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Numerical Solutions of a Differential System Considering a Pure Hybrid Fuzzy Neutral Delay Theory

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Abstract: In this paper, we propose and derive a new system called pure hybrid fuzzy neutral delay differential equations. We apply the classical fourth-order Runge–Kutta method (RK-4) to solve the proposed system of ordinary differential equations. First, we define the RK-4 method for hybrid fuzzy neutral delay differential equations and then establish the efficiency of this method by utilizing it to solve a particular type of fuzzy neutral delay differential equation. We provide a numerical example to verify the theoretical results. In addition, we compare the RK-4 and Euler solutions with the exact solutions. An error analysis is conducted to assess how much deviation from exactness is found in the two numerical methods. We arrive at the same conclusion for our hybrid fuzzy neutral delay differential system since the RK-4 method outperforms the classical Euler method.

Keywords: Euler method; fuzzy theory; hybrid differential equations; initial value problem; delay differential equations; Runge–Kutta method

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1. Introduction

Hybrid systems have been widely studied in different contexts [1]. Mathematically, hybrid systems correspond to continuous processes and are often disturbed, naturally or artificially, by the discreteness that arises in such continuous processes. Here, we are modeling these systems with the concept of delay in the derivative; that is, a neutral delay differential system. We use the concept of "fuzzy numbers" [2,3] to approximate the specific interval. The hybrid type and neutral type delay differential equations (DDEs) are important in the modeling of a natural system whose behavior is predicted by its history, which has undergone sudden discrete changes between continuous processes. The complete system is modeled and explained with a theory, while the modeling of any physical phenomenon does not occur all of a sudden. Some biological, chemical, or physical changes may occur internally or externally, disturbing the system. The basic concept of DDEs can be explained as: "The changes that happened yesterday will affect today's behavior of the system, and the changes that happen today will affect tomorrow's behavior". To control the extreme changes that may affect tomorrow's behavior, we introduce the neutral term. When only this term is included without the non-neutral delay term, it is called neutral DDEs. If the system is free from the solution term in the governing equations, we call it the pure form of differential equations.

Numerical solutions obtained by means of algorithms are very helpful for easily solving problems where it is difficult to find analytical solutions [4,5]. Fuzzy sets were introduced by Lofti A. Zadeh in 1965 [2], whereas elementary fuzzy calculus was introduced in [6]. The well-known fuzzy differential equations (FDEs) are utilized in modeling sciences and engineering problems, and a number of authors have studied the FDEs [7–15]. In [16,17], the problem of hybrid fuzzy differential equations (HFDEs) was rigorously studied in a numerical form, whereas in [18], perturbed Lyapunov-like functions and hybrid FDEs were analyzed. In [19], a numerical method for solving DDEs by the fourth-order Runge–Kutta (RK-4) method was discussed. In [20], numerical methods for DDEs were provided, and in [21], a fuzzy DDE by means of the RK-4 method was studied. The Runge–Kutta method was used to solve many forms of DDE in [22–25].

Recently, many authors have found an interest in nonlinear models for stochastic systems of differential equations [26–30], which have numerous applications. They also found both analytical and numerical solutions to their system using various methodologies.

Numerical solutions, often obtained by means of algorithms, are very helpful for easily solving problems where it is difficult to find analytical solutions [31,32]. There are several theoretical analyses and applications of hybrid and neutral systems. However, to the best of our knowledge, there are no studies that combine both systems. This is why we are interested in studying such a combination. Moreover, there is a major significance to studying the usage of both systems simultaneously, because one of them works on initial conditions at an initial time, the hybrid system. Nevertheless, the neutral delay system works on initial functions from not only the present time but also the past time. This has motivated us to frame a new system called hybrid fuzzy neutral delay differential equations (HFNDDEs) and contribute to the mathematical world. We built such an HFNDDE using the knowledge gained from preceding works. Since our system is fuzzy, the merits of vagueness are immense. The uncertainty that arises while modeling a physical problem will be eliminated, and it will not disturb the smoothness of the solutions. The solutions for each t will be limited to the real interval $[0,1]$. Since we added the hybrid term, which has both discrete and continuous parameters, the system has solutions, even when a negligible quantity of discontinuity arises. Therefore, our main contribution to this investigation is framing a non-fuzzy model and converting it to a fuzzy model. We first define the RK-4 method for HFNDDEs and then establish the efficiency of this method by utilizing it to solve a particular type of fuzzy neutral DDE. In this paper, we use the RK-4 method to propose a numerical solution to HFNDDEs.

The plan of the paper is as follows. In Section 2, we present the concept of hybrid fuzzy neutral delay differential systems. Then, in Section 3, the RK-4 method for approaching HFNDDEs is discussed, and then we define pure HFNDDEs. Section 4 describes a numerical example and elaborately finds its approximate and exact solutions to illustrate the theory presented in this investigation. Finally, in Section 5, we summarize the results of this study, which may be helpful for readers and researchers, as well as identify our findings, provide our final conclusions, and present ideas for future research.

