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Prescribed Time Fault-Tolerant Affine Formation Control for Multi-Agent Systems with Double-Integrator Dynamics

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Abstract: There is an increasing interest in the affine formation control of multi-agent systems, because it can change the centroid, orientation and scale of the formation by controlling only a few leaders. In this paper, the fault-tolerant affine formation control problem is addressed for double-integrator multi-agent systems with partial loss of efficiency and bias faults. Firstly, in order to track the leaders with dynamically changing accelerations, an acceleration observer with prescribed time convergence is proposed, which can estimate the ideal acceleration for each follower. Then, based on the acceleration observer, a fault-tolerant control algorithm is given. A new Lyapunov function candidate is constructed, based on which a sufficient condition to achieve the control objective is derived. Theoretical analysis shows that the formation tracking error can converge to zero within a prescribed time, and remain in a small neighborhood of zero after that time. Finally, numerical simulations are given to show the effectiveness of the proposed algorithm and compare it with existing results.

Keywords: affine formation control; prescribed-time convergence; multi-agent systems; fault-tolerant control

Citation: Tang, J.; Li, J.; Yang, X.; Du, Z.; Wu Y. Prescribed Time Fault-Tolerant Affine Formation Control for Multi-Agent Systems with Double-Integrator Dynamics. *Electronics* **2024**, *13*, 36. https://doi.org/10.3390/electronics13010036

Academic Editor: Marcin Witczak

Received: 9 November 2023
Revised: 14 December 2023
Accepted: 18 December 2023
Published: 20 December 2023

1. Introduction

The last two decades have witnessed ever-increasing research interests in the cooperative control of multi-agent systems [1–5]. This is mainly due to its wide applications in swarm robotics [6], wireless sensor networks [7], spacecraft systems [8], smart grids [9] and so on. Formation control is a significant research topic in the field of cooperative control of multi-agent systems. Recently, consensus theory has been successfully applied in formation control [10,11]. Different from the behavior-based approach [1], using the consensus-based formation control algorithm, system convergence can be proved mathematically.

Maneuver control is an important subtask of formation control, which aims to steer the formation as a whole, such that the geometric parameters like centroid, orientation and scale can be changed continuously. In order to solve the formation maneuvers problem, some methods have been proposed [12,13]. However, these methods add additional sensing and communication ability requirements to the agents. In [14], a method that can be used to track time-varying formations has been proposed. However, it needs to prescribe the desired maneuver of each agent in advance. Very recently, the affine formation control approach has been proposed in [15] to stabilize stationary target formations. The affine formation control approach defines the target formation using stress matrices. Different from most consensus-based formation control approaches, where the weights on the edges are positive, elements in the stress matrix can be negative. What is more, any translation, rotation and scaling change in the formation can be formulated as affine transformations of the nominal formation. Based on the work in [15], the affine formation maneuver control problem is then studied in [16] for both single-integrator and double-integrator dynamics.

By introducing the concept of affine localization, a leader–follower approach was proposed, which can track time-varying affine transformation of the nominal formation continuously.

Fast convergence is usually needed in the practice of multi-agent systems [17]. Many formation control algorithms with finite-time or fixed-time convergence have been proposed [18–20]. Nevertheless, using the finite-time and fixed-time formation control algorithms, the convergence time is dependent on the initial states. In recent years, the prescribe-time control problem has attracted more and more attention [21]. Using a time-varying feedback gain related to the appointed convergence time, the system can converge to zero within the time appointed by the user. A prescribe-time affine formation control approach has been proposed in [22] for multi-agent systems with single-integrator dynamics. Practical prescribed time affine formation control algorithms are given in [23], for the leaderless and the leader–follower affine formation control problems, respectively. In practice, some agents are modeled as double-integrator dynamics. As far as the multi-agent systems with double-integrator dynamics are concerned, prescribed time approaches have been reported for consensus and formation control problems [24,25].

In practical applications, some agents may be subject to actuator faults. Therefore, it is very important to design fault-tolerant control algorithms. Motivated by this, many researchers consider the fault-tolerant control problem for multi-agent systems [26–31]. Fault-tolerant formation control of multi-agent systems with finite-time convergence or fixed-time convergence has also been considered in [32–34]. The fault-tolerant affine formation control problem is investigated in [35] for heterogeneous multi-agent systems. A neural network-based adaptive fault-tolerant affine formation control algorithm is proposed for the followers, which can drive the formation tracking error to zero exponentially. To the authors' best knowledge, the prescribed time fault-tolerant affine formation control problem for multi-agent systems with double-integrator dynamics has not been addressed in the literature. This motivated this paper.

In this paper, the fault-tolerant affine formation control problem is investigated for double-integrator multi-agent systems with partial loss of efficiency and bias faults. An prescribed time observer-based control algorithm is proposed. Based on a newly designed Lyapunov function candidate, sufficient conditions are derived, under which the control objective is achieved. Theoretical analysis shows that, if the control parameters are properly chosen, the formation tracking error can converge to zero within a prescribed time, and remain in a small neighborhood of zero after that time. The algorithm proposed in this paper can be applied to the formation control of omnidirectional mobile robots. It can also be applied to the position loop of formation controllers for multiple quad-rotors. Compared with the existing results in the literature, the contributions of this paper are as follows:

- The algorithm proposed in this paper can guarantee prescribed time convergence, while the tracking error converges exponentially using the algorithm in [16]. Prescribed convergence time means that users can appoint the convergence time according to their demands. Therefore, using the proposed algorithm, the formation tracking error can converge faster than that in [16] if the designer chose a small convergence time. A comparison of the proposed algorithm with that in [16] is given in Section 4 by a simulation example. The simulation result shows that using controller (20) proposed in this paper, the convergence is faster than that of controller (25) in [16].
- Different from the work in [22], where multi-agent systems with single-integrator dynamics is considered, in this paper, double-integrator dynamics is investigated. It is more difficult to design prescribed-time convergence algorithms for double-integrator multi-agent systems because the system dynamics matrix is more complex than single integrator multi-agent systems. A new Lyapunov function is constructed in this paper, using which some cross-terms cancel each other.
- The fault-tolerant affine formation control problem is considered in this paper, and the work in [22] is a fault-free approach. To the best of the authors' knowledge, the prescribed time fault-tolerant affine formation control problem has not been addressed in the literature yet. When the loss of efficiency and bias faults is considered, the

closed-loop system model is more complex than the fault-free one. To deal with this problem, we add a matrix related to the efficiency of the actuators in the Lyapunov function, which simplified the theoretical analysis.

The rest of this paper is organized as follows: Section 2 gives some basic notations used in the paper, together with the problem formulation. Section 3 provides the main results of the paper, including the prescribed time observer and the observer-based fault-tolerant affine formation control algorithm. Theoretical analysis is also given in Section 3. Simulation results are given in Section 4 and the paper is concluded in Section 5.

Notations: In this paper, $\mathbf{1}_n$ indicates the vector with all entries equal to one. We use sgn, $\bullet$ and $\otimes$ to denote the signum function, the product and the Kronecker product, respectively. For a vector, $diag$ is the diagonal function, $\|\cdot\|_1$, $\|\cdot\|$ and $\|\cdot\|_\infty$ denote the 1-norm, 2-norm and ∞-norm, respectively. Given a positive definite matrix A, $\lambda_{min}(A)$ and $\lambda_{max}(A)$ denote the smallest and largest eigenvalue of A, respectively.

2. Preliminaries and Problem Formulation

2.1. Notations for Graph Theory and Formation

Consider a multi-agent system with N agents moving in $\mathbb{R}^d$. The communication topology among the agents can be described as a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, where $\mathcal{V}=\{1,2,\ldots,N\}$ is the node set and $\mathcal{E}\subset\mathcal{V}\times\mathcal{V}$ is the edge set. A directed edge $(i,j)\in\mathcal{E}$ indicates that agent j's information is available for agent i, but not vise versa. We say that agent j is a neighbor of agent i, if $(i,j)\in\mathcal{E}$. The neighbor set $\mathcal{N}_i$ of agent i is the set $\{j\in\mathcal{V}\mid(i,j)\in\mathcal{E}\}$. An undirected edge $(i,j)\in\mathcal{E}$ indicates that agent j's information is available for agent i; meanwhile, agent i's information is also available for agent j. If all the edges in a graph are undirected, it is an undirected graph. See Figure 1 as an example of a graph with seven nodes. In this graph, the node set is $\{1,2,3,4,5,6,7\}$. We can see that nodes 2 and 4 are connected by a line without arrow. This means that nodes 2 and 4 can obtain information from each other, i.e., there is an undirected edge between them. It is easy to see that there are 12 edges in this graph. Since all edges are undirected, this graph is an undirected graph.

