in 1. and Lemma 1, it can be established that the solution of Equation (15) has a global maximum in the operating range of the variable r, i.e., $0 \leq r \leq \bar{r}$.
3. Since the variable r can have a value within the interval $[0,\bar{r}]$, the constraint in Equation (15) is considered a compact and convex subset of $G(r;\gamma,\sigma,\bar{r},p)$.
4. Since Equation (15) is a constrained optimization problem whose objective function is nonlinear, then its solution can be found by means of the Karush–Kuhn–Tucker conditions. Therefore:

$$\begin{aligned} \max_{r} \quad & -\sigma\gamma^2 r^2 + 2\sigma\gamma^2\bar{r}r - rp \\ \text{s.t.} \quad & -r \leq 0 \ : \mu_1 \\ & r - \bar{r} \leq 0 \ : \mu_2 \end{aligned} \tag{A2}$$

$$\mathcal{L} = \sigma\gamma^2 r^2 - 2\sigma\gamma^2\bar{r}r + rp + \mu_1(-r) + \mu_2(r - \bar{r}) \tag{A3}$$

And, through the first-order conditions, yield to

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial r} &= 2\sigma\gamma^2 r - 2\sigma\gamma^2\bar{r} + p - \mu_1 + \mu_2 = 0 \\ \frac{\partial\mathcal{L}}{\partial\mu_1} &= -r \leq 0 \\ \frac{\partial\mathcal{L}}{\partial\mu_2} &= r - \bar{r} \leq 0 \\ \mu_1 &\leq 0 \\ \mu_2 &\leq 0 \\ \mu_1(-r) &= 0 \\ \mu_2(r - \bar{r}) &= 0 \end{aligned} \tag{A4}$$

Then, solving the system of Equation (A4) to find the feasible values of μ_1, μ_2 and r, it is obtained that $\mu_1 = \mu_2 = 0$ and $r = \bar{r} - \frac{p}{2\sigma\gamma^2}$

Therefore, the optimal policy of Equation (15) for the interval $0 \leq r \leq \bar{r}$ is given by Equation (A5)

$$r^* = \arg\max_r \; G(r;\gamma,\sigma,\bar{r},p) = \bar{r} - \frac{p}{2\sigma\gamma^2} \tag{A5}$$

5. Since the variable r can take values in the interval $[0,\bar{r}]$, the values that p can take, yield to
   - $r^* \leq \bar{r}$, then $\bar{r} - \frac{p}{2\sigma\gamma^2} \leq \bar{r}$ and $0 \leq \frac{p}{2\sigma\gamma^2}$ Therefore,

   $$p \geq 0 \tag{A6}$$

   - $r^* \geq 0$, then $\bar{r} - \frac{p}{2\sigma\gamma^2} \geq 0$ and $\bar{r} \geq \frac{p}{2\sigma\gamma^2}$ Therefore,

   $$p \leq 2\sigma\gamma^2\bar{r} \tag{A7}$$

□

Appendix C. Proof of the Theorem 2

Proof. Next, given Equation (19), the four steps to prove Theorem 2 are presented, namely:

1. Substituting the optimal policy of Equation (15) into the objective function of Equation (19), the profit function is established as:

$$\prod(p;r,\alpha,\phi,\beta,I_{tax},\tau) = [1+\alpha\phi\beta-(1+\alpha)\phi-I_{tax}\tau]\{\frac{-p^2}{2\sigma(1+\alpha)^2}+\bar{r}p\} \tag{A8}$$

Then, it can be verified that the objective function of Equation (A8) is concave through:

$$\frac{\partial^2\prod(p;r,\alpha,\phi,\beta,I_{tax},\tau)}{\partial p^2} < 0$$

Then given Equation (A8)

$$\frac{\partial\prod(p;r,\alpha,\phi,\beta,I_{tax},\tau)}{\partial p} = [1+\alpha\phi\beta-(1+\alpha)\phi-I_{tax}\tau]\{\frac{-p}{\sigma(1+\alpha)^2}+\bar{r}\}$$

$$\frac{\partial^2\prod(p;r,\alpha,\phi,\beta,I_{tax},\tau)}{\partial p^2} = \frac{-[1+\alpha\phi\beta-(1+\alpha)\phi-I_{tax}\tau]}{\sigma(1+\alpha)^2}$$

Then, to guarantee the concavity of

$$\frac{-[1+\alpha\phi\beta-(1+\alpha)\phi-I_{tax}\tau]}{\sigma(1+\alpha)^2} < 0$$