2. Hybrid Fuzzy Neutral Delay Differential Equations

Hybrid systems that involve discontinuous changes in a continuous process are employed in various fields, such as communication, signal processing, and transient analysis, among others. It is so called because of the hybrid term involved. This term transforms the ordinary differential system into a hybrid differential system. In Figure 1, we can see a scheme of a hybrid system.

The equation of any hybrid system is stated as

$$\begin{cases} y'(t) &= f(t, y(t), m(t)\rho(y_K(t_k))), \quad t \geq t_0; \\ y(t_0) &= y_0. \end{cases} \tag{1}$$

Note that the neutral delay differential system presents a solution that depends on the past change of the same system. The process also involves the present solution while studying its rate of change. If not, it is called the "pure neutral delay differential system". In Figure 2, we can see a scheme that shows the main idea of a neutral delay system.

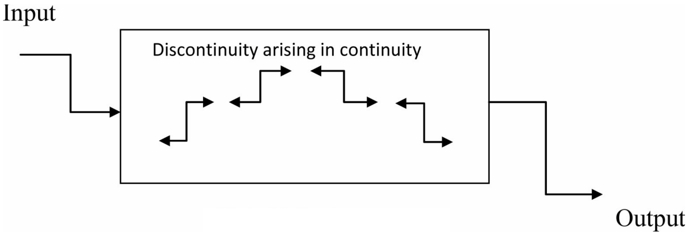

Figure 1. Scheme of the hybrid systems.

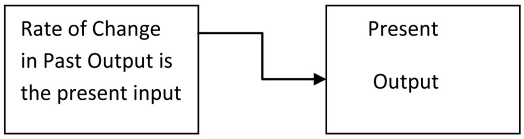

Figure 2. Scheme of the neutral delay systems.

Observe that the equation of any neutral delay system is formulated as

$$\begin{cases} y'(t) &= f(t, y(t), y'(t-\mu)), \quad t \geq t_0; \\ y(t) &= \phi(t), \quad -\mu \leq t \leq t_0. \end{cases} \tag{2}$$

Our new system is obtained by combining two different sides of differential equations; that is, the hybrid systems and the neutral delay systems, under one roof, called the hybrid neutral delay differential system, whose mathematical model is established in (3). From Figure 3, we can see a scheme of a hybrid neutral delay system.

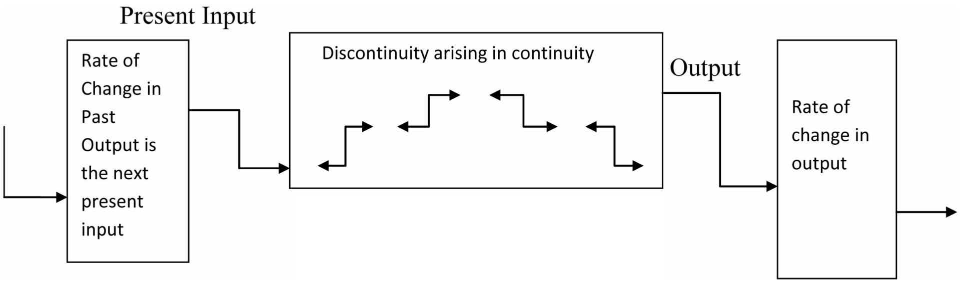

Figure 3. Scheme of the hybrid neutral delay systems.

Now, we consider a non-fuzzy model and convert it to a fuzzy model using suitable fuzzy numbers. To begin, consider $F_1 = (0.75 + 0.25\alpha)$ and $F_2 = (1.125 - 0.125\alpha)$ as the fuzzy numbers to be used throughout the paper. According to [19], the neutral type DDEs are defined in the form of $a_0 y'(t) + b_0 y(t) + a_1 y'(t-\mu) = g(t)$. Every first-order DDE becomes homogeneous when $g(t) = 0$. Here, $g(t)$ is employed as a hybrid term in this context and it is named "dubbed hybrid neutral delay differential equation", with the constants $a_0 = 1$, $b_0 = -1$, and $a_1 = -1$, whereas $g(t) = m(t)\rho(y_K(t_k))$. Let us consider the HFNDDE stated as

$$\begin{cases} y'(t) &= f(t, y(t), y'(t-\mu), m(t)\rho(y_K(t_k))), \quad t \geq t_0; \\ y(t) &= \phi(t), \quad -\mu \leq t \leq t_0; \\ y(t_0) &= y_0 \in \phi(t); \end{cases} \tag{3}$$

where f: $[0, \infty) \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \to \mathbb{E}$, ϕ is a continuous fuzzy mapping, and $y_0 \in \phi(t)$ is the essential initial condition. Then, we have $y_0(s) = y(s) = \phi(s)$, for $-\mu \leq s \leq 0$. Moreover, y_0 is a fuzzy number with α-level of intervals $[y_0]^\alpha = [\underline{y}_0^\alpha, \overline{y}_0^\alpha]$, where $0 \leq \alpha \leq 1$.

