

Article

Picture Fuzzy Soft Matrices and Application of Their Distance Measures to Supervised Learning: Picture Fuzzy Soft k -Nearest Neighbor (PFS- k NN)

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Abstract: This paper redefines picture fuzzy soft matrices (*pfs*-matrices) because of some of their inconsistencies resulting from Cuong’s definition of picture fuzzy sets. Then, it introduces several distance measures of *pfs*-matrices. Afterward, this paper proposes a new k NN-based classifier, namely the Picture Fuzzy Soft k -Nearest Neighbor (PFS- k NN) classifier. The proposed classifier utilizes the Minkowski’s metric of *pfs*-matrices to find the k -nearest neighbor. Thereafter, it performs an experimental study utilizing four UCI medical datasets and compares to the suggested approach using the state-of-the-art k NN-based classifiers. To evaluate the performance of the classification, it conducts ten iterations of five-fold cross-validation on all the classifiers. The findings indicate that PFS- k NN surpasses the state-of-the-art k NN-based algorithms in 72 out of 128 performance results based on accuracy, precision, recall, and F1-score. More specifically, the proposed method achieves higher accuracy and F1-score results compared to the other classifiers. Simulation results show that *pfs*-matrices and PFS- k NN are capable of modeling uncertainty and real-world problems. Finally, the applications of *pfs*-matrices to supervised learning are discussed for further research.

Keywords: soft sets; picture fuzzy sets; picture fuzzy soft matrices; distance measures; machine learning; k -nearest neighbor (k NN)

MSC: 03E72; 15B15; 62H30; 68T05



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1. Introduction

In daily life events, we frequently come across many intricate challenges that are full of uncertainties. Such uncertainties may be impossible to model using traditional mathematical approaches. As a result, state-of-the-art mathematical techniques are needed to model such uncertainties. To avoid ambiguities, Zadeh created the idea of fuzzy sets (f -sets) [1]. f -sets are common mathematical tools used in numerous domains, ranging from computer science [2,3] to pure mathematics [4–9]. Figure 1 shows some hybrid extensions of f -sets.

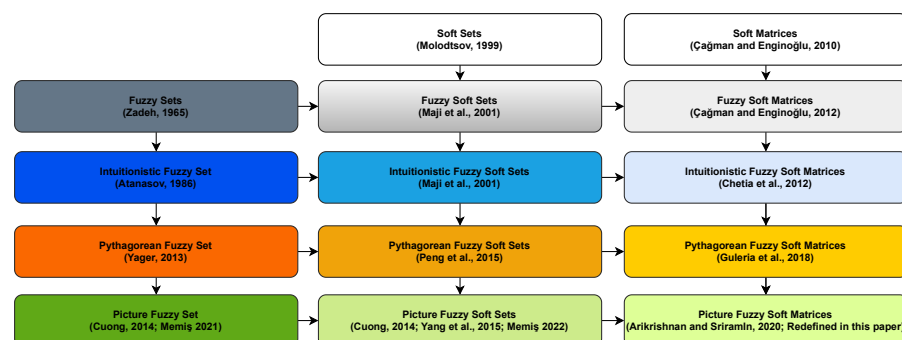


Figure 1. Some hybrid versions and extensions of fuzzy and soft sets [1,7,9–22].

An *f*-set has entries indicated by $\mu(x)$, i.e., a membership degree for x . Because $\mu(x) + \nu(x) = 1$, the non-membership degree $\nu(x)$ is calculated by subtracting the $\mu(x)$ from 1. However, if $\mu(x) + \nu(x) < 1$, it is not as simple, and there is additional uncertainty. As an extension of *f*-sets, intuitionistic fuzzy sets (*if*-sets) [10] have been proposed to model this form of uncertainty. An *if*-set has entries indicated by $\mu(x)$ and $\nu(x)$, namely membership and non-membership degrees, respectively, such that $0 \leq \mu(x) + \nu(x) \leq 1$ (Figure 2). In contrast to fuzzy sets, the idea of intuitionistic fuzzy sets can depict problems where $0 \leq \mu(x) + \nu(x) < 1$. In addition, the indeterminacy degree is determined as $1 - (\mu(x) + \nu(x))$.

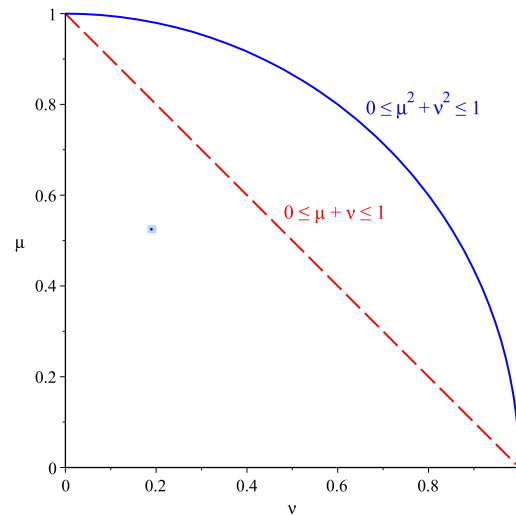


Figure 2. Comparison of space of intuitionistic and Pythagorean fuzzy membership.

Although *f*-sets and *if*-sets may overcome many difficulties and uncertainties [23], far more are encountered in practice. Consider the voting process for a presidential election. During this procedure, the electorate’s decisions can be divided into three categories: yes, no, and abstention. To represent such a process, Cuong proposed the notion of picture fuzzy sets (*pf*-sets) [16]. A *pf*-set has elements with the degrees of membership, non-membership, and neutral membership denoted by $\mu(x)$, $\nu(x)$, and $\eta(x)$, respectively. The refusal to vote or non-participation in voting leads to the indeterminacy described above. Furthermore, $1 - (\mu(x) + \eta(x) + \nu(x))$ reflects the degree of indeterminacy in *pf*-sets because $\mu(x) + \eta(x) + \nu(x) \leq 1$ in Cuong’s definition. Even though *pf*-sets model the aforementioned difficulties, the definitions and operations put forward by Cuong have conceptual errors. Memiş [21] revised the idea of *pf*-sets and associated operations to maintain consistency, where $\mu(x) + \eta(x) + \nu(x) \leq 2$.

Conversely, *pf*-sets are unable to model the problems comprising parameters and alternatives (objects) with a picture fuzzy membership (*pf*-membership) degree. In other words, *pfs*-sets [16,18,24] can represent problems with alternatives (objects) using *pf*-membership (Figure 3), with the expert voting on whether to accept, reject, or abstain from the alternatives.

Recently, various studies have been conducted on *pf*-sets and *pfs*-sets. The idea of a rough picture set has been introduced, and several of its topological features, including the lower and upper rough picture fuzzy approximation operators, have also been investigated [25]. The creation of clustering algorithms that can explore latent knowledge from a large number of datasets is an emerging research field in *pf*-sets. The distance and similarity measure is one of the most crucial tools in clustering that establishes the level of association between two objects. Therefore, generalized picture distance measure has been defined, and it has been applied to picture fuzzy clustering [26]. In addition to distance measure, picture fuzzy similarity has also been studied [27,28]. A technique for solving decision-making issues utilizing the generalized *pfs*-sets and an adjustable weighted soft discernibility matrix has been presented, and threshold functions have been defined [29].

A weighted soft discernibility matrix in the generalized pfs -sets has been employed to offer an illustrative example to demonstrate the superiority of the suggested approach therein. Matrix representations of mathematical concepts, such as pfs -sets are crucial in the context of computerizing [30,31]. Thus, Arikrishnan and Sriram [20] define picture fuzzy soft matrices and investigate their algebraic structures. Because the related study is based on Cuong's [16] study, there are some theoretical inconsistencies. Moreover, Arikrishnan and Sriram have only focused on the algebraic structures. The study of Sahu et al. [32] aims to analyze students' characteristics, such as career, memory, interest, knowledge, environment, and attitude, in order to predict the most suitable career path. This will enable students to explore and excel in their chosen field comfortably. A hybridized distance measure has been proposed, using picture fuzzy numbers to evaluate students, subjects, and students' characteristics for career selection. However, related studies only rely on fictitious problem data. A research study that integrates pfs -sets with Quality Function Deployment (QFD) to propose a Multiple Criteria Group Decision-Making (MCGDM) method has been discussed [33]. In this approach, the preferences of the decision-makers are collected in linguistic terms and transformed into Picture Fuzzy Numbers (PFNs). The study applies the proposed MCGDM method to rank social networking sites, specifically evaluating Facebook, Whatsapp, Instagram, and Twitter, providing valuable insights into their comparative performance. The study of Lu et al. [34] has introduced the concept of generalized pfs -sets by combining an image fuzzy soft set with a fuzzy parameter set. They discuss five main operations for generalized pfs -sets: subset, equality, union, intersection, and complement.

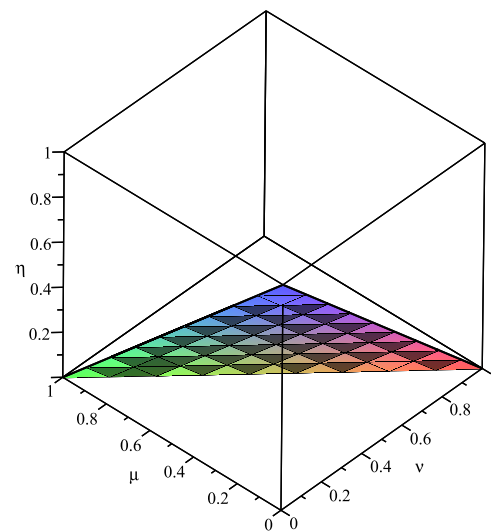


Figure 3. Space of picture fuzzy membership.

Suppose the problem has picture fuzzy uncertainty and a large number of data. In that case, pfs -sets cannot operate efficiently with a large number of data. Therefore, processing data through the computer is compulsory, and the matrix versions of the pfs -sets are needed. The concept of picture fuzzy soft matrices (pfs -matrices) was propounded in 2020 [20]; however, in the aforementioned study, only the algebraic structures of the concept have been investigated. To this end, this paper redefines the concept of pfs -matrices, defines the distance measures of the pfs -matrices, and applies them to supervised learning to manifest their modeling ability. The major contributions of this paper are as follows:

- pfs -matrices are redefined, and some of their basic properties are investigated.
- Distance measures of pfs -matrices are introduced.
- Picture fuzzy soft k -nearest neighbor (PFS- k NN) based on distance measure of pfs -matrices is proposed.
- An application of PFS- k NN to medical diagnosis is provided.

In Section 2 of the paper, definitions of *pf*-sets and *pfs*-sets are provided. In Section 3, the motivations of the redefining of *pfs*-matrices are detailed. In Section 4, the idea of *pfs*-matrices is redefined, and their properties are further examined. In Section 5, distance measures of *pfs*-matrices are introduced, and their basic properties are researched. In Section 6, a PFS-*k*NN classifier is proposed. In Section 7, the proposed classifier is applied to medical diagnosis and compared with the well-known *k*NN-based classifiers. Finally, we discuss *pfs*-matrices and PFS-*k*NN and provide conclusive remarks for further research.

2. Preliminaries

In this section, we present the concepts of *pf*-sets and *pfs*-sets by considering the notations used across this study. Across this paper, let E and U denote the parameter and alternative sets, respectively.

Definition 1 ([16,21]). Let f be a function such that $f : E \rightarrow [0, 1] \times [0, 1] \times [0, 1]$. Then, the graphic

$$\{(x, f(x)) : x \in E\} = \{(x, \mu(x), \eta(x), \nu(x)) : x \in E\}$$

is called a picture fuzzy set (*pf*-set) over E . Here, a *pf*-set is denoted by $\left\{ \left(\left\langle \begin{matrix} \mu(x) \\ \eta(x) \\ \nu(x) \end{matrix} \right\rangle x \right) : x \in E \right\}$ instead of $\{(x, \mu(x), \eta(x), \nu(x)) : x \in E\}$.

Moreover, for all $x \in E$, $\mu(x) + \nu(x) \leq 1$ and $\mu(x) + \eta(x) + \nu(x) \leq 2$. Furthermore, μ , η , and ν are the membership, neutral membership, and non-membership functions, respectively, and the indeterminacy degree of the element $x \in E$ is defined by $\pi(x) = 1 - (\mu(x) + \nu(x))$.

In the present paper, the set of all the *pf*-sets over E is symbolized by $PF(E)$ and $f \in PF(E)$.

Remark 1. In $PF(E)$, the notations $graph(f)$ and f are interchangeable since they have generated each other uniquely. Thus, we prefer the notation f to $graph(f)$ for brevity, provided that it results in no confusion.

Definition 2 ([16,22]). Let α be a function such that $\alpha : E \rightarrow PF(U)$. Then, the graphic

$$\{(x, \alpha((x, \mu(x), \eta(x), \nu(x)))) : x \in E\}$$

is called a picture fuzzy soft set (*pfs*-set) parameterized via E over U (or briefly over U).

