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On the Implications of $|U_{\mu i}| = |U_{\tau i}|$ in the Canonical Seesaw Mechanism

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Abstract: In the PMNS matrix, the relation $|U_{\mu i}| = |U_{\tau i}|$ (with $i = 1, 2, 3$) is experimentally favored at the present stage. The possible implications of this relation on some hidden flavor symmetry has attracted a lot of interest in the neutrino community. In this paper, we analyze the implications of $|U_{\mu i}| = |U_{\tau i}|$ (with $i = 1, 2, 3$) in the context of the canonical seesaw mechanism. We also show that the minimal $\mu - \tau$ symmetry proposed in JHEP 06 (2022) 034 is a possible but not necessary reason for the above-mentioned relation.

Keywords: neutrino physics; neutrino mass; neutrino mixing; canonical seesaw mechanism; flavor symmetry; Majorana neutrino

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1. Introduction

It has been more than 90 years since Wolfgang Pauli's proposal of the neutrino in his open letter to the "radiative ladies and gentlemen" attending the Gauverein meeting in Tübingen in 1930 [1,2]. However, the nature of these elementary particles is still largely shrouded in mystery. In the Standard Model of particle physics, neutrinos are understood to be massless fermions. This picture has been severely challenged by a large and increasing number of experimental results since the famous Homestake experiment on solar neutrinos [3]. It is now commonly accepted that at least two neutrino mass eigenvalues are nonzero and that there is mismatch between the neutrino mass eigenstates and flavor eigenstates [4]. These all hint at the existence of new physics beyond the Standard Model.

The fact that neutrinos (anti-neutrinos) are only observed to be left-handed (right-handed) is one reason for the inability of the Higgs mechanism to generate nonzero neutrino masses. Thus, a new mass generation mechanism is needed in the neutrino sector. Furthermore, we do not yet know whether massive neutrinos are Majorana particles or Dirac particles. In other words, the question whether massive neutrinos are their own anti-particles is still open. Considerable effort has been put into model-building, and we now have many candidates waiting to be tested (see, for example, S. F. King [5,6] and A. de Gouvêa [7]). At the present stage, the most promising class of neutrino mass models is the so-called seesaw mechanism, initiated by Peter Minkowski in 1977 [8]. In seesaw models, massive neutrinos are assumed to be Majorana particles, which are of course subject to the results of relevant experiments, especially those on neutrino-less double beta decay ($0\nu\beta\beta$) [9–13]. The small masses of active neutrinos come from the exchange of heavy messenger particles from the viewpoint of the seesaw mechanism. These heavy messenger particles can be right-handed singlet neutrinos such as the Type-I seesaw [8,14–17], triplet scalar bosons such as the Type-II seesaw [18–20], triplet fermions such as the Type-III seesaw [21], or some other possibilities in other seesaw models. For more details of the seesaw mechanism and Majorana neutrinos, one may refer to, for example, Cai et al. [22], Gluza [23], Barger et al. [24], Mohapatra and Smirnov [25], Rodejohann [26], Chen and Huang [27], Atre et al. [28], and Deppisch et al. [29].

Even limited to the seesaw family, there is still great richness to be explored and tested. It is the large number of degrees of freedom in model-building that leads to a lack of predictive power. As remarked by Witten in the opening talk at "Neutrino2000" [30]:

> For neutrino masses, the considerations have always been qualitative, and, despite some interesting attempts, there has never been a convincing quantitative model of the neutrino masses.

More than 20 years have passed, and a lot of data have been collected from neutrino experiments around the world, such as the Sudbury Neutrino Observatory (SNO) in Canada [31], Super-Kamiokande in Japan [32], Daya Bay in China [33], Double Chooz in France [34] and T2K in Japan [35]. Together with the results in the search for lepton number violating processes (see, for example, Dib et al. [36] and Drewes et al. [37]), they have provided significant constraints on the parameter space [7,22,38]. However, Witten's remark is still more or less true, and we are still far from a unique, quantitative, and satisfactory theory of massive neutrinos.

Based on those relevant experimental results, in addition to placing constraints on the relevant parameter space, we can also try to infer possible symmetries beneath the seesaw mechanism and constrain the flavor texture. In the 3×3 PMNS matrix U [39–41], there is one experimentally favored relation, viz. $|U_{\mu i}| = |U_{\tau i}|$ with $i = 1, 2, 3$, supported by a global analysis of the latest data on atmospheric, solar, reactor, and accelerator neutrino oscillations [38,42,43]. Recently, in [44], the author discusses the above-mentioned relation and claims that this relation necessarily implies $|R_{\mu i}| = |R_{\tau i}|$ (with $i = 1, 2, 3$), in which R is a 3×3 sub-matrix of the full 6×6 neutrino mixing matrix in the context of the canonical seesaw mechanism. The author further claims that, in the scenario $U = \mathcal{P}U^*$ with $\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, the relation $R = \mathcal{P}R^*$ is a necessary consequence. On this basis, it is argued that a minimal $\mu - \tau$ symmetry, viz. the invariance of the neutrino mass term under the transformation formed by $\nu_{e\mathrm{L}} \to (\nu_{e\mathrm{L}})^c$,$\nu_{\mu\mathrm{L}} \to (\nu_{\mu\mathrm{L}})^c$,$\nu_{\tau\mathrm{L}} \to (\nu_{\tau\mathrm{L}})^c$ on the left-handed neutrino fields and arbitrary unitary CP transformation on the right-handed neutrino fields, is expected to exist. In this paper, we analyze the implications of the relation $|U_{\mu i}| = |U_{\tau i}|$ (with $i = 1, 2, 3$) in the context of the canonical seesaw mechanism. We find that there exist some other nontrivial possibilities that can accommodate the above-mentioned relation in the PMNS matrix.

2. Some Basics of the Canonical Seesaw Mechanism

The canonical seesaw mechanism belongs to the Type-I seesaw. There are in total three right-handed neutrino fields, denoted by $N_{\alpha\mathrm{R}}$ with $\alpha = e, \mu, \tau$, being added into the Standard Model. The corresponding neutrino mass term with gauge invariance and Lorentz invariance is as follows [44]:

$$-\mathcal{L}_\nu = \overline{l_\mathrm{L}} Y_\nu \tilde{H} N_\mathrm{R} + \frac{1}{2}\overline{(N_\mathrm{R})^c} M_\mathrm{R} N_\mathrm{R} + \text{h.c.} \tag{1}$$

The notations in the above expression are explained here. l_L is the $\mathrm{SU}(2)_\mathrm{L}$ doublet formed by left-handed lepton fields. Y_ν is the 3×3 Yukawa coupling matrix. $\tilde{H}$ is defined as $i\sigma_2 H^*$, in which σ_2 is the second Pauli matrix, and H is the Higgs doublet. N_R is the column vector formed by those three right-handed neutrino fields $N_{\alpha\mathrm{R}}$. $(N_\mathrm{R})^c$ is defined as $\mathcal{C}\overline{N_\mathrm{R}}^T$ with the charge conjugation operator $\mathcal{C}$. M_R is the 3×3 symmetric Majorana mass matrix.

The three active neutrinos acquire masses after spontaneous electroweak gauge symmetry breaking, with the corresponding mass term being [44]:

$$-\mathcal{L}'_\nu = \frac{1}{2}\overline{\begin{pmatrix} \nu_\mathrm{L} & (N_\mathrm{R})^c \end{pmatrix}} \begin{pmatrix} \mathbf{0} & M_D \\ M_D^T & M_R \end{pmatrix} \begin{pmatrix} (\nu_\mathrm{L})^c \\ N_\mathrm{R} \end{pmatrix} + \text{h.c.} \tag{2}$$

The explanation of notations is as follows. $\nu_{\rm L}$ is the column vector formed by those three left-handed neutrino fields $\nu_{\alpha \rm L}$ with $\alpha = e, \mu, \tau$. M_D is defined as the product of the vacuum expectation value of the Higgs field $\langle H \rangle$ and the Yukawa coupling matrix Y_ν.

The masses of all six neutrinos can be retrieved by diagonalizing the whole 6×6 mass matrix using a 6×6 unitary matrix, viz.:

$$\begin{pmatrix} U & R \\ S & Q \end{pmatrix}^\dagger \begin{pmatrix} \mathbf{0} & M_D \\ M_D^T & M_R \end{pmatrix} \begin{pmatrix} U & R \\ S & Q \end{pmatrix}^* = \begin{pmatrix} D_\nu & \mathbf{0} \\ \mathbf{0} & D_N \end{pmatrix}, \tag{3}$$

in which $D_\nu = \mathrm{diag}(m_1, m_2, m_3)$ and $D_N = \mathrm{diag}(M_1, M_2, M_3)$ together contain all six neutrino masses. In this scenario, the submatrix U is generally not unitary, in contrast to the common scenario in some discussion of neutrino oscillations in which there are only three types of neutrinos. The latter is actually an effective theory after integrating out those heavy degrees of freedom (heavy neutrinos). Recently there have been some discussions on the so-called flavor invariants in this effective picture. It is shown that the polynomial ring formed by these flavor invariants is finitely generated [45–47]. Three sub-matrices R, S, and Q are incorporated to extend U to a 6×6 unitary matrix. From the unitarity of this 6×6 matrix, one can immediately obtain the following relations:

$$UU^\dagger + RR^\dagger = SS^\dagger + QQ^\dagger = I, \tag{4}$$

$$U^\dagger U + S^\dagger S = R^\dagger R + Q^\dagger Q = I, \tag{5}$$

$$US^\dagger + RQ^\dagger = U^\dagger R + S^\dagger Q = \mathbf{0}. \tag{6}$$

3. Implications of $|U_{\mu i}| = |U_{\tau i}|$

3.1. Six Classes of F Satisfying $RD_NR^T = (RF)D_N(RF)^T$

In the canonical seesaw mechanism, three light neutrino masses $\{m_1, m_2, m_3\}$ and three heavy neutrino masses $\{M_1, M_2, M_3\}$ are connected by the so-called exact seesaw formula

$$UD_\nu U^T + RD_NR^T = \mathbf{0}, \tag{7}$$

which can be easily obtained by focusing on the upper-left quadrant of the 6×6 matrix $\begin{pmatrix} U & R \\ S & Q \end{pmatrix} \begin{pmatrix} D_\nu & \mathbf{0} \\ \mathbf{0} & D_N \end{pmatrix} \begin{pmatrix} U & R \\ S & Q \end{pmatrix}^T$.

By simple observation, one can see that, for any 3×3 matrix R and 3×3 diagonal matrix $D_N = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_3 \end{pmatrix}$ with $M_1, M_2, M_3 \in \mathbb{R}^+$, there exist at least six distinct nontrivial classes of 3×3 matrices F, such that, for any of these choices, the relation $RD_NR^T = (RF)D_N(RF)^T$ is always true.

- The first class of F has the texture $\begin{pmatrix} 0 & \times & 0 \\ \times & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$:

  ■ $F_1 = \begin{pmatrix} 0 & +\frac{\sqrt{M_1}}{\sqrt{M_2}} & 0 \\ +\frac{\sqrt{M_2}}{\sqrt{M_1}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

- $F_2 = \begin{pmatrix} 0 & +\frac{\sqrt{M_1}}{\sqrt{M_2}} & 0 \\ -\frac{\sqrt{M_2}}{\sqrt{M_1}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,
- $F_3 = \begin{pmatrix} 0 & -\frac{\sqrt{M_1}}{\sqrt{M_2}} & 0 \\ +\frac{\sqrt{M_2}}{\sqrt{M_1}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,
- $F_4 = \begin{pmatrix} 0 & -\frac{\sqrt{M_1}}{\sqrt{M_2}} & 0 \\ -\frac{\sqrt{M_2}}{\sqrt{M_1}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

- The second class of F has the texture $\begin{pmatrix} 0 & 0 & \times \\ 0 & 1 & 0 \\ \times & 0 & 0 \end{pmatrix}$:
  - $F_5 = \begin{pmatrix} 0 & 0 & +\frac{\sqrt{M_1}}{\sqrt{M_3}} \\ 0 & 1 & 0 \\ +\frac{\sqrt{M_3}}{\sqrt{M_1}} & 0 & 0 \end{pmatrix}$,
  - $F_6 = \begin{pmatrix} 0 & 0 & +\frac{\sqrt{M_1}}{\sqrt{M_3}} \\ 0 & 1 & 0 \\ -\frac{\sqrt{M_3}}{\sqrt{M_1}} & 0 & 0 \end{pmatrix}$,
  - $F_7 = \begin{pmatrix} 0 & 0 & -\frac{\sqrt{M_1}}{\sqrt{M_3}} \\ 0 & 1 & 0 \\ +\frac{\sqrt{M_3}}{\sqrt{M_1}} & 0 & 0 \end{pmatrix}$,
  - $F_8 = \begin{pmatrix} 0 & 0 & -\frac{\sqrt{M_1}}{\sqrt{M_3}} \\ 0 & 1 & 0 \\ -\frac{\sqrt{M_3}}{\sqrt{M_1}} & 0 & 0 \end{pmatrix}$.

- The third class of F has the texture $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & 0 \end{pmatrix}$:
  - $F_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & +\frac{\sqrt{M_2}}{\sqrt{M_3}} \\ 0 & +\frac{\sqrt{M_3}}{\sqrt{M_2}} & 0 \end{pmatrix}$,
  - $F_{10} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & +\frac{\sqrt{M_2}}{\sqrt{M_3}} \\ 0 & -\frac{\sqrt{M_3}}{\sqrt{M_2}} & 0 \end{pmatrix}$,
  - $F_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{M_2}}{\sqrt{M_3}} \\ 0 & +\frac{\sqrt{M_3}}{\sqrt{M_2}} & 0 \end{pmatrix}$,
  - $F_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{M_2}}{\sqrt{M_3}} \\ 0 & -\frac{\sqrt{M_3}}{\sqrt{M_2}} & 0 \end{pmatrix}$.

- The fourth class of F has the texture $\begin{pmatrix} \times & 0 & \times \\ 0 & 1 & 0 \\ \times & 0 & \times \end{pmatrix}$:

- $F_{13} = \begin{pmatrix} \frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} & 0 & \lambda \\ 0 & 1 & 0 \\ -\frac{\lambda M_3}{M_1} & 0 & \frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}$,
- $F_{14} = \begin{pmatrix} \frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} & 0 & \lambda \\ 0 & 1 & 0 \\ \frac{\lambda M_3}{M_1} & 0 & -\frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}$,
- $F_{15} = \begin{pmatrix} -\frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} & 0 & \lambda \\ 0 & 1 & 0 \\ \frac{\lambda M_3}{M_1} & 0 & \frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}$,
- $F_{16} = \begin{pmatrix} -\frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} & 0 & \lambda \\ 0 & 1 & 0 \\ -\frac{\lambda M_3}{M_1} & 0 & -\frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}$,

where λ is an arbitrary real number.

- The fifth class of F has the texture $\begin{pmatrix} \times & \times & 0 \\ \times & \times & 0 \\ 0 & 0 & 1 \end{pmatrix}$:

- $F_{17} = \begin{pmatrix} -\frac{\sqrt{M_1 - \alpha^2 M_2}}{\sqrt{M_1}} & \alpha & 0 \\ -\frac{\alpha M_2}{M_1} & -\frac{\sqrt{M_1 - \alpha^2 M_2}}{\sqrt{M_1}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$,
- $F_{18} = \begin{pmatrix} -\frac{\sqrt{M_1 - \alpha^2 M_2}}{\sqrt{M_1}} & \alpha & 0 \\ \frac{\alpha M_2}{M_1} & \frac{\sqrt{M_1 - \alpha^2 M_2}}{\sqrt{M_1}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$,
- $F_{19} = \begin{pmatrix} \frac{\sqrt{M_1 - \alpha^2 M_2}}{\sqrt{M_1}} & \alpha & 0 \\ -\frac{\alpha M_2}{M_1} & \frac{\sqrt{M_1 - \alpha^2 M_2}}{\sqrt{M_1}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$,
- $F_{20} = \begin{pmatrix} \frac{\sqrt{M_1 - \alpha^2 M_2}}{\sqrt{M_1}} & \alpha & 0 \\ \frac{\alpha M_2}{M_1} & -\frac{\sqrt{M_1 - \alpha^2 M_2}}{\sqrt{M_1}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

where α is an arbitrary real number.