The extension principle given in [2] leads to a definition of $f(t, y(t), sy(t-\mu), \rho(y_K(t_k)))$ presented below, when y is a fuzzy number and $f\colon [t_0, t_f] \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \to \mathbb{E}$.

The HFNDDE stated in (3) can be transformed into the HFDDE formulated in (4). Then, we have the mathematical model expressed as

$$\begin{cases} y'(t) &= z(t); \\ z(t) &= f(t, y(t), z(t-\mu), m(t)\rho(y_K(t_k))), \quad t_0 \le t \le t_f; \\ y(t) &= \phi(t), \quad t \le t_0; \\ z(t) &= \phi'(t), \quad t \le t_0; \end{cases} \tag{4}$$

and

$$\begin{cases} y'(t) &= g(t, \phi(t), \psi(t), m(t)\rho(\phi_K(t_k))), \quad t_0 \le t \le t_f; \\ \psi(t) &= H(t, \phi(t), \psi(t-\mu)), \quad t-\mu \ge t_0; \\ y(t_0) &= \phi(t_0); \\ \psi(t) &= \mu(t), \quad t-\mu \le t_0; \end{cases} \tag{5}$$

where $\mu(t)$ is the initial function and $y(t)$ is the solution function, with $z(t)$ and $\psi(t)$ being auxiliary functions.

3. The RK-4 Method and Pure Hybrid Fuzzy Neutral Delay Differential Equations

The method that we present here is a new modified form of the existing RK-4 method. We propose this modified RK-4 method for solving HFNDDE systems. The existing methods can be used to solve FDEs, DDEs, hybrid differential systems, as well as fuzzy neutral and fuzzy mixed DDEs. Our new method combines fuzzy, hybrid, and delay differential systems modifying the existing RK-4 method and it is more suitable to solve these types of systems.

The solutions found by the new RK-4 method are compared with those of a new modified Euler method and the error analysis is provided. (Observe that one can easily find a modified Euler method for any system, but its algorithm is not provided here.) Note that a comparison of our modified RK-4 method with the existing RK-4 method (which is not suitable for solving our system) is not provided. However, one might compare it with the existing modified forms of other methods, such as in the case of the Euler method.

For an HFNDDE, as stated in (3), we apply the RK-4 method after transforming it to a hybrid fuzzy DDE by using the function $f(t, \phi(t), \phi(t-\mu), \rho(\phi(t_k)))$ as formulated in (4) and (5). Thus, we apply the Runge–Kutta method to an FDE [7], with f given in (3) being obtained via the well-known extension principle [2] from $f \in C(\mathbb{E} \times \mathbb{E} \times \mathbb{E} \times \mathbb{E})$. We assume that the existence and uniqueness of the solutions generated in (3) hold for each interval $[t_k, t_{k+1}]$. The exact solutions are denoted by means of $\underline{Y}(t;\alpha)$ and $\overline{Y}(t;\alpha)$. To estimate the approximate values $\underline{y}(t;\alpha)$ and $\overline{y}(t;\alpha)$, of $\underline{y}'(t;\alpha)$ and $\overline{y}'(t;\alpha)$, we utilize the RK-4 method defining

$$\begin{aligned} \underline{y}(t_{n+1};\alpha) - \underline{y}(t_n;\alpha) &= \sum_{i=1}^{4} w_i \underline{K}_i(t_n; y(t_n;\alpha)), \\ \overline{y}(t_{n+1};\alpha) - \overline{y}(t_n;\alpha) &= \sum_{i=1}^{4} w_i \overline{K}_i(t_n; y(t_n;\alpha)), \end{aligned}$$

with

$$\begin{aligned} K_1(t;\phi(t;\alpha)) = \min,\max \Big\{ hf\Big(t, \phi(t), \phi(t-\mu), \rho(\phi(t_k))\Big) \Big| \\ \phi(t) \in [\underline{\phi}(t_{k,n};\alpha), \overline{\phi}(t_{k,n};\alpha)], \phi(t_k) \in [\underline{\phi}(t_{k,0};\alpha), \overline{\phi}(t_{k,0};\alpha)] \\ \phi(t-\mu) \in [\underline{\phi}(t_{k,n}-\mu;\alpha), \overline{\phi}(t_{k,n}-\mu;\alpha)] \Big\}, \end{aligned}$$