Throughout this paper, the set of all the *pfs*-sets over U is symbolized by $PFS_E(U)$.

Remark 2. In $PFS_E(U)$, the notations $graph(\alpha)$ and α are interchangeable since they have generated each other uniquely. Thus, we prefer the notation α to $graph(\alpha)$ for brevity, provided that it results in no confusion.

Example 1. Let $E = \{x_1, x_2, x_3, x_4\}$ and $U = \{u_1, u_2, u_3, u_4, u_5\}$. Then,

$$\alpha = \left\{ \left(x_1, \left\{ \left\langle \begin{matrix} 0.8 \\ 0.1 \\ 0.1 \end{matrix} \right\rangle u_1, \left\langle \begin{matrix} 0.1 \\ 0.2 \\ 0.7 \end{matrix} \right\rangle u_3, \left\langle \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \right\rangle u_5 \right\} \right), \left(x_2, \left\{ \left\langle \begin{matrix} 0.6 \\ 0.4 \\ 0 \end{matrix} \right\rangle u_2, \left\langle \begin{matrix} 0 \\ 0.5 \\ 0.5 \end{matrix} \right\rangle u_4 \right\} \right), \left(x_3, \left\{ \left\langle \begin{matrix} 0.7 \\ 0 \\ 0.2 \end{matrix} \right\rangle u_3 \right\} \right), \right. \\ \left. \left(x_4, \left\{ \left\langle \begin{matrix} 0.1 \\ 0.3 \\ 0.2 \end{matrix} \right\rangle u_2, \left\langle \begin{matrix} 0.4 \\ 0.2 \\ 0.4 \end{matrix} \right\rangle u_5 \right\} \right) \right\}$$

is a *pfs*-set over U .

3. Motivations of the Redefining of *pf*s-Matrices

This section discusses the definition, fundamental operations, and counter-examples to Arikrishnan and Sriram’s definition [20], based on Cuong’s definition [16], considering the notations employed throughout the rest of the study.

Definition 3 ([16]). Let $\kappa : E \rightarrow [0, 1] \times [0, 1] \times [0, 1]$. Then, the graphic

$$\{(x, \kappa(x)) : x \in E\} = \left\{ \left\langle \begin{matrix} \mu(x) \\ \eta(x) \\ \nu(x) \end{matrix} \right\rangle x : x \in E \right\}$$

is called a picture fuzzy set (*pf*-set) over E such that $\mu(x) + \eta(x) + \nu(x) \leq 1$.

In this section, the set of all the *pf*-sets over E according to Cuong’s definition is denoted by $PF_C(E)$ and $\kappa \in PF_C(E)$.

Definition 4 ([16]). Let $\kappa_1, \kappa_2 \in PF_C(E)$. For all $x \in E$, if $\mu_1(x) \leq \mu_2(x)$, $\eta_1(x) \leq \eta_2(x)$, and $\nu_1(x) \geq \nu_2(x)$, then κ_1 is called a subset of κ_2 and is denoted by $\kappa_1 \subseteq \kappa_2$.

Definition 5 ([16]). Let $\kappa_1, \kappa_2 \in PF_C(E)$. If $\kappa_1 \subseteq \kappa_2$ and $\kappa_2 \subseteq \kappa_1$, then κ_1 and κ_2 are called equal *pf*-sets and are denoted by $\kappa_1 = \kappa_2$.

Definition 6 ([16]). Let $\kappa_1, \kappa_2, \kappa_3 \in PF_C(E)$. For all $x \in E$, if $\mu_3(x) = \max\{\mu_1(x), \mu_2(x)\}$, $\eta_3(x) = \min\{\eta_1(x), \eta_2(x)\}$, and $\nu_3(x) = \min\{\nu_1(x), \nu_2(x)\}$, then κ_3 is called union of κ_1 and κ_2 and is denoted by $\kappa_3 = \kappa_1 \cup \kappa_2$.

Definition 7 ([16]). Let $\kappa_1, \kappa_2, \kappa_3 \in PF_C(E)$. For all $x \in E$, if $\mu_3(x) = \min\{\mu_1(x), \mu_2(x)\}$, $\eta_3(x) = \min\{\eta_1(x), \eta_2(x)\}$, and $\nu_3(x) = \max\{\nu_1(x), \nu_2(x)\}$, then κ_3 is called intersection of κ_1 and κ_2 and is denoted by $\kappa_3 = \kappa_1 \cap \kappa_2$.

Definition 8 ([16]). Let $\kappa_1, \kappa_2 \in PF_C(E)$. For all $x \in E$, if $\mu_2(x) = \nu_1(x)$, $\eta_2(x) = \eta_1(x)$, and $\nu_2(x) = \mu_1(x)$, then κ_2 is called complement of κ_1 and is denoted by $\kappa_2 = \kappa_1^c$.

To hold the conditions “Empty *pf*-set over E is a subset of all the *pf*-set over E ” and “All *pf*-sets over E are the subset of universal *pf*-set over E ”, the definition and operations of *pf*-sets in [16] must be as follows [21]:

Definition 9 ([21]). Let $\kappa \in PF_C(E)$. For all $x \in E$, if $\mu(x) = 0$, $\eta(x) = 0$, and $\nu(x) = 1$, then κ is called empty *pf*-set and is denoted by $\left\langle \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \right\rangle_{E_C}$ or 0_{E_C} .

Definition 10 ([21]). Let $\kappa \in PF_C(E)$. For all $x \in E$, if $\mu(x) = 1$, $\eta(x) = 1$, and $\nu(x) = 0$, then κ is called universal *pf*-set and is denoted by $\left\langle \begin{matrix} 1 \\ 1 \\ 0 \end{matrix} \right\rangle_{E_C}$ or 1_{E_C} .

Cuong’s definitions have led to the inconsistencies in Examples 2 and 3 [21]:

Example 2 ([21]). There is a contradiction in Definition 10 since $1 + 1 + 0 \not\leq 1$, i.e., $1_{E_C} \notin PF_C(E)$. Moreover, even if $1_{E_C} \in PF_C(E)$, $(1_{E_C})^c \neq 0_{E_C}$.

Example 3 ([21]). Let $\kappa \in PF_C(E)$ such that $\kappa = \left\{ \left\langle \begin{matrix} 0.1 \\ 0.2 \\ 0.3 \end{matrix} \right\rangle x \right\}$. Then, $\kappa \cup 0_E \neq \kappa$ and $\kappa \cap 1_{E_C} \neq 1_{E_C}$.

Therefore, Memiş [21] has provided the definition and operations of *pf*-sets in [16] to overcome the aforementioned inconsistencies.

Definition 11 ([16,18]). $A \subseteq E$. The set

$$\left\{ \left(x, F_A \left(\left(\left\langle \begin{matrix} \mu(x) \\ \eta(x) \\ \nu(x) \end{matrix} \right\rangle x \right) \right) \right) : x \in A \subseteq E \right\}$$

is called a *pfs*-set over U , where F_A is a mapping given by $F : A \rightarrow PFC(U)$.

In this section, the set of all the *pfs*-sets over U according to Cuong’s definition is denoted by $PFS_C(U)$ and $F_A \in PFS_C(U)$.

Cuong [16] defined *pfs*-sets based on his own definition and operations of *pf*-sets. As a result, the inconsistencies mentioned earlier also apply to his concept of *pfs*-sets. Additionally, Yang et al. [18] claimed to have introduced the concept of *pf*-sets, even though Cuong had already defined it in [16]. Thus, the concept of *pfs*-sets has also similar inconsistencies therein. Hence, *pfs*-sets were redefined to deal with inconsistencies mentioned above [22].

Furthermore, the concept of *pfs*-matrices has similar inconsistencies therein, since Arikrishnan and Sriram [20] have introduced the *pfs*-matrices according to Cuong’s definition [16] and defined their union, intersection, and complement.

Definition 12 ([20]). Let $F_A \in PFS_C(U)$. Then, $[a_{ij}]$ is called *pfs*-matrix of F_A and defined by

$$[a_{ij}] := \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & \dots \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}$$

such that for $i \in \{1, 2, \dots\}$ and $j \in \{1, 2, \dots\}$,

$$a_{ij} := F_A \left(\left\langle \begin{matrix} \mu(x_j) \\ \eta(x_j) \\ \nu(x_j) \end{matrix} \right\rangle x_j \right) (u_i)$$

Here, if $|U| = m$ and $|E| = n$, then $[a_{ij}]$ has order $m \times n$.

In the present study, the membership, neutral membership, and non-membership degrees of $[a_{ij}]$, i.e., μ_{ij} , η_{ij} , and ν_{ij} , will be denoted by μ_{ij}^a , η_{ij}^a , and ν_{ij}^a , respectively, as long as they do not cause any confusion. Moreover, the set of all the *pfs*-matrices over U according to Arikrishnan and Sriram’s definition is denoted by $PFS_{AS}[U]$ and $F_A \in PFS_{AS}[U]$.

It must be noted that the following definitions from [20] expressed the notations employed throughout the present paper. Definitions of inclusion and equality in the *pfs*-matrices space is provided according to Arikrishnan and Sriram’s definitions.

Definition 13. Let $[a_{ij}], [b_{ij}] \in PFS_{AS}[U]$. For all i and j , if $\mu_{ij}^a \leq \mu_{ij}^b$, $\eta_{ij}^a \leq \eta_{ij}^b$, and $\nu_{ij}^a \geq \nu_{ij}^b$, then $[a_{ij}]$ is called a submatrix of $[b_{ij}]$ and is denoted by $[a_{ij}] \subseteq [b_{ij}]$.

Definition 14. Let $[a_{ij}], [b_{ij}] \in PFS_{AS}[U]$. For all i and j , if $\mu_{ij}^a = \mu_{ij}^b$, $\eta_{ij}^a = \eta_{ij}^b$, and $\nu_{ij}^a = \nu_{ij}^b$, then $[a_{ij}]$ and $[b_{ij}]$ are called equal *pfs*-matrices and denoted by $[a_{ij}] = [b_{ij}]$.

Definition 15 ([20]). Let $[a_{ij}], [b_{ij}], [c_{ij}] \in PFS_{AS}[U]$. For all i and j , if $\mu_{ij}^c = \max\{\mu_{ij}^a, \mu_{ij}^b\}$, $\eta_{ij}^c = \min\{\eta_{ij}^a, \eta_{ij}^b\}$, and $\nu_{ij}^c = \min\{\nu_{ij}^a, \nu_{ij}^b\}$, then $[c_{ij}]$ is called union of $[a_{ij}]$ and $[b_{ij}]$ and denoted by $[a_{ij}] \cup [b_{ij}]$.

Definition 16 ([20]). Let $[a_{ij}], [b_{ij}], [c_{ij}] \in PFS_{AS}[U]$. For all i and j , if $\mu_{ij}^c = \min\{\mu_{ij}^a, \mu_{ij}^b\}$, $\eta_{ij}^c = \min\{\eta_{ij}^a, \eta_{ij}^b\}$, and $\nu_{ij}^c = \max\{\nu_{ij}^a, \nu_{ij}^b\}$, then $[c_{ij}]$ is called intersection of $[a_{ij}]$ and $[b_{ij}]$ and denoted by $[a_{ij}] \hat{\cap} [b_{ij}]$.

Definition 17 ([20]). Let $[a_{ij}], [b_{ij}] \in PFS_{AS}[U]$. For all i and j , if $\mu_{ij}^b = \nu_{ij}^a$, $\eta_{ij}^b = \eta_{ij}^a$, and $\nu_{ij}^b = \mu_{ij}^a$, then $[b_{ij}]$ is complement of $[a_{ij}]$ and denoted by $[a_{ij}]^c$.

According to Arikrishnan and Sriram’s definitions, the definitions of empty and universal *pfs*-matrices must be defined as in Definitions 18 and 19, respectively, to hold the conditions “Empty *pfs*-matrices over U is a submatrix of all the *pfs*-matrices over U ” and “All *pfs*-matrices over U are the submatrix of universal *pfs*-matrix over U ”.

Definition 18. Let $[a_{ij}] \in PFS_{AS}[U]$. For all i and j , if $\mu_{ij} = 0$, $\eta_{ij} = 0$, and $\nu_{ij} = 1$, then $[a_{ij}]$ is empty *pfs*-matrix and is denoted by $\left[\begin{matrix} \langle 0 \\ 0 \\ 1 \rangle \end{matrix} \right]$.

Definition 19. Let $[a_{ij}] \in PFS_{AS}[U]$. For all i and j , if $\mu_{ij} = 1$, $\eta_{ij} = 1$, and $\nu_{ij} = 0$, then $[a_{ij}]$ is universal *pfs*-matrix and is denoted by $\left[\begin{matrix} \langle 1 \\ 1 \\ 0 \rangle \end{matrix} \right]$.