- The sixth class of F has the texture $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix}$:

- $F_{21} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} & \beta \\ 0 & -\frac{\beta M_3}{M_2} & -\frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}$,

- $F_{22} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & \beta \\ 0 & \frac{\beta M_3}{M_2} & \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}$,
- $F_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & \beta \\ 0 & -\frac{\beta M_3}{M_2} & \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}$,
- $F_{24} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & \beta \\ 0 & \frac{\beta M_3}{M_2} & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}$,

where β is an arbitrary real number.

Due to the existence of the free parameters λ, α, and β, the last three classes have some overlap. For example, by substituting $\lambda = 0$ in F_{13} of the fourth class, or $\alpha = 0$ in F_{19} of the fifth class, or $\beta = 0$ in F_{23} of the sixth class, we will obtain the identity matrix.

3.2. A Typical Scenario: $U = \mathcal{P}U^*$

The first scenario discussed in [44] is $U = \mathcal{P}U^*$, where $\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. By substituting this condition into the exact seesaw formula, we have:

$$(\mathcal{P}U^*)D_\nu(\mathcal{P}U^*)^T + RD_N R^T = \mathbf{0}. \tag{8}$$

By simultaneously left- and right-multiplying $\mathcal{P}$ in the above equation and then taking its complex conjugate, one obtains:

$$UD_\nu U^T + (\mathcal{P}R^*)D_N(\mathcal{P}R^*)^T = \mathbf{0}. \tag{9}$$

Note that we have made use of the properties that D_ν and D_N are both diagonal and real. Comparing the above equation with the previously mentioned exact seesaw formula, one immediately obtains:

$$RD_N R^T = (\mathcal{P}R^*)D_N(\mathcal{P}R^*)^T. \tag{10}$$

The author of [44] claims that the above equation necessarily implies $R = \mathcal{P}R^*$. However, this is obviously not correct, since $RD_N R^T = (\mathcal{P}R^*)D_N(\mathcal{P}R^*)^T$, as a matrix equation, is not a sufficient condition for $R = \mathcal{P}R^*$.

For any of the above-mentioned F_i, the relation $RF_i = \mathcal{P}R^*$ is consistent with $RD_N R^T = (\mathcal{P}R^*)D_N(\mathcal{P}R^*)^T$, since $RF_i = \mathcal{P}R^*$ implies $(RF_i)D_N(RF_i)^T = (\mathcal{P}R^*)D_N(\mathcal{P}R^*)^T$, and we also have $RD_N R^T = (RF_i)D_N(RF_i)^T$. When $\lambda = 0$ in F_{13} of the fourth class, or $\alpha = 0$ in F_{19} of the fifth class, or $\beta = 0$ in F_{23} of the sixth class, the matrix F becomes the identity matrix, with which the relation $RF = \mathcal{P}R^*$ reduces to $R = \mathcal{P}R^*$ and thus restores the result in [44]. Generally, the result in [44] is no more than a special case of all possibilities accommodating $U = \mathcal{P}U^*$.

In Appendix A, we analyze the implications corresponding to each possible F mentioned earlier, which is the core of this paper. The interested reader is strongly encouraged to jump to Appendix A before proceeding further.

4. Regarding the Possible Minimal Flavor Symmetry

In the analysis presented in Appendix A, we can see that there exist nontrivial possibilities that $RF = \mathcal{P}R^*$, with F being not equal to the identity matrix. In this section, we focus on its implications for flavor symmetry.

Note that all F we found earlier have the property that F^2 is the identity matrix. Thus, we have $R = \mathcal{P}R^*F$. By substituting $U = \mathcal{P}U^*$ and $R = \mathcal{P}R^*F$ into the unitary conditions, we find the following properties of S and Q:

$$\mathrm{S} = \mathcal{T}\mathrm{S}^*, \tag{11}$$

$$\mathrm{Q} = \mathcal{T}\mathrm{Q}^*F, \tag{12}$$

in which $\mathcal{T}$ is an arbitrary 3×3 unitary matrix. We substitute all these properties of $U, R, \mathrm{S}, \mathrm{Q}$ into:

$$\begin{pmatrix} U & R \\ \mathrm{S} & \mathrm{Q} \end{pmatrix}^\dagger \begin{pmatrix} \mathbf{0} & M_\mathrm{D} \\ M_\mathrm{D}^T & M_R \end{pmatrix} \begin{pmatrix} U & R \\ \mathrm{S} & \mathrm{Q} \end{pmatrix}^* = \begin{pmatrix} D_\nu & \mathbf{0} \\ \mathbf{0} & D_N \end{pmatrix}, \tag{13}$$

and then obtain:

$$\begin{pmatrix} \mathcal{P}U^* & \mathcal{P}R^*F \\ \mathcal{T}\mathrm{S}^* & \mathcal{T}\mathrm{Q}^*F \end{pmatrix}^\dagger \begin{pmatrix} \mathbf{0} & M_\mathrm{D} \\ M_\mathrm{D}^T & M_R \end{pmatrix} \begin{pmatrix} \mathcal{P}U^* & \mathcal{P}R^*F \\ \mathcal{T}\mathrm{S}^* & \mathcal{T}\mathrm{Q}^*F \end{pmatrix}^* = \begin{pmatrix} D_\nu & \mathbf{0} \\ \mathbf{0} & D_N \end{pmatrix}. \tag{14}$$

It is easy to notice that $\begin{pmatrix} \mathcal{P}U^* & \mathcal{P}R^*F \\ \mathcal{T}\mathrm{S}^* & \mathcal{T}\mathrm{Q}^*F \end{pmatrix} = \begin{pmatrix} \mathcal{P} & \mathbf{0} \\ \mathbf{0} & \mathcal{T} \end{pmatrix} \begin{pmatrix} U^* & R^* \\ \mathrm{S}^* & \mathrm{Q}^* \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & F \end{pmatrix}$. Thus, Equation (14) can be further rewritten as:

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & F \end{pmatrix}^T \begin{pmatrix} U & R \\ \mathrm{S} & \mathrm{Q} \end{pmatrix}^\dagger \begin{pmatrix} \mathbf{0} & \mathcal{P}M_\mathrm{D}^*\mathcal{T} \\ \mathcal{T}^T M_\mathrm{D}^\dagger \mathcal{P} & \mathcal{T}^T M_R^* \mathcal{T} \end{pmatrix} \begin{pmatrix} U & R \\ \mathrm{S} & \mathrm{Q} \end{pmatrix}^* \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & F \end{pmatrix} = \begin{pmatrix} D_\nu & \mathbf{0} \\ \mathbf{0} & D_N \end{pmatrix}. \tag{15}$$

Due to the existence of $\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & F \end{pmatrix}^T$ and $\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & F \end{pmatrix}$ in the left-hand side of Equation (15), we cannot make a direct comparison between Equations (13) and (15) to obtain constraint conditions for M_D and M_R, as claimed in [44].

Similar analysis can be applied to the case of $U = \mathcal{P}U\zeta$, with ζ being any of those eight diagonal matrices with $+1$ or -1 at its diagonal positions. Here, we choose F_1 as an example. By substituting $U = \mathcal{P}U\zeta$ into the exact seesaw formula, we have:

$$(\mathcal{P}U\zeta)D_\nu(\mathcal{P}U\zeta)^T + RD_N R^T = \mathbf{0}. \tag{16}$$

It is easy to see that $\zeta D_\nu \zeta^T = D_\nu$ and $\zeta D_N \zeta^T = D_N$. By simultaneously left- and right-multiplying $\mathcal{P}$ in the above equation, one obtains:

$$UD_\nu U^T + (\mathcal{P}R\zeta')D_N(\mathcal{P}R\zeta')^T = \mathbf{0}, \tag{17}$$

in which ζ' is any of those eight diagonal matrices with $+1$ or -1 at the diagonal positions, being independent on ζ.

Again, we cannot directly compare this with the original exact seesaw formula and conclude that it necessarily implies $R = \mathcal{P}R\zeta'$, since they are matrix equations. For any of the possibilities of F satisfying $FD_N F^T$, the relation $RF = \mathcal{P}R\zeta'$ is consistent with

$RD_NR^T = (\mathcal{P}R\zeta')D_N(\mathcal{P}R\zeta')^T$. For the sake of convenience, we denote the diagonal entries of ζ' as $\eta'_1, \eta'_2, \eta'_3$. For $F = F_1$, we have:

$$R\begin{pmatrix} 0 & +\frac{\sqrt{M_1}}{\sqrt{M_2}} & 0 \\ +\frac{\sqrt{M_2}}{\sqrt{M_1}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R \begin{pmatrix} \eta'_1 & 0 & 0 \\ 0 & \eta'_2 & 0 \\ 0 & 0 & \eta'_3 \end{pmatrix}, \tag{18}$$

which implies:

$$\begin{aligned} &\sqrt{M_2}R_{e2} = \eta'_1\sqrt{M_1}R_{e1},\ \sqrt{M_1}R_{e1} = \eta'_2\sqrt{M_2}R_{e2},\ R_{e3} = \eta'_3R_{e3}, \\ &\sqrt{M_2}R_{\mu 2} = \eta'_1\sqrt{M_1}R_{\tau 1},\ \sqrt{M_1}R_{\mu 1} = \eta'_2\sqrt{M_2}R_{\tau 2},\ R_{\mu 3} = \eta'_3R_{\tau 3}, \\ &\sqrt{M_2}R_{\tau 2} = \eta'_1\sqrt{M_1}R_{\mu 1},\ \sqrt{M_1}R_{\tau 1} = \eta'_2\sqrt{M_2}R_{\mu 2},\ R_{\tau 3} = \eta'_3R_{\mu 3}. \end{aligned} \tag{19}$$

There exist nontrivial possibilities when, for example, $\eta'_1 = \eta'_2 = -1$ and $\eta'_3 = 1$.

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U\zeta$, left- and right-multiplying both sides by $\mathcal{P}$, and substituting $\mathcal{P}R = RF_1\zeta'$, one can obtain the following relation from the unitary condition:

$$RF_1F_1^\dagger R^\dagger = RR^\dagger. \tag{20}$$

This relation can be satisfied by some nontrivial possibilities when $\eta'_1 = \eta'_2 = -1$ and $\eta'_3 = 1$, such as the first case, $R = \begin{pmatrix} 0 & 0 & R_{e3} \\ R_{\mu 1} & -\frac{\sqrt{M_1}}{\sqrt{M_2}}R_{\tau 1} & R_{\mu 3} \\ R_{\tau 1} & -\frac{\sqrt{M_1}}{\sqrt{M_2}}R_{\mu 1} & R_{\mu 3} \end{pmatrix}$ with $R_{\mu 1} = R^*_{\tau 1}$ and arbitrary positive M_1, M_2, or the second case, any R satisfying Equation (19) with $0 < M_1 = M_2$. The former automatically satisfies $|R_{\mu i}| = |R_{\tau i}|$ for $i = 1,2,3$ but not necessarily $R = \mathcal{P}R\zeta'$. Now, we focus on the latter. Such degeneracy between two heavy Majorana neutrinos is possible in the canonical seesaw mechanism. In this situation, $|R_{\mu i}|$ is not necessarily equal to $|R_{\tau i}|$ for $i = 1,2$. However, with the degree of freedom to choose eigenstates for degenerate eigenvalue $M_1 = M_2$, one can eventually obtain $|R_{\mu i}| = |R_{\tau i}|$ for $i = 1,2,3$. However, in order to reach $R = \mathcal{P}R\zeta'$, we need to have $R_{\mu 1} = R_{\mu 2} = R^*_{\tau 1} = R^*_{\tau 2}$ in some mass eigenbasis. This can happen *only if* the states $|\nu_\mu\rangle - \sum_{i=1}^3 U^*_{\mu i}|\nu_i\rangle - R^*_{\mu 3}|N_3\rangle$ and $|\nu_\tau\rangle - \sum_{i=1}^3 U^*_{\tau i}|\nu_i\rangle - R^*_{\tau 3}|N_3\rangle$ are the same state (up to an overall factor). Therefore, for a general situation, we only have $RF_1 = \mathcal{P}R\zeta'$. By substituting $U = \mathcal{P}U\zeta$ and $RF_1 = \mathcal{P}R\zeta'$ back into the unitary conditions, we can obtain:

$$S = \mathcal{T}'S\zeta, \tag{21}$$

$$Q = \mathcal{T}'Q\zeta'F_1. \tag{22}$$

Similar relations can be obtained for other possibilities of F, as in the scenario of $U = \mathcal{P}U^*$.

5. Discussion

In the analysis presented in the previous sections and Appendix A, we have shown that, although the experimentally favored relation $|U_{\mu i}| = |U_{\tau i}|$ with $i = 1,2,3$ can lead to the implications $|R_{\mu i}| = |R_{\tau i}|$ with $i = 1,2,3$ in the context of the canonical seesaw mechanism, the further implication that $R = \mathcal{P}R^*$ in the typical scenario $U = \mathcal{P}U^*$ is generally not guaranteed. For the sake of rigor, all possible cases should be considered. In order to support the previously mentioned minimal flavor symmetry claimed in [44], if indeed it exists, we need more evidence or hints from experiments.

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Appendix A

Appendix A.1. F_1

For $F_1 = \begin{pmatrix} 0 & +\frac{\sqrt{M_1}}{\sqrt{M_2}} & 0 \\ +\frac{\sqrt{M_2}}{\sqrt{M_1}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, the relation $RF_1 = \mathcal{P}R^*$ is:

$$R\begin{pmatrix} 0 & +\frac{\sqrt{M_1}}{\sqrt{M_2}} & 0 \\ +\frac{\sqrt{M_2}}{\sqrt{M_1}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R^*. \tag{A1}$$

The implications include:

$$\sqrt{M_1}R_{e1} = \sqrt{M_2}R^*_{e2},\ \sqrt{M_2}R_{\mu 2} = \sqrt{M_1}R^*_{\tau 1},\ \sqrt{M_1}R_{\mu 1} = \sqrt{M_2}R^*_{\tau 2},$$
$$R_{e3} = R^*_{e3},\ R_{\mu 3} = R^*_{\tau 3}. \tag{A2}$$

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U^*$, left- and right-multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_1 = \mathcal{P}R^*$, one can obtain the following relation from the unitary condition:

$$RF_1F_1^\dagger R^\dagger = RR^\dagger. \tag{A3}$$

This relation can be satisfied by some nontrivial possibilities, such as the first case, $R = \begin{pmatrix} 0 & 0 & R_{e3} \\ R_{\mu 1} & \frac{\sqrt{M_1}}{\sqrt{M_2}}R^*_{\tau 1} & R_{\mu 3} \\ R_{\tau 1} & \frac{\sqrt{M_1}}{\sqrt{M_2}}R^*_{\mu 1} & R^*_{\mu 3} \end{pmatrix}$ with $|R_{\mu 1}| = |R_{\tau 1}|$ and arbitrary positive M_1, M_2, or the second case, any R satisfying Equation (A2) with $0 < M_1 = M_2$. The former automatically satisfies $|R_{\mu i}| = |R_{\tau i}|$ for $i = 1,2,3$ but not necessarily $R = \mathcal{P}R^*$. Now, we focus on the latter. Such degeneracy between two heavy Majorana neutrinos is possible in the canonical seesaw mechanism. In this situation, $|R_{\mu i}|$ is not necessarily equal to $|R_{\tau i}|$ for $i = 1,2$. However, with the degree of freedom to choose eigenstates for degenerate eigenvalue $M_1 = M_2$, one can eventually obtain $|R_{\mu i}| = |R_{\tau i}|$ for $i = 1,2,3$. However, in order to reach $R = \mathcal{P}R^*$, we need to have $R_{\mu 1} = R_{\mu 2} = R^*_{\tau 1} = R^*_{\tau 2}$ in some mass eigenbasis. This can happen *only if* the states $|\nu_\mu\rangle - \sum_{i=1}^3 U^*_{\mu i}|\nu_i\rangle - R^*_{\mu 3}|N_3\rangle$ and $|\nu_\tau\rangle - \sum_{i=1}^3 U^*_{\tau i}|\nu_i\rangle - R^*_{\tau 3}|N_3\rangle$ are the same state (up to an overall factor). Therefore, for a general situation, we only have $RF_1 = \mathcal{P}R^*$.

