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Generalized Schwarzschild Spacetimes with a Linear Term and a Cosmological Constant

Orchidea Maria Lecian

Department of Clinical and Molecular Medicine, Sapienza University of Rome, 324-00185 Rome, Italy; orchideamaria.lecian@uniroma1.it

Abstract: Particular Kottler spacetimes are analytically investigated. The investigated spacetimes are spherically symmetric nonrotating spacetimes endowed with a Schwarzschild solid-angle element. SchwarzschildNairiai spacetimes, Schwarzschild spacetimes with a linear term, and Schwarzschild spacetimes with a linear term and a cosmological constant are studied. The infinite-redshift surfaces are analytically written. To this aim, the parameter spaces of the models are analytically investigated, and the conditions for which the analytical radii are reconducted to the physical horizons are used to set and to constrain the parameter spaces. The coordinate-singularity-avoiding coordinate extensions are newly written. Schwarzschild spacetimes with a linear term and a cosmological constant termare analytically studied, and the new singularity-avoiding coordinate extensions are detailed. The new roles of the linear term and of the cosmological constant term in characterizing the Schwarzschild radius are traced. The generalized Schwarzschild–deSitter case and generalized Schwarzschild–anti-deSitter case are characterized in a different manner. The weak field limit is newly recalled. The embeddings are newly provided. The quantum implementation is newly envisaged. The geometrical objects are newly calculated. As a result, for the Einstein field equations, the presence of quintessence is newly excluded. The Birkhoff theorem is newly proven to be obeyed.

Keywords: general relativity; generalized Schwarzschild black hole spacetimes; cosmological constant; linear term

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1. Introduction

The linearized perturbation problem of deSitter background spacetimes is solved in [1] for field perturbations.

The study of the quantum nature of generalized Schwarzschild–deSitter spacetimes and that of generalized Schwarzschild–anti-deSitter ones are required after the pioneering approach to the quantum perturbative description of the difference between the analytical radii (i.e., the surfaces of infinite redshift) of these spacetimes and their physical horizons after the implementation of the Schroedinger equation within the Regge–Wheeler formalism, as taken from [2].

The study of the parameter space of generalized Schwarzschild–deSitter spacetimes is required after the pioneering work of [2], in which some of the features of the duty of the cosmological constant term slightly modifying the Schwarzschild radius are partially presented.

The initial studies of generalized Schwarzschild spacetimes with a positive-valued cosmological constant term are accredited to Nariai [3–5]. In detail, the disposed Schwarzschild–deSitter spacetimes are found to be compatible with the position of singularity-avoiding coordinate extensions whose analytical expressions are set and whose remainder calculations are demonstrated straightforwardly to hold independent of the sign of the cosmological constant term.

A spherically symmetric generalized spacetime containing a Schwarzschild-like term, a linear term, and a cosmological constant term was introduced by [6] as an exact solution for locally conformal invariant Weyl gravity, which is apt at investigating galaxy rotation curves. Generalized spacetimes read as follows:

$$ds^2 = c^2\left(1 - \frac{\beta(2-3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - kr^2\right) magentadx_0^2 +$$

$$-\frac{1}{\left(1 - \frac{\beta(2-3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - kr^2\right)} dr^2 - r^2 d\theta^2 - r^2(\sin\theta)^2 d\phi^2. \quad (1)$$

The aim of this introductory example is the verification of nonrotating spherically symmetric spacetimes whose infinite-redshift surfaces can be studied with success [7], i.e., spacetimes in which the parameters involved in the generalization of the Schwarzschild spacetime are not independent, but are constrained from an analytical point of view. Furthermore, in the presented instance, further characterizations are present, arising from observational evidence.

The further implications of a linear term added to a Schwarzschild spacetime are studied in [8].

A more recent parameterization of spherically symmetric Schwarzschild spacetimes is discussed in [9], for which the characterizations are discussed in [10]. The allowed ranges of the parameters defining the exponents of the generalization addends are estimated in [11]. Further examination of the spacetime schemes is presented in [12].

Kottler spacetimes were introduced in [13]. Minnkowski spacetimes with a cosmological constant were investigated in [14] based on the motivations presented in [15]. The aim of the present work is to newly investigate some possible generalizations of Schwarzschild spacetimes endowed with a Schwarzschild solid-angle element in some pertinent instances for which the comparison with the phenomenological evidence is newly demonstrated and holds. In particular, the presence of a cosmological constant, that of a linear term, and that of both a cosmological constant term and a linear term are newly taken into account.

In all the instances, the infinite-redshift surfaces of the models are studied. The parameter spaces of the models are set and constrained. As results, the conditions required for the analytical radii to be reconducted to the physical horizons are newly analytically proven. The experimental validation techniques are introduced. The quantum implementation is approached.

In the case of generalized Schwarzschild spacetimes with a cosmological constant term, both the Schwarzschild–deSitter case and the Schwarzshild–anti-deSitter one are newly taken into account. The pioneering results of [2] were derived for the Schwarzschild–deSitter case only; the case of a negative value of the cosmological constant was considered in [16]. In the present paper, the cases are newly considered and newly completed in the study of the new constraints on the cosmological constant term, as well as for the new role of the cosmological constant in the new modifications of the Schwarzschild radius. The new constraints on the parameter space of the model are defined. Furthermore, the same new studies are applied to the (unexplored) Schwarzschild–anti-deSitter case. The newly found analytical expressions of the radii are scrutinized for the obtention of the new analytical expressions of the physical horizons from which the new parameter space is further newly investigated. The examined generalized spacetimes are newly demonstrated to exhibit different properties in the Schwarzschild–deSitter case and in the Schwarzschild–anti-deSitter one. The singularity-avoiding coordinate extensions are demonstrated to be compatible with the Nairai transformations [5] in a comparison with the case of a generalized Schwarzschild spacetime with a linear term and a cosmological constant, after which it is newly proven that, after the (well-posed) limit to a vanishing linear term, the sign of the value of the cosmological constant does not modify the orders of the remainders as far as the cosmological constant term is concerned.

Generalized Schwarzschild spacetimes with a linear term are newly prospected. The analytical radii are written, and the conditions for the position of the physical horizons are assessed. The character of the linear term in modifying the representations of the Schwarzschild radius is newly qualified. The coordinate-singularity-avoiding coordinates extensions are newly found; from this, the parameter space is newly further stamped. In particular, the new features of the linear term in constraining the parameter space of the model are newly typified.