Arikrishnan and Sriram’s definitions have resulted in the inconsistencies in Examples 4 and 5:

Example 4. There is a contradiction in Definition 19 since $1 + 1 + 0 \not\leq 1$, namely, $\left[\begin{matrix} \langle 1 \\ 1 \\ 0 \rangle \end{matrix} \right] \notin PFS_{AS}[U]$. Moreover, even if $\left[\begin{matrix} \langle 1 \\ 1 \\ 0 \rangle \end{matrix} \right] \in PFS_{AS}[U]$, $\left[\begin{matrix} \langle 1 \\ 1 \\ 0 \rangle \end{matrix} \right]^c = \left[\begin{matrix} \langle 0 \\ 1 \\ 1 \rangle \end{matrix} \right] \neq \left[\begin{matrix} \langle 0 \\ 0 \\ 1 \rangle \end{matrix} \right]$.

Example 5. Let $[a_{ij}] \in PFS_{AS}[U]$ such that $[a_{ij}] = \left[\begin{matrix} \langle 0.4 \\ 0.3 \\ 0.1 \rangle & \langle 0.2 \\ 0.4 \\ 0.3 \rangle \\ \langle 0.7 \\ 0.1 \rangle & \langle 0.1 \\ 0.5 \\ 0.2 \rangle \end{matrix} \right]$. Then,

$$[a_{ij}] \hat{\cup} \left[\begin{matrix} \langle 0 \\ 0 \\ 1 \rangle \end{matrix} \right] = \left[\begin{matrix} \langle 0.4 \\ 0 \\ 0.1 \rangle & \langle 0.2 \\ 0 \\ 0.3 \rangle \\ \langle 0.7 \\ 0 \\ 0.1 \rangle & \langle 0.1 \\ 0 \\ 0.2 \rangle \end{matrix} \right] \neq \left[\begin{matrix} \langle 0.4 \\ 0.3 \\ 0.1 \rangle & \langle 0.2 \\ 0.4 \\ 0.3 \rangle \\ \langle 0.7 \\ 0.1 \rangle & \langle 0.1 \\ 0.5 \\ 0.2 \rangle \end{matrix} \right] = [a_{ij}]$$

and

$$[a_{ij}] \hat{\cup} \left[\begin{matrix} \langle 1 \\ 1 \\ 0 \rangle \end{matrix} \right] = \left[\begin{matrix} \langle 1 \\ 0.3 \\ 0 \rangle & \langle 1 \\ 0.4 \\ 0 \rangle \\ \langle 1 \\ 0.1 \rangle & \langle 1 \\ 0.5 \\ 0 \rangle \end{matrix} \right] \neq \left[\begin{matrix} \langle 1 \\ 1 \\ 0 \rangle \end{matrix} \right]$$

Consequently, since the aforesaid definitions and operations of *pfs*-matrices and how they operate are inconsistent, this concept and its operations must be redefined.

4. Picture Fuzzy Soft Matrices (*pfs*-Matrices)

Cuong [16] and Yang et al. [18] have introduced the concept of *pfs*-sets to address the need for more general mathematical modeling of specific issues involving additional uncertainties. In addition, Yang et al. [18] have proposed an adjustable soft discernibility approach based on *pfs*-sets and applied it to a decision-making problem. Memiş [22] has redefined the concept of *pfs*-sets and applied it to a project selection problem. The applications described in the aforementioned studies demonstrate the successful use of

pfs-sets in addressing various issues with the uncertainties modeled by membership, non-membership, and neutral degrees, namely picture fuzzy uncertainties. These results suggest that researching the idea of *pfs*-sets is worthwhile. However, it is important to note that these ideas have drawbacks, such as complexity and lengthy computation times. Therefore, it is crucial to understand their matrix representations, i.e., *pfs*-matrices, and ensure their theoretical consistency in the context of computerizing the aforementioned problems. For instance, for utilizing *pfs*-sets in machine learning, *pfs*-matrices, which are matrix representation of *pfs*-sets, and their consistent theoretical definition and operations are needed.

Thus, in the present section, we make consistent the idea of *pfs*-matrices and present some of its fundamental properties. Since some of the propositions in this section have elementary proof, only the propositions with the complex proof are demonstrated.

Definition 20. Let $\alpha \in PFS_E(U)$ (See Definition 2). Then, $[a_{ij}]$ is called *pfs*-matrix of α and defined by

$$[a_{ij}] := \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & \dots \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}$$

such that for $i \in \{1, 2, \dots\}$ and $j \in \{1, 2, \dots\}$,

$$a_{ij} := \alpha \left(\left\langle \begin{matrix} \mu(x_j) \\ \eta(x_j) \\ \nu(x_j) \end{matrix} \right\rangle x_j \right) (u_i)$$

Here, if $|U| = m$ and $|E| = n$, then $[a_{ij}]$ has order $m \times n$.

In the present study, the membership, neutral membership, and non-membership degrees of $[a_{ij}]$, i.e., μ_{ij} , η_{ij} , and ν_{ij} , will be denoted by μ_{ij}^a , η_{ij}^a , and ν_{ij}^a , respectively, as long as they do not cause any confusion. Moreover, the set of all the *pfs*-matrices parameterized via E over U (briefly over U) is denoted by $PFS_E[U]$ and $[a_{ij}], [b_{ij}], [c_{ij}] \in PFS_E[U]$.

Example 6. The *pfs*-matrix of α given in Example 1 is as follows:

$$[a_{ij}] = \begin{bmatrix} \left\langle \begin{matrix} 0.8 \\ 0.1 \\ 0.1 \end{matrix} \right\rangle & \left\langle \begin{matrix} 0 \\ 1 \\ 1 \end{matrix} \right\rangle & \left\langle \begin{matrix} 0 \\ 1 \\ 1 \end{matrix} \right\rangle & \left\langle \begin{matrix} 0 \\ 1 \\ 1 \end{matrix} \right\rangle \\ \left\langle \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \right\rangle & \left\langle \begin{matrix} 0.6 \\ 0.4 \\ 0 \end{matrix} \right\rangle & \left\langle \begin{matrix} 0 \\ 1 \\ 1 \end{matrix} \right\rangle & \left\langle \begin{matrix} 0.1 \\ 0.3 \\ 0.2 \end{matrix} \right\rangle \\ \left\langle \begin{matrix} 0.1 \\ 0.2 \\ 0.7 \end{matrix} \right\rangle & \left\langle \begin{matrix} 0 \\ 1 \\ 1 \end{matrix} \right\rangle & \left\langle \begin{matrix} 0.7 \\ 0 \\ 0.2 \end{matrix} \right\rangle & \left\langle \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \right\rangle \\ \left\langle \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \right\rangle & \left\langle \begin{matrix} 0 \\ 1 \\ 1 \end{matrix} \right\rangle & \left\langle \begin{matrix} 0 \\ 1 \\ 1 \end{matrix} \right\rangle & \left\langle \begin{matrix} 0 \\ 1 \\ 1 \end{matrix} \right\rangle \\ \left\langle \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \right\rangle & \left\langle \begin{matrix} 0 \\ 0.5 \\ 0.5 \end{matrix} \right\rangle & \left\langle \begin{matrix} 0 \\ 1 \\ 1 \end{matrix} \right\rangle & \left\langle \begin{matrix} 0.4 \\ 0.2 \\ 0.4 \end{matrix} \right\rangle \end{bmatrix}$$

Definition 21. Let $[a_{ij}] \in PFS_E[U]$. For all i and j , if $\mu_{ij} = \lambda$, $\eta_{ij} = \epsilon$, and $\nu_{ij} = \omega$, then $[a_{ij}]$ is $(\lambda, \epsilon, \omega)$ -*pfs*-matrix and denoted by $\left[\left\langle \begin{matrix} \lambda \\ \epsilon \\ \omega \end{matrix} \right\rangle \right]$. Moreover, $\left[\left\langle \begin{matrix} 0 \\ 1 \\ 1 \end{matrix} \right\rangle \right]$ is empty *pfs*-matrix and $\left[\left\langle \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \right\rangle \right]$ is universal *pfs*-matrix.

Definition 22. Let $[a_{ij}], [b_{ij}], [c_{ij}] \in PFS_E[U]$, $I_E := \{j : x_j \in E\}$, and $R \subseteq I_E$. For all i and j , if

$$\mu_{ij}^c = \begin{cases} \mu_{ij}^a, & j \in R \\ \mu_{ij}^b, & j \in I_E \setminus R \end{cases}, \eta_{ij}^c = \begin{cases} \eta_{ij}^a, & j \in R \\ \eta_{ij}^b, & j \in I_E \setminus R \end{cases}, \text{ and } \nu_{ij}^c = \begin{cases} \nu_{ij}^a, & j \in R \\ \nu_{ij}^b, & j \in I_E \setminus R \end{cases}$$

then $[c_{ij}]$ is called *Rb-restriction* of $[a_{ij}]$ and is denoted by $[(a_{Rb})_{ij}]$. Briefly, if $[b_{ij}] = \left[\begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix} \right]$, then $[(a_R)_{ij}]$ can be used instead of $[(a_{Rb})_{ij}]$. It is clear that

$$(a_R)_{ij} = \begin{cases} \left\langle \begin{smallmatrix} \mu_{ij}^a \\ \eta_{ij}^a \\ \nu_{ij}^a \end{smallmatrix} \right\rangle, & j \in R \\ \left\langle \begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix} \right\rangle, & j \in I_E \setminus R \end{cases}$$

Definition 23. Let $[a_{ij}], [b_{ij}] \in PFS_E[U]$. For all i and j , if $\mu_{ij}^a \leq \mu_{ij}^b$, $\eta_{ij}^a \geq \eta_{ij}^b$, and $\nu_{ij}^a \geq \nu_{ij}^b$, then $[a_{ij}]$ is called a *submatrix* of $[b_{ij}]$ and denoted by $[a_{ij}] \subseteq [b_{ij}]$.

Definition 24. Let $[a_{ij}], [b_{ij}] \in PFS_E[U]$. For all i and j , if $\mu_{ij}^a = \mu_{ij}^b$, $\eta_{ij}^a = \eta_{ij}^b$, and $\nu_{ij}^a = \nu_{ij}^b$, then $[a_{ij}]$ and $[b_{ij}]$ are called *equal pfs-matrices* and denoted by $[a_{ij}] = [b_{ij}]$.

Proposition 1. Let $[a_{ij}], [b_{ij}], [c_{ij}] \in PFS_E[U]$. Then,

- i. $[a_{ij}] \subseteq \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix} \right]$
- ii. $\left[\begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix} \right] \subseteq [a_{ij}]$
- iii. $[a_{ij}] \subseteq [a_{ij}]$
- iv. $([a_{ij}] \subseteq [b_{ij}] \wedge [b_{ij}] \subseteq [a_{ij}]) \Leftrightarrow [a_{ij}] = [b_{ij}]$
- v. $([a_{ij}] \subseteq [b_{ij}] \wedge [b_{ij}] \subseteq [c_{ij}]) \Rightarrow [a_{ij}] \subseteq [c_{ij}]$
- vi. $([a_{ij}] = [b_{ij}] \wedge [b_{ij}] = [c_{ij}]) \Rightarrow [a_{ij}] = [c_{ij}]$

Proof. The proofs of *i-vi* are straightforward. \square

Remark 3. From Proposition 1, it is straightforward that the inclusion relation herein is a partial ordering relation in $PFS_E[U]$.

Definition 25. Let $[a_{ij}], [b_{ij}] \in PFS_E[U]$. If $[a_{ij}] \subseteq [b_{ij}]$ and $[a_{ij}] \neq [b_{ij}]$, then $[a_{ij}]$ is called a *proper submatrix* of $[b_{ij}]$ and denoted by $[a_{ij}] \subsetneq [b_{ij}]$.

Definition 26. Let $[a_{ij}], [b_{ij}], [c_{ij}] \in PFS_E[U]$. For all i and j , if $\mu_{ij}^c = \max\{\mu_{ij}^a, \mu_{ij}^b\}$, $\eta_{ij}^c = \min\{\eta_{ij}^a, \eta_{ij}^b\}$, and $\nu_{ij}^c = \min\{\nu_{ij}^a, \nu_{ij}^b\}$, then $[c_{ij}]$ is called *union* of $[a_{ij}]$ and $[b_{ij}]$ and denoted by $[a_{ij}] \cup [b_{ij}]$.

Definition 27. Let $[a_{ij}], [b_{ij}], [c_{ij}] \in PFS_E[U]$. For all i and j , if $\mu_{ij}^c = \min\{\mu_{ij}^a, \mu_{ij}^b\}$, $\eta_{ij}^c = \max\{\eta_{ij}^a, \eta_{ij}^b\}$, and $\nu_{ij}^c = \max\{\nu_{ij}^a, \nu_{ij}^b\}$, then $[c_{ij}]$ is called *intersection* of $[a_{ij}]$ and $[b_{ij}]$ and denoted by $[a_{ij}] \cap [b_{ij}]$.