Appendix A.2. F_2

For $F_2 = \begin{pmatrix} 0 & +\frac{\sqrt{M_1}}{\sqrt{M_2}} & 0 \\ -\frac{\sqrt{M_2}}{\sqrt{M_1}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, the relation $RF_2 = \mathcal{P}R^*$ is:

$$R\begin{pmatrix} 0 & +\frac{\sqrt{M_1}}{\sqrt{M_2}} & 0 \\ -\frac{\sqrt{M_2}}{\sqrt{M_1}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R^*. \tag{A4}$$

The implications include:

$$\begin{gathered} R_{e1} = R_{e2} = R_{\mu 1} = R_{\mu 2} = R_{\tau 1} = R_{\tau 2} = 0, \\ R_{e3} = R_{e3}^*,\ R_{\mu 3} = R_{\tau 3}^*. \end{gathered} \tag{A5}$$

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U^*$, multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_2 = \mathcal{P}R^*$, one can obtain the following relation from the unitary condition:

$$RF_2F_2^\dagger R^\dagger = RR^\dagger. \tag{A6}$$

Any R satisfying Equation (A5) will automatically satisfy this relation, with any positive M_1 and M_2. In this case, $|R_{\mu i}| = |R_{\tau i}|$ for $i = 1, 2, 3$ and, furthermore, $R = \mathcal{P}R^*$ is satisfied.

Appendix A.3. F_3

For $F_3 = \begin{pmatrix} 0 & -\frac{\sqrt{M_1}}{\sqrt{M_2}} & 0 \\ +\frac{\sqrt{M_2}}{\sqrt{M_1}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, the relation $RF_3 = \mathcal{P}R^*$ is:

$$R\begin{pmatrix} 0 & -\frac{\sqrt{M_1}}{\sqrt{M_2}} & 0 \\ +\frac{\sqrt{M_2}}{\sqrt{M_1}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R^*. \tag{A7}$$

The implications include:

$$\begin{gathered} R_{e1} = R_{e2} = R_{\mu 1} = R_{\mu 2} = R_{\tau 1} = R_{\tau 2} = 0, \\ R_{e3} = R_{e3}^*,\ R_{\mu 3} = R_{\tau 3}^*. \end{gathered} \tag{A8}$$

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U^*$, multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_3 = \mathcal{P}R^*$, one can obtain the following relation from the unitary condition:

$$RF_3F_3^\dagger R^\dagger = RR^\dagger. \tag{A9}$$

Any R satisfying Equation (A8) will automatically satisfy this relation, with any positive M_1 and M_2. In this case, $|R_{\mu i}| = |R_{\tau i}|$ for $i = 1, 2, 3$ and, furthermore, $R = \mathcal{P}R^*$ is satisfied.

Appendix A.4. F_4

For $F_4 = \begin{pmatrix} 0 & -\frac{\sqrt{M_1}}{\sqrt{M_2}} & 0 \\ -\frac{\sqrt{M_2}}{\sqrt{M_1}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, the relation $RF_4 = \mathcal{P}R^*$ is

$$R\begin{pmatrix} 0 & -\frac{\sqrt{M_1}}{\sqrt{M_2}} & 0 \\ -\frac{\sqrt{M_2}}{\sqrt{M_1}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R^* \tag{A10}$$

The implications include:

$$\sqrt{M_1}R_{e1} = -\sqrt{M_2}R^*_{e2},\ \sqrt{M_2}R_{\mu 2} = -\sqrt{M_1}R^*_{\tau 1},\ \sqrt{M_1}R_{\mu 1} = -\sqrt{M_2}R^*_{\tau 2},$$
$$R_{e3} = R^*_{e3},\ R_{\mu 3} = R^*_{\tau 3}. \tag{A11}$$

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U^*$, multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_4 = \mathcal{P}R^*$, one can obtain the following relation from the unitary condition:

$$RF_4F_4^\dagger R^\dagger = RR^\dagger. \tag{A12}$$

This relation can be satisfied by some nontrivial possibilities, such as the first case, $R = \begin{pmatrix} 0 & 0 & R_{e3} \\ R_{\mu 1} & -\frac{\sqrt{M_1}}{\sqrt{M_2}}R^*_{\tau 1} & R_{\mu 3} \\ R_{\tau 1} & -\frac{\sqrt{M_1}}{\sqrt{M_2}}R^*_{\mu 1} & R^*_{\mu 3} \end{pmatrix}$ with $|R_{\mu 1}| = |R_{\tau 1}|$ and arbitrary positive M_1, M_2, or the second case, any R satisfying Equation (A11) with $0 < M_1 = M_2$. The former automatically satisfies $|R_{\mu i}| = |R_{\tau i}|$ for $i = 1, 2, 3$ but not necessarily $R = \mathcal{P}R^*$. Now, we focus on the latter. Such degeneracy between two heavy Majorana neutrinos is possible in the canonical seesaw mechanism. In this situation, $|R_{\mu i}|$ is not necessarily equal to $|R_{\tau i}|$ for $i = 1, 2$. However, with the degree of freedom to choose eigenstates for degenerate eigenvalue $M_1 = M_2$, one can eventually obtain $|R_{\mu i}| = |R_{\tau i}|$ for $i = 1, 2, 3$. However, in order to reach $R = \mathcal{P}R^*$, we need to have $R_{\mu 1} = -R_{\mu 2} = R^*_{\tau 1} = -R^*_{\tau 2}$ in some mass eigenbasis. This can happen *only if* the states $|\nu_\mu\rangle - \sum_{i=1}^3 U^*_{\mu i}|\nu_i\rangle - R^*_{\mu 3}|N_3\rangle$ and $|\nu_\tau\rangle - \sum_{i=1}^3 U^*_{\tau i}|\nu_i\rangle - R^*_{\tau 3}|N_3\rangle$ are the same state (up to an overall factor). Therefore, for a general situation, we only have $RF_4 = \mathcal{P}R^*$.

Appendix A.5. F_5

For $F_5 = \begin{pmatrix} 0 & 0 & +\frac{\sqrt{M_1}}{\sqrt{M_3}} \\ 0 & 1 & 0 \\ +\frac{\sqrt{M_3}}{\sqrt{M_1}} & 0 & 0 \end{pmatrix}$, the relation $RF_5 = \mathcal{P}R^*$ is

$$R\begin{pmatrix} 0 & 0 & +\frac{\sqrt{M_1}}{\sqrt{M_3}} \\ 0 & 1 & 0 \\ +\frac{\sqrt{M_3}}{\sqrt{M_1}} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R^*. \tag{A13}$$

The implications include:

$$\sqrt{M_1}R_{e1} = \sqrt{M_3}R^*_{e3},\ \sqrt{M_3}R_{\mu 3} = \sqrt{M_1}R^*_{\tau 1},\ \sqrt{M_1}R_{\mu 1} = \sqrt{M_3}R^*_{\tau 3},$$
$$R_{e2} = R^*_{e2},\ R_{\mu 2} = R^*_{\tau 2}. \tag{A14}$$

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U^*$, multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_5 = \mathcal{P}R^*$, one can obtain the following relation from the unitary condition:

$$RF_5F_5^\dagger R^\dagger = RR^\dagger. \tag{A15}$$

This relation can be satisfied by some nontrivial possibilities, such as the first case, $R = \begin{pmatrix} 0 & R_{e2} & 0 \\ R_{\mu 1} & R_{\mu 2} & \frac{\sqrt{M_1}}{\sqrt{M_3}}R^*_{\tau 1} \\ R_{\tau 1} & R^*_{\mu 2} & \frac{\sqrt{M_1}}{\sqrt{M_3}}R^*_{\mu 1} \end{pmatrix}$ with $|R_{\mu 1}| = |R_{\tau 1}|$ and arbitrary positive M_1, M_3, or the second case, any R satisfying Equation (A14) with $0 < M_1 = M_3$. The former automatically satisfies $|R_{\mu i}| = |R_{\tau i}|$ for $i = 1,2,3$ but not necessarily $R = \mathcal{P}R^*$. Now, we focus on the latter. Such degeneracy between two heavy Majorana neutrinos is possible in the canonical seesaw mechanism. In this situation, $|R_{\mu i}|$ is not necessarily equal to $|R_{\tau i}|$ for $i = 1,3$. However, with the degree of freedom to choose eigenstates for degenerate eigenvalue $M_1 = M_3$, one can eventually obtain $|R_{\mu i}| = |R_{\tau i}|$ for $i = 1,2,3$. However, in order to reach $R = \mathcal{P}R^*$, we need to have $R_{\mu 1} = R_{\mu 3} = R^*_{\tau 1} = R^*_{\tau 3}$ in some mass eigenbasis. This can happen *only if* the states $|\nu_\mu\rangle - \sum_{i=1}^{3} U^*_{\mu i}|\nu_i\rangle - R^*_{\mu 2}|N_2\rangle$ and $|\nu_\tau\rangle - \sum_{i=1}^{3} U^*_{\tau i}|\nu_i\rangle - R^*_{\tau 2}|N_2\rangle$ are the same state (up to an overall factor). Therefore, for a general situation, we only have $RF_5 = \mathcal{P}R^*$.

Appendix A.6. F_6

For $F_6 = \begin{pmatrix} 0 & 0 & +\frac{\sqrt{M_1}}{\sqrt{M_3}} \\ 0 & 1 & 0 \\ -\frac{\sqrt{M_3}}{\sqrt{M_1}} & 0 & 0 \end{pmatrix}$, the relation $RF_6 = \mathcal{P}R^*$ is

$$R\begin{pmatrix} 0 & 0 & +\frac{\sqrt{M_1}}{\sqrt{M_3}} \\ 0 & 1 & 0 \\ -\frac{\sqrt{M_3}}{\sqrt{M_1}} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R^*. \tag{A16}$$

The implications include:

$$\begin{aligned} R_{e1} = R_{e3} = R_{\mu 1} = R_{\mu 3} = R_{\tau 1} = R_{\tau 3} = 0, \\ R_{e2} = R^*_{e2},\ R_{\mu 2} = R^*_{\tau 2}. \end{aligned} \tag{A17}$$

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U^*$, multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_6 = \mathcal{P}R^*$, one can obtain the following relation from the unitary condition:

$$RF_6F_6^\dagger R^\dagger = RR^\dagger. \tag{A18}$$

Any R satisfying Equation (A17) will automatically satisfy this relation, with any positive M_1 and M_3. In this case, $|R_{\mu i}| = |R_{\tau i}|$ for $i = 1,2,3$ and, furthermore, $R = \mathcal{P}R^*$ is satisfied.

Appendix A.7. F_7

For $F_7 = \begin{pmatrix} 0 & 0 & -\frac{\sqrt{M_1}}{\sqrt{M_3}} \\ 0 & 1 & 0 \\ +\frac{\sqrt{M_3}}{\sqrt{M_1}} & 0 & 0 \end{pmatrix}$, the relation $RF_7 = \mathcal{P}R^*$ is

$$R\begin{pmatrix} 0 & 0 & -\frac{\sqrt{M_1}}{\sqrt{M_3}} \\ 0 & 1 & 0 \\ +\frac{\sqrt{M_3}}{\sqrt{M_1}} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R^*. \tag{A19}$$

The implications include:

$$\begin{gathered} R_{e1} = R_{e3} = R_{\mu 1} = R_{\mu 3} = R_{\tau 1} = R_{\tau 3} = 0, \\ R_{e2} = R_{e2}^*, \; R_{\mu 2} = R_{\tau 2}^*. \end{gathered} \tag{A20}$$

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U^*$, multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_7 = \mathcal{P}R^*$, one can obtain the following relation from the unitary condition:

$$RF_7F_7^\dagger R^\dagger = RR^\dagger. \tag{A21}$$

Any R satisfying Equation (A20) will automatically satisfy this relation, with any positive M_1 and M_3. In this case, $|R_{\mu i}| = |R_{\tau i}|$ for $i = 1,2,3$ and furthermore $R = \mathcal{P}R^*$ is satisfied.

Appendix A.8. F_8

For $F_8 = \begin{pmatrix} 0 & 0 & -\frac{\sqrt{M_1}}{\sqrt{M_3}} \\ 0 & 1 & 0 \\ -\frac{\sqrt{M_3}}{\sqrt{M_1}} & 0 & 0 \end{pmatrix}$, the relation $RF_8 = \mathcal{P}R^*$ is

$$R\begin{pmatrix} 0 & 0 & -\frac{\sqrt{M_1}}{\sqrt{M_3}} \\ 0 & 1 & 0 \\ -\frac{\sqrt{M_3}}{\sqrt{M_1}} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R^* \tag{A22}$$

The implications include:

$$\begin{gathered} \sqrt{M_1}R_{e1} = -\sqrt{M_3}R_{e3}^*, \; \sqrt{M_3}R_{\mu 3} = -\sqrt{M_1}R_{\tau 1}^*, \; \sqrt{M_1}R_{\mu 1} = -\sqrt{M_3}R_{\tau 3}^*, \\ R_{e2} = R_{e2}^*, \; R_{\mu 2} = R_{\tau 2}^*. \end{gathered} \tag{A23}$$

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U^*$, multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_8 = \mathcal{P}R^*$, one can obtain the following relation from the unitary condition:

$$RF_8F_8^\dagger R^\dagger = RR^\dagger. \tag{A24}$$

This relation can be satisfied by some nontrivial possibilities, such as the first case, $R = \begin{pmatrix} 0 & R_{e2} & 0 \\ R_{\mu 1} & R_{\mu 2} & -\frac{\sqrt{M_1}}{\sqrt{M_3}}R_{\tau 1}^* \\ R_{\tau 1} & R_{\mu 2}^* & -\frac{\sqrt{M_1}}{\sqrt{M_3}}R_{\mu 1}^* \end{pmatrix}$ with $|R_{\mu 1}| = |R_{\tau 1}|$ and arbitrary positive M_1, M_3, or the second case, any R satisfying Equation (A23) with $0 < M_1 = M_3$. The former automatically satisfies $|R_{\mu i}| = |R_{\tau i}|$ for $i = 1,2,3$ but not necessarily $R = \mathcal{P}R^*$. Now, we focus on the latter. Such degeneracy between two heavy Majorana neutrinos is possible in the canonical seesaw mechanism. In this situation, $|R_{\mu i}|$ is not necessarily equal to $|R_{\tau i}|$ for $i = 1, 2$. However, with the degree of freedom to choose eigenstates for degenerate eigenvalue $M_1 = M_3$, one can eventually obtain $|R_{\mu i}| = |R_{\tau i}|$ for $i = 1, 2, 3$. However, in order to reach $R = \mathcal{P}R^*$, we need to have $R_{\mu 1} = R_{\mu 3} = -R^*_{\tau 1} = -R^*_{\tau 3}$ in some mass eigenbasis. This can happen *only if* the states $|\nu_\mu\rangle - \sum_{i=1}^{3} U^*_{\mu i}\,|\nu_i\rangle - R^*_{\mu 2}\,|N_2\rangle$ and $|\nu_\tau\rangle - \sum_{i=1}^{3} U^*_{\tau i}\,|\nu_i\rangle - R^*_{\tau 2}\,|N_2\rangle$ are the same state (up to an overall factor). Therefore, for a general situation, we only have $RF_8 = \mathcal{P}R^*$.