Generalized Schwarzschild spacetimes with a linear term and with a cosmological-constant term are newly surveyed. The role of the linear term and that of the cosmological-constant term in defining the parameter space of the scheme are newly portrayed. The analytical expressions of the four radii are newly found and newly discussed; the analytical expressions of the two physical horizons are newly found and newly discussed, from which new constraints on the parameter space of the scheme are established. The coordinate-singularity-avoiding coordinates extensions are newly written and newly reconnoitered; from this analysis, the remainders are newly proven not to be affected after the sign of the cosmological-constant term. The parameter space of generalized spacetime is newly researched. The aspects of the linear term and those of the cosmological-constant term are newly read as far as their tasks in modifying the Schwarzschild radius are concerned. The results are newly proven to hold in a different manner in the case of a positive sign of the cosmological constant and in that of a negative sign of the cosmological constant, i.e., such that generalized spacetimes exhibit new different properties. The limit of a vanishing value of the linear term is considered in a new manner for the comparison with the cases of a generalized Schwarzschild spacetime with a cosmological-constant term is newly considered; such a limit is newly probed to be well posed.

As a further new result, the study of the Einstein field equations allows one to exclude that the presence of the linear term in the metric tensor can be interpreted as mimicking quintessence matter.

The paper finds direct applications in the analytical study of the infinite-redshift surfaces of generalized Schwarzschild spacetimes. The paper is organized as follows.

In Section 1, generalized Schwarzschild spacetimes are introduced.

In Section 2, some of the possible generalizations of the Schwarzschild spacetimes are proposed.

In Section 3, the geometrical objects are written and the Einstein field equations (EFEs) are calculated from the Eisntein–Hilbert action according to [17]. As a further new result, the possibility of the presence of quintessence is ruled out after the analysis of the Einstein field equations.

In Section 4, generalized Schwarzschild spacetimes with a cosmological constant are newly analyzed. As results, the analytical radii are newly written; the role of the cosmological-constant term in modifying the Schwarzschild radius is newly defined; the Nariai coordinate-singularity-avoiding coordinates extensions are newly demonstrated to be well posed also in the Schwarzschild–anti-deSitter case.

In Section 5, generalized Schwarzschild spacetimes with a linear term are newly explored. The analytical expressions of the radii are found, and the analytical expressions of the physical horizons are written; the parameter space of the model is newly assessed. The coordinate-singularity-avoiding coordinates extensions are newly written and newly probed, form which new constraints on the parameter space of the scheme are learned.

In Section 6, generalized Schwarzschild spacetimes with a linear term and a cosmological constant are newly scrutinized. The analytical radii and the physical horizons are newly written, from which the parameter space of the spacetimes is newly set. The analytical expressions of the physical horizons are newly written, from which the new constraints on the parameter space of the scheme are newly obtained. The coordinate-singularity-avoiding coordinates transformations are newly implemented, after which the nontrivial initial value of the radial variable is newly calculated. From these analyses, the comparison with generalized Schwarzschild spacetimes with a cosmological constant is newly conducted as far as the pertinent aspects are concerned.

In Section 7, the embedding diagrams are newly analytically demonstrated, from which the new behaviors are exhibited from the $\mathcal{R}_{1212}$ of the embedding manifold; and the weak-field limit is newly addressed, from whose analyses the different roles of the linear term and that of the cosmological constant are newly spelled out.

In Section 8, the quantum features of generalized spacetimes are newly prospectively envisaged.

In Section 9, the concluding remarks are given.

2. About Some Generalizations of the Schwarzschild Spacetimes

In the present paper, the following specification of generalized Schwarzschild spacetimes is considered:

$$ds^2 = c^2\left(1 - \frac{r_s}{r} - \frac{k_1}{r^{1+3w_1}} - \frac{k_2}{r^{1+3w_2}}\right) magentadx_0^2 +$$
$$- \frac{1}{\left(1 - \frac{r_s}{r} - \frac{k_1}{r^{1+3w_1}} - \frac{k_2}{r^{1+3w_2}}\right)} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{2}$$

according to specific values of the exponents w_1 and w_2.

In [9], two particular spacetimes of generalized Reissner–Nordstrom–deSitter black hole spacetime, i.e.,

$$ds^2 = c^2\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} - kr^2\right) magentadx_0^2 +$$
$$- \frac{1}{\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} - kr^2\right)} dr^2 - r^2 d\theta^2 - r^2 (\sin\theta)^2 d\phi^2, \tag{3}$$

which, in their turn, correspond to a Reissner–Nordstrom–deSitter spacetime, i.e.,

$$ds^2 = c^2\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} + \frac{k_1}{r^{1+3w}}\right) magentadx_0^2 +$$
$$- \frac{1}{\left(1 - \frac{r_s}{r} + \frac{r_Q}{r^2}\frac{k_1}{r^{1+3w}}\right)} dr^2 - r^2 d\theta^2 - r^2 (\sin\theta)^2 d\phi^2 \tag{4}$$

were explored; as a result, an outer horizon (of deSitter type) at $r = r_Q$ with $-1 < w < -1/3$ was investigated, and an inner horizon (of the black hole type) was found at $r = r_Q$ within the choice $-\frac{1}{3} < w < 0$.

The physically relevant ranges of the parameters w_i of generalized metrics were studied in [11] as

$$-1 \leq w_i \leq\simeq 0.6 \tag{5}$$

and further scrutinized in [18–20].

The geometrical features and effects presented in the spacetimes are juxtaposed in [10].

A generalized Kiselev spacetimes endowed with a Schwarzschild solid-angle element is proposed in [9] as

$$ds^2 = c^2\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} - \sum_{n=1}^{n=N} k_n r^n\right) magentadx_0^2 +$$
$$- \frac{1}{\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} - \sum_{n=1}^{n=N} k_n r^n\right)} dr^2 - r^2 d\theta^2 - r^2 (\sin\theta)^2 d\phi^2. \tag{6}$$

Particular values from the summations Equation (6) are to be picked up.

In [21], the spacetimes

$$ds^2 = c^2\left(1 - \frac{r_s}{r} - \frac{\Lambda}{3}r^2 - \frac{K}{r^{3\omega+1}}\right) magentadx_0^2 +$$

$$-\frac{dr^2}{\left(1 - \frac{r_s}{r} - \frac{\Lambda}{3}r^2 - \frac{K}{r^{3\omega+1}}\right)} - r^2 d\theta^2 - r^2(sin\theta)^2 d\phi^2 \quad (7)$$

are studied.