Example 7. Assume that two pfs-matrices $[a_{ij}]$ and $[b_{ij}]$ are as follows:

$$[a_{ij}] = \begin{bmatrix} \langle 0.2 \\ 0.3 \\ 0.5 \rangle & \langle 0 \\ 1 \\ 1 \rangle & \langle 1 \\ 0 \\ 0 \rangle \\ \langle 0.1 \\ 0.5 \\ 0.2 \rangle & \langle 0.6 \\ 0.4 \\ 0 \rangle & \langle 0.9 \\ 0 \\ 0.1 \rangle \\ \langle 1 \\ 0 \\ 0 \rangle & \langle 0.8 \\ 0 \\ 0 \rangle & \langle 0.5 \\ 0.1 \\ 0.2 \rangle \end{bmatrix} \quad \text{and} \quad [b_{ij}] = \begin{bmatrix} \langle 0.6 \\ 0.2 \\ 0.1 \rangle & \langle 0.7 \\ 0.2 \\ 0 \rangle & \langle 0.5 \\ 0.4 \\ 0.1 \rangle \\ \langle 0 \\ 1 \\ 1 \rangle & \langle 1 \\ 0 \\ 0 \rangle & \langle 0.1 \\ 0.8 \\ 0.1 \rangle \\ \langle 0.4 \\ 0.3 \\ 0.3 \rangle & \langle 0.1 \\ 0.3 \\ 0.2 \rangle & \langle 1 \\ 0 \\ 0 \rangle \end{bmatrix}$$

Then,

$$[a_{ij}] \cup [b_{ij}] = \begin{bmatrix} \langle 0.6 \\ 0.2 \\ 0.1 \rangle & \langle 0.7 \\ 0.2 \\ 0 \rangle & \langle 1 \\ 0 \\ 0 \rangle \\ \langle 0.1 \\ 0.5 \\ 0.2 \rangle & \langle 1 \\ 0 \\ 0 \rangle & \langle 0.9 \\ 0 \\ 0.1 \rangle \\ \langle 1 \\ 0 \\ 0 \rangle & \langle 0.6 \\ 0 \\ 0 \rangle & \langle 1 \\ 0 \\ 0 \rangle \end{bmatrix} \quad \text{and} \quad [a_{ij}] \cap [b_{ij}] = \begin{bmatrix} \langle 0.2 \\ 0.3 \\ 0.5 \rangle & \langle 0 \\ 1 \\ 1 \rangle & \langle 0.5 \\ 0.4 \\ 0.1 \rangle \\ \langle 0 \\ 1 \\ 1 \rangle & \langle 0.6 \\ 0.4 \\ 0 \rangle & \langle 0.1 \\ 0.8 \\ 0.1 \rangle \\ \langle 0.4 \\ 0.3 \\ 0.3 \rangle & \langle 0.1 \\ 0.3 \\ 0.2 \rangle & \langle 0.5 \\ 0.1 \\ 0.2 \rangle \end{bmatrix}$$

Proposition 2. Let $[a_{ij}], [b_{ij}], [c_{ij}] \in PFS_E[U]$. Then,

- i. $[a_{ij}] \cup [a_{ij}] = [a_{ij}]$ and $[a_{ij}] \cap [a_{ij}] = [a_{ij}]$
- ii. $[a_{ij}] \cup \left[\begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix} \right] = [a_{ij}]$ and $[a_{ij}] \cap \left[\begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix} \right] = \left[\begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix} \right]$
- iii. $[a_{ij}] \cup \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix} \right] = \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix} \right]$ and $[a_{ij}] \cap \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix} \right] = [a_{ij}]$
- iv. $[a_{ij}] \cup [b_{ij}] = [b_{ij}] \cup [a_{ij}]$ and $[a_{ij}] \cap [b_{ij}] = [b_{ij}] \cap [a_{ij}]$
- v. $([a_{ij}] \cup [b_{ij}]) \cup [c_{ij}] = [a_{ij}] \cup ([b_{ij}] \cup [c_{ij}])$ and $([a_{ij}] \cap [b_{ij}]) \cap [c_{ij}] = [a_{ij}] \cap ([b_{ij}] \cap [c_{ij}])$
- vi. $[a_{ij}] \cup ([b_{ij}] \cap [c_{ij}]) = ([a_{ij}] \cup [b_{ij}]) \cap ([a_{ij}] \cup [c_{ij}])$ and $[a_{ij}] \cap ([b_{ij}] \cup [c_{ij}]) = ([a_{ij}] \cap [b_{ij}]) \cup ([a_{ij}] \cap [c_{ij}])$

Proof. vi. Let $[a_{ij}], [b_{ij}], [c_{ij}] \in PFS_E[U]$. Then,

$$\begin{aligned} [a_{ij}] \cup ([b_{ij}] \cap [c_{ij}]) &= [a_{ij}] \cup \left[\begin{smallmatrix} \min\{\mu_{ij}^b, \mu_{ij}^c\} \\ \max\{\eta_{ij}^b, \eta_{ij}^c\} \\ \max\{v_{ij}^b, v_{ij}^c\} \end{smallmatrix} \right] \\ &= \left[\begin{smallmatrix} \max\{\mu_{ij}^a, \min\{\mu_{ij}^b, \mu_{ij}^c\}\} \\ \min\{\eta_{ij}^a, \max\{\eta_{ij}^b, \eta_{ij}^c\}\} \\ \min\{v_{ij}^a, \max\{v_{ij}^b, v_{ij}^c\}\} \end{smallmatrix} \right] \\ &= \left[\begin{smallmatrix} \min\{\max\{\mu_{ij}^a, \mu_{ij}^b\}, \max\{\mu_{ij}^a, \mu_{ij}^c\}\} \\ \max\{\min\{\eta_{ij}^a, \eta_{ij}^b\}, \min\{\eta_{ij}^a, \eta_{ij}^c\}\} \\ \max\{\min\{v_{ij}^a, v_{ij}^b\}, \min\{v_{ij}^a, v_{ij}^c\}\} \end{smallmatrix} \right] \\ &= \left[\begin{smallmatrix} \max\{\mu_{ij}^a, \mu_{ij}^b\} \\ \min\{\eta_{ij}^a, \eta_{ij}^b\} \\ \min\{v_{ij}^a, v_{ij}^b\} \end{smallmatrix} \right] \cap \left[\begin{smallmatrix} \max\{\mu_{ij}^a, \mu_{ij}^c\} \\ \min\{\eta_{ij}^a, \eta_{ij}^c\} \\ \min\{v_{ij}^a, v_{ij}^c\} \end{smallmatrix} \right] \\ &= ([a_{ij}] \cup [b_{ij}]) \cap ([a_{ij}] \cup [c_{ij}]) \end{aligned}$$

The proof of $[a_{ij}] \cap ([b_{ij}] \cup [c_{ij}]) = ([a_{ij}] \cap [b_{ij}]) \cup ([a_{ij}] \cap [c_{ij}])$ is similar to the aforementioned proof. In addition, the proofs of i-v are straightforward. \square

Definition 28. Let $[a_{ij}], [b_{ij}], [c_{ij}] \in PFS_E[U]$. For all i and j , if $\mu_{ij}^c = \min\{\mu_{ij}^a, \nu_{ij}^b\}$, $\eta_{ij}^c = \max\{\eta_{ij}^a, 1 - \eta_{ij}^b\}$, and $\nu_{ij}^c = \max\{\nu_{ij}^a, \mu_{ij}^b\}$, then $[c_{ij}]$ is called difference between $[a_{ij}]$ and $[b_{ij}]$ and denoted by $[a_{ij}] \setminus [b_{ij}]$.

Proposition 3. Let $[a_{ij}] \in PFS_E[U]$. Then,

- i. $[a_{ij}] \setminus \left[\begin{matrix} 0 \\ 1 \\ 1 \end{matrix} \right] = [a_{ij}]$
- ii. $[a_{ij}] \setminus \left[\begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \right] = \left[\begin{matrix} 0 \\ 1 \\ 1 \end{matrix} \right]$

Proof. The proofs of *i* and *ii* are straightforward. \square

Remark 4. It must be emphasized that the difference operation herein is non-commutative and non-associative.

Definition 29. Let $[a_{ij}], [b_{ij}] \in PFS_E[U]$. For all i and j , if $\mu_{ij}^b = \nu_{ij}^a$, $\eta_{ij}^b = 1 - \eta_{ij}^a$, and $\nu_{ij}^b = \mu_{ij}^a$, then $[b_{ij}]$ is complement of $[a_{ij}]$ and denoted by $[a_{ij}]^{\tilde{c}}$ or $[a_{ij}^{\tilde{c}}]$. It is clear that $[a_{ij}]^{\tilde{c}} = \left[\begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \right] \setminus [a_{ij}]$.

Proposition 4. Let $[a_{ij}], [b_{ij}] \in PFS_E[U]$. Then,

- i. $([a_{ij}]^{\tilde{c}})^{\tilde{c}} = [a_{ij}]$
- ii. $\left[\begin{matrix} 0 \\ 1 \\ 1 \end{matrix} \right]^{\tilde{c}} = \left[\begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \right]$
- iii. $[a_{ij}] \setminus [b_{ij}] = [a_{ij}] \tilde{\cap} [b_{ij}]^{\tilde{c}}$
- iv. $[a_{ij}] \tilde{\subseteq} [b_{ij}] \Rightarrow [b_{ij}]^{\tilde{c}} \tilde{\subseteq} [a_{ij}]^{\tilde{c}}$

Proof. The proofs of *i-iv* are straightforward. \square

Proposition 5. Let $[a_{ij}], [b_{ij}] \in PFS_E[U]$. Then, the following De Morgan’s laws are valid.

- i. $([a_{ij}] \tilde{\cup} [b_{ij}])^{\tilde{c}} = [a_{ij}]^{\tilde{c}} \tilde{\cap} [b_{ij}]^{\tilde{c}}$
- ii. $([a_{ij}] \tilde{\cap} [b_{ij}])^{\tilde{c}} = [a_{ij}]^{\tilde{c}} \tilde{\cup} [b_{ij}]^{\tilde{c}}$

Proof. *i.* Let $[a_{ij}], [b_{ij}] \in PFS_E[U]$. Then,

$$\begin{aligned}
 ([a_{ij}] \tilde{\cup} [b_{ij}])^{\tilde{c}} &= \left[\begin{matrix} \max\{\mu_{ij}^a, \mu_{ij}^b\} \\ \min\{\eta_{ij}^a, \eta_{ij}^b\} \\ \min\{\nu_{ij}^a, \nu_{ij}^b\} \end{matrix} \right]^{\tilde{c}} \\
 &= \left[\begin{matrix} \min\{\nu_{ij}^a, \nu_{ij}^b\} \\ 1 - \min\{\eta_{ij}^a, \eta_{ij}^b\} \\ \max\{\mu_{ij}^a, \mu_{ij}^b\} \end{matrix} \right] \\
 &= \left[\begin{matrix} \min\{\nu_{ij}^a, \nu_{ij}^b\} \\ \max\{1 - \eta_{ij}^a, 1 - \eta_{ij}^b\} \\ \max\{\mu_{ij}^a, \mu_{ij}^b\} \end{matrix} \right] \\
 &= \left[\begin{matrix} \nu_{ij}^a \\ 1 - \eta_{ij}^a \\ \mu_{ij}^a \end{matrix} \right] \tilde{\cap} \left[\begin{matrix} \nu_{ij}^b \\ 1 - \eta_{ij}^b \\ \mu_{ij}^b \end{matrix} \right] \\
 &= [a_{ij}]^{\tilde{c}} \tilde{\cap} [b_{ij}]^{\tilde{c}}
 \end{aligned}$$

The proof of *ii* is similar to the aforementioned proof. \square

Definition 30. Let $[a_{ij}], [b_{ij}], [c_{ij}] \in PFS_E[U]$. For all i and j , if

$$\begin{aligned} \mu_{ij}^c &= \max\left\{\min\{\mu_{ij}^a, \nu_{ij}^b\}, \min\{\mu_{ij}^b, \nu_{ij}^a\}\right\} \\ \eta_{ij}^c &= \min\left\{\max\{\eta_{ij}^a, 1 - \eta_{ij}^b\}, \max\{\eta_{ij}^b, 1 - \eta_{ij}^a\}\right\} \end{aligned}$$

and

$$\nu_{ij}^c = \min\left\{\max\{\nu_{ij}^a, \mu_{ij}^b\}, \max\{\nu_{ij}^b, \mu_{ij}^a\}\right\}$$

then $[c_{ij}]$ is called symmetric difference between $[a_{ij}]$ and $[b_{ij}]$ and denoted $[a_{ij}] \tilde{\Delta} [b_{ij}]$.