Appendix A.9. F_9

For $F_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & +\frac{\sqrt{M_2}}{\sqrt{M_3}} \\ 0 & +\frac{\sqrt{M_3}}{\sqrt{M_2}} & 0 \end{pmatrix}$, the relation $RF_9 = \mathcal{P}R^*$ is

$$R\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & +\frac{\sqrt{M_2}}{\sqrt{M_3}} \\ 0 & +\frac{\sqrt{M_3}}{\sqrt{M_2}} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R^*. \tag{A25}$$

The implications include:

$$\sqrt{M_2}R_{e2} = \sqrt{M_3}R^*_{e3},\ \sqrt{M_3}R_{\mu 3} = \sqrt{M_2}R^*_{\tau 2},\ \sqrt{M_2}R_{\mu 2} = \sqrt{M_3}R^*_{\tau 3},$$
$$R_{e1} = R^*_{e1},\ R_{\mu 1} = R^*_{\tau 1}. \tag{A26}$$

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U^*$, multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_9 = \mathcal{P}R^*$, one can obtain the following relation from the unitary condition:

$$RF_9F_9^\dagger R^\dagger = RR^\dagger. \tag{A27}$$

This relation can be satisfied by some nontrivial possibilities, such as the first case, $R = \begin{pmatrix} R_{e1} & 0 & 0 \\ R_{\mu 1} & R_{\mu 2} & \frac{\sqrt{M_2}}{\sqrt{M_3}}R^*_{\tau 2} \\ R^*_{\mu 1} & R_{\tau 2} & \frac{\sqrt{M_2}}{\sqrt{M_3}}R^*_{\mu 2} \end{pmatrix}$ with $|R_{\mu 2}| = |R_{\tau 2}|$ and arbitrary positive M_2, M_3, or the second case, any R satisfying Equation (A26) with $0 < M_2 = M_3$. The former automatically satisfies $|R_{\mu i}| = |R_{\tau i}|$ for $i = 1, 2, 3$ but not necessarily $R = \mathcal{P}R^*$. Now, we focus on the latter. Such degeneracy between two heavy Majorana neutrinos is possible in the canonical seesaw mechanism. In this situation, $|R_{\mu i}|$ is not necessarily equal to $|R_{\tau i}|$ for $i = 2, 3$. However, with the degree of freedom to choose eigenstates for degenerate eigenvalue $M_2 = M_3$, one can eventually obtain $|R_{\mu i}| = |R_{\tau i}|$ for $i = 1, 2, 3$. However, in order to reach $R = \mathcal{P}R^*$, we need to have $R_{\mu 2} = R_{\mu 3} = R^*_{\tau 2} = R^*_{\tau 3}$ in some mass eigenbasis. This can happen *only if* the states $|\nu_\mu\rangle - \sum_{i=1}^{3} U^*_{\mu i}\,|\nu_i\rangle - R^*_{\mu 1}\,|N_1\rangle$ and $|\nu_\tau\rangle - \sum_{i=1}^{3} U^*_{\tau i}\,|\nu_i\rangle - R^*_{\tau 1}\,|N_1\rangle$ are the same state (up to an overall factor). Therefore, for a general situation, we only have $RF_9 = \mathcal{P}R^*$.

Appendix A.10. F_{10}

For $F_{10} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & +\frac{\sqrt{M_2}}{\sqrt{M_3}} \\ 0 & -\frac{\sqrt{M_3}}{\sqrt{M_2}} & 0 \end{pmatrix}$, the relation $RF_{10} = \mathcal{P}R^*$ is

$$R\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & +\frac{\sqrt{M_2}}{\sqrt{M_3}} \\ 0 & -\frac{\sqrt{M_3}}{\sqrt{M_2}} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R^*. \tag{A28}$$

The implications include:

$$\begin{aligned} R_{e2} = R_{e3} = R_{\mu 2} = \ R_{\mu 3} = R_{\tau 2} = R_{\tau 3} = 0, \\ R_{e1} = R^*_{e1},\ R_{\mu 1} = R^*_{\tau 1}. \end{aligned} \tag{A29}$$

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U^*$, multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_{10} = \mathcal{P}R^*$, one can obtain the following relation from the unitary condition:

$$RF_{10}F^\dagger_{10}R^\dagger = RR^\dagger. \tag{A30}$$

Any R satisfying Equation (A29) will automatically satisfy this relation, with any positive M_2 and M_3. In this case, $|R_{\mu i}| = |R_{\tau i}|$ for $i = 1,2,3$ and, furthermore, $R = \mathcal{P}R^*$ is satisfied.

Appendix A.11. F_{11}

For $F_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{M_2}}{\sqrt{M_3}} \\ 0 & +\frac{\sqrt{M_3}}{\sqrt{M_2}} & 0 \end{pmatrix}$, the relation $RF_{11} = \mathcal{P}R^*$ is

$$R\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{M_2}}{\sqrt{M_3}} \\ 0 & +\frac{\sqrt{M_3}}{\sqrt{M_2}} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R^*. \tag{A31}$$

The implications include:

$$\begin{aligned} R_{e2} = R_{e3} = R_{\mu 2} = \ R_{\mu 3} = R_{\tau 2} = R_{\tau 3} = 0, \\ R_{e1} = R^*_{e1},\ R_{\mu 1} = R^*_{\tau 1}. \end{aligned} \tag{A32}$$

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U^*$, multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_{11} = \mathcal{P}R^*$, one can obtain the following relation from the unitary condition:

$$RF_{11}F^\dagger_{11}R^\dagger = RR^\dagger. \tag{A33}$$

Any R satisfying Equation (A32) will automatically satisfy this relation, with any positive M_2 and M_3. In this case, $|R_{\mu i}| = |R_{\tau i}|$ for $i = 1,2,3$ and, furthermore, $R = \mathcal{P}R^*$ is satisfied.

Appendix A.12. F_{12}

For $F_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{M_2}}{\sqrt{M_3}} \\ 0 & -\frac{\sqrt{M_3}}{\sqrt{M_2}} & 0 \end{pmatrix}$, the relation $RF_{12} = \mathcal{P}R^*$ is

$$R\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{M_2}}{\sqrt{M_3}} \\ 0 & -\frac{\sqrt{M_3}}{\sqrt{M_2}} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R^*. \tag{A34}$$

The implications include:

$$\sqrt{M_2}R_{e2} = -\sqrt{M_3}R^*_{e3},\ \sqrt{M_3}R_{\mu 3} = -\sqrt{M_2}R^*_{\tau 2},\ \sqrt{M_2}R_{\mu 2} = -\sqrt{M_3}R^*_{\tau 3},$$
$$R_{e1} = R^*_{e1},\ R_{\mu 1} = R^*_{\tau 1}. \tag{A35}$$

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U^*$, multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_{12} = \mathcal{P}R^*$, one can obtain the following relation from the unitary condition:

$$RF_{12}F^\dagger_{12}R^\dagger = RR^\dagger. \tag{A36}$$

This relation can be satisfied by some nontrivial possibilities, such as the first case, $R = \begin{pmatrix} R_{e1} & 0 & 0 \\ R_{\mu 1} & R_{\mu 2} & -\frac{\sqrt{M_2}}{\sqrt{M_3}}R^*_{\tau 2} \\ R^*_{\mu 1} & R_{\tau 2} & -\frac{\sqrt{M_2}}{\sqrt{M_3}}R^*_{\mu 2} \end{pmatrix}$ with $|R_{\mu 2}| = |R_{\tau 2}|$ and arbitrary positive M_2, M_3, or the second case, any R satisfying Equation (A35) with $0 < M_2 = M_3$. The former automatically satisfies $|R_{\mu i}| = |R_{\tau i}|$ for $i = 1,2,3$ but not necessarily $R = \mathcal{P}R^*$. Now, we focus on the latter. Such degeneracy between two heavy Majorana neutrinos is possible in canonical seesaw mechanism. In this situation, $|R_{\mu i}|$ is not necessarily equal to $|R_{\tau i}|$ for $i = 2,3$. However, with the degree of freedom to choose eigenstates for degenerate eigenvalue $M_2 = M_3$, one can eventually obtain $|R_{\mu i}| = |R_{\tau i}|$ for $i = 1,2,3$. However, in order to reach $R = \mathcal{P}R^*$, we need to have $R_{\mu 2} = R_{\mu 3} = -R^*_{\tau 2} = -R^*_{\tau 3}$ in some mass eigenbasis. This can happen *only if* the states $|\nu_\mu\rangle - \sum_{i=1}^3 U^*_{\mu i}|\nu_i\rangle - R^*_{\mu 1}|N_1\rangle$ and $|\nu_\tau\rangle - \sum_{i=1}^3 U^*_{\tau i}|\nu_i\rangle - R^*_{\tau 1}|N_1\rangle$ are the same state (up to an overall factor). Therefore, for a general situation, we only have $RF_{12} = \mathcal{P}R^*$.

Appendix A.13. F_{13}

For $F_{13} = \begin{pmatrix} \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & 0 & \lambda \\ 0 & 1 & 0 \\ -\frac{\lambda M_3}{M_1} & 0 & \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}$, the relation $RF_{13} = \mathcal{P}R^*$ is

$$R\begin{pmatrix} \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & 0 & \lambda \\ 0 & 1 & 0 \\ -\frac{\lambda M_3}{M_1} & 0 & \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R^*. \tag{A37}$$

The implications include:

$$
\begin{aligned}
&R_{e2} = R^*_{e2},\ R_{\mu 2} = R^*_{\tau 2},\\
&\begin{pmatrix} \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & -\frac{\lambda M_3}{M_1} \\ \lambda & \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} R_{e1} \\ R_{e3} \end{pmatrix} = \begin{pmatrix} R^*_{e1} \\ R^*_{e3} \end{pmatrix},\\
&\begin{pmatrix} \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & -\frac{\lambda M_3}{M_1} \\ \lambda & \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} R_{\mu 1} \\ R_{\mu 3} \end{pmatrix} = \begin{pmatrix} R^*_{\tau 1} \\ R^*_{\tau 3} \end{pmatrix},\\
&\begin{pmatrix} \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & -\frac{\lambda M_3}{M_1} \\ \lambda & \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} R_{\tau 1} \\ R_{\tau 3} \end{pmatrix} = \begin{pmatrix} R^*_{\mu 1} \\ R^*_{\mu 3} \end{pmatrix}.
\end{aligned}
\tag{A38}
$$

If $\lambda = 0$, then F_{13} will reduce to the identity matrix, which corresponds to $R = \mathcal{P}R^*$.

For the case with $0 < \lambda^2 \leq \frac{M_1}{M_3}$, the matrix $\begin{pmatrix} \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & -\frac{\lambda M_3}{M_1} \\ \lambda & \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}$ is a real matrix but not an identity matrix. From the last three equations of Equation (A38), we have:

$$
\begin{pmatrix} \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & -\frac{\lambda M_3}{M_1} \\ \lambda & \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}^2 \begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix} = \begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix}. \tag{A39}
$$

By analyzing the eigenvalues and determinant of $\begin{pmatrix} \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & -\frac{\lambda M_3}{M_1} \\ \lambda & \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}^2$, we can see that, if any column of $\begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix}$ is not a zero column matrix, then λ^2 must be equal to $\frac{M_1}{M_3}$. However, by substituting $\lambda^2 = \frac{M_1}{M_3}$ back into the last three equations of Equation (A38), we eventually obtain $\begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Therefore, for $0 < \lambda^2 \leq \frac{M_1}{M_3}$, $R = \mathcal{P}R^*$ is satisfied.

For the case with $\lambda^2 > \frac{M_1}{M_3}$, the entry $\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}}$ is purely imaginary. By taking the complex conjugate of the last three equations in Equation (A38), we obtain:

$$
\begin{aligned}
&\begin{pmatrix} -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & -\frac{\lambda M_3}{M_1} \\ \lambda & -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & -\frac{\lambda M_3}{M_1} \\ \lambda & \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix}\\
&= \begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix},
\end{aligned}
\tag{A40}
$$

which immediately implies that $R_{e1} = R_{e3} = R_{\mu 1} = R_{\mu 3} = R_{\tau 1} = R_{\tau 3} = 0$, and thus $R = \mathcal{P}R^*$ is satisfied.

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U^*$, multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_{13} = \mathcal{P}R^*$, one can obtain the following relation from the unitary condition:

$$
RF_{13}F_{13}^\dagger R^\dagger = RR^\dagger. \tag{A41}
$$

For the case with $\lambda = 0$, F_{13} is the identity matrix, and thus the above relation is satisfied.

For the cases with $0 < \lambda^2 \leq \frac{M_1}{M_3}$ and with $\lambda^2 > \frac{M_1}{M_3}$, we have shown that $R_{e2} = R^*_{e2}$, $R_{\mu 2} = R^*_{\tau 2}$, and $R_{e1} = R_{e3} = R_{\mu 1} = R_{\mu 3} = R_{\tau 1} = R_{\tau 3} = 0$. By substituting $R = \begin{pmatrix} 0 & R_{e2} & 0 \\ 0 & R_{\mu 2} & 0 \\ 0 & R^*_{\mu 2} & 0 \end{pmatrix}$, we can see that Equation (A41) is always satisfied.

Therefore, for $F = F_{13}$, we recover the conclusions of [44].

Appendix A.14. F_{14}

For $F_{14} = \begin{pmatrix} \frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} & 0 & \lambda \\ 0 & 1 & 0 \\ \frac{\lambda M_3}{M_1} & 0 & -\frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}$, the relation $RF_{14} = \mathcal{P}R^*$ is

$$R \begin{pmatrix} \frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} & 0 & \lambda \\ 0 & 1 & 0 \\ \frac{\lambda M_3}{M_1} & 0 & -\frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R^*. \tag{A42}$$

The implications include:

$$\begin{gathered} R_{e2} = R^*_{e2},\ R_{\mu 2} = R^*_{\tau 2}, \\ \begin{pmatrix} \frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} & \frac{\lambda M_3}{M_1} \\ \lambda & -\frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} R_{e1} \\ R_{e3} \end{pmatrix} = \begin{pmatrix} R^*_{e1} \\ R^*_{e3} \end{pmatrix}, \\ \begin{pmatrix} \frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} & \frac{\lambda M_3}{M_1} \\ \lambda & -\frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} R_{\mu 1} \\ R_{\mu 3} \end{pmatrix} = \begin{pmatrix} R^*_{\tau 1} \\ R^*_{\tau 3} \end{pmatrix}, \\ \begin{pmatrix} \frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} & \frac{\lambda M_3}{M_1} \\ \lambda & -\frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} R_{\tau 1} \\ R_{\tau 3} \end{pmatrix} = \begin{pmatrix} R^*_{\mu 1} \\ R^*_{\mu 3} \end{pmatrix}. \end{gathered} \tag{A43}$$

If $\lambda = 0$, then F_{14} will reduce to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, with implications including:

$$\begin{gathered} R_{e1} = R^*_{e1},\ R_{e2} = R^*_{e2},\ R_{e3} = -R^*_{e3}, \\ R_{\mu 1} = R^*_{\tau 1},\ R_{\mu 2} = R^*_{\tau 2},\ R_{\mu 3} = -R^*_{\tau 3}. \end{gathered} \tag{A44}$$

It is easy to see that, for $\lambda = 0$, the relation $R = \mathcal{P}R^*$ is generally not satisfied.