The cosmological constant is parameterized after the methods exposed in [22,23] as a function of ω, of the positive normalization factor of the density of the fluid, and of r. The different qualities of the physical horizons are hinted at but not parameterized; in particular, no constraints on the phase space are investigated for the merging of the analytical radii. Particular cases of ω are considered. The interaction between incident plane wave and black hole are studied at the classical level and at the semiclassical one after the analysis of the scattering cross section, of the absorption section, and of the polarization. The tools of geometric optics are applied. The study of the Ljapunov exponents is performed.

In the present work, generalized Schwarzschild spacetimes with a cosmological constant, generalized Schwarzschild spacetimes with a linear term, and generalized Schwarzschild spacetimes with a linear term and with a cosmological-constant term are further studied, for which the constraints on the parameter space are refined as far as the inequalities connecting the Schwarzschild radius, the coefficient of the linear term, and the cosmological constant are concerned. In particular, the conditions to obtain the physical horizons after the definitions of the analytical radii are newly set. The coordinate-singularity-avoiding coordinates extensions are newly written; from the expressions of the coordinates extensions, the roles of the parameters in modifying the features of the Schwarszchild radius are newly described analytically.

The main results in the analysis of the particular spacetime

$$ds^2 = \left(1 - \frac{r_s}{r} + \tilde{\beta}r + \frac{\tilde{\Lambda}}{3}r^2\right) magentadx_0^2 - \frac{1}{1 - \frac{r_s}{r} + \tilde{\beta}r + \frac{\tilde{\Lambda}}{3}r^2} dr^2 - r^2 sin\theta d\phi^2 + r^2 d\theta^2 \quad (8)$$

were scrutinized in [12]. It is recalled to be a solution of the EFEs of several models, i.e., conformal gravity [6,24,25], de Rham–Gabadadze–Tolley (dRGT) massive gravity [26], and modified gravity [27,28].

As stressed in [12], the relevance of studying the metric is being outlined in the new era of the LIGO experiment [29] and in the Event Horizon experiment [30,31]; more in detail, in the latter case, the possibility of comparing the shadow of the black hole in the Sagittarius A^* is being considered in the case of a Kerr–Newman black hole and in the case of other spacetimes, as far as the difficulties in the fitting of the parameters are concerned.

The roles of the linear terms are recapitulated in [12] from a geometrical point of view. It is the aim of the present paper to convey the geometrical meaning of the parameters, as they are on the left-hand-side of the EFEs. In this case, the parameter $\tilde{\beta}$ imposes algebraic constraints between the curvature invariants at any generic spacetime point [32]. The procedures followed in [12] are based on tracing the local approach; more in detail, the scalar polynomial curvature invariants, the Cartan curvature invariants, and the Newman–Penrose curvature scalars are reported.

The local measurements of the parameters is performed as far as the size of the area of the horizon [33]. Before proceeding further, it is the aim of the present paper to estimate the values of the physical radii of the horizons as a function of the Schwarzschild radius, the parameter of the linear term, and that of the cosmological constant before continuing with the research lines proposed in [12] in order to define the infinite-redshift surfaces.

It is, nevertheless, possible to infer form the EFEs that the geometrical aspects of the $\tilde{\beta}$ parameter can mimic matter features (i.e., features which are due to the addends on the right-hand-side of the EFEs): in [12], it is inferred that this addend can mimic the aspects of the Kerr black hole as far as the accretion is concerned.

Similarly, the effects of the $\tilde{\beta}$ term on the shadow of the black hole are also inferred. As determined in [34], the geometrical features of the $\tilde{\beta}$ term do not mimic any dark energy component interacting with the black hole.

The extensive items of the bibliography are provided within [12].

Some of the main differences in the models of which the metric is solution are outlined in [12] as the features after the evaporation.

3. Methodologies

In the present approach, the guidelines of [17] are followed. In particular, the Einstein–Hilbert action is taken. The matter content is never placed into the metric tensor, not even in the ultra-relativistic limit; this way, the addends in the components of the metric tensor do not refer to matter fields, but the geometrical qualities of the spacetime. Furthermore, the addends qualifying the geometrical properties give rise to further addends in the left-hand side of the EFEs and do not refer to matter content as follows.

The relevant geometric objects for a spherically symmetric metric generalizing the Schwarzschild metric, endowed with a Schwarzschild solid angle, whose line element is spelled as

$$ds^2 = c^2(1 - \frac{r_s}{r} + \psi(r)) magentadx_0^2 + \tag{9}$$

$$-\frac{1}{(1 - \frac{r_s}{r} + \psi(r))} dr^2 - r^2 d\theta^2 - r^2 (sin\theta)^2 d\phi^2, \tag{10}$$

where $(1 - \frac{r_s}{r} + \psi(r))$ is the generalization of the Schwarzschild term, qualified after the functional dependence on the function $\psi(r)$ and after the parameters qualifying it are here listed.

The Ricci tensor $R_{\mu\nu}$ is written as

$$R_{tt} = \frac{1}{r^2}[r - r_s + r\psi]\left[r\frac{d^2\psi}{dr^2} + 2\frac{d\psi}{dr}\right], \tag{11a}$$

$$R_{rr} = -\frac{1}{2[r - r_s + r\psi]}\left[r\frac{d^2\psi}{dr^2} + 2\frac{d\psi}{dr}\right], \tag{11b}$$

$$R_{\theta\theta} = -\psi + r\frac{d\psi}{dr}, \tag{11c}$$

$$R_{\phi\phi} = \left[-\psi + r\frac{d\psi}{dr}\right](sin\theta)^2. \tag{11d}$$

The Ricci scalar R is written as

$$R = \frac{1}{r^2}\left[r^2\frac{d^2\psi}{dr^2} + 4r\frac{d\psi}{dr} + 2\psi\right]. \tag{12}$$

The geometrical objects therefore contain the terms corresponding to the linear term and those corresponding to the cosmological-constant term.

The terms arising due to the linear term have the qualities of a fluid.

The cosmological-constant terms are qualified after the sign of the value of the cosmological constant as generalized deSitter model or generalized anti-deSitter model.

The geometrical objects can be expanded as series of the parameters qualifying the modifying function ψ only if the parameters are infinitesimal; even though the role of the parameters is to modify the Schwarzschild radius only slightly, this duty can be accomplished also within the portions of the parameter spaces, where the parameters are not infinitesimal.

The EFEs are written as

$$G_{ab} - \frac{1}{2}g_{ab}R = 0 \tag{13}$$

Indeed, the presence of matter is not placed in the metric tensor, as from [17].