$$
\begin{aligned}
K_2(t;\phi(t;\alpha)) \quad &= \quad \min,\max\left\{hf\left(t+\frac{h}{2},\phi(t),\phi(t-\mu),\rho(\phi(t_k))\right)\right| \\
&\phi(t)\in[\underline{z}_1(t_{k,n},\phi(t_{k,n};\alpha)),\overline{z}_1(t_{k,n},\phi(t_{k,n};\alpha))], \\
&\phi(t_k)\in[\underline{\phi}(t_{k,0};\alpha),\overline{\phi}(t_{k,0};\alpha)] \\
&\left.\phi(t-\mu)\in[\underline{z}_1(t_{k,n}-\mu,\phi(t_{k,n}-\mu;\alpha)),\overline{z}_1(t_{k,n}-\mu,\phi(t-\mu;\alpha))]\right\}, \\
K_3(t;\phi(t;\alpha)) \quad &= \quad \min,\max\left\{hf\left(t+\frac{h}{2},\phi(t),\phi(t-\mu),\rho(\phi(t_k))\right)\right| \\
&\phi(t)\in[\underline{z}_2(t_{k,n},\phi(t_{k,n};\alpha)),\overline{z}_2(t_{k,n},\phi(t_{k,n};\alpha))], \\
&\phi(t_k)\in[\underline{\phi}(t_{k,0};\alpha),\overline{\phi}(t_{k,0};\alpha)] \\
&\left.\phi(t-\mu)\in[\underline{z}_2(t_{k,n}-\mu,\phi(t_{k,n}-\mu;\alpha)),\overline{z}_2(t_{k,n}-\mu,\phi(t-\mu;\alpha))]\right\}, \\
K_4(t;\phi(t;\alpha)) \quad &= \quad \min,\max\left\{hf\left(t+h,\phi(t),\phi(t-\mu),\rho(\phi(t_k))\right)\right| \\
&\phi(t)\in[\underline{z}_3(t_{k,n},\phi(t_{k,n};\alpha)),\overline{z}_3(t_{k,n},\phi(t_{k,n};\alpha))], \\
&\phi(t_k)\in[\underline{\phi}(t_{k,0};\alpha),\overline{\phi}(t_{k,0};\alpha)] \\
&\left.\phi(t-\mu)\in[\underline{z}_3(t_{k,n}-\mu,\phi(t_{k,n}-\mu;\alpha)),\overline{z}_3(t_{k,n}-\mu,\phi(t-\mu;\alpha))]\right\},
\end{aligned}
$$

where h is the size of every step and $t_{k,n}=t_{k,0}+nh$. Now, we state

$$
\begin{aligned}
z_1(t_{k,n},\phi(t_{k,n};\alpha)) \quad &= \quad \underline{\phi}(t_{k,n};\alpha)+\frac{1}{2}\underline{K}_1(t_{k,n},\phi(t_{k,n};\alpha)), \\
z_2(t_{k,n},\phi(t_{k,n};\alpha)) \quad &= \quad \underline{\phi}(t_{k,n};\alpha)+\frac{1}{2}\underline{K}_2(t_{k,n},\phi(t_{k,n};\alpha)), \\
z_3(t_{k,n},\phi(t_{k,n};\alpha)) \quad &= \quad \underline{\phi}(t_{k,n};\alpha)+\underline{K}_3(t_{k,n},\phi(t_{k,n};\alpha)).
\end{aligned}
$$

Then, we establish

$$
\begin{aligned}
P_1[(t,\underline{\phi}(t;\alpha),\overline{y}(t;\alpha))] &= \underline{K}_1(t,\phi(t;\alpha))+2\underline{K}_2(t,\phi(t;\alpha))+2\underline{K}_3(t,\phi(t;\alpha))+\underline{K}_4(t,\phi(t;\alpha)), \\
P_2[(t,\underline{\phi}(t;\alpha),\overline{y}(t;\alpha))] &= \overline{K}_1(t,\phi(t;\alpha))+2\overline{K}_2(t,\phi(t;\alpha))+2\overline{K}_3(t,\phi(t;\alpha))+\overline{K}_4(t,\phi(t;\alpha)).
\end{aligned}
$$

Therefore, the fuzzy-valued approximate solution is given by

$$
\begin{aligned}
\underline{y}(t_{n+1}) \quad &= \quad \underline{\phi}(t_{n+1})+\underline{y}(t_n-\mu), \\
\overline{y}(t_{n+1}) \quad &= \quad \overline{\phi}(t_{n+1})+\overline{y}(t_n-\mu),
\end{aligned}
$$

where

$$
\begin{cases}
\underline{\phi}(t_{n+1};\alpha)=\underline{\phi}(t_n;\alpha)+\dfrac{1}{6}S[(t_n,\underline{\phi}(t_n;\alpha),\overline{\phi}_n(t;\alpha))], \\
\overline{\phi}(t_{n+1};\alpha)=\overline{\phi}(t_n;\alpha)+\dfrac{1}{6}T[(t_n,\underline{\phi}(t_n;\alpha),\overline{\phi}_n(t;\alpha))].
\end{cases} \tag{6}
$$

Note that the expressions stated in (6) allow the final computation of $\underline{y}(t_{n+1})$ and $\overline{y}(t_{n+1})$.