Proposition 6. Let $[a_{ij}], [b_{ij}] \in PFS_E[U]$. Then,

- i. $[a_{ij}] \tilde{\Delta} \left[\left\langle \begin{matrix} 0 \\ 1 \\ 1 \end{matrix} \right\rangle \right] = [a_{ij}]$
- ii. $[a_{ij}] \tilde{\Delta} \left[\left\langle \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \right\rangle \right] = [a_{ij}]^c$
- iii. $[a_{ij}] \tilde{\Delta} [b_{ij}] = [b_{ij}] \tilde{\Delta} [a_{ij}]$
- iv. $[a_{ij}] \tilde{\Delta} [b_{ij}] = ([a_{ij}] \tilde{\setminus} [b_{ij}]) \tilde{\cup} ([b_{ij}] \tilde{\setminus} [a_{ij}])$

Proof. iv. Let $[a_{ij}], [b_{ij}] \in PFS_E[U]$. Then,

$$\begin{aligned} [a_{ij}] \tilde{\Delta} [b_{ij}] &= \left[\left\langle \begin{matrix} \max\{\min\{\mu_{ij}^a, \nu_{ij}^b\}, \min\{\mu_{ij}^b, \nu_{ij}^a\}\} \\ \min\{\max\{\eta_{ij}^a, 1 - \eta_{ij}^b\}, \max\{\eta_{ij}^b, 1 - \eta_{ij}^a\}\} \\ \min\{\max\{\nu_{ij}^a, \mu_{ij}^b\}, \max\{\nu_{ij}^b, \mu_{ij}^a\}\} \end{matrix} \right\rangle \right] \\ &= \left[\left\langle \begin{matrix} \min\{\mu_{ij}^a, \nu_{ij}^b\} \\ \max\{\eta_{ij}^a, 1 - \eta_{ij}^b\} \\ \max\{\nu_{ij}^a, \mu_{ij}^b\} \end{matrix} \right\rangle \right] \tilde{\cup} \left[\left\langle \begin{matrix} \min\{\mu_{ij}^b, \nu_{ij}^a\} \\ \max\{\eta_{ij}^b, 1 - \eta_{ij}^a\} \\ \max\{\nu_{ij}^b, \mu_{ij}^a\} \end{matrix} \right\rangle \right] \\ &= ([a_{ij}] \tilde{\setminus} [b_{ij}]) \tilde{\cup} ([b_{ij}] \tilde{\setminus} [a_{ij}]) \end{aligned}$$

The proofs of *i-iii* are similar to the proof mentioned above. \square

Remark 5. It must be emphasized that the symmetric difference operation herein is non-associative.

5. Distance Measures of pfs-Matrices

This section, firstly, defines the concept of metrics over $PFS_E[U]$. One of the significant goals herein is to contribute to *pf*-sets and soft sets theoretically. The other goal is to improve the modeling skill of *pfs*-matrices for classification problems in machine learning owing to the aforementioned theoretical contribution. Throughout this study, let $I_n = \{1, 2, \dots, n\}$.

Definition 31. Let $d : PFS_E[U] \times PFS_E[U] \rightarrow \mathbb{R}$ be a function. Then, d is a metric over $PFS_E[U]$ for all $[a_{ij}], [b_{ij}], [c_{ij}] \in PFS_E[U]$ if d satisfies the following properties,

- i. $d([a_{ij}], [b_{ij}]) = 0 \Leftrightarrow [a_{ij}] = [b_{ij}]$
- ii. $d([a_{ij}], [b_{ij}]) = d([b_{ij}], [a_{ij}])$
- iii. $d([a_{ij}], [b_{ij}]) \leq d([a_{ij}], [c_{ij}]) + d([c_{ij}], [b_{ij}])$

Secondly, Minkowski, Euclidean, and Hamming metrics over $PFS_E[U]$ are propounded. Thereafter, their three properties are investigated.

Proposition 7. The function $d_M^p : PFS_E[U] \times PFS_E[U] \rightarrow \mathbb{R}$ defined by

$$d_M^p([a_{ij}], [b_{ij}]) := \left(\frac{1}{3} \sum_{i=1}^m \sum_{j=1}^n (|\mu_{ij}^a - \mu_{ij}^b|^p + |\eta_{ij}^a - \eta_{ij}^b|^p + |\nu_{ij}^a - \nu_{ij}^b|^p + |\pi_{ij}^a - \pi_{ij}^b|^p) \right)^{\frac{1}{p}}$$

such that $p \in \mathbb{N}^+$ is Minkowski metric over $PFS_E[U]$. Its normalized version, namely normalized Minkowski metric, is defined as follows:

$$\hat{d}_M^p([a_{ij}], [b_{ij}]) := \left(\frac{1}{3mn} \sum_{i=1}^m \sum_{j=1}^n (|\mu_{ij}^a - \mu_{ij}^b|^p + |\eta_{ij}^a - \eta_{ij}^b|^p + |\nu_{ij}^a - \nu_{ij}^b|^p + |\pi_{ij}^a - \pi_{ij}^b|^p) \right)^{\frac{1}{p}}$$

such that $p \in \mathbb{N}^+$.

Specifically, d_M^1 and d_M^2 are Hamming and Euclidean metrics and represented by d_H and d_E , respectively. Moreover, \hat{d}_M^1 and \hat{d}_M^2 are normalized Hamming and Euclidean metrics and are represented by \hat{d}_H and \hat{d}_E , respectively.

Proof. Let $[a_{ij}], [b_{ij}], [c_{ij}] \in PFS_E[U]$ and $p \in \mathbb{N}^+$. Satisfying of d_M^p the conditions *i* and *ii* is straightforward from Definition 31. Then,

$$\begin{aligned} \text{iii. } d_M^p([a_{ij}], [b_{ij}]) &= \left(\frac{1}{3} \sum_{i=1}^m \sum_{j=1}^n (|\mu_{ij}^a - \mu_{ij}^b|^p + |\eta_{ij}^a - \eta_{ij}^b|^p + |\nu_{ij}^a - \nu_{ij}^b|^p + |\pi_{ij}^a - \pi_{ij}^b|^p) \right)^{\frac{1}{p}} \\ &= \left(\frac{1}{3} \sum_{i=1}^m \sum_{j=1}^n (|\mu_{ij}^a - \mu_{ij}^c + \mu_{ij}^c - \mu_{ij}^b|^p + |\eta_{ij}^a - \eta_{ij}^c + \eta_{ij}^c - \eta_{ij}^b|^p \right. \\ &\quad \left. + |\nu_{ij}^a - \nu_{ij}^c + \nu_{ij}^c - \nu_{ij}^b|^p + |\pi_{ij}^a - \pi_{ij}^c + \pi_{ij}^c - \pi_{ij}^b|^p) \right)^{\frac{1}{p}} \\ &\leq \left(\frac{1}{3} \sum_{i=1}^m \sum_{j=1}^n (|\mu_{ij}^a - \mu_{ij}^c|^p + |\mu_{ij}^c - \mu_{ij}^b|^p + |\eta_{ij}^a - \eta_{ij}^c|^p + |\eta_{ij}^c - \eta_{ij}^b|^p \right. \\ &\quad \left. + |\nu_{ij}^a - \nu_{ij}^c|^p + |\nu_{ij}^c - \nu_{ij}^b|^p + |\pi_{ij}^a - \pi_{ij}^c|^p + |\pi_{ij}^c - \pi_{ij}^b|^p) \right)^{\frac{1}{p}} \\ &\leq \left(\frac{1}{3} \sum_{i=1}^m \sum_{j=1}^n (|\mu_{ij}^a - \mu_{ij}^c|^p + |\eta_{ij}^a - \eta_{ij}^c|^p + |\nu_{ij}^a - \nu_{ij}^c|^p + |\pi_{ij}^a - \pi_{ij}^c|^p) \right)^{\frac{1}{p}} \\ &\quad + \left(\frac{1}{3} \sum_{i=1}^m \sum_{j=1}^n (|\mu_{ij}^c - \mu_{ij}^b|^p + |\eta_{ij}^c - \eta_{ij}^b|^p + |\nu_{ij}^c - \nu_{ij}^b|^p + |\pi_{ij}^c - \pi_{ij}^b|^p) \right)^{\frac{1}{p}} \\ &= d_M^p([a_{ij}], [c_{ij}]) + d_M^p([c_{ij}], [b_{ij}]) \end{aligned}$$

Moreover, $0 \leq |\mu_{ij}^a - \mu_{ij}^b| \leq 1, 0 \leq |\eta_{ij}^a - \eta_{ij}^b| \leq 1, 0 \leq |\nu_{ij}^a - \nu_{ij}^b| \leq 1$, and $0 \leq |\pi_{ij}^a - \pi_{ij}^b| \leq 1$ because $0 \leq \mu_{ij}^a, \mu_{ij}^b, \eta_{ij}^a, \eta_{ij}^b, \nu_{ij}^a, \nu_{ij}^b, \pi_{ij}^a, \pi_{ij}^b \leq 1$, for all $i \in I_m$ and $j \in I_n$. Hence,

$$\begin{aligned} 0 &\leq |\mu_{ij}^a - \mu_{ij}^b|^p + |\eta_{ij}^a - \eta_{ij}^b|^p + |\nu_{ij}^a - \nu_{ij}^b|^p + |\pi_{ij}^a - \pi_{ij}^b|^p \\ &\leq |\mu_{ij}^a - \mu_{ij}^b| + |\eta_{ij}^a - \eta_{ij}^b| + |\nu_{ij}^a - \nu_{ij}^b| + |\pi_{ij}^a - \pi_{ij}^b| \\ &\leq |\mu_{ij}^a| + |\mu_{ij}^b| + |\eta_{ij}^a - \eta_{ij}^b| + |\nu_{ij}^a| + |\nu_{ij}^b| + |\pi_{ij}^a| + |\pi_{ij}^b| \end{aligned}$$

$$\begin{aligned}
 &= \mu_{ij}^a + \mu_{ij}^b + |\eta_{ij}^a - \eta_{ij}^b| + \nu_{ij}^a + \nu_{ij}^b + \pi_{ij}^a + \pi_{ij}^b \\
 &= \mu_{ij}^a + \mu_{ij}^b + |\eta_{ij}^a - \eta_{ij}^b| + \nu_{ij}^a + \nu_{ij}^b + (1 - \mu_{ij}^a - \nu_{ij}^a) + (1 - \mu_{ij}^b - \nu_{ij}^b) \\
 &= 2 + |\eta_{ij}^a - \eta_{ij}^b| \\
 &\leq 3
 \end{aligned}$$

Then,

$$\begin{aligned}
 \left(\frac{1}{3mn} \sum_{i=1}^m \sum_{j=1}^n 0\right)^{\frac{1}{p}} &\leq \hat{d}_M^p([a_{ij}], [b_{ij}]) \leq \left(\frac{1}{3mn} \sum_{i=1}^m \sum_{j=1}^n 3\right)^{\frac{1}{p}} \\
 0 &\leq \hat{d}_M^p([a_{ij}], [b_{ij}]) \leq \left(\frac{1}{3mn} 3mn\right)^{\frac{1}{p}} \\
 0 &\leq \hat{d}_M^p([a_{ij}], [b_{ij}]) \leq 1
 \end{aligned}$$

□

Proposition 8. Let $\left[\begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix}\right]_{m \times n}, \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right]_{m \times n} \in PFS_E[U]$ and $p \in \mathbb{N}^+$. Then,

$$d_M^p\left(\left[\begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right]\right) = \sqrt[p]{mn} \quad \text{and} \quad \hat{d}_M^p\left(\left[\begin{smallmatrix} 0 \\ 1 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right]\right) = 1$$

Proof. The proof is straightforward. □

Proposition 9. Let $[a_{ij}]_{m \times n}, [b_{ij}]_{m \times n} \in PFS_E[U]$ and $p \in \mathbb{N}^+$. Then, $d_M^p([a_{ij}], [b_{ij}]) \leq \sqrt[p]{mn}$.

Proof. The proof is straightforward. □

Proposition 10. Let $[a_{ij}]_{m \times n}, [b_{ij}]_{m \times n}, [c_{ij}]_{m \times n} \in PFS_E[U]$ and $p \in \mathbb{N}^+$. Then,

- i. $[a_{ij}] \tilde{\subseteq} [b_{ij}] \tilde{\subseteq} [c_{ij}] \Rightarrow \left(d_M^p([a_{ij}], [b_{ij}]) \leq d_M^p([a_{ij}], [c_{ij}]) \wedge d_M^p([b_{ij}], [c_{ij}]) \leq d_M^p([a_{ij}], [c_{ij}])\right)$
- ii. $[a_{ij}] \tilde{\subseteq} [b_{ij}] \tilde{\subseteq} [c_{ij}] \Rightarrow \left(\hat{d}_M^p([a_{ij}], [b_{ij}]) \leq \hat{d}_M^p([a_{ij}], [c_{ij}]) \wedge \hat{d}_M^p([b_{ij}], [c_{ij}]) \leq \hat{d}_M^p([a_{ij}], [c_{ij}])\right)$

Proof. The proofs of *i* and *ii* are straightforward. □

6. Picture Fuzzy Soft *k*-Nearest Neighbor Classifier: PFS-*k*NN

In this section, firstly, the basic expressions and notations to be required for the suggested PFS-*k*NN based on *pfs*-matrices are provided. Throughout the paper, let $D = [d_{ij}]_{m \times (n+1)}$ represent a data matrix. The last column of D consists of class labels of the data. Here, m and n are the numbers of samples and attributes in D , respectively. Moreover, let $(D_{train})_{m_1 \times n}, C_{m_1 \times 1}$, and $(D_{test})_{m_2 \times n}$ derived from attained D denote a training matrix, class matrix of the training matrix, and the testing matrix, respectively, such that $m_1 + m_2 = m$. Moreover, let $U_{k \times 1}$ be a matrix comprising of unique class labels of $C_{m_1 \times 1}$. Further, let $D_{i-train}$ and D_{i-test} represent *i*th rows of D_{train} and D_{test} , respectively. In a similar manner, $D_{train-j}$ and D_{test-j} represent *j*th rows of D_{train} and D_{test} , respectively. Furthermore, let $T'_{m_2 \times 1}$ stand for the predicted classes of the testing queries.