The EFE Equation (13) can be spelled out according to Equations (11) and (12). The following constraints on the modification addend ψ are obtained.

From the $_{tt}$ component of the EFEs and from the $_{rr}$ component of the EFEs, the new constraints are found:

$$r\left[r\frac{d^2\psi}{dr^2}+2\frac{d\psi}{dr}\right]\equiv\frac{1}{2}\left[r^2\frac{d^2\psi}{dr^2}+4r\frac{d\psi}{dr}+2\psi\right]. \tag{14}$$

From the $_{\theta\theta}$ component of the EFEs and from the $_{\phi\phi}$ component of the EFEs, the new constraint are found:

$$-\psi+r\frac{d\psi}{dr}=\frac{1}{2}\left[r^2\frac{d^2\psi}{dr^2}+4r\frac{d\psi}{dr}+2\psi\right] \tag{15}$$

The constraints Equations (14) and (15) ensure that the Birkhoff theorem is obeyed, to which the condition

$$1-\frac{r_s}{r}+\psi(r)\neq 0 \tag{16}$$

must be imposed. The surfaces

$$1-\frac{r_s}{r}+\psi(r)=0 \tag{17}$$

individuate the infinite-redshift surfaces of the models, which are investigated in the present work.

3.1. Geometrical Qualities of a Linear Term

It is possible to inquire about the geometrical qualities of the the addends possibly obtained in $\psi(r)$.

More in detail, as an example, it is possible to inquire about the geometrical interpretation of a linear addend as $\psi(r=k_1/r)$. In particular, it is possible to investigate whether a linear term modifying the EFEs can mimic (in the sense that the addends due to the geometric term are on the left-hand side of the EFEs, while the addends referring to matter contributions should be on the right-hand side of the EFEs) the presence of quintessence: it is straightforwardly calculated that, in general, the radial and transverse pressures differ; for this reason, the linear term in the metric tensor is, thus, not mimicking quintessence. It might be interpreted as mimicking some kind of anisotropic fluid matter: therefore, quintessence can be excluded as "geometrically" mimicked as a source since the equation of state $p=\omega\rho$ assumes that the pressure is isotropic.

Methodologies to Investigate the Role of the Linear Term

In [24], some aspects of conformal gravity are studied. The Birkhoff theorem in conformal gravity is enunciated: "the most general spherically symmetric (electrovacuum) solution is stationary, i.e., "absolutely stable" with respect to spherically symmetric perturbations". Furthermore, the conjecture is taken that any field theory in which the linearized spin-0 modes are absent will let the Birkhoff theorem hold even in the case in which the solution is not asymptotically flat. It is our aim to provide one of the possible most straightforward examples of the corresponding Bach gravity model as follows.

The spacetime is taken

$$g_{tt}=c_1r^2+c_2r+c_3+\frac{c_4}{r}, \tag{18a}$$

$$\tilde{A}=\tilde{A}_\mu dx^\mu \tag{18b}$$

with

$$3c_2c_4-c_3^2+3\frac{q^2}{2\beta}=0 \tag{19}$$

Ibidem, there proves that a spherically symmetric solution of the Bach–Maxwell equation always admits at least the fourth Killing vector ξ; given ξ nonnull and timelike, the diagonalized line element is written as

$$ds^2 = e^{2v}(dx^0)^2 + e^{2w}(dx_1)^2, \tag{20}$$

with

$$v \equiv v(x^0, x^1), \tag{21a}$$
$$w \equiv w(x^0, x^1), \tag{21b}$$
$$\xi \equiv \xi(x^0, x^1). \tag{21c}$$

The corresponding Killing equations brings

$$w = w(x_1), \tag{22a}$$
$$v = \alpha_1(x_0) + \alpha_2(x_1). \tag{22b}$$

with $w(x_1)$, α_1 and α_2 arbitrary functions.

From Equation (18), the Buchdal solution [35] is taken as

$$c_2 = q = 0, \tag{23a}$$
$$c_3 = 1 \tag{23b}$$

The Bach gravity [36] is based on the extension of the Weyl gravity and about the Weyl extension of the concept of curvature tensor; it is based on the investigation of the invariant I as

$$I = g_{\alpha\beta}\Xi^\alpha\Xi^\beta e^{\tilde{F}_\gamma dx^\gamma} \tag{24}$$

under the need of Ξ^μ an arbitrary vector.

The new example provided here is of Bach gravity from Equation (18), with

$$c_1 = c_3 = 0 \tag{25}$$

which is a first straightforward example of Bach gravity comprehending the linear term (and a "Schwarzschild-like" term), which exemplifies one of the schemes proposed for investigation in the present paper.

The absence of spin-0 model provides one with motivation of a conjecture according to which the linearized theory might contain a mass-less spin-2 ghost which corresponds to quadrupole radiation emission.

In [6], the action corresponding to the same-indices saturation of two conformal Weyl tensors is taken. The Einstein–Hilbert (part of the) action is absent. The theory is strictly conformally invariant, i.e., the masses of the particles are due to dynamical effects only. The solution is found as the metric Equation (1). The deSitter solution is recalled to be a vacuum solution. The lienar term γ is placed in correspondence with the inverse of the Hubble length. The linear terms are also ibidem described within the framework of galactic dynamics. The γ term can be described as due to the effects of some "background geometry" on star trajectories.

The linear term is ibidem also commented to be referring to the effect of "other galaxies" on a test particle.

The homogeneity of distribution of galaxies implies the shaping of the linear term.

The dimensionless product $\gamma\beta$ is used for fitting galaxy rotation curves.

In [25], topological black holes are studied, for which the curvature at infinity is calculated after a line element compatible with one of Equation (1), and the spacetimes are compared with topological black hole spacetimes in anti-deSitter gravity.

The case of spacetimes with nonnegative curvature at infinity is also presented ibidem. The thermodynamical properties of the found black holes are discussed as "induced phenomena".

The linear term is ibidem used to discriminate the lapse function of the spherical black hole from that of a black hole in anti-deSitter gravity.

The analytical continuations of the line element are considered, in which the linear term plays a role.

The possibility of such gravity having "consistent interaction with massless higher spin fields" is considered ibidem from [37].

In [26], nonlinear theories of "massive gravity" are considered. The metric tensor is taken as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{26}$$

with $\eta_{\mu\nu}$ being the Minkowski metric and $h_{\mu\nu}$ the perturbations; the metric Equation (26) is expanded as a function of the Stueckelberg field Φ^a (which transforms as a scalar) as

$$\Phi^a \to x^a - \eta^{a\mu}\partial_\mu \pi \tag{27a}$$

$$g_{\mu\nu} = \tilde{H}_{\mu\nu} + \eta_{\mu\nu}\partial^\mu \Phi^a \partial_\nu \Phi^b \tag{27b}$$

with π being the helicity-0 mode of the graviton.