Next, we introduce the pure HFNDDE by reformulating the HFNDDE as

$$
\begin{cases}
y'(t) \quad = \quad Ay(t)+By'(t-\mu)+Cm(t)\rho(y(t_k)), \qquad 0\le t\le 3; \\
y(t) \quad = \quad \phi(t), \quad -\mu\le t\le 0; \\
y(t_0) \quad = \quad y_0\in\phi(t);
\end{cases} \tag{7}
$$

where A, B, and C are constant coefficients. When the coefficient of $y(t)$ is equal to zero; that is, $A = 0$, the equation leads to the pure HFNDDE. It obeys all the properties of HFNDDEs and using the RK-4 method, we solve the problem as illustrated in the next section.

4. Numerical Example

In the following example, we solve the pure HFNDDE by employing the RK-4 method given in Section 3. We find numerical solutions by utilizing both the RK-4 and classical Euler methods. The fuzzy-valued numerical solutions obtained by the RK-4 and Euler methods are compared to exact solutions, and their error analysis is also conducted.

The plots are displayed for both ordinary ($t \in [0,3]$) and fuzzy ($t = 3$, $\alpha = 0$ and $\alpha = 1$-2D plot; with $t \in [0,3]$ and $\alpha \in [0,1]$-3D plot) values. As it is well-known, the Euler method is given by $y_{n+1} = y_n + hf(t_n, y_n)$. We use the same step size of the RK-4 method for comparison of its fuzzy values, as reported in Tables 1 and 2.

Consider the PHFNDDE, whose mathematical model is formulated as

$$\begin{cases} y'(t) &= y'(t-1) + m(t)\rho(y(t_k)), & 0 \le t \le 3; \\ y(t) &= [(F_1)\mathrm{e}^t, (F_2)\mathrm{e}^t], & -1 \le t \le 0; \end{cases} \tag{8}$$

where $m(t) = |\sin(\pi t)|$, for $k \in \{0, 1, \dots\}$, and

$$\rho_k(\mu) = \begin{cases} \hat{0}, & k = 0; \\ \mu, & k \in \{1, \dots\}. \end{cases}$$

Throughout the problem stated here, we take $h = 0.1$ as the step size. To obtain $y(3.0;\alpha)$, the approximate solution is tabulated and plotted to show the accuracy of the method for the given problem.

The exact solution of the expression defined in (8) is given by

$$Y(t;\alpha) = \begin{cases} [(F_1)\mathrm{e}^t, (F_2)\mathrm{e}^t], & -1 \le t \le 0; \\ [(F_1)\mathrm{e}^{t-1} - \frac{\mathrm{e}\cos(\pi t)}{\pi} - \frac{1}{\mathrm{e}} + \frac{\mathrm{e}}{\pi} + (F_2)\mathrm{e}^{t-1} - \frac{\mathrm{e}\cos(\pi t)}{\pi} - \frac{1}{\mathrm{e}} + \frac{\mathrm{e}}{\pi} + 1], & 0 \le t \le 1; \\ [(F_1)\mathrm{e}^{t-2} + 2 - \frac{2}{\mathrm{e}} + \frac{2\mathrm{e}}{\pi}, (F_2)\mathrm{e}^{t-2} + 2 - \frac{2}{\mathrm{e}} + \frac{2\mathrm{e}}{\pi}], & 1 \le t \le 2; \\ [(F_1)\mathrm{e}^{t-3} - \frac{\mathrm{e}\cos(\pi t)}{\pi} - \frac{3}{\mathrm{e}} + \frac{3\mathrm{e}}{\pi} + 3, (F_2)\mathrm{e}^{t-3} - \frac{\mathrm{e}\cos(\pi t)}{\pi} - \frac{3}{\mathrm{e}} + \frac{3\mathrm{e}}{\pi} + 3], & 2 \le t \le 3; \end{cases} \tag{9}$$

where $Y(t;\alpha) = [\underline{Y}(t;\alpha), \overline{Y}(t;\alpha)]$.