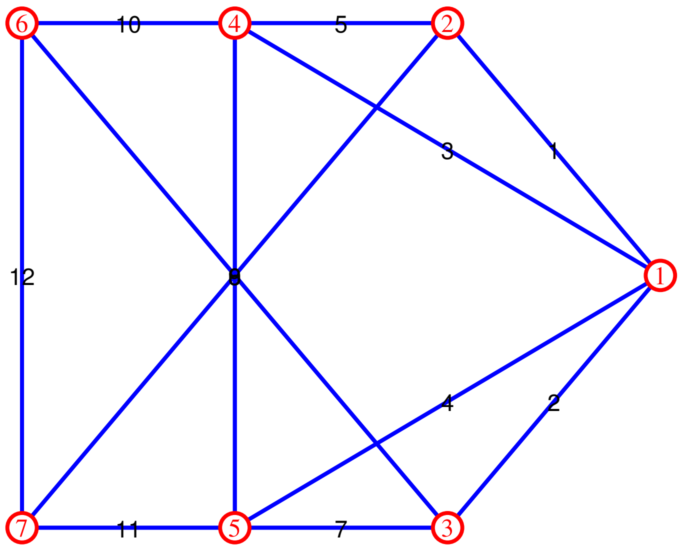

Figure 1. An example of an undirected graph with 7 nodes.

Let $p_i(t)\in\mathbb{R}^d$ be the position of agent i and $p(t)\triangleq[p_1^T(t),\ldots,p_N^T(t)]^T$, the formation can be denoted as $(\mathcal{G},p)$. Assume that N_l agents are selected as leaders, and the rest $N-N_l$ agents are followers. Without loss of generality, let $\mathcal{V}_l=\{1,2,\ldots,N_l\}$ denote the leader set and $\mathcal{V}_f=\mathcal{V}/\mathcal{V}_l$ denote the follower set. The positions of leaders and followers are, respectively, $p_l(t)=[p_1^T(t),\ldots,p_{N_l}^T(t)]^T$ and $p_f(t)=[p_{N_l+1}^T(t),\ldots,p_N^T(t)]^T$.

For a formation $(\mathcal{G},p)$, the stress matrix $\Omega\in\mathbb{R}^{N\times N}$ is defined as

$$[\Omega]_{ij} = \begin{cases} \sum_{j=1}^{N} \omega_{ij}, & i = j, \\ -\omega_{ij}, & i \neq j, (i,j) \in \mathcal{E} \\ 0, & i \neq j, (i,j) \notin \mathcal{E} \end{cases}$$

where ω_{ij} is the weight on edge (i,j) satisfying $\omega_{ij} = \omega_{ji}$. With the leader–follower communication structure, Ω can be divided as

$$\Omega = \begin{bmatrix} \Omega_{ll} & \Omega_{lf} \\ \Omega_{fl} & \Omega_{ff} \end{bmatrix}. \tag{1}$$

2.2. Notations for Affine Formation Control

Let $r = [r_1^T, \ldots, r_N^T]^T = [r_l^T, r_f^T]^T$ be the nominal configuration, where $r_i \in \mathbb{R}^d$ is the nominal position of agent i. The nominal formation of multi-agent systems can be denoted as (G, r). Given a set of points $\{p_i\}_{i=1}^n$, the affine span of these points is defined as

$$S = \left\{ \sum_{i=1}^{n} a_i p_i : a_i \in \mathbb{R}, \sum_{i=1}^{n} a_i p_i = 1 \right\}$$

If the dimension of S is d, we say that $\{p_i\}_{i=1}^n$ affinely span $\mathbb{R}^d$. The affine image of the nominal configuration is defined as [15]

$$\mathcal{A}(r) = \left\{ p \in \mathbb{R}^{dN} : p = (I_N \otimes A)r + 1_N \otimes b, A \in \mathbb{R}^{d \times d}, b \in \mathbb{R}^d \right\}.$$

The objective of affine formation control is to steer the multi-agent system to track the time-varying target formation

$$p^*(t) = (I_N \otimes A(t))r + 1_N \otimes b(t) \tag{2}$$

where $A(t) \in \mathbb{R}^{d \times d}$ and $b(t) \in \mathbb{R}^d$ are continuous of t. It is easy to see that the target formation always belongs to $\mathcal{A}(r)$.

In this paper, we consider multi-agent systems with a leader–follower structure, the objective is to control the whole formation by controlling the leaders. To achieve this objective, it is necessary to make sure that the positions of the followers are uniquely determined by the positions of the leaders. To tackle this problem, we need the following definitions and assumptions.

Definition 1 ([16])**.** *Affine localizability: The nominal formation (G, r) is affinely localizable by the leaders, if for any $p = [p_l^T, p_f^T]^T \in \mathcal{A}(r)$, p_f can be uniquely determined by p_l.*

The following assumptions are essential for affine localizability of the nominal formation.

Assumption 1. *Assume that $\{r_i\}_{i=1}^N$ in the nominal formation (G, r) affinely span $\mathbb{R}^d$.*

Assumption 2. *Assume that $\{r_i\}_{i=1}^{N_l}$ in the nominal formation (G, r) affinely span $\mathbb{R}^d$.*

Lemma 1 ([16])**.** *If Assumptions 1 and 2 are satisfied, the nominal formation (G, r) is affinely localizable.*

At least $d + 1$ points are needed to span $\mathbb{R}^d$, so the number of leaders must satisfy $N_l \geq n + 1$. For more details about affine span, please refer to [16].

Assumption 3. *Assume that the nominal formation (G, r) has a positive definite stress matrix Ω satisfying rank$(\Omega) = N - d - 1$.*

Lemma 2 ([16])**.** *Under Assumptions 1 and 3, the nominal formation (G, r) is affinely localizable if and only if Ω_{ff} is non-singular. When Ω_{ff} is non-singular, for any $p = [p_l^T, p_f^T]^T \in \mathcal{A}(r)$, p_f can be uniquely calculated as $p_f = -(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_d)p_l$.*

Figure 2 gives an example of leader–follower-based affine formation control. As is mentioned above, at least three leaders are needed in $\mathbb{R}^2$. Without loss of generality, we chose agents 1–3 as the leaders, as shown in Figure 2. Suppose that the nominal configuration is

$$r = \begin{bmatrix} 4 & 2 \\ 3 & 3 \\ 3 & 1 \\ 2 & 3 \\ 2 & 1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}.$$

It is easy to see that Assumptions 1 and 2 are satisfied. By the method provided in [16], we can obtain the following stress matrix satisfying Assumption 3

$$\Omega = \begin{bmatrix} 0.2741 & -0.2741 & -0.2741 & 0.1370 & 0.1370 & 0 & 0 \\ -0.2741 & 0.6852 & 0 & -0.5482 & 0 & 0 & 0.1370 \\ -0.2741 & 0 & 0.6852 & 0 & -0.5482 & 0.1370 & 0 \\ 0.1370 & -0.5482 & 0 & 0.7537 & -0.0685 & -0.2741 & 0 \\ 0.1370 & 0 & -0.5482 & -0.0685 & 0.7537 & 0 & -0.2741 \\ 0 & 0 & 0.1370 & -0.2741 & 0 & 0.2741 & -0.1370 \\ 0 & 0.1370 & 0 & 0 & -0.2741 & -0.1370 & 0.2741 \end{bmatrix}.$$

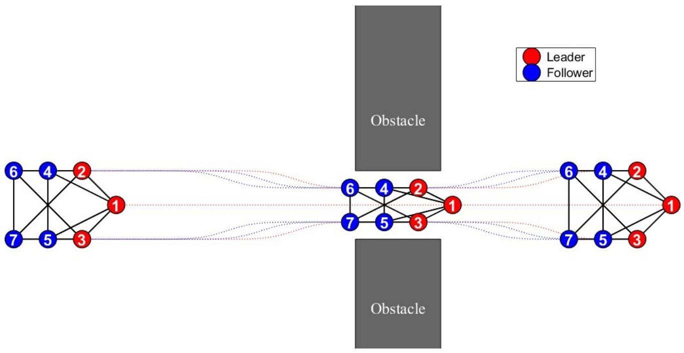

Figure 2. An example of leader–follower-based affine formation control.

We can see that Ω_{ff} is non-singular. By Lemma 2, we have that r_f is affinely localizable by r_l, and the relationship between them is $r_f = -(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_2)r_l$. Assume that $p(t) \in \mathcal{A}(r)$, and by the definition of affine image we have $p_f(t) = -(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_2)p_l(t)$.

The main idea of the leader–follower-based affine formation control is, if we can drive the leaders to satisfy $p_l(t) = (I_N \otimes A(t))r_l + 1_N \otimes b(t)$, and make sure that $p_f(t) = -(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_2)p_l(t)$, then we have $p_f(t) = (I_N \otimes A(t))r_f + 1_N \otimes b(t)$. Therefore, the formation of the multi-agent systems is always in the affine image of the nominal configuration.

The affine image of the nominal configuration includes its moving, rotation, scaling and shear [16]. When the team needs to pass through a narrow passage, as is shown in Figure 2, we can drive the leaders to change their positions according to $p_l(t) = (I_N \otimes A(t))r_l + 1_N \otimes b(t)$. If the control law for the followers is designed to guarantee that $p_f(t) = -(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_2)p_l(t)$, then the formation is kept in the affine image of the nominal configuration, and all agents can pass though the narrow passage with a desired formation.

2.3. Problem Formulation

In this paper, we consider that multi-agent systems consist of N agents with the following double-integrator dynamics:

$$\begin{aligned} \dot{p}_i(t) &= v_i(t) \\ \dot{v}_i(t) &= u_i(t), \quad i \in \mathcal{V}, \end{aligned} \tag{3}$$

where $p_i(t)$, $v_i(t)$ and $u_i(t) \in \mathbb{R}^d$ are, respectively, the position, velocity and control input of agent i.