As a methodology, quartic orders of the metric perturbations are considered. Lorentz invariance is not broken.

In [27], within a generalized scheme, the line element is taken as

$$ds^2 = -\tilde{B}(r) magentadx_0^2 + \tilde{A}(r)dr^2 + r^2(sin\theta)^2 d\phi^2 + r^2 d\theta^2. \tag{28}$$

After [38], in [27], the component of the metric tensor $\tilde{B}(r)$ is considered as

$$\tilde{B}(r) = 1 - \frac{2m}{r} + \frac{\tilde{\alpha}}{\tilde{d}} r; \tag{29}$$

in Equation (29), $al\tilde{p}ha$ was a "small dimensionless constant" qualifying the general model, and $\tilde{d}$ was a "characteristic length scale" of the order of galactic scales. At the Solar System m scales, $r << d$, Equation (29) was apt at describing the "Pioneer anomaly".

In [28], the spherically symmetric line element is considered from Equation (28), i.e., one with $\tilde{A}(r) \equiv \tilde{B}(r)$, where a cosmological-constant term is added in Equation (29); the numerical value of $\tilde{\alpha}$ is taken as $\tilde{\alpha} \simeq 10^{-6}$, and $\tilde{d} \simeq 10\, kpc$; the choice is motivated after the analysis of [27]. The analysis of [27] is effective in describing flat rotation curves of spiral galaxies.

In [12], the linear-term parameter of the redshift function is written as a function of the scalar polynomial curvature invariants. The calculations of the accretion and that of the shadow are hinted.

Ibidem, the photon sphere of the black hole spacetime whose g_{tt} component of the metric tensor contains a linear term is found to be of a radius small than that of the Schwarzschild spacetime.

Ibidem, the comparison of the black hole spacetime containing a linear term in the redshift function and the Kerr spacetime are compared. The linear-term parameter is expressed as a function of the radii of the innermost stable orbits which depend on spin parameter of the Kerr black hole spacetime, as delineated in the analysis of [39]. Accordingly, the cosmological constant is also expressed as a function of these variables: the effects of the linear term of the redshift function on the definition of the black hole mass is defined as still under study. The calculations of the geodesics are proposed, after which the possibility of a degeneracy in the chosen definition arises. Further investigation is ibidem proposed about which degeneracies are to be solved after evaporation. Ibidem, more investigation is reported to be needed after the findings of [40–42].

More in detail, the definition of the mass of the black hole spacetimes with a linear term in the redshift functions is remarked to be of novel aspects.

It is one of the purposes of the present paper to analyze and to constrain the parameter space of the black hole spacetimes that also contain a cosmological constant term as a function of the linear-term parameter for the purposes indicated in [12].

3.2. Applications of Methodologies

The infinite-redshift surfaces are studied as obeying the equations $g_{tt} \equiv 0$; the results are the geometrical radii, on which the physical implementation of the model is performed; as a result of the study of the functions, the physical horizons are found.

The coordinate-singularity-avoiding coordinates extensions are newly analytically written. The Schwarzschild–Nairiai spacetime is taken as an example. The further cases are newly studied. More in detail, the coordinate-singularity-avoiding coordinates extensions are written in order to avoid the coordinate singularity; their role is apt to remove the coordinates singularity in a physical manner analogous to that interpreted for the metric to become regular. The methodologies followed for these purposes are the series expansions of the measures which take into account the orders of the parameters of the series (which are the linear-term parameter and the cosmological constant); indeed, for the analysis of the physical horizons, the role of these terms is understood as modifying the role of the Schwarzschild radius only slightly: this way, each of the addends of the orders of magnitude in the series expansions are kept under control directly.

As results, the spacetimes examined are described as endowed with the proper physical horizons, and with the proper coordinate-singularity-avoiding coordinate extensions; furthermore, the parameters involved in the schemes are physically implemented: from the study of the parameter spaces available for the models, the linear term and the cosmological-constant term are proven to be admitted as modifying the role of the Schwarzschild radius only slightly.

4. The Schwarzschild Spacetimes with a Cosmological Constant

4.1. The Schwarzschild–deSitter Spacetimes

The spherically symmetric Schwarzschild–deSitter (Nariai) spacetimes, endowed with a Schwarzschild solid-angle element, are defined after the g_{tt} element $g_{tt} = 1 - \frac{r_s}{r} - \frac{\Lambda}{3}r^2$ as

$$ds^2 = c^2\left(1 - \frac{r_s}{r} - \frac{\Lambda}{3}r^2\right) magentadx_0^2 - \frac{dr^2}{\left(1 - \frac{r_s}{r} - \frac{\Lambda}{3}r^2\right)} - r^2 d\theta^2 - r^2(sin\theta)^2 d\phi^2 \quad (30)$$

An upper bound of the mass M of a black-hole solution of a Schwarzschild–deSitter spacetimes from the value of the cosmological-constant term was found in [2] as a function of Λ as $M \leq \sqrt{1/9\Lambda}$.

Selected qualitative characterizations of the horizons of a black-hole solution of Schwarzschild–deSitter spacetimes are explored in [43].

4.1.1. Schwarzschild–deSitter Radii

The Schwarzschild–deSitter radii are recalled in [43] as

$$R_1 = \frac{2}{\sqrt{\Lambda}} cos \frac{\tilde{\theta}}{3}, \quad (31a)$$

$$R_2 = \frac{2}{\sqrt{\Lambda}} cos \left[\frac{\tilde{\theta}}{3} + \frac{2\pi}{3}\right], \quad (31b)$$

$$R_3 = \frac{2}{\sqrt{\Lambda}} cos \left[\frac{\tilde{\theta}}{3} + \frac{4\pi}{3}\right], \quad (31c)$$

with $\tilde{\theta} = -3\sqrt{\Lambda}M$, after the parameterization recalled in [43].

4.1.2. Schwarzschild–deSitter Coordinate-Singularity-Avoiding Coordinates Extension

The results of [5] are recalled here for comparison with the further findings of the present paper, and opportunely commented upon.