The approximate solution of the expression defined in (8) is stated as

$$y(n;\alpha) = \begin{cases} [(F_1), (F_2)], & -10 \le n \le 0; \\ [(F_1)(y_0 + h(A_1 + A_2)), (F_2)(y_0 + h(A_1 + A_2))], & 1 \le n \le 10; \\ [(F_1)(y_0 + h(A_2 + B_1 + B_2)), (F_2)(y_0 + h(A_2 + B_1 + B_2))], & 11 \le n \le 20; \\ [(F_1)(y_0 + h(A_3 + A_4 + B_1 + B_3 + B_4)), (F_2)(y_0 + h(A_3 + A_4 + B_1 + B_3 + B_4))], & 21 \le n \le 30; \end{cases} \tag{10}$$

where n takes only integer values and $y(n;\alpha) = [\underline{y}(n;\alpha), \overline{y}(n;\alpha)]$.

Now, consider

$$A_1 = c_0 \sum_{s=0}^{2n} \mathrm{e}^{-1+sh/2}, \tag{11}$$

$$A_2 = c_0 \sum_{s=0}^{2n} H \sin\left(\frac{sh\pi}{2}\right), \tag{12}$$

$$A_3 = c_1 \sum_{s=0}^{20} H \sin\left(\frac{sh\pi}{2}\right), \tag{13}$$

$$A_4 = c_4 \sum_{s=40}^{2n} H \sin\left(\frac{sh\pi}{2}\right), \tag{14}$$

$$B_1 = c_1 \sum_{s=0}^{20} \mathrm{e}^{-1+sh/2}, \quad (15)$$

$$B_2 = c_2 \sum_{s=20}^{2n} \mathrm{e}^{-2+sh/2}, \quad (16)$$

$$B_3 = c_3 \sum_{s=20}^{40} \mathrm{e}^{-2+sh/2}, \quad (17)$$

$$B_4 = c_4 \sum_{s=40}^{2n} \mathrm{e}^{-3+sh/2}, \quad (18)$$

where c_0, c_1, c_2, c_3, and c_4 are the coefficients given by

$$c_0 = \begin{cases} 1/6, & s \in \{0, 2n\}; \\ 2/3, & s \in \{1, 3, \ldots, 2n-1\}; \\ 1/3, & s \in \{2, 4, \ldots, 2n-2\}; \end{cases}$$

$$c_1 = \begin{cases} 1/6, & s \in \{0, 20\}; \\ 2/3, & s \in \{1, 3, \ldots, 19\}; \\ 1/3, & s \in \{2, 4, \ldots, 18\}; \end{cases}$$

$$c_2 = \begin{cases} 1/6, & s \in \{20, 2n\}; \\ 2/3, & s \in \{21, 23, \ldots, 2n-1\}; \\ 1/3, & s \in \{22, 24, \ldots, 2n-2\}; \end{cases}$$

$$c_3 = \begin{cases} 1/6, & s \in \{20, 40\}; \\ 2/3, & s \in \{21, 23, \ldots, 39\}; \\ 1/3, & s \in \{22, 24, \ldots, 38\}; \end{cases}$$

$$c_4 = \begin{cases} 1/6, & s \in \{40, 2n\}; \\ 2/3, & s \in \{41, 43, \ldots, 2n-1\}; \\ 1/3, & s \in \{42, 44, \ldots, 2n-2\}; \end{cases}$$

with $H = (1 + h + h^2/2 + h^3/6, h^4/24)^{10}$, for $t \in [t_0, t_n]$; that is, $t \in [0, 3]$, $h = 0.1$, $n = 10t$ and $y(t) = y(n)$.

An error analysis is conducted by using the formula

$$E = |\text{exact solution} - \text{approximate solution}|,$$

with the absolute errors E being reported in Table 3. Figures 4 and 5 compare the exact and approximate solutions graphically for different values of α and t. Figure 6 shows the approximate solution for values of α and t in the interval $[0, 3]$.

Table 1. Values of exact and approximate solutions by using the RK-4 method for the indicated configuration.

	Approximate		**Exact**	
α	$\underline{y}(n;\alpha)$	$\overline{y}(n;\alpha)$	$\underline{Y}(t;\alpha)$	$\overline{Y}(t;\alpha)$
0.0	4.76804606005959	7.15206909008938	4.76803919566105	7.15205879349158
0.1	4.92698092872824	7.07260165575506	4.92697383551642	7.07259147356389
0.2	5.08591579739689	6.99313422142073	5.08590847537179	6.99312415363621
0.3	5.24485066606555	6.91366678708640	5.24484311522715	6.91365683370852
0.4	5.40378553473420	6.83419935275208	5.40377775508252	6.83418951378084
0.5	5.56272040340285	6.75473191841775	5.56271239493789	6.75472219385315
0.6	5.72165527207151	6.67526448408342	5.72164703479326	6.67525487392547
0.7	5.88059014074016	6.59579704974910	5.88058167464863	6.59578755399779
0.8	6.03952500940881	6.51632961541477	6.03951631450400	6.51632023407010
0.9	6.19845987807746	6.43686218108044	6.19845095435937	6.43685291414242
1.0	6.35739474674612	6.35739474674612	6.35738559421473	6.35738559421473

Table 2. Values of exact and approximate solutions by using the Euler method for the indicated configuration.