For the case with $0 < \lambda^2 \leq \frac{M_1}{M_3}$, the matrix $\begin{pmatrix} \frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} & \frac{\lambda M_3}{M_1} \\ \lambda & -\frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}$ is a real matrix but not an identity matrix. From the last three equations of Equation (A43), we have:

$$\begin{pmatrix} \frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} & \frac{\lambda M_3}{M_1} \\ \lambda & -\frac{\sqrt{M_1 - \lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}^2 \begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix} = \begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix}. \tag{A45}$$

This is trivially true since $\begin{pmatrix} \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & \frac{\lambda M_3}{M_1} \\ \lambda & -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}^2$ is equal to the identity matrix for any positive M_1, M_3, and any real λ.

For the case with $\lambda^2 > \frac{M_1}{M_3}$, the entry $\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}}$ is purely imaginary. By taking the complex conjugate of the last three equations in Equation (A43), we obtain:

$$\begin{pmatrix} -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & \frac{\lambda M_3}{M_1} \\ \lambda & \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & \frac{\lambda M_3}{M_1} \\ \lambda & -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix}$$
$$= \begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix}. \tag{A46}$$

From the eigenvalues of $\begin{pmatrix} -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & \frac{\lambda M_3}{M_1} \\ \lambda & \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & \frac{\lambda M_3}{M_1} \\ \lambda & -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}$, we can see that, if any column of $\begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix}$ is not a zero column matrix, then we must have $2\frac{\lambda^2 M_3}{M_1} - 1 = 1$ and $\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \frac{\sqrt{\lambda^2 M_3}}{\sqrt{M_1}} = 0$, which are impossible when $\lambda^2 > \frac{M_1}{M_3} > 0$. We eventually obtain $\begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Therefore, for $\lambda^2 > \frac{M_1}{M_3}$, $R = \mathcal{P}R^*$ is satisfied.

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U^*$, multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_{14} = \mathcal{P}R^*$, one can obtain the following relation from the unitary condition:

$$RF_{14}F_{14}^\dagger R^\dagger = RR^\dagger. \tag{A47}$$

For the case with $\lambda = 0$, $F_{14}F_{14}^\dagger$ is the identity matrix; thus, the above relation is satisfied for any R satisfying Equation (A44).

For the case with $0 < \lambda^2 \leq \frac{M_1}{M_3}$, there exist nontrivial possibilities satisfying Equation (A47) but not $R = \mathcal{P}R^*$. For example, when $\lambda = \frac{\sqrt{M_1}}{\sqrt{M_3}}$, F_{14} will reduce to $\begin{pmatrix} 0 & 0 & \frac{\sqrt{M_1}}{\sqrt{M_3}} \\ 0 & 1 & 0 \\ \frac{\sqrt{M_3}}{\sqrt{M_1}} & 0 & 0 \end{pmatrix}$, and $R = \begin{pmatrix} \frac{\sqrt{M_3}}{\sqrt{M_1}} R_{e3} & R_{e2} & R_{e3} \\ 0 & R_{\mu 2} & 0 \\ 0 & R_{\mu 2}^* & 0 \end{pmatrix}$ with real R_{e3} and R_{e2} and complex $R_{\mu 2}$ is a solution.

For the case with $\lambda^2 > \frac{M_1}{M_3}$, we have shown that $R_{e2} = R_{e2}^*$, $R_{\mu 2} = R_{\tau 2}^*$, and $R_{e1} = R_{e3} = R_{\mu 1} = R_{\mu 3} = R_{\tau 1} = R_{\tau 3} = 0$. By substituting $R = \begin{pmatrix} 0 & R_{e2} & 0 \\ 0 & R_{\mu 2} & 0 \\ 0 & R_{\mu 2}^* & 0 \end{pmatrix}$, we can see that Equation (A47) is always satisfied.

Therefore, for $F = F_{14}$, the relation $R = \mathcal{P}R^*$ is generally not satisfied.

Appendix A.15. F_{15}

For $F_{15} = \begin{pmatrix} -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & 0 & \lambda \\ 0 & 1 & 0 \\ \frac{\lambda M_3}{M_1} & 0 & \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}$, the relation $RF_{15} = \mathcal{P}R^*$ is

$$R\begin{pmatrix} -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & 0 & \lambda \\ 0 & 1 & 0 \\ \frac{\lambda M_3}{M_1} & 0 & \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R^*. \tag{A48}$$

The implications include:

$$\begin{gathered} R_{e2} = R^*_{e2},\ R_{\mu2} = R^*_{\tau2}, \\ \begin{pmatrix} -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & \frac{\lambda M_3}{M_1} \\ \lambda & \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}\begin{pmatrix} R_{e1} \\ R_{e3} \end{pmatrix} = \begin{pmatrix} R^*_{e1} \\ R^*_{e3} \end{pmatrix}, \\ \begin{pmatrix} -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & \frac{\lambda M_3}{M_1} \\ \lambda & \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}\begin{pmatrix} R_{\mu1} \\ R_{\mu3} \end{pmatrix} = \begin{pmatrix} R^*_{\tau1} \\ R^*_{\tau3} \end{pmatrix}, \\ \begin{pmatrix} -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & \frac{\lambda M_3}{M_1} \\ \lambda & \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}\begin{pmatrix} R_{\tau1} \\ R_{\tau3} \end{pmatrix} = \begin{pmatrix} R^*_{\mu1} \\ R^*_{\mu3} \end{pmatrix}. \end{gathered} \tag{A49}$$

If $\lambda = 0$, then F_{15} will reduce to $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, with implications including:

$$\begin{gathered} R_{e1} = -R^*_{e1},\ R_{e2} = R^*_{e2},\ R_{e3} = R^*_{e3}, \\ R_{\mu1} = -R^*_{\tau1},\ R_{\mu2} = R^*_{\tau2},\ R_{\mu3} = R^*_{\tau3}. \end{gathered} \tag{A50}$$

It is easy to see that, for $\lambda = 0$, the relation $R = \mathcal{P}R^*$ is generally not satisfied.

For the case with $0 < \lambda^2 \leq \frac{M_1}{M_3}$, the matrix $\begin{pmatrix} -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & \frac{\lambda M_3}{M_1} \\ \lambda & \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}$ is a real matrix but not an identity matrix. From the last three equations of Equation (A49), we have:

$$\begin{pmatrix} -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & \frac{\lambda M_3}{M_1} \\ \lambda & \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}^2 \begin{pmatrix} R_{e1} & R_{\mu1} & R_{\tau1} \\ R_{e3} & R_{\mu3} & R_{\tau3} \end{pmatrix} = \begin{pmatrix} R_{e1} & R_{\mu1} & R_{\tau1} \\ R_{e3} & R_{\mu3} & R_{\tau3} \end{pmatrix}. \tag{A51}$$

This is trivially true since $\begin{pmatrix} -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & \frac{\lambda M_3}{M_1} \\ \lambda & \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}^2$ is equal to the identity matrix for any positive M_1, M_3, and any real λ.

For the case with $\lambda^2 > \frac{M_1}{M_3}$, the entry $-\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}}$ is purely imaginary. By taking the complex conjugate of the last three equations in Equation (A49), we obtain:

$$\begin{pmatrix} \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & \frac{\lambda M_3}{M_1} \\ \lambda & -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & \frac{\lambda M_3}{M_1} \\ \lambda & \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix}$$
$$= \begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix}. \tag{A52}$$

From the eigenvalues of $\begin{pmatrix} \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & \frac{\lambda M_3}{M_1} \\ \lambda & -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & \frac{\lambda M_3}{M_1} \\ \lambda & \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}$, we can see that, if any column of $\begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix}$ is not a zero column matrix, then we must have $2\frac{\lambda^2 M_3}{M_1} - 1 = 1$ and $\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \frac{\sqrt{\lambda^2 M_3}}{\sqrt{M_1}} = 0$, which are impossible when $\lambda^2 > \frac{M_1}{M_3} > 0$. We eventually obtain $\begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Therefore, for $\lambda^2 > \frac{M_1}{M_3}$, $R = \mathcal{P}R^*$ is satisfied.

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U^*$, multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_{15} = \mathcal{P}R^*$, one can obtain the following relation from the unitary condition:

$$RF_{15}F_{15}^\dagger R^\dagger = RR^\dagger. \tag{A53}$$

For the case with $\lambda = 0$, $F_{15}F_{15}^\dagger$ is the identity matrix; thus, the above relation is satisfied for any R satisfying Equation (A50).

For the case with $0 < \lambda^2 \le \frac{M_1}{M_3}$, there exist nontrivial possibilities satisfying Equation (A53) but not $R = \mathcal{P}R^*$. For example, when $\lambda = \frac{\sqrt{M_1}}{\sqrt{M_3}}$, F_{15} will reduce to $\begin{pmatrix} 0 & 0 & \frac{\sqrt{M_1}}{\sqrt{M_3}} \\ 0 & 1 & 0 \\ \frac{\sqrt{M_3}}{\sqrt{M_1}} & 0 & 0 \end{pmatrix}$, and $R = \begin{pmatrix} \frac{\sqrt{M_3}}{\sqrt{M_1}} R_{e3} & R_{e2} & R_{e3} \\ 0 & R_{\mu 2} & 0 \\ 0 & R_{\mu 2}^* & 0 \end{pmatrix}$ with real R_{e3} and R_{e2} and complex $R_{\mu 2}$ is a solution.

For the case with $\lambda^2 > \frac{M_1}{M_3}$, we have shown that $R_{e2} = R_{e2}^*$, $R_{\mu 2} = R_{\tau 2}^*$, and $R_{e1} = R_{e3} = R_{\mu 1} = R_{\mu 3} = R_{\tau 1} = R_{\tau 3} = 0$. By substituting $R = \begin{pmatrix} 0 & R_{e2} & 0 \\ 0 & R_{\mu 2} & 0 \\ 0 & R_{\mu 2}^* & 0 \end{pmatrix}$, we can see that Equation (A53) is always satisfied.

Therefore, for $F = F_{15}$, the relation $R = \mathcal{P}R^*$ is generally not satisfied.

Appendix A.16. F_{16}

For $F_{16} = \begin{pmatrix} -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & 0 & \lambda \\ 0 & 1 & 0 \\ -\frac{\lambda M_3}{M_1} & 0 & -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}$, the relation $RF_{16} = \mathcal{P}R^*$ is

$$R \begin{pmatrix} -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & 0 & \lambda \\ 0 & 1 & 0 \\ -\frac{\lambda M_3}{M_1} & 0 & -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R^*. \tag{A54}$$

The implications include:

$$
\begin{aligned}
&R_{e2} = R_{e2}^*,\ R_{\mu2} = R_{\tau2}^*, \\
&\begin{pmatrix} -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & -\frac{\lambda M_3}{M_1} \\ \lambda & -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} R_{e1} \\ R_{e3} \end{pmatrix} = \begin{pmatrix} R_{e1}^* \\ R_{e3}^* \end{pmatrix}, \\
&\begin{pmatrix} -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & -\frac{\lambda M_3}{M_1} \\ \lambda & -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} R_{\mu1} \\ R_{\mu3} \end{pmatrix} = \begin{pmatrix} R_{\tau1}^* \\ R_{\tau3}^* \end{pmatrix}, \\
&\begin{pmatrix} -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & -\frac{\lambda M_3}{M_1} \\ \lambda & -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} R_{\tau1} \\ R_{\tau3} \end{pmatrix} = \begin{pmatrix} R_{\mu1}^* \\ R_{\mu3}^* \end{pmatrix}.
\end{aligned} \tag{A55}
$$

If $\lambda = 0$, then F_{16} will reduce to $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, with implications including:

$$
\begin{aligned}
&R_{e1} = -R_{e1}^*,\ R_{e2} = R_{e2}^*,\ R_{e3} = -R_{e3}^*, \\
&R_{\mu1} = -R_{\tau1}^*,\ R_{\mu2} = R_{\tau2}^*,\ R_{\mu3} = -R_{\tau3}^*.
\end{aligned} \tag{A56}
$$

It is easy to see that, for $\lambda = 0$, the relation $R = \mathcal{P}R^*$ is generally not satisfied.

For the case with $0 < \lambda^2 \leq \frac{M_1}{M_3}$, the matrix $\begin{pmatrix} -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & -\frac{\lambda M_3}{M_1} \\ \lambda & -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}$ is a real matrix but not an identity matrix. From the last three equations of Equation (A55), we have

$$
\begin{pmatrix} -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & -\frac{\lambda M_3}{M_1} \\ \lambda & -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}^2 \begin{pmatrix} R_{e1} & R_{\mu1} & R_{\tau1} \\ R_{e3} & R_{\mu3} & R_{\tau3} \end{pmatrix} = \begin{pmatrix} R_{e1} & R_{\mu1} & R_{\tau1} \\ R_{e3} & R_{\mu3} & R_{\tau3} \end{pmatrix}. \tag{A57}
$$

By analyzing the eigenvalues and determinant of $\begin{pmatrix} -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & -\frac{\lambda M_3}{M_1} \\ \lambda & -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix}^2$, we can see that, if any column of $\begin{pmatrix} R_{e1} & R_{\mu1} & R_{\tau1} \\ R_{e3} & R_{\mu3} & R_{\tau3} \end{pmatrix}$ is not a zero column matrix, then λ^2 must be equal to $\frac{M_1}{M_3}$. However, by substituting $\lambda^2 = \frac{M_1}{M_3}$ back into the last three equations of Equation (A55), we eventually obtain $\begin{pmatrix} R_{e1} & R_{\mu1} & R_{\tau1} \\ R_{e3} & R_{\mu3} & R_{\tau3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Therefore, for $0 < \lambda^2 \leq \frac{M_1}{M_3}$, $R = \mathcal{P}R^*$ is satisfied.

For the case with $\lambda^2 > \frac{M_1}{M_3}$, the entry $-\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}}$ is purely imaginary. By taking the complex conjugate of the last three equations in Equation (A55), we obtain:

$$
\begin{aligned}
&\begin{pmatrix} \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & -\frac{\lambda M_3}{M_1} \\ \lambda & \frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} & -\frac{\lambda M_3}{M_1} \\ \lambda & -\frac{\sqrt{M_1-\lambda^2 M_3}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} R_{e1} & R_{\mu1} & R_{\tau1} \\ R_{e3} & R_{\mu3} & R_{\tau3} \end{pmatrix} \\
&= \begin{pmatrix} R_{e1} & R_{\mu1} & R_{\tau1} \\ R_{e3} & R_{\mu3} & R_{\tau3} \end{pmatrix},
\end{aligned} \tag{A58}
$$

which immediately implies that $R_{e1} = R_{e3} = R_{\mu1} = R_{\mu3} = R_{\tau1} = R_{\tau3} = 0$ and, thus, $R = \mathcal{P}R^*$ is satisfied.

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U^*$, multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_{16} = \mathcal{P}R^*$, one can obtain the following relation from the unitary condition:

$$RF_{16}F_{16}^\dagger R^\dagger = RR^\dagger. \tag{A59}$$

For the case with $\lambda = 0$, $F_{16}F_{16}^\dagger$ is the identity matrix; thus, the above relation is satisfied for any R satisfying Equation (A56).

For the cases with $0 < \lambda^2 \le \frac{M_1}{M_3}$ and with $\lambda^2 > \frac{M_1}{M_3}$, we have shown that $R_{e2} = R_{e2}^*$, $R_{\mu 2} = R_{\tau 2}^*$, and $R_{e1} = R_{e3} = R_{\mu 1} = R_{\mu 3} = R_{\tau 1} = R_{\tau 3} = 0$. By substituting $R = \begin{pmatrix} 0 & R_{e2} & 0 \\ 0 & R_{\mu 2} & 0 \\ 0 & R_{\mu 2}^* & 0 \end{pmatrix}$, we can see that Equation (A59) is always satisfied.