The Kruskal coordinates allow one to rewrite the maximal extension of the metric Equation (30). The coordinates singularity is shown to be avoided in [5] after the choice of the coordinates (u, v) obeying the system

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial \rho}, \tag{32a}$$

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial \rho}, \tag{32b}$$

with the radial variable defied after the differential $d\rho$ as

$$d\rho \equiv \frac{dr}{1 - \frac{rs}{r} - \frac{\lambda}{3}r^2}, \tag{33}$$

therefore with $\rho_0 = 0$, $r_0 = 0$.

The Nariai coordinates Equation (32) are written as

$$u = e^{k\rho/T} ch \frac{kt}{T}, \tag{34a}$$

$$v = e^{k\rho/T} sh \frac{kt}{T}, \tag{34b}$$

where the variable ρ is obtained after Equation (33) as

$$\rho = r + 2Mln\left(\frac{r}{2M} - 1\right), \tag{35}$$

with the initial values of the variables already fixed.

The quantity T from Equation (34) is hypothesized as

$$T^{-1} = \sqrt{\Lambda/3} \tag{36}$$

for the modification of the Schwarzschild radius to be small, i.e., for the demonstration of the integration of Equation (33) as Equation (34) within the chosen order(s) of infinitesimals.

k is calculated after the equality

$$u^2 - v^2 = e^{2k\rho/T}; \tag{37}$$

The different orders of infinitesimals can also be calculated for the purpose of the modifications of the Schwarzschild radius to be small.

The latter condition is reflected after

$$\frac{\Lambda}{k} = 32M^2. \tag{38}$$

4.2. Generalized Schwarzschild Spacetimes with a Cosmological Constant

The generalized spherically symmetric Schwarzschild spacetimes with a cosmological-constant term, endowed with a Schwarzschild solid-angle element, are written as

$$ds^2 = c^2\left(1 - \frac{r_s}{r} - k_2 r^2\right) magentadx_0^2 + \\ - \frac{1}{\left(1 - \frac{r_s}{r} - k_2 r^2\right)} dr^2 - r^2 d\theta^2 - r^2 (\sin\theta)^2 d\phi^2. \tag{39}$$

4.2.1. Analytical Study of the Solutions: The Radii

The $g_{tt} = 0$ equation is solved after the radii

$$r_1 = \frac{1}{6}\frac{(\beta k_2^2)^{(1/3)}}{k_2} + \frac{2}{(\beta k_2^2)^{1/3}}, \tag{40a}$$

$$r_2 = -\frac{1}{12}\frac{(\beta k_2^2)^{(1/3)}}{k_2} - \frac{1}{(\beta k_2^2)^{1/3}} + i\frac{\sqrt{3}}{2}\left[\frac{1}{6}\frac{(\beta k_2^2)^{1/3}}{k_2} - \frac{2}{(\beta k_2^2)^{1/3}}\right], \tag{40b}$$

$$r_3 = -\frac{1}{12}\frac{(\beta k_2^2)^{(1/3)}}{k_2} - \frac{1}{(\beta k_2^2)^{1/3}} - i\frac{\sqrt{3}}{2}\left[\frac{1}{6}\frac{(\beta k_2^2)^{1/3}}{k_2} - \frac{2}{(\beta k_2^2)^{1/3}}\right], \tag{40c}$$

where the functions

$$\beta \equiv 12\sqrt{3}\sqrt{\frac{\alpha}{k2}} - 108 r_s \tag{41}$$

and

$$\alpha \equiv 27 k_2 r_s^2 - 4 \tag{42}$$

are defined. The choice of this parameterization, and, in particular, the choice of the parameterization Equation (42), are made for comparison of the new general results obtained here with the result in [2].

The existence of the radii Equation (40) is analytically discussed after the realness of the square root of the function α/k_2

$$\frac{27k_2 r_s^2 - 4}{k2} \geq 0 \tag{43}$$

from which the result of [2] is refined and extended; in particular, r_1 is found to be well defined in the new intervals

$$-\infty < k_2 < 0, \tag{44a}$$

$$k_2 > \frac{4}{27 r_s^2}. \tag{44b}$$

The conditions on k_2 are found, as well as the new conditions on the modifications of the Schwarzschild radius.

The request that the denominators be different from zero yields

$$12\sqrt{3}\sqrt{\frac{27k_2 r_s^2 - 4}{k_2}} - 108 r_s \neq 0. \tag{45}$$

The request of the denominator of the function multiplying the imaginary unit be different from zero requires

$$k_2^2\left[\sqrt{3}\sqrt{\frac{27k_2 r_s^2 - 4}{k_2}} - 9 r_s\right] \neq 0. \tag{46}$$

4.2.2. Merging of Two of the Three Radii

The two radii r_2 and r_3 of Equation (40) are identified as function multiplying the imaginary unit vanishes, i.e., $12^{1/3}k_2 - \left[\left(\sqrt{3}\sqrt{\frac{27k_2r_s^2-4}{k2}} - 9r_s\right)k_2^2\right] = 0$.

The two radii are found as

$$r_a = \frac{1}{6}\frac{(\beta k_2^2)^{(1/3)}}{k_2} + \frac{2}{(\beta k_2^2)^{1/3}}, \tag{47a}$$

$$r_b = -\frac{1}{12}\frac{(\beta k_2^2)^{(1/3)}}{k_2} - \frac{1}{(\beta k_2^2)^{1/3}}. \tag{47b}$$

The coordinates singularity is removed after the choice of the coordinates [5].

The new constraints on the Schwarzschild radius imposed after the cosmological-constant term are therefore summoned.

4.3. Comparison with Generalized Schwarzschild Spacetimes with a Linear Term and a Cosmological Constant

The parameterization of the three analytical radii can be also achieved in the equivalent manner as the equation $g_{tt} = 0$ is solved by the three radii

$$r_1 = \frac{1}{6}\frac{B^{1/3}}{k_2} + \frac{2}{3}\frac{+3k_2^2}{k_2 B^{1/3}}, \tag{48a}$$

$$r_2 = -\frac{1}{12}\frac{B^{1/3}}{k_2} - \frac{1}{3}\frac{3k_2}{k_2 B^{1/3}} + i\frac{\sqrt{3}}{2}\left(\frac{B^{1/3}}{6k_2} - \frac{2}{3}\frac{3k_2}{k_2 B^{1/3}}\right), \tag{48b}$$

$$r_3 = -\frac{1}{12}\frac{B^{1/3}}{k_2} - \frac{1}{3}\frac{3k_2}{B^{1/3}} - i\frac{\sqrt{3}}{2}\left(\frac{B^{1/3}}{6k_2} - \frac{2}{3}\frac{3k_2}{k_2 B^{1/3}}\right), \tag{48c}$$

where the function B is defined as

$$B \equiv 12\sqrt{3}\sqrt{+27k_2 r_s - 4k_2}k_2 - 108k_2^2 r_s. \tag{49}$$

The analysis of Equation (48) is analogous to the study of Equation (68), from which the qualifying features of the Schwarzschild spacetimes with a cosmological-constant term can be appreciated to lead to the analysis of two very different geometries in the deSitter case and in the anti-deSitter case.