Definition 32. Let $u \in \mathbb{R}^n$. Then, the vector $\hat{u} \in \mathbb{R}^n$ such that $j \in I_n$ defined by

$$\hat{u}_j := \begin{cases} \frac{u_j - \min_{k \in I_n} \{u_k\}}{\max_{k \in I_n} \{u_k\} - \min_{k \in I_n} \{u_k\}}, & \max_{k \in I_n} \{u_k\} \neq \min_{k \in I_n} \{u_k\} \\ 1, & \max_{k \in I_n} \{u_k\} = \min_{k \in I_n} \{u_k\} \end{cases}$$

is called normalized u , i.e., normalizing vector of u .

Definition 33. Consider the training matrix $(D_{train})_{m_1 \times n}$ attained from $D = [d_{ij}]_{m \times (n+1)}$, $i \in I_{m_1}$, and $j \in I_n$. Then, the matrix defined by

$$\tilde{d}_{ij-train} := \begin{cases} \frac{d_{ij-train} - \min_{k \in I_m} \{d_{kj}\}}{\max_{k \in I_m} \{d_{kj}\} - \min_{k \in I_m} \{d_{kj}\}}, & \max_{k \in I_m} \{d_{kj}\} \neq \min_{k \in I_m} \{d_{kj}\} \\ 1, & \max_{k \in I_m} \{d_{kj}\} = \min_{k \in I_m} \{d_{kj}\} \end{cases}$$

is called feature-fuzzification matrix of D_{train} , namely column normalized matrix of D_{train} , and it is denoted by $\tilde{D}_{train} = [\tilde{d}_{ij-train}]_{m_1 \times n}$.

Definition 34. Consider the testing matrix $(D_{test})_{m_2 \times n}$ attained from $D = [d_{ij}]_{m \times (n+1)}$, $i \in I_{m_2}$, and $j \in I_n$. Then, the matrix defined by

$$\tilde{d}_{ij-test} := \begin{cases} \frac{d_{ij-test} - \min_{k \in I_m} \{d_{kj}\}}{\max_{k \in I_m} \{d_{kj}\} - \min_{k \in I_m} \{d_{kj}\}}, & \max_{k \in I_m} \{d_{kj}\} \neq \min_{k \in I_m} \{d_{kj}\} \\ 1, & \max_{k \in I_m} \{d_{kj}\} = \min_{k \in I_m} \{d_{kj}\} \end{cases}$$

is called feature-fuzzification matrix of D_{test} , namely column normalized matrix of D_{test} , and it is denoted by $\tilde{D}_{test} = [\tilde{d}_{ij-test}]_{m_2 \times n}$.

Definition 35. Let $\tilde{D}_{train} = [\tilde{d}_{ij-train}]_{m_1 \times n}$ be a feature-fuzzification matrix of $(D_{train})_{m_1 \times n}$. Then, the matrix

$$\tilde{D}_{train}^\lambda = [\tilde{d}_{train-ij}^\lambda] = \left[\left\langle \begin{matrix} \mu_{ij-train}^{\tilde{D}^\lambda} \\ \eta_{ij-train}^{\tilde{D}^\lambda} \\ \nu_{ij-train}^{\tilde{D}^\lambda} \end{matrix} \right\rangle \right]_{m_1 \times n}$$

is called feature picture fuzzification of \tilde{D}_{train} and is defined by

$$\mu_{ij-train}^{\tilde{D}^\lambda} := 1 - (1 - \tilde{d}_{ij-train})^\lambda, \quad \eta_{ij-train}^{\tilde{D}^\lambda} := \frac{\tilde{d}_{ij-train}}{\lambda}, \quad \text{and} \quad \nu_{ij-train}^{\tilde{D}^\lambda} := (1 - \tilde{d}_{ij-train})^{\lambda(\lambda+1)}$$

such that $i \in I_{m_1}$, $j \in I_n$, and $\lambda \in [0, \infty)$.

Definition 36. Let $\tilde{D}_{test} = [\tilde{d}_{ij-test}]_{m_2 \times n}$ be a feature-fuzzification matrix of $(D_{test})_{m_2 \times n}$. Then, the matrix

$$\tilde{D}_{test}^\lambda = [\tilde{d}_{test-ij}^\lambda] = \left[\left\langle \begin{matrix} \mu_{ij-test}^{\tilde{D}^\lambda} \\ \eta_{ij-test}^{\tilde{D}^\lambda} \\ \nu_{ij-test}^{\tilde{D}^\lambda} \end{matrix} \right\rangle \right]_{m_2 \times n}$$

is called feature picture fuzzification of \tilde{D}_{test} and is defined by

$$\mu_{ij-test}^{\tilde{D}^\lambda} := 1 - (1 - \tilde{d}_{ij-test})^\lambda, \quad \eta_{ij-test}^{\tilde{D}^\lambda} := \frac{\tilde{d}_{ij-test}}{\lambda}, \quad \text{and} \quad \nu_{ij-test}^{\tilde{D}^\lambda} := (1 - \tilde{d}_{ij-test})^{\lambda(\lambda+1)}$$

such that $i \in I_{m_2}$, $j \in I_n$, and $\lambda \in [0, \infty)$.

Definition 37. Let $(\tilde{D}_{train})_{m_1 \times n}$ be a feature-fuzzification matrix of $(D_{train})_{m_1 \times n}$ and $\tilde{D}_{train}^\lambda = [\tilde{a}_{train-ij}^\lambda] = \left[\left\langle \begin{matrix} \mu_{ij}^{\tilde{D}_{train}^\lambda} \\ \eta_{ij}^{\tilde{D}_{train}^\lambda} \\ \nu_{ij}^{\tilde{D}_{train}^\lambda} \end{matrix} \right\rangle \right]_{m_1 \times n}$ be the picture fuzzification of \tilde{D}_{train} . Then, the pfs-matrix $\left[b_{ij}^{\tilde{D}_{train}^\lambda} \right]_{1 \times n}$ is the training pfs-matrix attained by k^{th} row of $\tilde{D}_{train}^\lambda$ and is defined by $b_{1j}^{\tilde{D}_{train}^\lambda} := \left\langle \begin{matrix} \mu_{kj}^{\tilde{D}_{train}^\lambda} \\ \eta_{kj}^{\tilde{D}_{train}^\lambda} \\ \nu_{kj}^{\tilde{D}_{train}^\lambda} \end{matrix} \right\rangle$ such that $k \in I_{m_1}$ and $j \in I_n$.

Definition 38. Let $(\tilde{D}_{test})_{m_2 \times n}$ be a feature-fuzzification matrix of $(D_{test})_{m_2 \times n}$ and $\tilde{D}_{test}^\lambda = [\tilde{a}_{test-ij}^\lambda] = \left[\left\langle \begin{matrix} \mu_{ij}^{\tilde{D}_{test}^\lambda} \\ \eta_{ij}^{\tilde{D}_{test}^\lambda} \\ \nu_{ij}^{\tilde{D}_{test}^\lambda} \end{matrix} \right\rangle \right]_{m_2 \times n}$ be the picture fuzzification of \tilde{D}_{test} . Then, the pfs-matrix $\left[a_{ij}^{\tilde{D}_{test}^\lambda} \right]_{1 \times n}$ is called the testing pfs-matrix attained by k^{th} row of \tilde{D}_{test}^λ and is defined by $a_{1j}^{\tilde{D}_{test}^\lambda} := \left\langle \begin{matrix} \mu_{kj}^{\tilde{D}_{test}^\lambda} \\ \eta_{kj}^{\tilde{D}_{test}^\lambda} \\ \nu_{kj}^{\tilde{D}_{test}^\lambda} \end{matrix} \right\rangle$ such that $k \in I_{m_1}$ and $j \in I_n$.

Secondly, a new classifier named PFS-kNN employing the Minkowski metric of pfs-matrices is suggested (Algorithm 1). Pseudocode of the proposed PFS-kNN is presented in Algorithm 1. In Line 1, it obtains feature fuzzification of testing and training matrices required for feature picture fuzzification. In Line 2, the feature picture fuzzification of testing and training matrices utilizing their feature fuzzification versions. The aim herein is to make the data ready in a way that can be used in the distance calculation of pfs-matrices. In Lines 3–4, the i th testing pfs-matrix is constructed by extracting i th sample from the feature picture fuzzification of the testing matrix. Similarly, in Lines 5–6, the j th training pfs-matrix is constructed by extracting j th sample from the feature picture fuzzification of the training matrix. In Line 7, the distance between the i th test sample and the j th training sample is calculated utilizing the Minkowski metric over the pfs-matrices in accordance with Proposition 7, and Dm_{j1} is attained. In Line 9, k -nearest neighbor according to the matrix of picture fuzzy soft distances, namely Dm_{j1} , is determined. In Line 10, the most repetitive class label (predicted class label) of the determined k -nearest neighbor is obtained. In Line 11, the predicted class label, particularly diagnosis label in medical diagnosis, is assigned to the test sample. In Line 12–13, finally, the predicted label (class) matrix is created for the test queries.

Algorithm 1 PFS-kNN’s pseudocode

Input: $(D_{train})_{m_1 \times n}$, $C_{m_1 \times 1}$, $(D_{test})_{m_2 \times n}$, k , λ , and p

Output: $T'_{m_2 \times 1}$

PFS-kNN(\tilde{D}_{train} , C , D_{test} , k , λ , p)

- 1: Calculate feature fuzzification of D_{test} and D_{train} , i.e., \tilde{D}_{test} and \tilde{D}_{train} ▷ See Definition 33 and 34
- 2: Calculate feature picture fuzzification of \tilde{D}_{test} and \tilde{D}_{train} , i.e., \tilde{D}_{test}^λ and $\tilde{D}_{train}^\lambda$ ▷ See Definition 35 and 36
- 3: **for** i from 1 to m_2 **do**
- 4: Calculate the testing pfs-matrix $\left[a_{ij}^{\tilde{D}_{test}^\lambda} \right]_{1 \times n}$ employing $\tilde{D}_{i-test}^\lambda$
- 5: **for** j from 1 to m_1 **do**
- 6: Calculate the training pfs-matrix $\left[b_{ij}^{\tilde{D}_{train}^\lambda} \right]_{1 \times n}$ employing $\tilde{D}_{j-train}^\lambda$
- 7: $Dm_{j1} \leftarrow d_M^p \left(\left[a_{ij}^{\tilde{D}_{test}^\lambda} \right], \left[b_{ij}^{\tilde{D}_{train}^\lambda} \right] \right)$ ▷ See Proposition 7
- 8: **end for**
- 9: Find k -nearest neighbor using $[Dm_{j1}]$
- 10: Find the most repetitive class label in the considered k -nearest neighbor
- 11: $t'_{k1} \leftarrow$ most repetitive class label (predicted class label)
- 12: **end for**
- 13: **return** $T'_{m_2 \times 1}$

7. Application of PFS-kNN to Medical Diagnosis

In this section, firstly, details of the datasets used in simulation and the setting of the compared classifiers are provided according to the methodology presented in Figure 4. Afterward, the performance metrics for classification problems are introduced. Finally, simulation results for several medical datasets in UC Irvine Machine Learning Repository (UCI-MLR) [35] are presented, and the discussion of the results are provided.