This is:

$$\begin{Bmatrix} [1+\alpha\phi\beta-(1+\alpha)\phi-I_{tax}\tau] > 0 \\ \sigma(1+\alpha)^2 > 0 \end{Bmatrix} \tag{A9}$$

Substituting Equations (2) and (3) into Equation (A9) given $\sigma > 0$, then Equation (A10) is the condition to guarantee the concavity of the objective function of Equation (19)

$$\phi < \frac{1+4\alpha}{1+5\alpha}\left[1-\frac{I_{tax}}{1+\alpha}\right] \tag{A10}$$

Therefore, if condition Equation (A10) is fulfilled, then $\prod(p;r,\alpha,\phi,\beta,I_{tax},\tau)$ can be considered to be a strictly concave function.

2. Based on the results obtained in 1. and Theorem 1, it can be established that the solution of Equation (19) has a global maximum in the operating range of the variable p, i.e., $0 \leq p \leq 2\sigma(1+\alpha)^2\bar{r}$.
3. Since the variable p can have a value within the interval $[0, 2\sigma(1+\alpha)^2\bar{r}]$, the constraint in (19), subject to the variable p, is considered a compact and convex subset of $\prod(p; r, \alpha, \phi, \beta, I_{tax}, \tau)$.
4. Since Equation (19) is a constrained optimization problem whose objective function is nonlinear, then its solution can be found by means of the Karush–Kuhn–Tucker conditions. Therefore:

$$\begin{array}{ll} \max\limits_{p} & [1+\alpha\phi\beta-(1+\alpha)\phi-I_{tax}\tau]\{\frac{-p^2}{2\sigma(1+\alpha)^2}+\bar{r}p\} \\ \text{s.t.} & -p \leq 0 \ : \mu_1 \\ & p-2\sigma(1+\alpha)^2\bar{r} \leq 0 \ : \mu_2 \end{array} \quad \text{(A11)}$$

$$\mathcal{L} = [1+\alpha\phi\beta-(1+\alpha)\phi-I_{tax}\tau]\{\frac{p^2}{2\sigma(1+\alpha)^2}-\bar{r}p\}+\mu_1(-p)+\mu_2(p-2\sigma(1+\alpha)^2\bar{r}) \quad \text{(A12)}$$

And, through the first-order conditions, yield to

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial p} &= [1+\alpha\phi\beta-(1+\alpha)\phi-I_{tax}\tau]\{\frac{p}{\sigma(1+\alpha)^2}-\bar{r}\}-\mu_1+\mu_2=0 \\ \frac{\partial\mathcal{L}}{\partial\mu_1} &= -p \leq 0 \\ \frac{\partial\mathcal{L}}{\partial\mu_2} &= p-2\sigma(1+\alpha)^2\bar{r} \leq 0 \\ \mu_1 &\leq 0 \\ \mu_2 &\leq 0 \\ \mu_1(-p) &= 0 \\ \mu_2(p-2\sigma(1+\alpha)^2\bar{r}) &= 0 \end{aligned} \quad \text{(A13)}$$

Then, solving the system of Equation (A13) to find the feasible values of μ_1, μ_2 and p, it is obtained that $\mu_1 = \mu_2 = 0$ and $p = \sigma(1+\alpha)^2\bar{r}$
Therefore, the optimal policy of Equation (19) for the interval $0 \leq p \leq 2\sigma(1+\alpha)^2\bar{r}$ is given by Equation (A14)

$$p^* = \arg\max_{p} \prod(p; r, \alpha, \phi, \beta, I_{tax}, \tau) = \sigma(1+\alpha)^2\bar{r} \quad \text{(A14)}$$

□

Appendix D. Proof of the Theorem 3

Proof. Next, given Equation (22), the four steps to prove Theorem 3 are presented, namely:

1. Substituting the optimal policies of Equations (15) and (19) into the objective function of Equation (22), the social welfare function is established as:

$$SW(\alpha; p, r, \gamma, \sigma, \bar{r}, \phi, I_{tax}, \tau) = \sigma\bar{r}^2\{\frac{1}{4}[3-2\phi-2I_{tax}]+\frac{3}{2}[1-\phi-\frac{1}{3}I_{tax}]\alpha+\frac{3}{4}[1-2\phi]\alpha^2-\frac{1}{2}[\phi]\alpha^3\} \quad \text{(A15)}$$