Furthermore, the constraints obtained also compare with those obtained after Equation (40), as needed.

4.4. Coordinate-Singularity-Avoiding Coordinates Extensions of Generalized Schwarzschild Spacetimes with a Cosmological-Constant Term

From the calculations of the orders of the remainders in Equation (33), it is here newly probed that the sign of the cosmological constant does not affect the series expansion of the denominator of Equation (33).

As a new result, the sign of the cosmological constant does not affect the choice of the coordinate-singularity-avoiding coordinates extension. Differently stated, the choice of [5] for a Schwarzschild–deSitter spacetime holds also in the case of Schwarzschild–anti-deSitter spacetimes because also in this case there are no powers with even-denominator exponent of the cosmological-constant term.

5. The Schwarzschild Spacetimes with a Linear Term

The generalized spherically symmetric Schwarzschild spacetimes, endowed with a Schwarzschild solid-angle element, are specified after the g_{tt} element

$$g_{tt} = 1 - \frac{r_s}{r} - k_1 r \tag{50}$$

as

$$ds^2 = c^2\left(1 - \frac{r_s}{r} - k_1 r\right) magentadx_0^2 +$$

$$-\frac{1}{(1-\frac{r_s}{r}-k_1 r)}dr^2 - r^2 d\theta^2 - r^2(\sin\theta)^2 d\phi^2, \tag{51}$$

Some aspects of generalized metric have been investigated in [44,45].
The two distinct radii are analyzed at the condition

$$k_1 = \frac{1}{r_s}; \tag{52}$$

the two radii are found to merge as

$$r_\pm \equiv r_h = \frac{1}{2k_1} = 2r_s. \tag{53}$$

The coordinates for the spacetimes were analyzed after [46] in the cases

$$k_1 < \frac{1}{4r_s}, \tag{54}$$

which leads to

$$r_\pm = \frac{1}{2}\frac{1 \pm \sqrt{1-4k_1 r_s}}{k_1} \tag{55}$$

and in the case of the naked singularity

$$k_1 > \frac{1}{4r_s}. \tag{56}$$

5.1. Generalized Schwarzschild Spacetimes with a Linear Term

5.1.1. The Two Radii

The solution of the equation $g_{rr} = 0$ is found as the two radii

$$r_1 = \frac{1}{2}\frac{1 + \sqrt{1-4k_1 r_s}}{k_1}, \tag{57a}$$

$$r_2 = \frac{1}{2}\frac{1 - \sqrt{1-4k_1 r_s}}{k_1}. \tag{57b}$$

5.1.2. Coordinate-Singularity-Avoiding Coordinates Extension

The coordinates extension (u, v) able to avoid the coordinates singularity are defined with new radial coordinate ρ

$$\rho - \rho_0 \simeq \frac{r + r_s \ln\left(\frac{r}{r_s} - 1\right)}{(1-k_1 r)\left(1-\frac{r}{r_s}\right)(1+k_1 r)}. \tag{58}$$

The radial coordinate ρ is rewritten as

$$\rho = r + r_s \ln\left(\frac{r}{r_s} - 1\right) + O(r^2) + O(k_1^2) + R_3 + R_4, \tag{59}$$

with $+O(r^2) + O(k_1^2)$ as the remainders of the series expansions of Equation (59). The remainder R_3 is

$$R_3 = O\left(r_s^6 \frac{\ln\frac{r-r_s}{r_s}}{r-r_s}\right); \tag{60}$$

the remainder R_4 is

$$R_4 = O\left(r_s^5 \ln\frac{r-r_s}{r_s}\right). \tag{61}$$

Thus, from Equation (58), the initial value of the new radial coordinate ρ is newly calculated as

$$\rho_0 = 0. \tag{62}$$

The new expression of T is found from Equation (59) as

$$T^{-1} = \frac{1}{2}\frac{r_s^2 k_1^2}{1 - k_1^2 r_s^2} \tag{63}$$

i.e., such that its expansions respect the orders of infinitesimals of r^2 in Equation (59).

The new further constraint on the linear r from Equation (58) is found as

$$\frac{k_1 r_s}{1 - k_1 1^2 r_s^2} \equiv 1. \tag{64}$$

The new expression of k is fixed here after the request Equation (37). The extensions Equation (34) therefore holds with the definition Equation (58), after having provided that the modification(s) of the Schwarzschild radius are small, i.e., from Equations (60) and (61).

The set of coordinates extension allows one to merge the two solutions without posing $k_1 = 0$. From $zero - th$ order in r of the coordinates extension, the new condition

$$(r - r_s)(1 - k_1 r) \neq 0 \tag{65}$$

is found.

6. Generalized Schwarzschild Spacetimes with a Linear Term and a Cosmological Constant

The generalized spherically symmetric Schwarzschild spacetimes with a linear term and a cosmological constant are specified after the g_{tt} element

$$g_{tt} = 1 - \frac{r_s}{r} - k_1 r - k_2 r^2 \tag{66}$$

as

$$ds^2 = c^2\left(1 - \frac{r_s}{r} - k_1 r - k_2 r^2\right) magentadx_0^2 + \\ - \frac{1}{\left(1 - \frac{r_s}{r} - k_1 r - k_2 r^2\right)} dr^2 - r^2 d\theta^2 - r^2 (\sin\theta)^2 d\phi^2. \tag{67}$$