Therefore, for $F = F_{16}$, the relation $R = \mathcal{P}R^*$ is generally not satisfied.

Appendix A.17. F_{17}

For $F_{17} = \begin{pmatrix} -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & \alpha & 0 \\ -\frac{\alpha M_2}{M_1} & -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, the relation $RF_{17} = \mathcal{P}R^*$ is

$$R\begin{pmatrix} -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & \alpha & 0 \\ -\frac{\alpha M_2}{M_1} & -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R^*. \tag{A60}$$

The implications include:

$$\begin{aligned} & R_{e3} = R_{e3}^*,\ R_{\mu 3} = R_{\tau 3}^*, \\ & \begin{pmatrix} -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & -\frac{\alpha M_2}{M_1} \\ \alpha & -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix}\begin{pmatrix} R_{e1} \\ R_{e2} \end{pmatrix} = \begin{pmatrix} R_{e1}^* \\ R_{e2}^* \end{pmatrix}, \\ & \begin{pmatrix} -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & -\frac{\alpha M_2}{M_1} \\ \alpha & -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix}\begin{pmatrix} R_{\mu 1} \\ R_{\mu 2} \end{pmatrix} = \begin{pmatrix} R_{\tau 1}^* \\ R_{\tau 2}^* \end{pmatrix}, \\ & \begin{pmatrix} -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & -\frac{\alpha M_2}{M_1} \\ \alpha & -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix}\begin{pmatrix} R_{\tau 1} \\ R_{\tau 2} \end{pmatrix} = \begin{pmatrix} R_{\mu 1}^* \\ R_{\mu 2}^* \end{pmatrix}. \end{aligned} \tag{A61}$$

If $\alpha = 0$, then F_{17} will reduce to $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, with implications including:

$$\begin{aligned} & R_{e1} = -R_{e1}^*,\ R_{e2} = -R_{e2}^*,\ R_{e3} = R_{e3}^*, \\ & R_{\mu 1} = -R_{\tau 1}^*,\ R_{\mu 2} = -R_{\tau 2}^*,\ R_{\mu 3} = R_{\tau 3}^*. \end{aligned} \tag{A62}$$

It is easy to see that, for $\alpha = 0$, the relation $R = \mathcal{P}R^*$ is generally not satisfied.

For the case with $0 < \alpha^2 \leq \frac{M_1}{M_2}$, the matrix $\begin{pmatrix} -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & -\frac{\alpha M_2}{M_1} \\ \alpha & -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix}$ is a real matrix but not an identity matrix. From the last three equations of Equation (A61), we have:

$$\begin{pmatrix} -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & -\frac{\alpha M_2}{M_1} \\ \alpha & -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix}^2 \begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e2} & R_{\mu 2} & R_{\tau 2} \end{pmatrix} = \begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e2} & R_{\mu 2} & R_{\tau 2} \end{pmatrix}. \tag{A63}$$

By analyzing the eigenvalues and determinant of $\begin{pmatrix} -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & -\frac{\alpha M_2}{M_1} \\ \alpha & -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix}^2$, we can see that, if any column of $\begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e2} & R_{\mu 2} & R_{\tau 2} \end{pmatrix}$ is not a zero column matrix, then α^2 must be equal to $\frac{M_1}{M_2}$. However, by substituting $\alpha^2 = \frac{M_1}{M_2}$ back into the last three equations of Equation (A61), we eventually obtain $\begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e2} & R_{\mu 2} & R_{\tau 2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Therefore, for $0 < \alpha^2 \leq \frac{M_1}{M_2}$, $R = \mathcal{P}R^*$ is satisfied.

For the case with $\alpha^2 > \frac{M_1}{M_2}$, the entry $-\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}}$ is purely imaginary. By taking the complex conjugate of the last three equations in Equation (A61), we obtain:

$$\begin{pmatrix} \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & -\frac{\alpha M_2}{M_1} \\ \alpha & \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & -\frac{\alpha M_2}{M_1} \\ \alpha & -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e2} & R_{\mu 2} & R_{\tau 2} \end{pmatrix}$$
$$= \begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e2} & R_{\mu 2} & R_{\tau 2} \end{pmatrix}, \tag{A64}$$

which immediately implies that $R_{e1} = R_{e2} = R_{\mu 1} = R_{\mu 2} = R_{\tau 1} = R_{\tau 2} = 0$ and, thus, $R = \mathcal{P}R^*$ is satisfied.

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U^*$, multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_{17} = \mathcal{P}R^*$, one can obtain the following relation from the unitary condition:

$$RF_{17}F_{17}^\dagger R^\dagger = RR^\dagger. \tag{A65}$$

For the case with $\alpha = 0$, $F_{17}F_{17}^\dagger$ is the identity matrix; thus, the above relation is satisfied for any R satisfying Equation (A62).

For the cases with $0 < \alpha^2 \leq \frac{M_1}{M_2}$ and with $\alpha^2 > \frac{M_1}{M_2}$, we have shown that $R_{e3} = R_{e3}^*$, $R_{\mu 3} = R_{\tau 3}^*$, and $R_{e1} = R_{e2} = R_{\mu 1} = R_{\mu 2} = R_{\tau 1} = R_{\tau 2} = 0$. By substituting $R = \begin{pmatrix} 0 & 0 & R_{e3} \\ 0 & 0 & R_{\mu 3} \\ 0 & 0 & R_{\mu 3}^* \end{pmatrix}$, we can see that Equation (A65) is always satisfied.

Therefore, for $F = F_{17}$, the relation $R = \mathcal{P}R^*$ is generally not satisfied.

Appendix A.18. F_{18}

For $F_{18} = \begin{pmatrix} -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & \alpha & 0 \\ \frac{\alpha M_2}{M_1} & \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, the relation $RF_{18} = \mathcal{P}R^*$ is

$$R\begin{pmatrix} -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & \alpha & 0 \\ \frac{\alpha M_2}{M_1} & \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R^*. \tag{A66}$$

The implications include:

$$\begin{gathered} R_{e3} = R^*_{e3},\ R_{\mu 3} = R^*_{\tau 3}, \\ \begin{pmatrix} -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & \frac{\alpha M_2}{M_1} \\ \alpha & \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix}\begin{pmatrix} R_{e1} \\ R_{e2} \end{pmatrix} = \begin{pmatrix} R^*_{e1} \\ R^*_{e2} \end{pmatrix}, \\ \begin{pmatrix} -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & \frac{\alpha M_2}{M_1} \\ \alpha & \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix}\begin{pmatrix} R_{\mu 1} \\ R_{\mu 2} \end{pmatrix} = \begin{pmatrix} R^*_{\tau 1} \\ R^*_{\tau 2} \end{pmatrix}, \\ \begin{pmatrix} -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & \frac{\alpha M_2}{M_1} \\ \alpha & \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix}\begin{pmatrix} R_{\tau 1} \\ R_{\tau 2} \end{pmatrix} = \begin{pmatrix} R^*_{\mu 1} \\ R^*_{\mu 2} \end{pmatrix}. \end{gathered} \tag{A67}$$

If $\alpha = 0$, then F_{18} will reduce to $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, with implications including:

$$\begin{gathered} R_{e1} = -R^*_{e1},\ R_{e2} = R^*_{e2},\ R_{e3} = R^*_{e3}, \\ R_{\mu 1} = -R^*_{\tau 1},\ R_{\mu 2} = R^*_{\tau 2},\ R_{\mu 3} = R^*_{\tau 3}. \end{gathered} \tag{A68}$$

It is easy to see that, for $\alpha = 0$, the relation $R = \mathcal{P}R^*$ is generally not satisfied.

For the case with $0 < \alpha^2 \leq \frac{M_1}{M_2}$, the matrix $\begin{pmatrix} -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & \frac{\alpha M_2}{M_1} \\ \alpha & \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix}$ is a real matrix but not an identity matrix. From the last three equations of Equation (A67), we have:

$$\begin{pmatrix} -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & \frac{\alpha M_2}{M_1} \\ \alpha & \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix}^2 \begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e2} & R_{\mu 2} & R_{\tau 2} \end{pmatrix} = \begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e2} & R_{\mu 2} & R_{\tau 2} \end{pmatrix}. \tag{A69}$$

This is trivially true since $\begin{pmatrix} -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & \frac{\alpha M_2}{M_1} \\ \alpha & \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix}^2$ is equal to the identity matrix for any positive M_1, M_2, and any real α.

For the case with $\alpha^2 > \frac{M_1}{M_2}$, the entry $-\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}}$ is purely imaginary. By taking the complex conjugate of the last three equations in Equation (A67), we obtain:

$$\begin{pmatrix} \frac{\sqrt{M_1-\alpha^2M_2}}{\sqrt{M_1}} & \frac{\alpha M_2}{M_1} \\ \alpha & -\frac{\sqrt{M_1-\alpha^2M_2}}{\sqrt{M_1}} \end{pmatrix}\begin{pmatrix} -\frac{\sqrt{M_1-\alpha^2M_2}}{\sqrt{M_1}} & \frac{\alpha M_2}{M_1} \\ \alpha & \frac{\sqrt{M_1-\alpha^2M_2}}{\sqrt{M_1}} \end{pmatrix}\begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e2} & R_{\mu 2} & R_{\tau 2} \end{pmatrix}$$
$$= \begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e2} & R_{\mu 2} & R_{\tau 2} \end{pmatrix}. \tag{A70}$$

From the eigenvalues of $\begin{pmatrix} \frac{\sqrt{M_1-\alpha^2M_2}}{\sqrt{M_1}} & \frac{\alpha M_2}{M_1} \\ \alpha & -\frac{\sqrt{M_1-\alpha^2M_2}}{\sqrt{M_1}} \end{pmatrix}\begin{pmatrix} -\frac{\sqrt{M_1-\alpha^2M_2}}{\sqrt{M_1}} & \frac{\alpha M_2}{M_1} \\ \alpha & \frac{\sqrt{M_1-\alpha^2M_2}}{\sqrt{M_1}} \end{pmatrix}$, we can see that, if any column of $\begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e2} & R_{\mu 2} & R_{\tau 2} \end{pmatrix}$ is not a zero column matrix, then we must have $2\frac{\alpha^2M_2}{M_1} - 1 = 1$ and $\frac{\sqrt{M_1-\alpha^2M_2}}{\sqrt{M_1}}\frac{\sqrt{\alpha^2M_2}}{\sqrt{M_1}} = 0$, which are impossible when $\alpha^2 > \frac{M_1}{M_2} > 0$. We eventually obtain $\begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e2} & R_{\mu 2} & R_{\tau 2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Therefore, for $\alpha^2 > \frac{M_1}{M_2}$, $R = \mathcal{P}R^*$ is satisfied.

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U^*$, multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_{18} = \mathcal{P}R^*$, one can obtain the following relation from the unitary condition:

$$RF_{18}F_{18}^\dagger R^\dagger = RR^\dagger. \tag{A71}$$

For the case with $\alpha = 0$, $F_{18}F_{18}^\dagger$ is the identity matrix; thus, the above relation is satisfied for any R satisfying Equation (A68).

For the case with $0 < \alpha^2 \leq \frac{M_1}{M_2}$, there exist nontrivial possibilities satisfying Equation (A71) but not $R = \mathcal{P}R^*$. For example, when $\alpha = \frac{\sqrt{M_1}}{\sqrt{M_2}}$, F_{18} will reduce to $\begin{pmatrix} 0 & \frac{\sqrt{M_1}}{\sqrt{M_2}} & 0 \\ \frac{\sqrt{M_2}}{\sqrt{M_1}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $R = \begin{pmatrix} \frac{\sqrt{M_2}}{\sqrt{M_1}}R_{e2} & R_{e2} & R_{e3} \\ 0 & 0 & R_{\mu 3} \\ 0 & 0 & R_{\mu 3}^* \end{pmatrix}$ with real R_{e2} and R_{e3} and complex $R_{\mu 3}$ is a solution.

For the case with $\alpha^2 > \frac{M_1}{M_2}$, we have shown that $R_{e3} = R_{e3}^*$, $R_{\mu 3} = R_{\tau 3}^*$, and $R_{e1} = R_{e2} = R_{\mu 1} = R_{\mu 2} = R_{\tau 1} = R_{\tau 2} = 0$. By substituting $R = \begin{pmatrix} 0 & 0 & R_{e3} \\ 0 & 0 & R_{\mu 3} \\ 0 & 0 & R_{\mu 3}^* \end{pmatrix}$, we can see that Equation (A71) is always satisfied.

Therefore, for $F = F_{18}$, the relation $R = \mathcal{P}R^*$ is generally not satisfied.

Appendix A.19. F_{19}

For $F_{19} = \begin{pmatrix} \frac{\sqrt{M_1-\alpha^2M_2}}{\sqrt{M_1}} & \alpha & 0 \\ -\frac{\alpha M_2}{M_1} & \frac{\sqrt{M_1-\alpha^2M_2}}{\sqrt{M_1}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, the relation $RF_{19} = \mathcal{P}R^*$ is

$$R\begin{pmatrix} \frac{\sqrt{M_1-\alpha^2M_2}}{\sqrt{M_1}} & \alpha & 0 \\ -\frac{\alpha M_2}{M_1} & \frac{\sqrt{M_1-\alpha^2M_2}}{\sqrt{M_1}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}R^*. \tag{A72}$$

The implications include:

$$
\begin{aligned}
&R_{e3} = R_{e3}^*,\ R_{\mu 3} = R_{\tau 3}^*,\\
&\begin{pmatrix} \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & -\frac{\alpha M_2}{M_1} \\ \alpha & \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} R_{e1} \\ R_{e2} \end{pmatrix} = \begin{pmatrix} R_{e1}^* \\ R_{e2}^* \end{pmatrix},\\
&\begin{pmatrix} \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & -\frac{\alpha M_2}{M_1} \\ \alpha & \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} R_{\mu 1} \\ R_{\mu 2} \end{pmatrix} = \begin{pmatrix} R_{\tau 1}^* \\ R_{\tau 2}^* \end{pmatrix},\\
&\begin{pmatrix} \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & -\frac{\alpha M_2}{M_1} \\ \alpha & \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} R_{\tau 1} \\ R_{\tau 2} \end{pmatrix} = \begin{pmatrix} R_{\mu 1}^* \\ R_{\mu 2}^* \end{pmatrix}.
\end{aligned} \tag{A73}
$$

If $\alpha = 0$, then F_{19} will reduce to the identity matrix, which corresponds to $R = \mathcal{P}R^*$.

For the case with $0 < \alpha^2 \leq \frac{M_1}{M_2}$, the matrix $\begin{pmatrix} \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & -\frac{\alpha M_2}{M_1} \\ \alpha & \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix}$ is a real matrix but not an identity matrix. From the last three equations of Equation (A73), we have:

$$
\begin{pmatrix} \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & -\frac{\alpha M_2}{M_1} \\ \alpha & \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix}^2 \begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e2} & R_{\mu 2} & R_{\tau 2} \end{pmatrix} = \begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e2} & R_{\mu 2} & R_{\tau 2} \end{pmatrix}. \tag{A74}
$$

By analyzing the eigenvalues and determinant of $\begin{pmatrix} \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & -\frac{\alpha M_2}{M_1} \\ \alpha & \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix}^2$, we can see that, if any column of $\begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e2} & R_{\mu 2} & R_{\tau 2} \end{pmatrix}$ is not a zero column matrix, then α^2 must be equal to $\frac{M_1}{M_2}$. However, by substituting $\alpha^2 = \frac{M_1}{M_2}$ back into the last three equations of Equation (A73), we eventually obtain $\begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e2} & R_{\mu 2} & R_{\tau 2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Therefore, for $0 < \alpha^2 \leq \frac{M_1}{M_2}$, $R = \mathcal{P}R^*$ is satisfied.