Actuator failures can not be ignored in practice. With partial loss of efficiency and bias faults, the control input can be written as

$$\tilde{u}_i(t) = \theta_i u_i(t) + \varepsilon_i(t), \tag{4}$$

where $\theta_i \in (0, 1]$ is a constant scalar, $\varepsilon_i(t) \in \mathbb{R}^m$.

Assumption 4. *Assume that* $\| u_i(t) \| < U_i$ *and* $\| \dot{u}_i(t) \|_\infty < d_i$, $i \in \mathcal{V}_l$, *where* $U_i, d_i > 0$ *are positive scalars.*

Similar to [16], we assume that the leaders are well-controlled.

Assumption 5. *Assume that* $p_i(t) = p_i^*(t)$, $v_i(t) = v_i^*(t)$ *and* $u_i(t) = \dot{v}_i^*(t)$, *for* $i \in \mathcal{V}_l$, *where* $p_i^*(t)$, $v_i^*(t)$ *and* $\dot{v}_i^*(t)$ *are objective position, velocity and accelerations of agent* i *in the objective formation.*

From Lemmas 1 and 2 we can see that, if Assumptions 1–3 are satisfied, Ω_{ff} is positive definite, and for any $p(t) = [p_l^T(t), p_f^T(t)]^T \in \mathcal{A}(r)$, $p_f(t)$ can be uniquely calculated as $p_f(t) = -(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_d)p_l(t)$. Therefore, when the leaders are well-controlled, the objective position of the followers is $p_f^*(t) = -(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_d)p_l(t)$. It is easy to see that the ideal velocity of the followers is $v_f^*(t) = -(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_d)v_l(t)$. Define $\delta_p(t) \triangleq p_f(t) - p_f^*(t)$, $\delta_v(t) \triangleq v_f(t) - v_f^*(t)$, and $\delta(t) \triangleq [\delta_p^T(t), \delta_v^T(t)]^T$.

Definition 2. *Fault-tolerant prescribed-time affine formation control: For a given double-integrator multi-agent systems* (3) *subject to actuator faults* (4)*, the objective of fault-tolerant prescribed-time affine formation tracking control is to design a control protocol* $u_i(t)$ *such that* $\delta(t)$ *converge to* $S(\epsilon) = \{\delta : \| \delta \| < \epsilon\}$ *as* $t \to t_0 + T$*, and* $\delta(t)$ *remains in* S *after* $t > t_0 + T$*, where* ϵ *and* T *are positive scalars.*

The following time-varying function is commonly used in prescribed time control [21]:

$$\mu(t) = \begin{cases} \left(\frac{T}{t_0+T-t}\right)^\rho, & t_0 \le t < t_0 + T, \\ 1, & t \ge t_0 + T, \end{cases} \tag{5}$$

where $\rho > 1$ is a constant scalar, t_0 is the initial time and T is the convergence time defined by the user. The derivative of $\mu(t)$ is

$$\dot{\mu}(t) = \begin{cases} \frac{\rho}{T}\mu^{(1+\frac{1}{\rho})}, & t_0 \le t < t_0 + T, \\ 0, & t \ge t_0 + T, \end{cases} \tag{6}$$

where the right-hand derivative is used at $t = T$. To analyze the prescribed time convergence property, the following lemmas are needed:

Lemma 3 ([24])**.** *Consider system*

$$\dot{y}(t) = f(t, y(t)), \quad y_0 = y(0), \tag{7}$$

if there exists a Lyapunov function $V(y)$ such that

$$\dot{V}(y) + cV(y) + h\phi(t)V(y) \leq 0, \quad t \in [t_0, \infty), \tag{8}$$

where $c \geq 0$, $h > 0$, and $\phi(t)$ is defined as

$$\phi(t) = \begin{cases} \frac{\dot{\mu}(t)}{\mu(t)}, & t_0 \leq t < t_0 + T, \\ \frac{\rho}{T}, & t \geq t_0 + T, \end{cases} \tag{9}$$

then, system (7) is globally prescribed time stable with a prescribed time $t_0 + T$. In particular, the following solution can be obtained:

$$\begin{cases} V(y) \leq \mu^{-h} exp^{-c(t-t_0)} V(y_0), & t_0 \leq t < t_0 + T, \\ V(y) \equiv 0, & t \geq t_0 + T. \end{cases} \tag{10}$$

Lemma 4 ([33]). *Assume $V(t) : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is a continuously differentiable function satisfying*

$$\dot{V}(t) \leq -cV(t) - h\frac{\dot{\mu}(t)}{\mu(t)}V(t) + \zeta, \quad t \in [t_0, \infty), \tag{11}$$

where $c, h, \zeta > 0$. Then,

$$V(t) \begin{cases} \leq \mu^{-h} e^{-c(t-t_0)} V(t_0) + \epsilon(t), & t \in [t_0, t_0 + T), \\ \leq \frac{\zeta}{c}, & t \in [t_0 + T, \infty), \end{cases} \tag{12}$$

where

$$\epsilon(t) = \left(\frac{t_0 + T - t}{ch - 1} - \frac{\mu^{-h}(t)}{ch - 1} \right) \zeta. \tag{13}$$

3. Main Results

3.1. Prescribed-Time Acceleration Observer

Due to the fact that the accelerations of the leaders are non-zero, each follower needs an acceleration estimator to estimate its ideal acceleration. Assume that the followers can communicate with each other, and part of them can receive information from some leaders. The followers can adopt the following observer to estimate their ideal acceleration:

$$\begin{aligned} \dot{\xi}_i(t) &= -[c_1 + h_1\phi(t)] \sum_{j \in \mathcal{N}_i} \omega_{ij}(\xi_i(t) - \xi_j(t)) \\ &\quad -\gamma sgn\left[\sum_{j \in \mathcal{N}_i} \omega_{ij}(\xi_i(t) - \xi_j(t)) \right], \quad i \in \mathcal{V}_f, \end{aligned} \tag{14}$$

where c_1, h_1 and γ are positive scalars, $\xi_j(t) = u_j(t)$ if $j \in \mathcal{V}_L$, and $\phi(t)$ is defined in Lemma 3. In the remainder of this paper, subscript t is omitted for simplicity if no confusions occur.

Remark 1. *In (14), the time-varying feedback gain $\phi(t)$ plays an important role. It is noticed that $\phi(t)$ increases as $t \to t_0 + T$. This avoids the feedback term converging too fast, and hence guarantees a fast convergence speed when $t \in [t_0, t_0 + T]$.*

Theorem 1. *Define $\xi \triangleq [\xi_{n_l+1}^T, ..., \xi_N^T]^T$, $u_l \triangleq [u_1^T, ..., u_{n_l}^T]^T$. If Assumptions 1–5 are satisfied, and $\gamma > \max_{i \in \mathcal{V}_l} d_i \bullet \parallel \Omega_{ff}^{-1}\Omega_{fl} \parallel_\infty$, using observer (14), ξ will converge to $-(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_m)u_l$ within $t_0 + T$, and $\xi \equiv -(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_m)u_l$ for $t > t_0 + T$.*

Proof. Equation (14) can be rewritten in a compact form as

$$\begin{aligned}\dot{\xi} &= -(c_1 + h_1\phi)\Big[(\Omega_{ff} \otimes I_m)\xi + (\Omega_{fl} \otimes I_m)u_l\Big] \\ &\quad -\gamma sgn\Big[(\Omega_{ff} \otimes I_m)\xi + (\Omega_{fl} \otimes I_m)u_l\Big]. \end{aligned} \tag{15}$$

From Lemmas 1 and 2, we can see that, if Assumptions 1–3 are satisfied, Ω_{ff} is positive definite. Let $\eta = \xi - (-\Omega_{ff}^{-1}\Omega_{fl} \otimes I_m)u_l$. From (15), we have

$$\begin{aligned}\dot{\eta} &= \dot{\xi} - (-\Omega_{ff}^{-1}\Omega_{fl} \otimes I_m)\dot{u}_l \\ &= -(c_1 + h_1\phi)\Big[(\Omega_{ff} \otimes I_m)\xi + (\Omega_{fl} \otimes I_m)u_l\Big] \\ &\quad -\gamma sgn\Big[(\Omega_{ff} \otimes I_m)\xi + (\Omega_{fl} \otimes I_m)u_l\Big] + (\Omega_{ff}^{-1}\Omega_{fl} \otimes I_m)\dot{u}_l. \end{aligned} \tag{16}$$