Then, it can be verified that the objective function of Equation (A15) is concave through:

$$\frac{\partial^2 SW(\alpha; p, r, \gamma, \sigma, \bar{r}, \phi, I_{tax}, \tau)}{\partial\alpha^2} < 0$$

Then given Equation (A15)

$$\frac{\partial SW(\alpha;p,r,\gamma,\sigma,\bar{r},\phi,I_{tax},\tau)}{\partial\alpha} = \frac{3}{2}\sigma\bar{r}^2\{[1-\phi-\frac{1}{3}I_{tax}]+[1-2\phi]\alpha-[\phi]\alpha^2\}$$
$$\frac{\partial^2 SW(\alpha;p,r,\gamma,\sigma,\bar{r},\phi,I_{tax},\tau)}{\partial\alpha^2} = \frac{3}{2}\sigma\bar{r}^2\{[1-2\phi]-2[\phi]\alpha\}$$

Then, to guarantee the concavity of

$$\frac{3}{2}\sigma\bar{r}^2\{[1-2\phi]-2[\phi]\alpha\}<0 \tag{A16}$$

Then Equation (A17) is the condition to guarantee the concavity of the objective function of Equation (22)

$$\frac{1-2\phi}{2\phi}<\alpha \tag{A17}$$

Therefore, if condition Equation (A17) is fulfilled, then $SW(\alpha;p,r,\gamma,\sigma,\bar{r},\phi,I_{tax},\tau)$ can be considered to be a strictly concave function.

2. Based on the results obtained in 1., Theorem 1, and Theorem 2, it can be established that the solution of Equation (22) has a global maximum in the operating range of the variable α, i.e., $0\le\alpha\le 1$.
3. Since the variable α can have a value within the interval $[0,1]$, the constraint in Equation (22), subject to the variable α, is considered a compact and convex subset of $SW(\alpha;p,r,\gamma,\sigma,\bar{r},\phi,I_{tax},\tau)$.
4. Since Equation (22) is a constrained optimization problem whose objective function is nonlinear, then its solution can be found by means of the Karush–Kuhn–Tucker conditions. Therefore:

$$\begin{array}{ll}\max\limits_{\alpha} & \sigma\bar{r}^2\{\frac{1}{4}[3-2\phi-2I_{tax}]+\frac{3}{2}[1-\phi-\frac{1}{3}I_{tax}]\alpha+\frac{3}{4}[1-2\phi]\alpha^2-\frac{1}{2}[\phi]\alpha^3\} \\ \text{s.t.} & -\alpha\le 0\ :\mu_1 \\ & \alpha-1\le 0\ :\mu_2\end{array} \tag{A18}$$

$$\mathcal{L}=-\sigma\bar{r}^2\{\frac{1}{4}[3-2\phi-2I_{tax}]+\frac{3}{2}[1-\phi-\frac{1}{3}I_{tax}]\alpha+\frac{3}{4}[1-2\phi]\alpha^2-\frac{1}{2}[\phi]\alpha^3\}+\mu_1(-\alpha)+\mu_2(\alpha-1) \tag{A19}$$

And, through the first-order conditions, yield to

$$\begin{gathered}\frac{\partial\mathcal{L}}{\partial\alpha}=-\frac{3}{2}\sigma\bar{r}^2\{[1-\phi-\frac{1}{3}I_{tax}]+[1-2\phi]\alpha-[\phi]\alpha^2\}-\mu_1+\mu_2=0 \\ \frac{\partial\mathcal{L}}{\partial\mu_1}=-\alpha\le 0 \\ \frac{\partial\mathcal{L}}{\partial\mu_2}=\alpha-1\le 0 \\ \mu_1\le 0 \\ \mu_2\le 0 \\ \mu_1(-\alpha)=0 \\ \mu_2(\alpha-1)=0\end{gathered} \tag{A20}$$

Then, solving the system of Equation (A20) to find the feasible values of μ_1,μ_2 and α, it is obtained that $\mu_1=\mu_2=0$ and $\alpha=\frac{\sqrt{1-\frac{4\phi I_{tax}}{3}}-2\phi+1}{2\phi}$

Therefore, the optimal policy of Equation (22) for the interval $0\le\alpha\le 1$ is given by Equation (A21)

$$\alpha^*=\arg\max_{\alpha}\ SW(\alpha;p,r,\gamma,\sigma,\bar{r},\phi,I_{tax},\tau)=\frac{\sqrt{1-\frac{4\phi I_{tax}}{3}}-2\phi+1}{2\phi} \tag{A21}$$

□

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