For the case with $\alpha^2 > \frac{M_1}{M_2}$, the entry $\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}}$ is purely imaginary. By taking the complex conjugate of the last three equations in Equation (A73), we obtain:

$$
\begin{aligned}
&\begin{pmatrix} -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & -\frac{\alpha M_2}{M_1} \\ \alpha & -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & -\frac{\alpha M_2}{M_1} \\ \alpha & \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e2} & R_{\mu 2} & R_{\tau 2} \end{pmatrix}\\
&= \begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e2} & R_{\mu 2} & R_{\tau 2} \end{pmatrix},
\end{aligned} \tag{A75}
$$

which immediately implies that $R_{e1} = R_{e2} = R_{\mu 1} = R_{\mu 2} = R_{\tau 1} = R_{\tau 2} = 0$ and, thus, $R = \mathcal{P}R^*$ is satisfied.

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U^*$, multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_{19} = \mathcal{P}R^*$, one can obtain the following relation from the unitary condition:

$$
RF_{19}F_{19}^\dagger R^\dagger = RR^\dagger. \tag{A76}
$$

For the case with $\alpha = 0$, F_{19} is the identity matrix; thus, the above relation is satisfied.

For the cases with $0 < \alpha^2 \leq \frac{M_1}{M_2}$ and with $\alpha^2 > \frac{M_1}{M_2}$, we have shown that $R_{e3} = R^*_{e3}$, $R_{\mu3} = R^*_{\tau3}$, and $R_{e1} = R_{e2} = R_{\mu1} = R_{\mu2} = R_{\tau1} = R_{\tau2} = 0$. By substituting $R = \begin{pmatrix} 0 & 0 & R_{e3} \\ 0 & 0 & R_{\mu3} \\ 0 & 0 & R^*_{\mu3} \end{pmatrix}$, we can see that Equation (A76) is always satisfied.

Therefore, for $F = F_{19}$, we recover the conclusions of [44].

Appendix A.20. F_{20}

For $F_{20} = \begin{pmatrix} \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & \alpha & 0 \\ \frac{\alpha M_2}{M_1} & -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, the relation $RF_{20} = \mathcal{P}R^*$ is

$$R \begin{pmatrix} \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & \alpha & 0 \\ \frac{\alpha M_2}{M_1} & -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R^*. \tag{A77}$$

The implications include:

$$\begin{gathered} R_{e3} = R^*_{e3},\ R_{\mu3} = R^*_{\tau3}, \\ \begin{pmatrix} \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & \frac{\alpha M_2}{M_1} \\ \alpha & -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} R_{e1} \\ R_{e2} \end{pmatrix} = \begin{pmatrix} R^*_{e1} \\ R^*_{e2} \end{pmatrix}, \\ \begin{pmatrix} \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & \frac{\alpha M_2}{M_1} \\ \alpha & -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} R_{\mu1} \\ R_{\mu2} \end{pmatrix} = \begin{pmatrix} R^*_{\tau1} \\ R^*_{\tau2} \end{pmatrix}, \\ \begin{pmatrix} \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & \frac{\alpha M_2}{M_1} \\ \alpha & -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} R_{\tau1} \\ R_{\tau2} \end{pmatrix} = \begin{pmatrix} R^*_{\mu1} \\ R^*_{\mu2} \end{pmatrix}. \end{gathered} \tag{A78}$$

If $\alpha = 0$, then F_{20} will reduce to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, with implications including:

$$\begin{gathered} R_{e1} = R^*_{e1},\ R_{e2} = -R^*_{e2},\ R_{e3} = R^*_{e3}, \\ R_{\mu1} = R^*_{\tau1},\ R_{\mu2} = -R^*_{\tau2},\ R_{\mu3} = R^*_{\tau3}. \end{gathered} \tag{A79}$$

It is easy to see that, for $\alpha = 0$, the relation $R = \mathcal{P}R^*$ is generally not satisfied.

For the case with $0 < \alpha^2 \leq \frac{M_1}{M_2}$, the matrix $\begin{pmatrix} \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & \frac{\alpha M_2}{M_1} \\ \alpha & -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix}$ is a real matrix but not an identity matrix. From the last three equations of Equation (A78), we have:

$$\begin{pmatrix} \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & \frac{\alpha M_2}{M_1} \\ \alpha & -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix}^2 \begin{pmatrix} R_{e1} & R_{\mu1} & R_{\tau1} \\ R_{e2} & R_{\mu2} & R_{\tau2} \end{pmatrix} = \begin{pmatrix} R_{e1} & R_{\mu1} & R_{\tau1} \\ R_{e2} & R_{\mu2} & R_{\tau2} \end{pmatrix}. \tag{A80}$$

This is trivially true since $\begin{pmatrix} \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & \frac{\alpha M_2}{M_1} \\ \alpha & -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix}^2$ is equal to the identity matrix for any positive M_1, M_2, and any real α.

For the case with $\alpha^2 > \frac{M_1}{M_2}$, the entry $\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}}$ is purely imaginary. By taking the complex conjugate of the last three equations in Equation (A78), we obtain:

$$\begin{pmatrix} -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & \frac{\alpha M_2}{M_1} \\ \alpha & \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & \frac{\alpha M_2}{M_1} \\ \alpha & -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e2} & R_{\mu 2} & R_{\tau 2} \end{pmatrix} = \begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e2} & R_{\mu 2} & R_{\tau 2} \end{pmatrix}. \tag{A81}$$

From the eigenvalues of $\begin{pmatrix} -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & \frac{\alpha M_2}{M_1} \\ \alpha & \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} & \frac{\alpha M_2}{M_1} \\ \alpha & -\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \end{pmatrix}$, we can see that, if any column of $\begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e2} & R_{\mu 2} & R_{\tau 2} \end{pmatrix}$ is not a zero column matrix, then we must have $2\frac{\alpha^2 M_2}{M_1} - 1 = 1$ and $\frac{\sqrt{M_1-\alpha^2 M_2}}{\sqrt{M_1}} \frac{\sqrt{\alpha^2 M_2}}{\sqrt{M_1}} = 0$, which are impossible when $\alpha^2 > \frac{M_1}{M_2} > 0$. We eventually obtain $\begin{pmatrix} R_{e1} & R_{\mu 1} & R_{\tau 1} \\ R_{e2} & R_{\mu 2} & R_{\tau 2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Therefore, for $\alpha^2 > \frac{M_1}{M_2}$, $R = \mathcal{P}R^*$ is satisfied.

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U^*$, multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_{20} = \mathcal{P}R^*$, one can obtain the following relation from the unitary condition:

$$RF_{20}F_{20}^\dagger R^\dagger = RR^\dagger. \tag{A82}$$

For the case with $\alpha = 0$, $F_{20}F_{20}^\dagger$ is the identity matrix; thus, the above relation is satisfied for any R satisfying Equation (A79).

For the case with $0 < \alpha^2 \le \frac{M_1}{M_2}$, there exist nontrivial possibilities satisfying Equation (A82) but not $R = \mathcal{P}R^*$. For example, when $\alpha = \frac{\sqrt{M_1}}{\sqrt{M_2}}$, F_{20} will reduce to $\begin{pmatrix} 0 & \frac{\sqrt{M_1}}{\sqrt{M_2}} & 0 \\ \frac{\sqrt{M_2}}{\sqrt{M_1}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $R = \begin{pmatrix} \frac{\sqrt{M_2}}{\sqrt{M_1}} R_{e2} & R_{e2} & R_{e3} \\ 0 & 0 & R_{\mu 3} \\ 0 & 0 & R_{\mu 3}^* \end{pmatrix}$ with real R_{e2} and R_{e3} and complex $R_{\mu 3}$ is a solution.

For the case with $\alpha^2 > \frac{M_1}{M_2}$, we have shown that $R_{e3} = R_{e3}^*$, $R_{\mu 3} = R_{\tau 3}^*$, and $R_{e1} = R_{e2} = R_{\mu 1} = R_{\mu 2} = R_{\tau 1} = R_{\tau 2} = 0$. By substituting $R = \begin{pmatrix} 0 & 0 & R_{e3} \\ 0 & 0 & R_{\mu 3} \\ 0 & 0 & R_{\mu 3}^* \end{pmatrix}$, we can see that Equation (A82) is always satisfied.

Therefore, for $F = F_{20}$, the relation $R = \mathcal{P}R^*$ is generally not satisfied.

Appendix A.21. F_{21}

For $F_{21} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & \beta \\ 0 & -\frac{\beta M_3}{M_2} & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}$, the relation $RF_{21} = \mathcal{P}R^*$ is

$$R\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & \beta \\ 0 & -\frac{\beta M_3}{M_2} & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R^*. \tag{A83}$$

The implications include:

$$\begin{aligned} &R_{e1} = R^*_{e1},\ R_{\mu1} = R^*_{\tau1}, \\ &\begin{pmatrix} -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & -\frac{\beta M_3}{M_2} \\ \beta & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}\begin{pmatrix} R_{e2} \\ R_{e3} \end{pmatrix} = \begin{pmatrix} R^*_{e2} \\ R^*_{e3} \end{pmatrix}, \\ &\begin{pmatrix} -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & -\frac{\beta M_3}{M_2} \\ \beta & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}\begin{pmatrix} R_{\mu2} \\ R_{\mu3} \end{pmatrix} = \begin{pmatrix} R^*_{\tau2} \\ R^*_{\tau3} \end{pmatrix}, \\ &\begin{pmatrix} -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & -\frac{\beta M_3}{M_2} \\ \beta & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}\begin{pmatrix} R_{\tau2} \\ R_{\tau3} \end{pmatrix} = \begin{pmatrix} R^*_{\mu2} \\ R^*_{\mu3} \end{pmatrix}. \end{aligned} \tag{A84}$$

If $\beta = 0$, then F_{21} will reduce to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, with implications including:

$$\begin{aligned} R_{e1} = R^*_{e1},\ R_{e2} = -R^*_{e2},\ R_{e3} = -R^*_{e3}, \\ R_{\mu1} = R^*_{\tau1},\ R_{\mu2} = -R^*_{\tau2},\ R_{\mu3} = -R^*_{\tau3}. \end{aligned} \tag{A85}$$

It is easy to see that, for $\beta = 0$, the relation $R = \mathcal{P}R^*$ is generally not satisfied.

For the case with $0 < \beta^2 \leq \frac{M_2}{M_3}$, the matrix $\begin{pmatrix} -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & -\frac{\beta M_3}{M_2} \\ \beta & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}$ is a real matrix but not an identity matrix. From the last three equations of Equation (A84), we have:

$$\begin{pmatrix} -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & -\frac{\beta M_3}{M_2} \\ \beta & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}^2 \begin{pmatrix} R_{e2} & R_{\mu2} & R_{\tau2} \\ R_{e3} & R_{\mu3} & R_{\tau3} \end{pmatrix} = \begin{pmatrix} R_{e2} & R_{\mu2} & R_{\tau2} \\ R_{e3} & R_{\mu3} & R_{\tau3} \end{pmatrix}. \tag{A86}$$

By analyzing the eigenvalues and determinant of $\begin{pmatrix} -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & -\frac{\beta M_3}{M_2} \\ \beta & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}^2$, we can see that, if any column of $\begin{pmatrix} R_{e2} & R_{\mu2} & R_{\tau2} \\ R_{e3} & R_{\mu3} & R_{\tau3} \end{pmatrix}$ is not a zero column matrix, then β^2 must be equal to $\frac{M_2}{M_3}$. However, by substituting $\beta^2 = \frac{M_2}{M_3}$ back into the last three equations of Equation (A84), we eventually obtain $\begin{pmatrix} R_{e2} & R_{\mu2} & R_{\tau2} \\ R_{e3} & R_{\mu3} & R_{\tau3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Therefore, for $0 < \beta^2 \leq \frac{M_2}{M_3}$, $R = \mathcal{P}R^*$ is satisfied.

For the case with $\beta^2 > \frac{M_2}{M_3}$, the entry $-\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}}$ is purely imaginary. By taking the complex conjugate of the last three equations in Equation (A84), we obtain:

$$\begin{pmatrix} \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & -\frac{\beta M_3}{M_2} \\ \beta & \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & -\frac{\beta M_3}{M_2} \\ \beta & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix} \begin{pmatrix} R_{e2} & R_{\mu 2} & R_{\tau 2} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix}$$
$$= \begin{pmatrix} R_{e2} & R_{\mu 2} & R_{\tau 2} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix}, \tag{A87}$$

which immediately implies that $R_{e2} = R_{e3} = R_{\mu 2} = R_{\mu 3} = R_{\tau 2} = R_{\tau 3} = 0$ and, thus, $R = \mathcal{P}R^*$ is satisfied.

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U^*$, multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_{21} = \mathcal{P}R^*$, one can obtain the following relation from the unitary condition:

$$RF_{21}F_{21}^\dagger R^\dagger = RR^\dagger. \tag{A88}$$

For the case with $\beta = 0$, $F_{21}F_{21}^\dagger$ is the identity matrix; thus, the above relation is satisfied for any R satisfying Equation (A85).

For the cases with $0 < \beta^2 \leq \frac{M_2}{M_3}$ and with $\beta^2 > \frac{M_2}{M_3}$, we have shown that $R_{e1} = R_{e1}^*$, $R_{\mu 1} = R_{\tau 1}^*$, and $R_{e2} = R_{e3} = R_{\mu 2} = R_{\mu 3} = R_{\tau 2} = R_{\tau 3} = 0$. By substituting $R = \begin{pmatrix} R_{e1} & 0 & 0 \\ R_{\mu 1} & 0 & 0 \\ R_{\mu 1}^* & 0 & 0 \end{pmatrix}$, we can see that Equation (A88) is always satisfied.

Therefore, for $F = F_{21}$, the relation $R = \mathcal{P}R^*$ is generally not satisfied.

Appendix A.22. F_{22}

For $F_{22} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & \beta \\ 0 & \frac{\beta M_3}{M_2} & \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}$, the relation $RF_{22} = \mathcal{P}R^*$ is

$$R \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & \beta \\ 0 & \frac{\beta M_3}{M_2} & \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R^*. \tag{A89}$$

The implications include:

$$R_{e1} = R_{e1}^*,\ R_{\mu 1} = R_{\tau 1}^*,$$
$$\begin{pmatrix} -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & \frac{\beta M_3}{M_2} \\ \beta & \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix} \begin{pmatrix} R_{e2} \\ R_{e3} \end{pmatrix} = \begin{pmatrix} R_{e2}^* \\ R_{e3}^* \end{pmatrix},$$
$$\begin{pmatrix} -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & \frac{\beta M_3}{M_2} \\ \beta & \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix} \begin{pmatrix} R_{\mu 2} \\ R_{\mu 3} \end{pmatrix} = \begin{pmatrix} R_{\tau 2}^* \\ R_{\tau 3}^* \end{pmatrix},$$
$$\begin{pmatrix} -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & \frac{\beta M_3}{M_2} \\ \beta & \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix} \begin{pmatrix} R_{\tau 2} \\ R_{\tau 3} \end{pmatrix} = \begin{pmatrix} R_{\mu 2}^* \\ R_{\mu 3}^* \end{pmatrix}. \tag{A90}$$

If $\beta = 0$, then F_{22} will reduce to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, with implications including:

$$\begin{aligned} R_{e1} = R_{e1}^*,\ R_{e2} = -R_{e2}^*,\ R_{e3} = R_{e3}^*, \\ R_{\mu 1} = R_{\tau 1}^*,\ R_{\mu 2} = -R_{\tau 2}^*,\ R_{\mu 3} = R_{\tau 3}^*. \end{aligned} \tag{A91}$$

It is easy to see that, for $\beta = 0$, the relation $R = \mathcal{P} R^*$ is generally not satisfied.