Choose the Lyapunov function as $V_1 = \frac{1}{2}\eta^T(\Omega_{ff} \otimes I_m)\eta$. It follows from (16) that

$$\begin{aligned}\dot{V}_1 &= \eta^T(\Omega_{ff} \otimes I_m)\dot{\eta} \\ &= \eta^T(\Omega_{ff} \otimes I_m)\Big\{-(c_1 + h_1\phi)\Big[(\Omega_{ff} \otimes I_m)\xi + (\Omega_{fl} \otimes I_m)u_l\Big] \\ &\quad -\gamma sgn\Big[(\Omega_{ff} \otimes I_m)\xi + (\Omega_{fl} \otimes I_m)u_l\Big] + (\Omega_{ff}^{-1}\Omega_{fl} \otimes I_m)\dot{u}_l\Big\} \\ &= -(c_1 + h_1\phi)\eta^T(\Omega_{ff} \otimes I_m)(\Omega_{ff} \otimes I_m)\Big[\xi - (-\Omega_{ff}^{-1}\Omega_{fl} \otimes I_m)u_l\Big] \\ &\quad -\gamma\eta^T(\Omega_{ff} \otimes I_m)sgn\Big[(\Omega_{ff} \otimes I_m)\eta\Big] + \eta^T(\Omega_{ff} \otimes I_m)(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_m)\dot{u}_l \\ &= -(c_1 + h_1\phi)\eta^T(\Omega_{ff}^2 \otimes I_m)\eta - \gamma\eta^T(\Omega_{ff} \otimes I_m)sgn\Big[(\Omega_{ff} \otimes I_m)\eta\Big] \\ &\quad +\eta^T(\Omega_{ff} \otimes I_m)(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_m)\dot{u}_l \\ &\leq -(c_1 + h_1\phi)\eta^T(\Omega_{ff}^2 \otimes I_m)\eta - \gamma \parallel (\Omega_{ff} \otimes I_m)\eta \parallel_1 \\ &\quad +\eta^T(\Omega_{ff} \otimes I_m)(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_m)\dot{u}_l. \end{aligned} \tag{17}$$

By the Holders inequality, it can be obtained that

$$\begin{aligned}\eta^T(\Omega_{ff} \otimes I_m)(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_m)\dot{u}_l &\leq \parallel (\Omega_{ff}^{-1}\Omega_{fl} \otimes I_m)\dot{u}_l \parallel_\infty \parallel (\Omega_{ff} \otimes I_m)\eta \parallel_1 \\ &\leq \parallel \Omega_{ff}^{-1}\Omega_{fl} \otimes I_m \parallel_\infty \parallel \dot{u}_l \parallel_\infty \parallel (\Omega_{ff} \otimes I_m)\eta \parallel_1 \\ &\leq \max_{i \in \mathcal{V}_l} d_i \bullet \parallel \Omega_{ff}^{-1}\Omega_{fl} \parallel_\infty \parallel (\Omega_{ff} \otimes I_m)\eta \parallel_1 . \end{aligned} \tag{18}$$

If $\gamma > \max_{i \in \mathcal{V}_l} d_i \bullet \parallel \Omega_{ff}^{-1}\Omega_{fl} \parallel_\infty$, it follows from (17) and (18) that

$$\begin{aligned}\dot{V}_1 &\leq -(c_1 + h_1\phi)\eta^T(\Omega_{ff}^2 \otimes I_m)\eta \\ &\leq -(c_1 + h_1\phi)\lambda_{min}\eta^T(\Omega_{ff} \otimes I_m)\eta \\ &= -(\tilde{c} + \tilde{h}_1\phi)V_1, \end{aligned} \tag{19}$$

where $\tilde{c}_1 = c_1\lambda_{min}$, $\tilde{h}_1 = h_1\lambda_{min}$ and $\lambda_{min} = \lambda_{min}(\Omega_{ff})$. By Lemma 3, we have V_1 which will converge to zero within $t_0 + T$ and $V_1 \equiv 0$ for $t \geq t_0 + T$. It then follows that η will converge to zero within $t_0 + T$ and $\eta \equiv 0$ for $t \geq t_0 + T$, which implies that ξ will converge to $-(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_m)u_l$ within $t_0 + T$, and $\xi \equiv -(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_m)u_l$ for $t \geq t_0 + T$. □

Remark 2. *As is mentioned in Section 2.3, the ideal velocity of the followers is* $v_f^*(t) = -(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_d)v_l(t)$*. Therefore, the ideal acceleration of the followers is* $-(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_d)u_l(t)$*. Theorem 1 shows that, using observer (14),* ξ *will converge to* $-(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_m)u_l$ *within* $t_0 + T$*, and* $\xi \equiv -(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_m)u_l$ *for* $t > t_0 + T$*. So the followers can receive their ideal accelerations.*

3.2. Fault-Tolerant Affine Formation Control with a Prescribed Convergence Time

Based on the acceleration observer (14), the follower can adopt the following control algorithm:

$$u_i(t) = -k_1\phi^2 \sum_{j\in\mathcal{N}_i} \omega_{ij}(p_i(t) - p_j(t)) - k_2\phi \sum_{j\in\mathcal{N}_i} \omega_{ij}(v_i(t) - v_j(t)) + \xi_i(t), \quad i \in \mathcal{V}_f, \tag{20}$$

where $k_1, k_2 > 0$ are constant parameters to be designed and ϕ is defined in Lemma 3.

Actuator failures cannot be ignored in practice. Controller (20), with partial loss of efficiency and bias faults, can be written as [33,35]

$$\begin{aligned}\tilde{u}_i(t) &= \theta_i u_i(t) + \varepsilon_i(t)\\ &= -\theta_i k_1 \phi^2(t) \sum_{j\in\mathcal{N}_i} \omega_{ij}(p_i(t) - p_j(t))\\ &\quad -\theta_i k_2 \phi(t) \sum_{j\in\mathcal{N}_i} \omega_{ij}(v_i(t) - v_j(t)) + \theta_i\xi_i(t) + \varepsilon_i(t), \quad i \in \mathcal{V}_f,\end{aligned} \tag{21}$$

where $\theta_i \in (0, 1]$ is a constant scalar, $\varepsilon_i(t) \in \mathbb{R}^m$.

Assumption 6. *Assume that the efficiency factor and unknown output bias are bounded, and there exist positive scalars θ^* and $\tilde{\varepsilon}_i$ such that $0 < \theta^* \le \theta_i \le 1$, and $\| \varepsilon_i(t) \| < \tilde{\varepsilon}_i$.*

Remark 3. *Same with [33], we assume that the efficiency factor and unknown output bias are bounded. When $\theta_i(t) = 1$ and $\varepsilon_i(t) = 0$, the actuator is fault-free. The lower bound θ^* means that the actuator will not lose all efficiency.*

Theorem 2. *Under Assumptions 1–6, using observer (14) and the fault-tolerant affine formation control algorithm (21) with user-defined convergence time T and control parameters c_1, h_1, γ, k_1, k_2 and ρ, if $\gamma > \max_{i\in\mathcal{V}_l} d_i \bullet \| \Omega_{ff}^{-1}\Omega_{fl} \|_\infty$, and there exists positive scalars α, c and h such that the following conditions are satisfied,*

$$k_1 < 2k_2^2\lambda_{min}\theta^*, \tag{22}$$

$$\frac{k_1\lambda_{min}}{k_2} - \frac{2\lambda_{min}}{\rho} - \frac{1}{2k_2\rho\theta^*} - h\lambda_{max} - \frac{h}{2k_2\theta^*} > 0, \tag{23}$$

$$\frac{\rho}{T}\left(\frac{k_1\lambda_{min}}{k_2} - \frac{2\lambda_{min}}{\rho} - \frac{1}{2k_2\rho\theta^*} - h\lambda_{max} - \frac{h}{2k_2\theta^*}\right) - \left(c\lambda_{max} + \frac{c}{2k_2\theta^*}\right) - \frac{\alpha}{k_2\theta^*} > 0, \tag{24}$$

$$k_2\lambda_{min} - \frac{k_1}{k_2\theta^*} - \frac{k_1}{2k_2\rho\theta^*} - \frac{h}{2\theta^*} - \frac{hk_1}{2k_2\theta^*} > 0, \tag{25}$$

$$\frac{\rho}{T}\left(k_2\lambda_{min} - \frac{k_1}{k_2\theta^*} - \frac{k_1}{2k_2\rho\theta^*} - \frac{h}{2\theta^*} - \frac{hk_1}{2k_2\theta^*}\right) - \left(\frac{c}{2\theta^*} + \frac{ck_1}{2k_2\theta^*}\right) - \frac{\alpha}{\theta^*} > 0, \tag{26}$$

the formation tracking error δ will converge to zero within $t_0 + T$, and remain in the following neighborhood of zero:

$$\left\{ \delta : \| \delta \| \le \sqrt{\frac{\zeta_0}{c\lambda_{min}(\Gamma)}\left[1 + \left(\frac{T}{\rho}\right)^2\right]} \right\}$$

where $\zeta_0 = \frac{1}{\theta^*\alpha}(\frac{k_1}{k_2} + 1)((1-\theta^*)^2 \| \Omega_{ff}^{-1}\Omega_{fl} \|_2^2 \sum_{i\in\mathcal{V}_l} U_i^2 + \sum_{i\in\mathcal{V}_f} \tilde{\varepsilon}_i^2)$, and δ is defined before Definition 2.