	Approximate		Exact	
α	$\underline{y}(n;\alpha)$	$\overline{y}(n;\alpha)$	$\underline{Y}(t;\alpha)$	$\overline{Y}(t;\alpha)$
0.0	4.676726124815156	7.015089187222735	4.76803919566105	7.15205879349158
0.1	4.832616995642328	6.937143751809149	4.92697383551642	7.07259147356389
0.2	4.988507866469501	6.859198316395563	5.08590847537179	6.99312415363621
0.3	5.144398737296671	6.781252880981976	5.24484311522715	6.91365683370852
0.4	5.300289608123844	6.703307445568391	5.40377775508252	6.83418951378084
0.5	5.456180478951015	6.625362010154805	5.56271239493789	6.75472219385315
0.6	5.612071349778188	6.547416574741219	5.72164703479326	6.67525487392547
0.7	5.767962220605360	6.469471139327633	5.88058167464863	6.59578755399779
0.8	5.923853091432531	6.391525703914047	6.03951631450400	6.51632023407010
0.9	6.079743962259703	6.313580268500461	6.19845095435937	6.43685291414242
1.0	6.235634833086875	6.235634833086875	6.35738559421473	6.35738559421473

Table 3. Error analysis of approximate solutions by using the RK-4 and Euler methods for the indicated configuration.

	RK-4 Method		Euler Method	
α	$\underline{y}(n;\alpha)$	$\overline{y}(n;\alpha)$	$\underline{y}(t;\alpha)$	$\overline{y}(t;\alpha)$
0.0	0.000006864	0.000010297	0.091313071	0.136969606
0.1	0.000007093	0.000010182	0.094356840	0.135447722
0.2	0.000007322	0.000010068	0.097400609	0.133925837
0.3	0.000007551	0.000009953	0.100444378	0.132403953
0.4	0.000007780	0.000009839	0.103488147	0.130882068
0.5	0.000008008	0.000009725	0.106531916	0.129360184
0.6	0.000008237	0.000009610	0.109575685	0.127838299
0.7	0.000008466	0.000009496	0.112619454	0.126316415
0.8	0.000008695	0.000009381	0.115663223	0.124794530
0.9	0.000008924	0.000009267	0.118706992	0.123272646
1.0	0.000009153	0.000009153	0.121750761	0.121750761

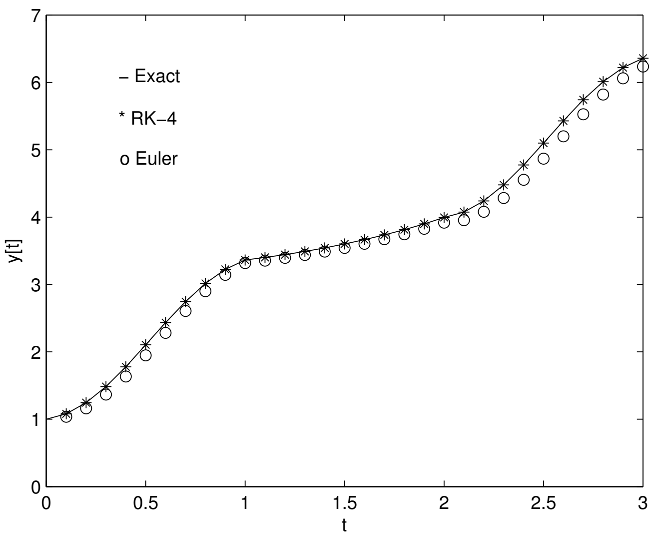

Figure 4. Graphical comparison of the approximate and exact solutions for $h = 0.1$, $\alpha = 1$ and $t \in [0,3]$.

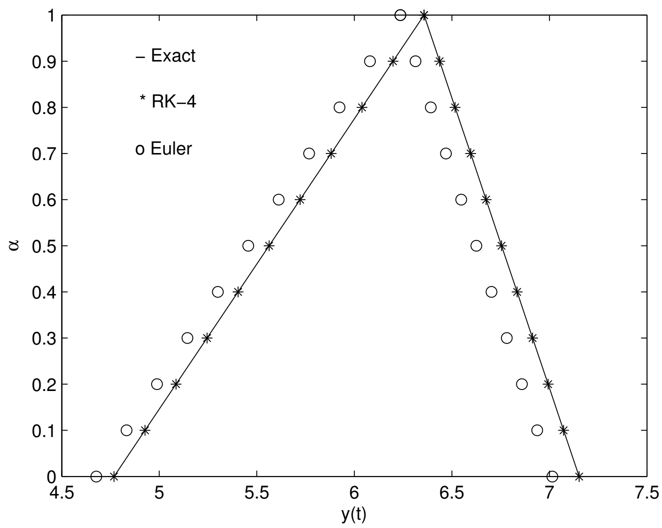

Figure 5. Graphical comparison between the approximate and exact solutions for $h = 0.1$, $\alpha \in [0, 1]$ and $t = 3$.