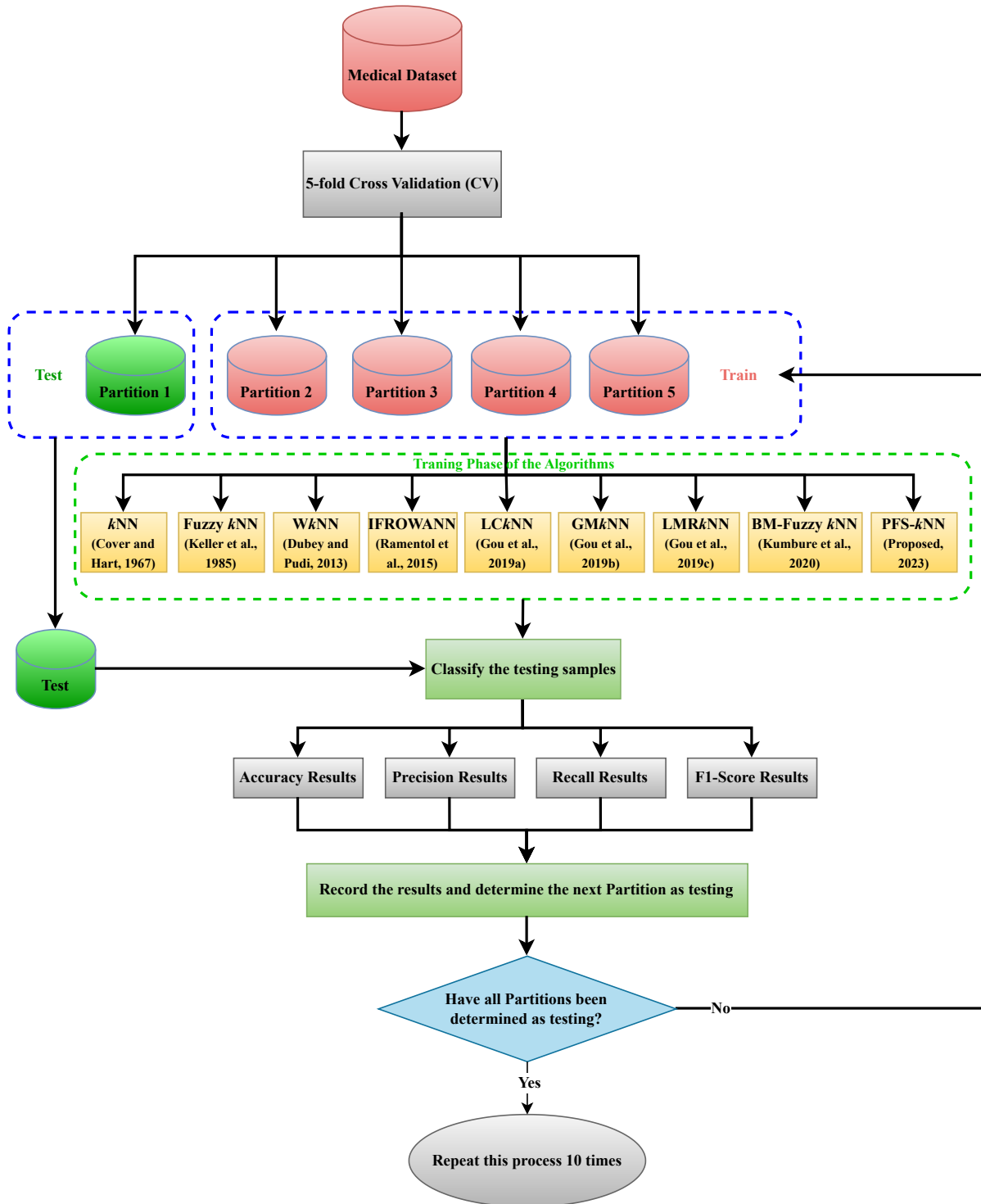


Figure 4. Simulation methodology of the present study via kNN-based classifiers [36–43].

7.1. Medical Datasets

One of the major motivations of this paper is the applicability of PFS- k NN in medical diagnosis. Therefore, the well-known and commonly used four medical diagnosis datasets in UCI-MLR [35] were chosen. This subsection offers descriptions of the following medical datasets employed in the simulation, provided in Table 1: “Breast Tissue”, “Parkinsons[sic]”, “Breast Cancer Wisconsin”, and “Indian Liver”.

Breast Tissue [35]: This dataset measured impedance frequencies: 15.625, 31.25, 62.5, 125, 250, 500, and 1000 KHz. The aforesaid frequencies were used to test the impedance of freshly removed breast tissue. The impedance spectrum is formed by these data plotted in the (actual, imaginary) plane, from which the features of the breast tissue are calculated. The dataset can be used to predict the categorization of either the original six classes or four classes by combining the mastopathy, fibro-adenoma, and glandular types whose distinction is unnecessary (they cannot be differentiated accurately).

Parkinsons[sic] [35]: The dataset consists of a range of biological voice measurements from 31 patients, 23 of whom have Parkinson’s disease. Each column in the dataset stands for a separate vocal measure, and each row corresponds to one of these people’s 195 voice recordings (“name” column). The major purpose of the data is to differentiate between healthy and Parkinson’s disease patients by utilizing the “status” column, which is set to 0 for healthy and 1 for Parkinson’s disease patients.

Breast Cancer Wisconsin (Diagnostic) [35]: This dataset uses a digitized picture of a fine needle aspirate (FNA) of a breast mass to construct characteristics. They describe the characteristics of the cell nuclei shown in the photograph. The separation plane mentioned above was created using the Multisurface approach-Tree (MSM-T), a classification approach that constructs a decision tree using linear programming [44]. To locate relevant features, an exhaustive search in the space of 1–4 features and 1–3 separation planes was utilized. The exact linear program used to obtain the separation plane in 3-dimensional space is described in [45].

Indian Liver Patient (ILPD) [35]: This data collection contains 416 records for liver patients and 167 for non-liver patients. The dataset was gathered in the northeastern state of Andhra Pradesh, India. The selector is a class label categorizing people (liver sick or not). This data collection has 441 male and 142 female patients records. Any patient over the age of 89 is labeled as “90”.

Table 1. Properties of several medical datasets in UCI.

| No. | Name | Instance Number | Attribute Number | Class Number | Imbalance |
|-----|-------------------------|-----------------|------------------|--------------|-----------|
| 1 | Breast Tissue | 106 | 9 | 6 | ✓ |
| 2 | Parkinsons[sic] | 195 | 22 | 2 | ✓ |
| 3 | Breast Cancer Wisconsin | 569 | 30 | 2 | ✓ |
| 4 | Indian Liver | 583 | 10 | 2 | ✓ |

7.2. Quality Metrics for Classification Performance

In this subsection, the mathematical expressions of the quality metrics for binary and multi classification [46], i.e., Accuracy, Precision, Sensitivity (or Recall), and F1-Score, are presented to make a comparison of the considered classifiers. Assume that $D_{test} = \{y_1, y_2, \dots, y_n\}$ is n queries to be classified, $T = \{t_1, t_2, \dots, t_n\}$ is their ground truth class sets, $T' = \{t'_1, t'_2, \dots, t'_n\}$ is their prediction class sets, and l is their number of the class. The quality metrics for binary classification are as follows:

$$\text{Accuracy}(T, T') := \frac{TP + TN}{TP + TN + FP + FN}$$

$$\text{Precision}(T, T') := \frac{TP}{TP + FP}$$

$$\text{Recall}(T, T') := \frac{TP}{TP + FN}$$

$$\text{F1-Score}(T, T') := \frac{2TP}{2TP + FP + FN}$$

where true positive (TP), true negative (TN), false positive (FP), and false negative (FN) are defined as follows:

$$TP := \left| \left\{ y_j | 1 \in T_j \wedge 1 \in T'_j, 1 \leq j \leq l \right\} \right|$$

$$TN := \left| \left\{ y_j | 0 \notin T_j \wedge 0 \notin T'_j, 1 \leq j \leq l \right\} \right|$$

$$FP := \left| \left\{ y_j | 0 \notin T_j \wedge 1 \in T'_j, 1 \leq j \leq l \right\} \right|$$

$$FN := \left| \left\{ y_j | 1 \in T_j \wedge 0 \notin T'_j, 1 \leq j \leq l \right\} \right|$$

such that $|\cdot|$ stands for the cardinality of a set.

The performance metrics for multi classification are as follows:

$$\text{Accuracy}(T, T') := \frac{1}{l} \sum_{i=1}^l \frac{TP_i + TN_i}{TP_i + TN_i + FP_i + FN_i}$$

$$\text{Precision}(T, T') := \frac{1}{l} \sum_{i=1}^l \frac{TP_i}{TP_i + FP_i}$$

$$\text{Recall}(T, T') := \frac{1}{l} \sum_{i=1}^l \frac{TP_i}{TP_i + FN_i}$$

$$\text{F1-Score}(T, T') := \frac{1}{l} \sum_{i=1}^l \frac{2TP_i}{2TP_i + FP_i + FN_i}$$

where i th true positive (TP_i), i th true negative (TN_i), i th false positive (FP_i), and i th false negative (FN_i) for the class i are defined as follows:

$$TP_i := \left| \left\{ x_j | i \in T_j \wedge i \in T'_j, 1 \leq k \leq l \right\} \right|$$

$$TN_i := \left| \left\{ x_j | i \notin T_j \wedge i \notin T'_j, 1 \leq k \leq l \right\} \right|$$

$$FP_i := \left| \left\{ x_j | i \notin T_j \wedge i \in T'_j, 1 \leq k \leq l \right\} \right|$$

$$FN_i := \left| \left\{ x_j | i \in T_j \wedge i \notin T'_j, 1 \leq k \leq l \right\} \right|$$

such that $|\cdot|$ stands for the cardinality of a set.

7.3. Diagnosis Results for Medical Diagnosis

In this subsection, the comparison of PFS- k NN with the well-known and state-of-the-art k NN-based classifiers (Table 2), i.e., k NN [36], Fuzzy k NN [37], WkNN [38], IFROWANN [39], LCkNN [40], GMkNN [41], LMRkNN [42], and BM-Fuzzy k NN [43], is performed by employing a computer with I(R)Core(TM) I5-4200H CPU@2.80GHz and 8 GB RAM and MATLAB R2021b software. Random 10 runs rely on the five-fold cross-validation (CV) [47,48], generating the classifiers' performance results in which each CV, of which four parts are

selected for training and the other for testing (for more details about CV, see [47]), randomly split the considered dataset into five parts. Table 3 presents the average Accuracy, Precision, Recall, and F1-Score results of PFS-*k*NN, *k*NN, Fuzzy *k*NN, *Wk*NN, IFROWANN, *Lck*NN, *GMk*NN, *LMRk*NN, and BM-Fuzzy *k*NN for the datasets.

Table 2. Details of the *k*NN-based classifier.

| Ref. | Year | Classifier | Number of Nearest Neighbors Fixed | Nearest Neighbors Adaptive | Crisp | Employed-Fuzzy Set | Concept- <i>fr</i> -Set | <i>pfs</i> -Matrix | Distance | Inverse Distance | Class Distribution Impact | Class Imbalance Impact |
|----------|------|----------------------|-----------------------------------|----------------------------|-------|--------------------|-------------------------|--------------------|----------|------------------|---------------------------|------------------------|
| [36] | 1967 | <i>k</i> NN | ✓ | | ✓ | | | | ✓ | | | |
| [37] | 1985 | Fuzzy <i>k</i> NN | ✓ | | ✓ | ✓ | | | | ✓ | | |
| [38] | 2013 | <i>Wk</i> NN | ✓ | | ✓ | | | | | | ✓ | |
| [39] | 2015 | IFROWANN | | ✓ | | | ✓ | | | | | |
| [40] | 2019 | <i>Lck</i> NN | ✓ | | ✓ | | | | ✓ | | | ✓ |
| [41] | 2019 | <i>GMk</i> NN | ✓ | | | | | | ✓ | | ✓ | |
| [42] | 2019 | <i>LMRk</i> NN | ✓ | | | | | | ✓ | | ✓ | |
| [43] | 2020 | BM-Fuzzy <i>k</i> NN | ✓ | | | ✓ | | | | ✓ | | |
| Proposed | 2023 | PFS- <i>k</i> NN | ✓ | | | | | ✓ | ✓ | | | |

Table 3. Diagnosis performance results of the *k*NN-based classifiers.