Proof. Let $\tilde{\delta}_p = \phi\delta_p$, and we have

$$\dot{\tilde{\delta}}_p = \dot{\phi}\delta_p + \phi\dot{\delta}_p = \frac{\dot{\phi}}{\phi}\tilde{\delta}_p + \phi(\dot{p}_f - \dot{p}_f^*) = \bar{\phi}\tilde{\delta}_p + \phi\delta_v \tag{27}$$

where $\bar{\phi} = \frac{\dot{\phi}}{\phi} = \begin{cases} \frac{\phi}{\rho}, & t_0 \le t < t_0 + T, \\ 0, & t \ge t_0 + T. \end{cases}$

Let $\tilde{u}_f = [\tilde{u}^T_{n_l+1}, \dots, \tilde{u}^T_N]^T$, $\varepsilon = [\varepsilon^T_{n_l+1}, \dots, \varepsilon^T_N]^T$, $\Theta = diag\{\theta_{n_l}, \dots, \theta_N\}$, and $\omega = ((\Theta - I_{N-n_l}) \otimes I_m)\xi + \varepsilon$. Equation (21) can be written in a compact form as

$$\begin{aligned} \tilde{u}_f &= -k_1\phi^2\Big[(\Theta\Omega_{ff} \otimes I_m)p_f + (\Theta\Omega_{fl} \otimes I_m)p_l\Big] \\ &\quad -k_2\phi\Big[(\Theta\Omega_{ff} \otimes I_m)v_f + (\Theta\Omega_{fl} \otimes I_m)v_l\Big] + (\Theta \otimes I_m)\xi + \varepsilon \end{aligned} \tag{28}$$

From (28) and the definition of v^*_f, we have

$$\begin{aligned} \dot{\delta}_v &= \dot{v}_f - \dot{v}^*_f \\ &= \tilde{u}_f - (-\Omega^{-1}_{ff}\Omega_{fl} \otimes I_m)u_l \\ &= -k_1\phi^2\Big[(\Theta\Omega_{ff} \otimes I_m)p_f + (\Theta\Omega_{fl} \otimes I_m)p_l\Big] \\ &\quad -k_2\phi\Big[(\Theta\Omega_{ff} \otimes I_m)v_f + (\Theta\Omega_{fl} \otimes I_m)v_l\Big] \\ &\quad +(\Theta \otimes I_m)\xi - (-\Omega^{-1}_{ff}\Omega_{fl} \otimes I_m)u_l + \varepsilon \\ &= -k_1\phi^2(\Theta\Omega_{ff} \otimes I_m)\Big[p_f - (-\Omega^{-1}_{ff}\Omega_{fl} \otimes I_m)p_l\Big] \\ &\quad -k_2\phi(\Theta\Omega_{ff} \otimes I_m)\Big[v_f - (-\Omega^{-1}_{ff}\Omega_{fl} \otimes I_m)v_l\Big] \\ &\quad +\xi - (-\Omega^{-1}_{ff}\Omega_{fl} \otimes I_m)u_l + [(\Theta - I_{N-n_l}) \otimes I_m]\xi + \varepsilon \\ &= -k_1\phi(\Theta\Omega_{ff} \otimes I_m)\tilde{\delta}_p - k_2\phi(\Theta\Omega_{ff} \otimes I_m)\delta_v + \eta + \omega, \end{aligned} \tag{29}$$

where we have used the fact that $\tilde{\delta}_p = \phi\delta_p$. Let $\tilde{\delta} \triangleq [\tilde{\delta}^T_p, \delta^T_v]^T$ and choose the Lyapunov candidate as

$$V_2 = \frac{1}{2}\tilde{\delta}^T(\Gamma \otimes I_m)\tilde{\delta}, \tag{30}$$

where

$$\Gamma = \begin{bmatrix} 2k_1\Omega_{ff} & \frac{k_1}{k_2}\Theta^{-1} \\ \frac{k_1}{k_2}\Theta^{-1} & \Theta^{-1} \end{bmatrix}.$$

If condition (22) is satisfied, it is trivial to prove that $2k_1\Omega_{ff} - \frac{k_1^2}{k_2^2}\Theta^{-1} > 0$. Therefore, Γ is positive-definite. From (27) and (29), we have

$$\begin{aligned} \dot{V}_2 &= 2k_1\tilde{\delta}^T_p(\Omega_{ff} \otimes I_m)\dot{\tilde{\delta}}_p + \frac{k_1}{k_2}\delta^T_v(\Theta^{-1} \otimes I_m)\dot{\tilde{\delta}}_p + \frac{k_1}{k_2}\tilde{\delta}^T_p(\Theta^{-1} \otimes I_m)\dot{\delta}_v + \delta^T_v(\Theta^{-1} \otimes I_m)\dot{\delta}_v \\ &= 2k_1\tilde{\delta}^T_p(\Omega_{ff} \otimes I_m)(\bar{\phi}\tilde{\delta}_p + \phi\delta_v) + \frac{k_1}{k_2}\delta^T_v(\Theta^{-1} \otimes I_m)(\bar{\phi}\tilde{\delta}_p + \phi\delta_v) \\ &\quad +\frac{k_1}{k_2}\tilde{\delta}^T_p(\Theta^{-1} \otimes I_m)\Big[-k_1\phi(\Theta\Omega_{ff} \otimes I_m)\tilde{\delta}_p - k_2\phi(\Theta\Omega_{ff} \otimes I_m)\delta_v + \eta + \omega\Big] \\ &\quad +\delta^T_v(\Theta^{-1} \otimes I_m)\Big[-k_1\phi(\Theta\Omega_{ff} \otimes I_m)\tilde{\delta}_p - k_2\phi(\Theta\Omega_{ff} \otimes I_m)\delta_v + \eta + \omega\Big] \\ &= 2k_1\bar{\phi}\tilde{\delta}^T_p(\Omega_{ff} \otimes I_m)\tilde{\delta}_p + \frac{k_1}{k_2}\delta^T_v(\Theta^{-1} \otimes I_m)(\bar{\phi}\tilde{\delta}_p + \phi\delta_v) \\ &\quad -\frac{k_1^2}{k_2}\phi\tilde{\delta}^T_p(\Omega_{ff} \otimes I_m)\tilde{\delta}_p + \frac{k_1}{k_2}\tilde{\delta}^T_p(\Theta^{-1} \otimes I_m)(\eta + \omega) \\ &\quad -k_2\phi\delta^T_v(\Omega_{ff} \otimes I_m)\delta_v + \delta^T_v(\Theta^{-1} \otimes I_m)(\eta + \omega) \end{aligned} \tag{31}$$

For $t \in [t_0, t_0 + T]$, notice that $\bar{\phi} = \frac{\phi}{\rho}$, and from (31) we have

$$
\begin{aligned}
\dot{V}_2 &= (\frac{2k_1}{\rho} - \frac{k_1^2}{k_2})\phi\tilde{\delta}_p^T(\Omega_{ff} \otimes I_m)\tilde{\delta}_p + \frac{k_1}{k_2\rho}\phi\delta_v^T(\Theta^{-1} \otimes I_m)\tilde{\delta}_p - k_2\phi\delta_v^T(\Omega_{ff} \otimes I_m)\delta_v \\
&+ \frac{k_1}{k_2}\phi\delta_v^T(\Theta^{-1} \otimes I_m)\delta_v + \frac{k_1}{k_2}\tilde{\delta}_p^T(\Theta^{-1} \otimes I_m)(\eta + \omega) + \delta_v^T(\Theta^{-1} \otimes I_m)(\eta + \omega).
\end{aligned} \tag{32}
$$

If condition (23) is satisfied, we have $\frac{2}{\rho} - \frac{k_1}{k_2} < 0$. Noting that

$$
\begin{aligned}
-\tilde{\delta}_p^T(\Omega_{ff} \otimes I_m)\tilde{\delta}_p &\leq -\lambda_{min}\tilde{\delta}_p^T\tilde{\delta}_p, && (33) \\
-\delta_v^T(\Omega_{ff} \otimes I_m)\delta_v &\leq -\lambda_{min}\delta_v^T\delta_v, && (34) \\
\tilde{\delta}_p^T(\Omega_{ff} \otimes I_m)\tilde{\delta}_p &\leq \lambda_{max}\tilde{\delta}_p^T\tilde{\delta}_p, && (35) \\
\delta_v^T(\Theta^{-1} \otimes I_m)\delta_v &\leq \frac{1}{\theta^*}\delta_v^T\delta_v, && (36) \\
\tilde{\delta}_p^T(\Theta^{-1} \otimes I_m)\delta_v &\leq \frac{1}{2\theta^*}(\tilde{\delta}_p^T\tilde{\delta}_p + \delta_v^T\delta_v), && (37)
\end{aligned}
$$

and, for any $\alpha > 0$,

$$
\begin{aligned}
\tilde{\delta}_p^T(\Theta^{-1} \otimes I_m)\eta &\leq \frac{1}{2\theta^*}(\alpha\tilde{\delta}_p^T\tilde{\delta}_p + \frac{1}{\alpha}\eta^T\eta), && (38) \\
\delta_v^T(\Theta^{-1} \otimes I_m)\eta &\leq \frac{1}{2\theta^*}(\alpha\delta_v^T\delta_v + \frac{1}{\alpha}\eta^T\eta), && (39) \\
\tilde{\delta}_p^T(\Theta^{-1} \otimes I_m)\omega &\leq \frac{1}{2\theta^*}(\alpha\tilde{\delta}_p^T\tilde{\delta}_p + \frac{1}{\alpha}\omega^T\omega), && (40) \\
\delta_v^T(\Theta^{-1} \otimes I_m)\omega &\leq \frac{1}{2\theta^*}(\alpha\delta_v^T\delta_v + \frac{1}{\alpha}\omega^T\omega), && (41)
\end{aligned}
$$

it follows from (32)–(41) that

$$
\begin{aligned}
\dot{V}_2 \leq\; & -k_1\left[\left(\frac{k_1\lambda_{min}}{k_2} - \frac{2\lambda_{min}}{\rho} - \frac{1}{2k_2\rho\theta^*}\right)\phi - \frac{\alpha}{k_2\theta^*}\right]\tilde{\delta}_p^T\tilde{\delta}_p \\
& -\left[\phi(k_2\lambda_{min} - \frac{k_1}{k_2\theta^*} - \frac{k_1}{2k_2\rho\theta^*}) - \frac{\alpha}{\theta^*}\right]\delta_v^T\delta_v + \frac{1}{2\theta^*\alpha}(\frac{k_1}{k_2} + 1)(\eta^T\eta + \omega^T\omega).
\end{aligned} \tag{42}
$$