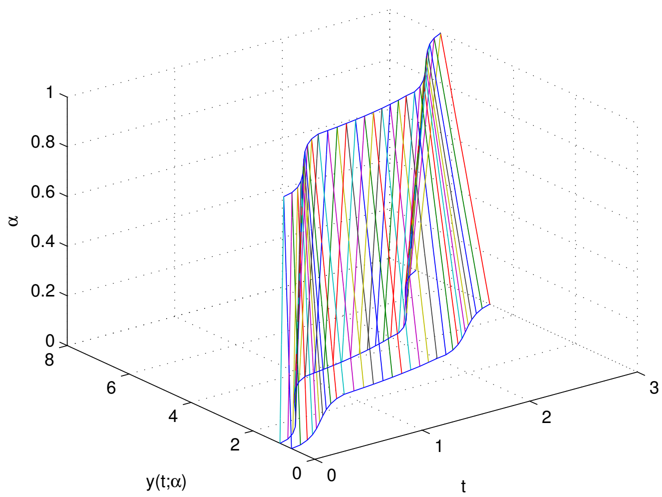

Figure 6. Approximate solution obtained with the RK-4 method for $h = 0.14$, $\alpha \in [0, 1]$ and $t \in [0, 3]$.

5. Results, Discussion and Conclusions

Next, we summarize the significance and novelty of the paper, so that it may be helpful for readers and other researchers:

- We developed hybrid fuzzy neutral delay differential equations and pure hybrid fuzzy neutral delay differential equations as governing equations for new systems based on fuzzy differential equations.
- Many authors have extended the fuzzy differential equations so far to include fuzzy hybrid differential equations and fuzzy delay differential equations. However, in this study, the fuzzy differential equations are extended to combinations of hybrid and delay differential equations, particularly neutral delay differential equations.
- The theoretical details were applied to a numerical example, and both analytical and numerical solutions were found. In this way, we generalized the approximate solutions algebraically.
- Though it is dealing with fuzzy solutions, it is enough to provide fuzzy plots. Nevertheless, we provided both non-fuzzy and fuzzy types of solutions. In both the non-fuzzy and fuzzy types of solutions, the coincidence of the exact and approximate solutions was shown graphically. For different values of t, the fuzzy valued plots were provided.

- The most important part of the paper involved using the Runge–Kutta method for solving non-pure (that is, including $y(t)$) hybrid fuzzy neutral delay differential equations, which we employed to solve a pure form of it (that is, without $y(t)$).
- We evaluated analytical solutions to the problem that we offered in the study, even though we dealt with numerical answers. The numerical solutions obtained by means of the fourth-order Runge–Kutta method were generalized and mentioned in the problem. Thus, when increasing the order to solve a different problem, the readers themselves can find the numerical solution. For the numerical results, we compared the numerical solutions obtained by means of the fourth-order Runge–Kutta and Euler methods with the exact solutions.
- Note that the fourth-order Runge–Kutta method gives better accuracy than the Euler method. Nonetheless, we compared the result with the Euler method to establish that the system obeys even the lower order methods.
- Though it is a more complicated form of a fuzzy differential equation, we also provided a numerical example to verify the theoretical results.

The application of numerical methods obtained by the fourth-order Runge–Kutta method for finding numerical solutions to hybrid fuzzy neutral delay differential equations has been viewed with an illustrative example. The comparison of solutions represented in Figure 4, for the non-fuzzy initial value problem, and in Figure 5, for the fuzzy initial value problem, established the accuracy of the fourth-order Runge–Kutta method in relation to the exact solution. The 3D graphical representation of the approximate solution given in Figure 6 shows all the values in $t \in [0,3]$ and $\alpha \in [0,1]$ obtained by the fourth-order Runge–Kutta method, which allowed us to understand that the exact solution coincides with this plot displayed in Figure 5. The same system of hybrid fuzzy neutral delay differential equations can be adopted for the case of fractional order. Therefore, as a future direction of the present investigation, we will try to find applications, such as particle motion in a circular cavity, spring pendulum, coupled oscillator, and new physical systems with non-singular derivatives, such as that provided in [33–36]. Also as a future work, we could model a hybrid system with a rate of change with respect to time for the delay response in signal processing.

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