For the case with $0 < \beta^2 \leq \frac{M_2}{M_3}$, the matrix $\begin{pmatrix} -\frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} & \frac{\beta M_3}{M_2} \\ \beta & \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}$ is a real matrix but not an identity matrix. From the last three equations of Equation (A90), we have:

$$\begin{pmatrix} -\frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} & \frac{\beta M_3}{M_2} \\ \beta & \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}^2 \begin{pmatrix} R_{e2} & R_{\mu 2} & R_{\tau 2} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix} = \begin{pmatrix} R_{e2} & R_{\mu 2} & R_{\tau 2} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix}. \tag{A92}$$

This is trivially true since $\begin{pmatrix} -\frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} & \frac{\beta M_3}{M_2} \\ \beta & \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}^2$ is equal to the identity matrix for any positive M_2, M_3, and any real β.

For the case with $\beta^2 > \frac{M_2}{M_3}$, the entry $-\frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}}$ is purely imaginary. By taking the complex conjugate of the last three equations in Equation (A90), we obtain:

$$\begin{aligned} &\begin{pmatrix} \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} & \frac{\beta M_3}{M_2} \\ \beta & -\frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} & \frac{\beta M_3}{M_2} \\ \beta & \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix} \begin{pmatrix} R_{e2} & R_{\mu 2} & R_{\tau 2} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix} \\ &= \begin{pmatrix} R_{e2} & R_{\mu 2} & R_{\tau 2} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix}. \end{aligned} \tag{A93}$$

From the eigenvalues of $\begin{pmatrix} \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} & \frac{\beta M_3}{M_2} \\ \beta & -\frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} & \frac{\beta M_3}{M_2} \\ \beta & \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}$, we can see that, if any column of $\begin{pmatrix} R_{e2} & R_{\mu 2} & R_{\tau 2} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix}$ is not a zero column matrix, then we must have $2\frac{\beta^2 M_3}{M_2} - 1 = 1$ and $\frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} \frac{\sqrt{\beta^2 M_3}}{\sqrt{M_2}} = 0$, which are impossible when $\beta^2 > \frac{M_2}{M_3} > 0$. We eventually obtain $\begin{pmatrix} R_{e2} & R_{\mu 2} & R_{\tau 2} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Therefore, for $\beta^2 > \frac{M_2}{M_3}$, $R = \mathcal{P} R^*$ is satisfied.

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P} U^*$, multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_{22} = \mathcal{P} R^*$, one can obtain the following relation from the unitary condition:

$$RF_{22} F_{22}^\dagger R^\dagger = RR^\dagger. \tag{A94}$$

For the case with $\beta = 0$, $F_{22} F_{22}^\dagger$ is the identity matrix; thus, the above relation is satisfied for any R satisfying Equation (A91).

For the case with $0 < \beta^2 \leq \frac{M_2}{M_3}$, there exist nontrivial possibilities satisfying Equation (A94) but not $R = \mathcal{P}R^*$. For example, when $\beta = \frac{\sqrt{M_2}}{\sqrt{M_3}}$, F_{22} will reduce to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{M_2}}{\sqrt{M_3}} \\ 0 & \frac{\sqrt{M_3}}{\sqrt{M_2}} & 0 \end{pmatrix}$, and $R = \begin{pmatrix} R_{e1} & \frac{\sqrt{M_3}}{\sqrt{M_2}} R_{e3} & R_{e3} \\ R_{\mu 1} & 0 & 0 \\ R_{\mu 1}^* & 0 & 0 \end{pmatrix}$ with real R_{e1} and R_{e3} and complex $R_{\mu 1}$ is a solution.

For the case with $\beta^2 > \frac{M_2}{M_3}$, we have shown that $R_{e1} = R_{e1}^*$, $R_{\mu 1} = R_{\tau 1}^*$, and $R_{e2} = R_{e3} = R_{\mu 2} = R_{\mu 3} = R_{\tau 2} = R_{\tau 3} = 0$. By substituting $R = \begin{pmatrix} R_{e1} & 0 & 0 \\ R_{\mu 1} & 0 & 0 \\ R_{\mu 1}^* & 0 & 0 \end{pmatrix}$, we can see that Equation (A94) is always satisfied.

Therefore, for $F = F_{22}$, the relation $R = \mathcal{P}R^*$ is generally not satisfied.

Appendix A.23. F_{23}

For $F_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} & \beta \\ 0 & -\frac{\beta M_3}{M_2} & \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}$, the relation $RF_{23} = \mathcal{P}R^*$ is

$$R \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} & \beta \\ 0 & -\frac{\beta M_3}{M_2} & \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R^*. \tag{A95}$$

The implications include:

$$\begin{gathered} R_{e1} = R_{e1}^*,\ R_{\mu 1} = R_{\tau 1}^*, \\ \begin{pmatrix} \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} & -\frac{\beta M_3}{M_2} \\ \beta & \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix} \begin{pmatrix} R_{e2} \\ R_{e3} \end{pmatrix} = \begin{pmatrix} R_{e2}^* \\ R_{e3}^* \end{pmatrix}, \\ \begin{pmatrix} \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} & -\frac{\beta M_3}{M_2} \\ \beta & \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix} \begin{pmatrix} R_{\mu 2} \\ R_{\mu 3} \end{pmatrix} = \begin{pmatrix} R_{\tau 2}^* \\ R_{\tau 3}^* \end{pmatrix}, \\ \begin{pmatrix} \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} & -\frac{\beta M_3}{M_2} \\ \beta & \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix} \begin{pmatrix} R_{\tau 2} \\ R_{\tau 3} \end{pmatrix} = \begin{pmatrix} R_{\mu 2}^* \\ R_{\mu 3}^* \end{pmatrix}. \end{gathered} \tag{A96}$$

If $\beta = 0$, then F_{23} will reduce to the identity matrix, which corresponds to $R = \mathcal{P}R^*$.

For the case with $0 < \beta^2 \leq \frac{M_2}{M_3}$, the matrix $\begin{pmatrix} \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} & -\frac{\beta M_3}{M_2} \\ \beta & \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}$ is a real matrix but not an identity matrix. From the last three equations of Equation (A96), we have:

$$\begin{pmatrix} \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} & -\frac{\beta M_3}{M_2} \\ \beta & \frac{\sqrt{M_2 - \beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}^2 \begin{pmatrix} R_{e2} & R_{\mu 2} & R_{\tau 2} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix} = \begin{pmatrix} R_{e2} & R_{\mu 2} & R_{\tau 2} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix}. \tag{A97}$$

By analyzing the eigenvalues and determinant of $\begin{pmatrix} \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & -\frac{\beta M_3}{M_2} \\ \beta & \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}^2$, we can see that, if any column of $\begin{pmatrix} R_{e2} & R_{\mu 2} & R_{\tau 2} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix}$ is not a zero column matrix, then β^2 must be equal to $\frac{M_2}{M_3}$. However, by substituting $\beta^2 = \frac{M_2}{M_3}$ back into the last three equations of Equation (A96), we eventually obtain $\begin{pmatrix} R_{e2} & R_{\mu 2} & R_{\tau 2} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Therefore, for $0 < \beta^2 \leq \frac{M_2}{M_3}$, $R = \mathcal{P}R^*$ is satisfied.

For the case with $\beta^2 > \frac{M_2}{M_3}$, the entry $\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}}$ is purely imaginary. By taking the complex conjugate of the last three equations in Equation (A96), we obtain:

$$\begin{pmatrix} -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & -\frac{\beta M_3}{M_2} \\ \beta & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & -\frac{\beta M_3}{M_2} \\ \beta & \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix} \begin{pmatrix} R_{e2} & R_{\mu 2} & R_{\tau 2} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix}$$
$$= \begin{pmatrix} R_{e2} & R_{\mu 2} & R_{\tau 2} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix}, \tag{A98}$$

which immediately implies that $R_{e2} = R_{e3} = R_{\mu 2} = R_{\mu 3} = R_{\tau 2} = R_{\tau 3} = 0$ and, thus, $R = \mathcal{P}R^*$ is satisfied.

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U^*$, multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_{23} = \mathcal{P}R^*$, one can obtain the following relation from the unitary condition:

$$RF_{23}F_{23}^\dagger R^\dagger = RR^\dagger. \tag{A99}$$

For the case with $\beta = 0$, F_{23} is the identity matrix; thus, the above relation is satisfied.

For the cases with $0 < \beta^2 \leq \frac{M_2}{M_3}$ and with $\beta^2 > \frac{M_2}{M_3}$, we have shown that $R_{e1} = R_{e1}^*$, $R_{\mu 1} = R_{\tau 1}^*$, and $R_{e2} = R_{e3} = R_{\mu 2} = R_{\mu 3} = R_{\tau 2} = R_{\tau 3} = 0$. By substituting $R = \begin{pmatrix} R_{e1} & 0 & 0 \\ R_{\mu 1} & 0 & 0 \\ R_{\mu 1}^* & 0 & 0 \end{pmatrix}$, we can see that Equation (A99) is always satisfied.

Therefore, for $F = F_{23}$, we recover the conclusions of [44].

Appendix A.24. F_{24}

For $F_{24} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & \beta \\ 0 & \frac{\beta M_3}{M_2} & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}$, the relation $RF_{24} = \mathcal{P}R^*$ is

$$R \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & \beta \\ 0 & \frac{\beta M_3}{M_2} & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} R^*. \tag{A100}$$

The implications include:

$$
\begin{aligned}
&R_{e1} = R_{e1}^*,\ R_{\mu 1} = R_{\tau 1}^*,\\
&\begin{pmatrix} \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & \frac{\beta M_3}{M_2} \\ \beta & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix} \begin{pmatrix} R_{e2} \\ R_{e3} \end{pmatrix} = \begin{pmatrix} R_{e2}^* \\ R_{e3}^* \end{pmatrix},\\
&\begin{pmatrix} \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & \frac{\beta M_3}{M_2} \\ \beta & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix} \begin{pmatrix} R_{\mu 2} \\ R_{\mu 3} \end{pmatrix} = \begin{pmatrix} R_{\tau 2}^* \\ R_{\tau 3}^* \end{pmatrix},\\
&\begin{pmatrix} \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & \frac{\beta M_3}{M_2} \\ \beta & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix} \begin{pmatrix} R_{\tau 2} \\ R_{\tau 3} \end{pmatrix} = \begin{pmatrix} R_{\mu 2}^* \\ R_{\mu 3}^* \end{pmatrix}.
\end{aligned} \tag{A101}
$$

If $\beta = 0$, then F_{24} will reduce to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, with implications including:

$$
\begin{aligned}
R_{e1} = R_{e1}^*,\ R_{e2} = R_{e2}^*,\ R_{e3} = -R_{e3}^*,\\
R_{\mu 1} = R_{\tau 1}^*,\ R_{\mu 2} = R_{\tau 2}^*,\ R_{\mu 3} = -R_{\tau 3}^*.
\end{aligned} \tag{A102}
$$

It is easy to see that, for $\beta = 0$, the relation $R = \mathcal{P}R^*$ is generally not satisfied.

For the case with $0 < \beta^2 \leq \frac{M_2}{M_3}$, the matrix $\begin{pmatrix} \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & \frac{\beta M_3}{M_2} \\ \beta & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}$ is a real matrix but not an identity matrix. From the last three equations of Equation (A101), we have:

$$
\begin{pmatrix} \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & \frac{\beta M_3}{M_2} \\ \beta & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}^2 \begin{pmatrix} R_{e2} & R_{\mu 2} & R_{\tau 2} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix} = \begin{pmatrix} R_{e2} & R_{\mu 2} & R_{\tau 2} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix}. \tag{A103}
$$

This is trivially true since $\begin{pmatrix} \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & \frac{\beta M_3}{M_2} \\ \beta & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}^2$ is equal to the identity matrix for any positive M_2, M_3, and any real β.

For the case with $\beta^2 > \frac{M_2}{M_3}$, the entry $\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}}$ is purely imaginary. By taking the complex conjugate of the last three equations in Equation (A101), we obtain:

$$
\begin{aligned}
&\begin{pmatrix} -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & \frac{\beta M_3}{M_2} \\ \beta & \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & \frac{\beta M_3}{M_2} \\ \beta & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix} \begin{pmatrix} R_{e2} & R_{\mu 2} & R_{\tau 2} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix}\\
&= \begin{pmatrix} R_{e2} & R_{\mu 2} & R_{\tau 2} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix}.
\end{aligned} \tag{A104}
$$

From the eigenvalues of $\begin{pmatrix} -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & \frac{\beta M_3}{M_2} \\ \beta & \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} & \frac{\beta M_3}{M_2} \\ \beta & -\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \end{pmatrix}$, we can see that, if any column of $\begin{pmatrix} R_{e2} & R_{\mu 2} & R_{\tau 2} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix}$ is not a zero column matrix, then we must have $2\frac{\beta^2 M_3}{M_2} - 1 = 1$ and $\frac{\sqrt{M_2-\beta^2 M_3}}{\sqrt{M_2}} \frac{\sqrt{\beta^2 M_3}}{\sqrt{M_2}} = 0$, which are impossible when $\beta^2 > \frac{M_2}{M_3} > 0$.

We eventually obtain $\begin{pmatrix} R_{e2} & R_{\mu 2} & R_{\tau 2} \\ R_{e3} & R_{\mu 3} & R_{\tau 3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Therefore, for $\beta^2 > \frac{M_2}{M_3}$, $R = \mathcal{P}R^*$ is satisfied.

In addition to the exact seesaw formula, we also need to pay attention to the unitary condition $UU^\dagger + RR^\dagger = I$. By substituting $U = \mathcal{P}U^*$, multiplying both sides by $\mathcal{P}$, taking the complex conjugate, and substituting $RF_{24} = \mathcal{P}R^*$, one can obtain the following relation from the unitary condition:

$$RF_{24}F_{24}^\dagger R^\dagger = RR^\dagger. \tag{A105}$$

For the case with $\beta = 0$, $F_{24}F_{24}^\dagger$ is the identity matrix; thus, the above relation is satisfied for any R satisfying Equation (A102).

For the case with $0 < \beta^2 \le \frac{M_2}{M_3}$, there exist nontrivial possibilities satisfying Equation (A105) but not $R = \mathcal{P}R^*$. For example, when $\beta = \frac{\sqrt{M_2}}{\sqrt{M_3}}$, F_{24} will reduce to $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{M_2}}{\sqrt{M_3}} \\ 0 & \frac{\sqrt{M_3}}{\sqrt{M_2}} & 0 \end{pmatrix}$, and $R = \begin{pmatrix} R_{e1} & \frac{\sqrt{M_3}}{\sqrt{M_2}}R_{e3} & R_{e3} \\ R_{\mu 1} & 0 & 0 \\ R_{\mu 1}^* & 0 & 0 \end{pmatrix}$ with real R_{e1} and R_{e3} and complex $R_{\mu 1}$ is a solution.

For the case with $\beta^2 > \frac{M_2}{M_3}$, we have shown that $R_{e1} = R_{e1}^*$, $R_{\mu 1} = R_{\tau 1}^*$, and $R_{e2} = R_{e3} = R_{\mu 2} = R_{\mu 3} = R_{\tau 2} = R_{\tau 3} = 0$. By substituting $R = \begin{pmatrix} R_{e1} & 0 & 0 \\ R_{\mu 1} & 0 & 0 \\ R_{\mu 1}^* & 0 & 0 \end{pmatrix}$, we can see that Equation (A105) is always satisfied.

Therefore, for $F = F_{24}$, the relation $R = \mathcal{P}R^*$ is generally not satisfied.

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