From (30) and (35)–(37), we have

$$
\begin{aligned}
V_2 &= k_1\tilde{\delta}_p^T(\Omega_{ff} \otimes I_m)\tilde{\delta}_p + \frac{1}{2}\delta_v^T(\Theta^{-1} \otimes I_m)\delta_v + \frac{k_1}{k_2}\tilde{\delta}_p^T(\Theta^{-1} \otimes I_m)\delta_v \\
&\leq (k_1\lambda_{max} + \frac{k_1}{2k_2\theta^*})\tilde{\delta}_p^T\tilde{\delta}_p + (\frac{1}{2\theta^*} + \frac{k_1}{2k_2\theta^*})\delta_v^T\delta_v.
\end{aligned} \tag{43}
$$

Suppose that conditions (23) and (25) are satisfied, combined with the fact that $\phi \geq \frac{\rho}{T}$, from (42) and (43) we have for any $c, h > 0$,

$$
\begin{aligned}
& \dot{V}_2 + (c + h\phi)V_2 \\
\leq\; & -k_1\left[\left(\frac{k_1\lambda_{min}}{k_2} - \frac{2\lambda_{min}}{\rho} - \frac{1}{2k_2\rho\theta^*} - h\lambda_{max} - \frac{h}{2k_2\theta^*}\right)\frac{\rho}{T}\right. \\
& \left. -\left(c\lambda_{max} + \frac{c}{2k_2\theta^*}\right) - \frac{\alpha}{k_2\theta^*}\right]\tilde{\delta}_p^T\tilde{\delta}_p \\
& -\left[\frac{\rho}{T}\left(k_2\lambda_{min} - \frac{k_1}{k_2\theta^*} - \frac{k_1}{2k_2\rho\theta^*} - \frac{h}{2\theta^*} - \frac{hk_1}{2k_2\theta^*}\right) - \left(\frac{c}{2\theta^*} + \frac{ck_1}{2k_2\theta^*}\right) - \frac{\alpha}{\theta^*}\right]\delta_v^T\delta_v \\
& + \frac{1}{2\theta^*\alpha}(\frac{k_1}{k_2} + 1)(\eta^T\eta + \omega^T\omega).
\end{aligned} \tag{44}
$$

Suppose we can find $c, h > 0$ such that conditions (24) and (26) are also satisfied, we can see that $\dot{V}_2 + (c + h\phi_2)V_2 \leq \frac{1}{2\theta^*\alpha}(\frac{k_1}{k_2} + 1)(\eta^T\eta + \omega^T\omega)$. Therefore,

$$\dot{V}_2 \leq -(c + h\phi)V_2 + \frac{1}{2\theta^*\alpha}(\frac{k_1}{2k_2} + 1)(\eta^T\eta + \omega^T\omega). \tag{45}$$

By the definition of ω, we have

$$\begin{aligned} \omega^T\omega &\leq 2\xi^T((\Theta - I_{N-n_l})^2 \otimes I_m)\xi + 2\varepsilon^T\varepsilon \\ &\leq 2(1-\theta^*)^2\xi^T\xi + 2\sum_{i\in\mathcal{V}_f} \tilde{\varepsilon}_i^2. \end{aligned} \tag{46}$$

If $\gamma > \max_{i\in\mathcal{V}_l} d_i \bullet \parallel \Omega_{ff}^{-1}\Omega_{fl} \parallel_\infty$, from Theorem 1 we have that ξ converge to $-(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_m)u_l$ and η converge to zero within $t_0 + T$. Therefore, $\frac{1}{2\theta^*\alpha}(\frac{k_1}{k_2} + 1)(\eta^T\eta + \omega^T\omega)$ is bounded during $[t_0, t_0 + T)$. Assume that $\frac{1}{2\theta^*\alpha}(\frac{k_1}{k_2} + 1)(\eta^T\eta + \omega^T\omega) \leq \zeta_1$ during $[t_0, t_0 + T)$, and we have

$$\dot{V}_2 \leq -(c + h\phi)V_2 + \zeta_1. \tag{47}$$

By Lemma 4, we have

$$V_2 \leq \mu^{-h}e^{-c(t-t_0)}V(t_0) + \epsilon_2(t), \tag{48}$$

where

$$\epsilon_2(t) = \left(\frac{t_0 + T - t}{ch - 1} - \frac{\mu^{-h}(t)}{ch - 1}\right)\zeta_1. \tag{49}$$

Due to the fact that $\mu^{-h} \to 0$ and $\frac{t_0+T-t}{ch-1} - \frac{\mu^{-h}(t)}{ch-1} \to 0$ as $t \to t_0 + T$, we have $V_2 \to 0$ as $t \to t_0 + T$. It follows that $\tilde{\delta}_p$ and δ_v converge to zero as $t \to t_0 + T$. Notice that ϕ is positive, and we have δ_p converge to zero as $t \to t_0 + T$. Therefore, δ converge to zero as $t \to t_0 + T$.

For $t \in [t_0 + T, \infty)$, from Theorem 1 we know that $\eta = 0$. Noting that $\bar{\phi} = 0$ when $t \geq t_0 + T$ and $\phi = \frac{\rho}{T}$, by a similar process we can obtain that

$$\begin{aligned} &\dot{V}_2 + (c + h\phi)V_2 \\ \leq\; & -k_1\left[\left(\frac{k_1\lambda_{min}}{k_2} - h\lambda_{max} - \frac{h}{2k_2\theta^*}\right)\frac{\rho}{T} - \left(c\lambda_{max} + \frac{c}{2k_2\theta^*}\right) - \frac{\alpha}{k_2\theta^*}\right]\tilde{\delta}_p^T\tilde{\delta}_p \\ & -\left[\frac{\rho}{T}\left(k_2\lambda_{min} - \frac{k_1}{k_2\theta^*} - \frac{h}{2\theta^*} - \frac{hk_1}{2k_2\theta^*}\right) - \left(\frac{c}{2\theta^*} + \frac{ck_1}{2k_2\theta^*}\right) - \frac{\alpha}{\theta^*}\right]\delta_v^T\delta_v \\ & + \frac{1}{2\theta^*\alpha}(\frac{k_1}{k_2} + 1)\omega^T\omega. \end{aligned} \tag{50}$$

Suppose that conditions (22)–(26) are satisfied, it follows that

$$\dot{V}_2 \leq -(c + h\frac{T}{\rho})V_2 + \frac{1}{2\theta^*\alpha}(\frac{k_1}{k_2} + 1)\omega^T\omega. \tag{51}$$

From Theorem 1, we know that $\xi \equiv -(\Omega_{ff}^{-1}\Omega_{fl} \otimes I_m)u_l$ for $t \geq t_0 + T$. It follows from (46) that

$$\begin{aligned} \omega^T\omega &\leq 2(1-\theta^*)^2 \parallel (\Omega_{ff}^{-1}\Omega_{fl} \otimes I_m)u_l \parallel^2 + 2\sum_{i\in\mathcal{V}_f} \tilde{\varepsilon}_i^2 \\ &\leq 2(1-\theta^*)^2 \parallel \Omega_{ff}^{-1}\Omega_{fl} \parallel_2^2 \sum_{i\in\mathcal{V}_l} U_i^2 + 2\sum_{i\in\mathcal{V}_f} \tilde{\varepsilon}_i^2. \end{aligned} \tag{52}$$

It follows from (51) that

$$\dot{V}_2 \leq -(c + h\frac{T}{\rho})V_2 + \zeta_0. \tag{53}$$

Noting that $c, h, T, \rho > 0$, and $V_2(t_0 + T) = 0$, we have $V_2 \leq \frac{\zeta_0}{c}$ for $t > t_0 + T$. Therefore, $\tilde{\delta}_p^T \tilde{\delta}_p + \delta_v^T \delta_v$ is bounded by $\frac{\zeta_0}{c\lambda_{min}(\Gamma)}$ for $t > t_0 + T$. It then follows that $\delta_p^T \delta_p \leq \frac{\zeta_0 T^2}{c\lambda_{min}(\Gamma)\rho^2}$ for $t > t_0 + T$. It is easy to see that $\| \delta \| \leq \sqrt{\frac{\zeta_0}{c\lambda_{min}(\Gamma)}[1 + (\frac{T}{\rho})^2]}$ for $t > t_0 + T$. □