| Medical Datasets | Classifiers | Accuracy | Precision | Recall | F1-Score |
|------------------|----------------------|--------------|--------------|--------------|--------------|
| Breast Tissue | <i>k</i> NN | 86.59 | 61.12 | 57.98 | 62.94 |
| | Fuzzy <i>k</i> NN | 85.34 | 59.32 | 54.39 | 59.18 |
| | <i>Wk</i> NN | 86.72 | 61.24 | 58.59 | 62.87 |
| | IFROWANN | 85.65 | 64.38 | 56.02 | 67.18 |
| | <i>Lck</i> NN | 75.37 | 20.07 | 23.89 | 40.15 |
| | <i>GMk</i> NN | 88.45 | 66.72 | 63.55 | 66.45 |
| | <i>LMRk</i> NN | 84.09 | 54.66 | 51.62 | 58.62 |
| | BM-Fuzzy <i>k</i> NN | 85.57 | 60.82 | 57.79 | 62.09 |
| | PFS- <i>k</i> NN | 88.51 | 65.34 | 60.84 | 67.95 |
| Parkinsons[sic] | <i>k</i> NN | 85.03 | 74.57 | 62.16 | 66.53 |
| | Fuzzy <i>k</i> NN | 84.92 | 73.14 | 64.49 | 67.23 |
| | <i>Wk</i> NN | 84.92 | 73.14 | 64.49 | 67.23 |
| | IFROWANN | 68.26 | 44.18 | 100 | 61.08 |
| | <i>Lck</i> NN | 68.05 | 42.07 | 68.80 | 51.38 |
| | <i>GMk</i> NN | 83.85 | 68.53 | 68.96 | 67.31 |
| | <i>LMRk</i> NN | 68.97 | 42.33 | 70.82 | 52.64 |
| | BM-Fuzzy <i>k</i> NN | 78.10 | 55.91 | 63.20 | 58.42 |
| | PFS- <i>k</i> NN | 91.18 | 89.38 | 73.51 | 79.80 |
| Breast Cancer | <i>k</i> NN | 92.90 | 92.92 | 87.79 | 90.18 |
| | Fuzzy <i>k</i> NN | 92.46 | 92.29 | 87.18 | 89.57 |
| | <i>Wk</i> NN | 92.46 | 92.29 | 87.18 | 89.57 |
| | IFROWANN | 78.07 | 63.25 | 99.53 | 77.27 |
| | <i>Lck</i> NN | 79.03 | 67.13 | 87.98 | 75.98 |
| | <i>GMk</i> NN | 93.09 | 91.60 | 89.91 | 90.64 |
| | <i>LMRk</i> NN | 86.82 | 81.50 | 83.85 | 82.54 |
| | BM-Fuzzy <i>k</i> NN | 91.95 | 91.00 | 87.27 | 88.97 |
| | PFS- <i>k</i> NN | 93.59 | 90.18 | 93.17 | 91.56 |
| Indian Liver | <i>k</i> NN | 66.55 | 75.08 | 79.55 | 77.22 |
| | Fuzzy <i>k</i> NN | 65.97 | 76.15 | 76.20 | 76.12 |
| | <i>Wk</i> NN | 65.97 | 76.15 | 76.20 | 76.12 |
| | IFROWANN | 30.21 | 100 | 2.19 | 4.24 |
| | <i>Lck</i> NN | 67.07 | 75.40 | 79.96 | 77.58 |
| | <i>GMk</i> NN | 67.36 | 77.72 | 76.16 | 76.87 |
| | <i>LMRk</i> NN | 60.93 | 77.33 | 64.04 | 69.97 |
| | BM-Fuzzy <i>k</i> NN | 65.73 | 75.88 | 76.28 | 76.01 |
| | PFS- <i>k</i> NN | 67.46 | 75.57 | 77.98 | 77.69 |

Accuracy, Precision, Recall, and F1-Score results are offered in percentage. The best results are shown in bold.

Based on the results obtained from Accuracy, it is evident that PFS-*k*NN surpasses all other *k*NN-based classifiers that were compared. This is similarly observed when it comes to F1-Score results. However, it should be noted that the proposed approach has lower Precision and Recall results when compared to the other classifiers. Nevertheless, the results are still close to the highest score in general.

These simulation results manifest that *pfs*-matrices and PFS-*k*NN can model uncertainty and real-world problems, such as medical diagnosis and machine learning. It is important to note that applying these models can significantly impact the accuracy of such issues, leading to more reliable and effective solutions. Therefore, using PFS-*k*NN and *pfs*-matrices is recommended when dealing with similar problems.

In this study, we evaluated the Accuracy performance values of various algorithms on four medical datasets. To obtain a comprehensive understanding of the algorithms' performance, we ran each algorithm 50 times (10 times five-fold cross-validation) and plotted the results as box plots in Figure 5.

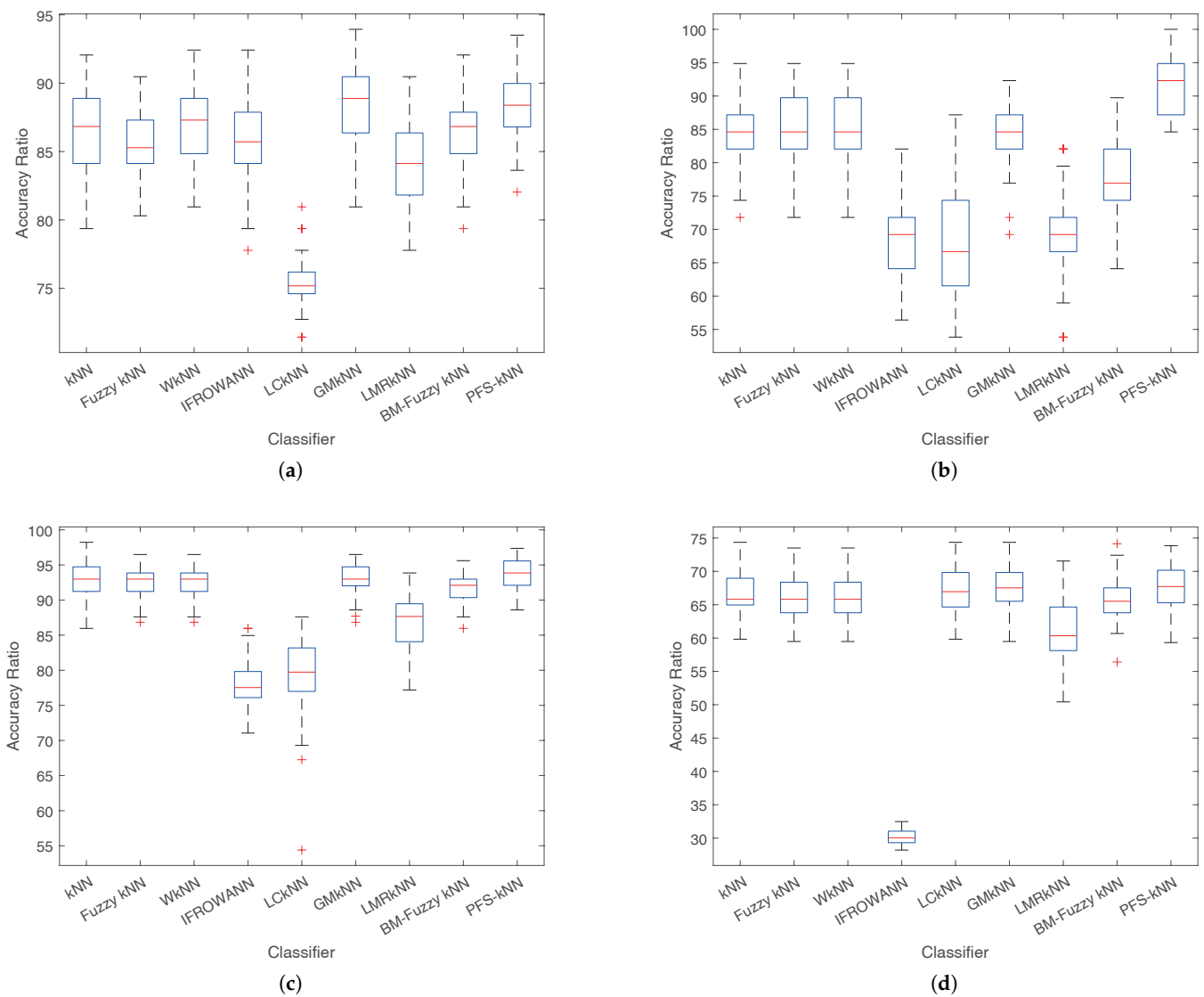


Figure 5. Box plot of Accuracy results of 50 runs for the classifiers: (a) Breast Tissue, (b) Parkinson's, (c) Breast Cancer, (d) Indian Liver.

From the visual results in Figure 5a–d, we can observe that PFS-*k*NN outperforms the other algorithms, with the highest performance value and a performance value distribution that is close to normal distribution. This indicates that PFS-*k*NN is a reliable algorithm for these medical datasets.

Similarly, in Figure 5b, we see that PFS-*k*NN produces the highest performance results, with the 50 performance values almost following a normal distribution. Moreover, the distance between quartiles is relatively low, suggesting that PFS-*k*NN is consistent in performance.

Overall, the box plots in Figure 5 demonstrate that PFS-*k*NN is a superior algorithm compared to the others evaluated in this study, and it is a promising option for medical data analysis.

8. Discussion on PFS-*k*NN in Medical Diagnosis and Supervised Learning

This section discusses the significance of the proposed PFS-*k*NN classifier's performance on medical diagnosis datasets herein.

Accuracy and F1-Score Dominance: The achievement of PFS-*k*NN outperforming all other *k*NN-based classifiers in terms of Accuracy and F1-Score is remarkable. Accuracy measures the overall correctness of the classifier's predictions, while the F1-Score considers both precision and recall. These metrics are crucial in medical diagnosis, where accurately identifying and classifying medical conditions can be a life-or-death matter. The superior performance of PFS-*k*NN in these areas indicates its potential as a valuable tool for enhancing the accuracy and effectiveness of medical diagnoses.

Precision and Recall Trade-Off: While PFS-*k*NN performs well in terms of Accuracy and F1-Score, it is observed to have slightly lower Precision and Recall compared to other classifiers. Precision measures the ratio of correctly predicted positive cases to all predicted positive cases, while Recall measures the ratio of correctly predicted positive cases to all actual positive cases. In medical diagnosis, Precision is vital for minimizing false positive errors, and Recall is crucial for reducing false negatives. The slightly lower Precision and Recall values suggest that PFS-*k*NN might be more cautious when making positive predictions, possibly to reduce false positive errors. However, the results are still close to the highest scores overall, indicating a reasonable balance between these metrics.

Modeling Uncertainty and Real-World Problems: Addressing the concept of *pfs*-matrices and their role in modeling uncertainty in practical scenarios, such as medical diagnosis, is significant. Medical diagnosis frequently deals with intricate and uncertain data, and the capability of PFS-*k*NN to model uncertainty is a valuable advantage. This indicates that the classifier is flexible and resilient in handling various demanding datasets, making it suitable for real-world applications where data are inherently uncertain and noisy.

Impact on Accuracy and Reliability: The practical importance of using PFS-*k*NN and *pfs*-matrices in areas, such as medical diagnosis mentioned in the previous section indicates that they can notably affect accuracy. By enhancing accuracy in medical diagnosis, they can provide more dependable and efficient solutions, decrease misdiagnosis rates, and improve patient outcomes. This emphasizes the potential of PFS-*k*NN to make a valuable contribution to the healthcare industry, where precision and accuracy are crucial.

Recommendation for Similar Problems: The suggestion to utilize PFS-*k*NN and *pfs*-matrices as a conclusion highlights the belief in the effectiveness of this approach. This indicates that the advantages demonstrated in the research are not restricted to the dataset employed for assessment but can also apply to other medical diagnosis scenarios or related fields.

In brief, the performance of the proposed PFS-*k*NN classifier on medical diagnosis datasets, assessed using Minkowski metrics over *pfs*-matrices, demonstrates its potential to enhance the accuracy and dependability of medical diagnoses. While there are some trade-offs in Precision and Recall, the overall superiority in Accuracy and F1-Score, coupled with its capability to model uncertainty, positions PFS-*k*NN as a promising tool for improving healthcare and addressing real-world challenges in supervised learning.

9. Conclusions

This paper redefined the idea of *pfs*-matrices, and their fundamental properties were examined extensively. Afterward, distance measures of *pfs*-matrices were introduced. Then, PFS-*k*NN, via the aforementioned distance measures, was suggested and applied to medical diagnosis. The results manifested that the concept of *pfs*-matrices and the proposed PFS-*k*NN approach can model uncertainty and real-world problems such as medical diagnosis.

The current study, which focuses on soft sets, has significantly contributed to the literature in both theoretical and practical aspects. This study has introduced three crucial additions that redefine the mathematics underlying *pfs*-matrices and proposed new distance measures between *pfs*-matrices and PFS-*k*NN. By doing so, this paper has expanded the understanding of this field and enhanced its applicability in real-world problems. In addition, this research has gained prominence in the literature due to its innovative contributions, which have opened up new avenues for further exploration and research in the field.

In future works, there is potential for further investigation into the algebraic and topological structures of *pfs*-matrices and the exploration of new distance and similarity measures. While *pfs*-matrices have proven effective in addressing specific problems, it is essential to acknowledge their limitations when dealing with picture fuzzy parameters. To overcome this issue, research can be conducted on several related concepts, such as intuitionistic fuzzy parameterized intuitionistic fuzzy soft matrices (*ifpifs*-matrices) [49,50], aggregation operators of *pfs*-matrices [51,52], picture fuzzy parameterized picture fuzzy soft sets (*pfppfs*-sets) [53], and picture fuzzy parameterized picture fuzzy soft matrices (*pfppfs*-matrices). Additionally, interval-valued intuitionistic fuzzy parameterized interval-valued intuitionistic fuzzy soft sets (*d*-sets) [4] and interval-valued intuitionistic fuzzy parameterized interval-valued intuitionistic fuzzy soft matrices (*d*-matrices) [5] are other related concepts that may be worth exploring. We can better understand their potential applications and limitations by studying and applying these concepts to different real-world problems. For instance, different real-world problems, such as trend prediction of component stock [54], remote sensing image fusion [55], and Landsat image fusion [56] can be investigated, and the applications of *pfs*-matrices to them can be focused.

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