Remark 4. *Algorithms* (20) *and* (21) *are designed based on the acceleration estimator* (14)*. Maybe some sliding mode control algorithms without disturbance observer can be proposed, by considering the leaders' accelerations as disturbances. However, sliding mode control algorithms may cause chatters of the tracking error. If the leaders' accelerations are not compensated for using the observer-based method, the disturbance will be very large when the formation changes rapidly. This will lead to large chatters. Therefore, even though the sliding mode control algorithms are adopted, it would be better to use an observer to estimate the ideal acceleration of the followers to compensate for the leaders' accelerations.*

Remark 5. *From Theorem 2, we can see that, in algorithm* (21)*, the formation tracking error converges with a prescribed convergence time. As a comparison, using controller (25) in* [16]*, only exponential convergence can be achieved. Therefore, the algorithm proposed in this paper can guarantee faster convergence. This will be confirmed in Section 4.2 by a numerical simulation.*

Remark 6. *Of note, c, h and α in Theorem 2 are auxiliary variables for problem analysis. These parameters are not used in the control algorithm* (21)*. It is also worth noting that c and h can be chosen differently from c_1 and h_1 in the observer* (14)*. When selecting control parameters to satisfy conditions* (22)–(26)*, c h and α can be chosen freely.*

Remark 7. *It is easy to see that, if the following conditions are satisfied:*

$$k_1 < 2k_2^2 \lambda_{min} \theta^*, \tag{54}$$

$$\frac{k_1 \lambda_{min}}{k_2} - \frac{2\lambda_{min}}{\rho} - \frac{1}{2k_2 \rho \theta^*} > 0, \tag{55}$$

$$k_2 \lambda_{min} - \frac{k_1}{k_2 \theta^*} - \frac{k_1}{2k_2 \rho \theta^*} > 0, \tag{56}$$

one can always find $c, h, \alpha > 0$ satisfying (23)–(26)*.*

4. Simulation Studies

In this section, simulation examples are given to show the effectiveness of the proposed algorithm and compare it with existing results.

4.1. Fault-Tolerant Affine Formation Control

Consider a multi-agent system with three leaders and four followers moving in two-dimensional space with dynamics given in (3). Assume that agents 1–3 are leaders and agents 4–7 are followers. The communication topology is shown in Figure 1, which is the same with [16]. The nominal configuration and the stress matrix are given in Section 2.2. By some calculation, we can obtain $\lambda_{min}(\Omega_{ff}) = 0.0235$, $\lambda_{max}(\Omega_{ff}) = 0.9593$. The parameters used in the simulation are chosen as $k_1 = 10$, $k_2 = 25$, $T = 2$, $\rho = 10$, $c_1 = 10$, $h_1 = 0.1$ and $\gamma = 5$.

Let $\theta^* = 0.8$ and $\tilde{\varepsilon}_i = 0.01$. Most of the time, only a part of the agents are subject to partial loss of efficiency and bias faults. In order to test the effectiveness of the proposed algorithm, we assume that all the followers are subject to loss of efficiency faults. Using the fault-tolerant controller (21) with observer (14), the simulation results are shown in Figures 3 and 4.

We assume that the multi-agent system needs to go through two narrow gates, as is shown in Figure 3. In this process, the formation needs to change the scale four times and rotate two times. Figure 3 shows trajectories of agents 1–7. It can be seen that the desired formation is achieved and when the desired formation is changed, and the followers can track the formation appropriately even when all the followers are subject to loss of efficiency faults. Figure 4 shows the formation tracking error $\| \ \delta(t) \ \|$. It can be seen that the formation tracking error converges to zero within $T = 2$ and remains in a small neighborhood of zero. When the formation is changing, the tracking error increases. This is mainly caused by the actuator fault. Once the formation change is completed, the tracking error converges rapidly.

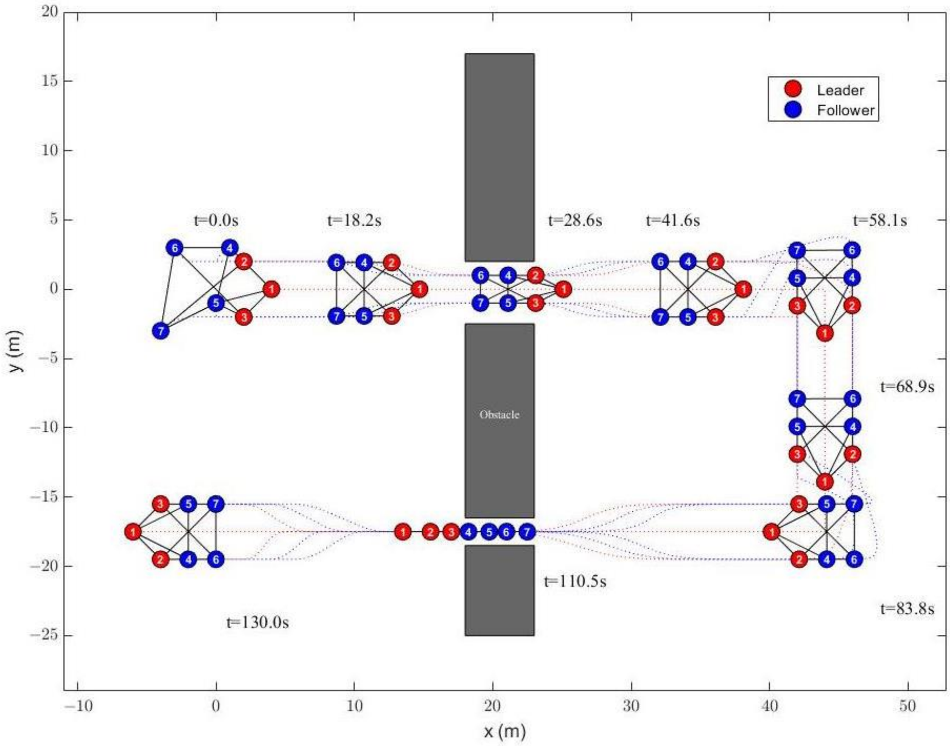

Figure 3. Trajectories of multi-agent systems with 4 followers. Red balls denote the leaders and blue balls denote the followers.

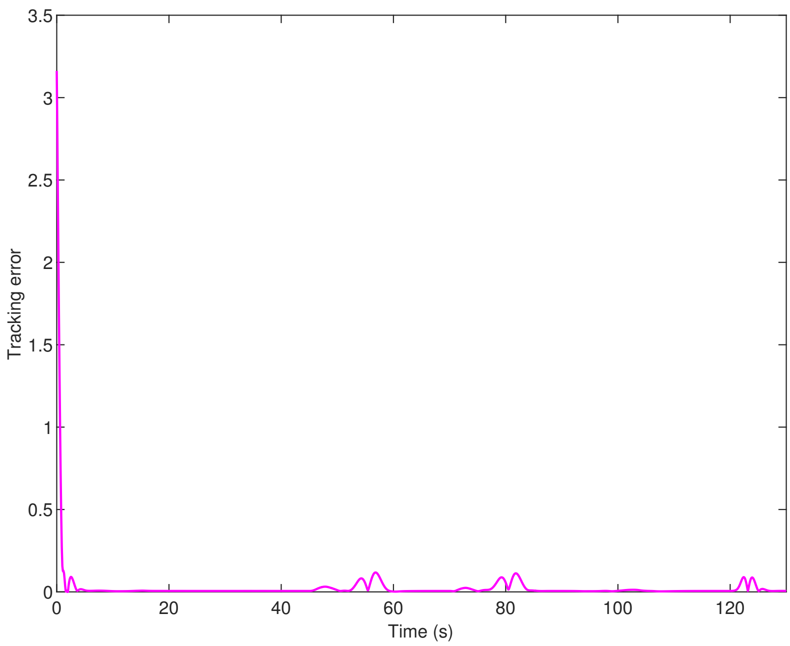

Figure 4. Formation tracking error $\| \ \delta(t) \ \|$.

To test the effectiveness of the proposed controller, we consider the multi-agent systems with more followers. Figures 5 and 6 show the trajectories of the multi-agent system with six followers and eight followers, respectively. We can see that the followers can also track the formation well when the number of followers increases.

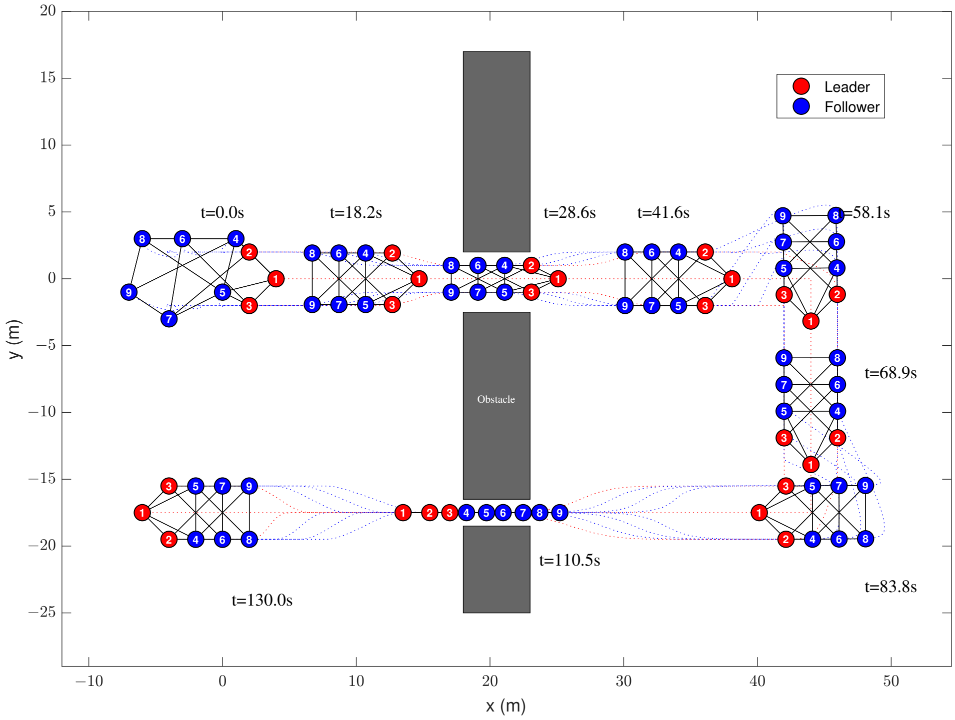

Figure 5. Trajectories of multi-agent systems with 6 followers. Red balls denote the leaders and blue balls denote the followers.

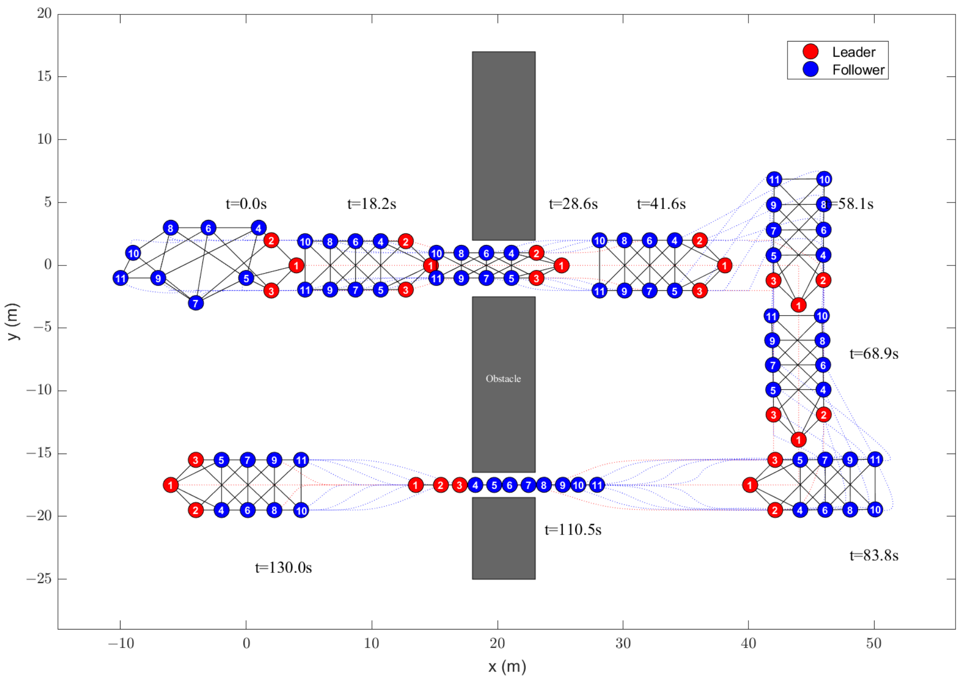

Figure 6. Trajectories of multi-agent systems with 8 followers. Red balls denote the leaders and blue balls denote the followers.

4.2. Comparison with Existing Results

Since the fault-tolerant affine formation control problem of multi-agent systems with double-integrator dynamics has not been considered in the literature, we compare the proposed controller with controller (25) in [16] in a fault-free manner. Let $\theta^* = 1$, $\tilde{\varepsilon}_i = 0$, and controller (21) becomes controller (20), which is a fault-free version of controller (21).

Figure 7 shows the formation tracking error $\| \delta(t) \|$ using controller (25) in [16], and controller (20) proposed in this paper with T equal to 2, 10 and 20, respectively. It can been seen that controller (25) in [16] needs about 20 s to stabilize the tracking error. As a comparison, using controller (20) proposed in this paper, the tracking error converges within prescribed time. Therefore, the proposed algorithm can guarantee faster convergence than controller (25) in [16]. It can also be seen that, when the prescribed time is 10, the tracking error converges within 5 s. Therefore, the prescribed time convergence does not mean that the tracking error will converge to zero until the prescribed time. It can converge much earlier than that time.

Since Figures 4 and 7 show the fault-tolerant case and fault-free case of the same multi-agent systems, we can compare them to see the effect of actuator faults. By comparison, we can see that the tracking error is smaller when no actuator fault occurs. This confirms that the increased tracking error during formation changing is mainly caused by the actuator faults.

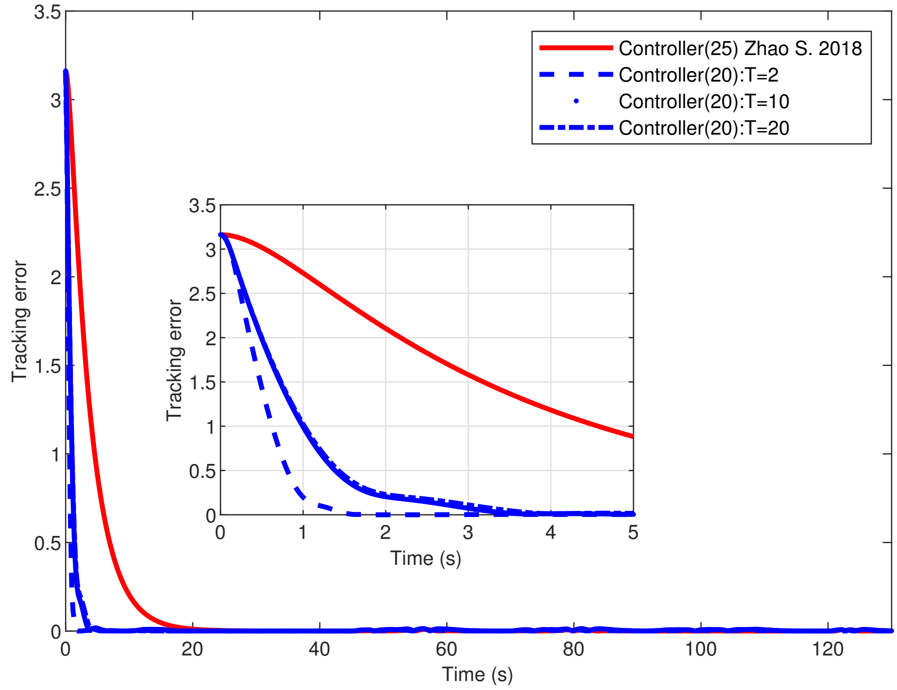

Figure 7. Tracking error $\| \delta(t) \|$ of controller (25) in [16]) (red solid line) and controller (20) proposed in this paper with T = 2 (dashed line), T = 10 (dotted line), T = 20 (dash-dot line).

5. Conclusions

This paper proposes a fault-tolerant affine formation control algorithm for double-integrator multi-agent systems with partial loss of efficiency and bias faults. Firstly, an acceleration observer is proposed for each follower to estimate its ideal acceleration. It is proved that the estimation error can converge to zero in a prescribed time T. Then, based on the acceleration estimator, a control algorithm is proposed to solve the fault-tolerant affine formation control problem. A Lyapunov function candidate is constructed to analysis the convergence of tracking errors of the followers. Theoretical analysis shows that, if parameters in the algorithm are properly chosen, the formation tracking error can converge to zero within a user appointed time, and remain in a small region near zero after that time. Finally, a numerical simulation is given to show the effectiveness of the proposed algorithm. Comparison with existing results is also made by a simulation. The simulation result shows that, by using the algorithm proposed in this paper, the formation tracking error converges faster than the existing results.

This paper provides an efficient attempt on fault-tolerant affine formation control of multi-agent systems with double-integrator dynamics. There are several meaningful research directions for the future. For example, the results derived in this paper can be further extended by investigating multi-agent systems with directed communication topologies [36], collision avoidance [37] and input saturation [38]. In addition, in this paper, we consider the double-integrator multi-agent systems with partial loss of efficiency and bias faults only. Therefore, it is also a research direction to investigate the effect of other type of faults such as sensor faults, communication failures and so on.

The algorithm proposed in this paper can be applied to formation control of omnidirectional mobile robots. It can also be applied to the position loop of formation controllers for multiple quadrotors. The future work can also be focused on testing the proposed algorithm by formation control experiments using omnidirectional mobile robots, quad-rotors or other types of robotics.

Author Contributions: Methodology, J.L., X.Y. and J.T.; software, J.T.; validation, Z.D. and Y.W.; writing—original draft preparation, J.T.; writing—review and editing, J.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by National Natural Science Foundation of China: 62173164.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

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