6.1. The Three Radii

The equation $g_{rr} = 0$ is solved by the three radii

$$r_1 = \frac{1}{6}\frac{b^{1/3}}{k_2} + \frac{2}{3}\frac{k_1^2 + 3k_2^2}{k_2 b^{1/3}} - \frac{1}{3}\frac{k_1}{k_2}, \tag{68a}$$

$$r_2 = -\frac{1}{12}\frac{b^{1/3}}{k_2} - \frac{1}{3}\frac{k_1^2 + 3k_2}{k_2 b^{1/3}} - \frac{1}{3}\frac{k_1}{k_2} + i\frac{\sqrt{3}}{2}\left(\frac{b^{1/3}}{6k_2} - \frac{2}{3}\frac{k_1^2 + 3k_2}{k_2 b^{1/3}}\right), \tag{68b}$$

$$r_3 = -\frac{1}{12}\frac{b^{1/3}}{k_2} - \frac{1}{3}\frac{k_1^2 + 3k_2}{b^{1/3}} - \frac{1}{3}\frac{k_1}{k_2} - i\frac{\sqrt{3}}{2}\left(\frac{b^{1/3}}{6k_2} - \frac{2}{3}\frac{k_1^2 + 3k_2}{k_2 b^{1/3}}\right), \tag{68c}$$

where the function b is defined as

$$b \equiv 12\sqrt{3}\sqrt{4r_s k_1^3 - k_1^2 + 18k_1 k_2 r_s + 27k_2 r_s - 4k_2 k_2 +} \\ -8k_1^3 - 108k_2^2 r_s - 36k_1 k_2. \tag{69}$$

The request for the denominator to be different from zero $b \neq 0$ implies the new constraints of the Schwarzschild radius

$$r_s \neq \frac{1}{54k_2^2}\left(\frac{4}{35}\sqrt{3}\sqrt{-105k_1^2 - 315k_2}\left(\frac{k_1^2 + 3k_2^2}{3}\right) - 4k_1^3 - 18k_1k_2\right), \tag{70a}$$

$$r_s \neq \frac{1}{54k_2^2}\left(-\frac{4}{35}\sqrt{3}\sqrt{-105k_1^2 - 315k_2}\left(\frac{k_1^2 + 3k_2^2}{3}\right) - 4k_1^3 - 18k_1k_2\right), \tag{70b}$$

with the new condition on the parameters

$$k2 \neq -\frac{1}{3}k_1^2. \tag{71}$$

Merging of Two of the Three Solutions

The expressions of the two radii r_2 and r_3 are discussed as

$$\Bigg(12\sqrt{3}\sqrt{4r_sk_1^3 - k_1^2 + 18k_1k_2r_s + 27k_2r_s - 4k_2k_2 +}$$
$$-8k_1^3 - 108k_2^2r_s - 36k_1k_2\Bigg)^{\frac{2}{3}} - 4k_1^2 - 12k_2 = 0, \tag{72}$$

for which the new constraint on the Schwarzschild radius is found:

$$r_s = -\frac{1}{27}\frac{2(k_1^2 + 3k_2)^{3/2} + 2k_1^3 + 9k_1k_2}{k_2^2}. \tag{73}$$

A solution with no real roots of of $r_1 = 0$ is found only at the (hypothesized, not taken into account) value only at $r_s = 0$.

6.2. Coordinate-Singularity-Avoiding Coordinates Extension

The coordinate-singularity-avoiding coordinates extension (u, v) is found after the hypotheses that the cosmological-constant term should modify the Schwarzschild term not strongly, and that the the k_1 term induce a modification of next order after choosing the new differential

$$d\rho \equiv \frac{dr}{(1 + k_1r + k_2r^2)(1 - \frac{r_s}{r})(1 + k_1r - k_2r^2)}. \tag{74}$$

The new differential Equation (74) is integrated as

$$\rho - \rho_0 \simeq \frac{1}{4}\frac{r_s(k_1 + 2k_2r_s)(k_1^2 - k_2 + k_1k_2r_s)}{k_2(k_2r_s^2 + k_1r_s + 1)^2} + r +$$
$$+ r_s ln\left(\frac{r}{r_s} - 1\right) + R1(k_1, k_2, r_s) + R_2(k_1, k_2, r_s). \tag{75}$$

In Equation (75) $R(k_1, k_2, r_s)$ is the remainder of the series of the ln term function of the Schwarzschild-related term, and $R_2(k_1, k_2, r_s)$ is the remainder of the series of the term containing the contributions related to the parameters k_1 and k_2 ones, in the vicinity of $r \simeq r_s$.

Thus, the initial value of the new radial coordinate ρ_0 in Equation (75) is set. The remainder $R_1(k_1, k_2, r_s)$ is of order $R_1(k_1, k_2, r_s) \equiv O(k_2^2)$. The remainder R_2 is of order

$$R_2 \simeq \sum_i f_i\left(k_1, k_2, r_s, \frac{1}{\sqrt{k_1^2 - 4k_2}}\right)O(r^2) \tag{76}$$

with f_i being the pertinent set of functions.

The remainder R_2 encounters the conditions requested in [5]. The new further condition is found:

$$k_1^2 - 4k_2 \geq 0. \tag{77}$$

As the calculation Equation (75) is spelled in $O(r^2)$ term, the following orders are calculated in even powers of the k_2 term. The $zero - th$ order in r of Equation (75), i.e., the constant term, implies the new condition $k_2 r_s^2 + k_1 r_s + 1 \neq 0$, that is

$$k_2 \neq -\frac{1 + k_1 r_s}{r_s^2}. \tag{78}$$

More in detail, from Equation (74), the following new constraint is found:

$$\frac{1}{4} r_s \frac{k1}{k_2} \frac{k_1^2 - k_2 + r_s k_1 k_2}{1 + k_1 r_s + k_2 r_s^2} = 1; \tag{79}$$

the new definition of T is found as

$$T^{-1} = 8 \frac{k_2}{r_s} \frac{1 + r_s k_1 + r_s^2 k_2}{(k_1^2 - 2k_2)(k_1^2 - k_2 + r_s k_1 k_2)}. \tag{80}$$

The new constraint of the parameter space of the model is learned from Equation (79); it is straightforwardly newly demonstrated to hold in different manners according to the values of the linear term k_1 and to those of the cosmological-constant term k_2.

7. Outlook

7.1. Embeddings

For the spherically symmetric considered spacetimes, the segment $d\sigma^2 = g_{rr}(r)dr^2 + g_{\varphi\varphi}(r)d\phi^2$ is rewritten in a three-dimensional Euclidean space coordinatized in the cylindrical coordinates as $d\sigma^2 = \left[1 + \left(\frac{dz}{d\varrho}\right)^2\right] d\varrho^2 + \varrho^2 d\varphi^2$ with $\varrho(r^2) = g_{\varphi\varphi}(r) \geq 0$. The embedding functions $z(\varrho)_{k_i}$ are found as

$$g_{\varrho\varrho} = [1 + \frac{1}{4}(\frac{1}{\varrho} - \frac{rs}{\varrho^{3/2}} - \frac{k_1}{\sqrt{\varrho}} - k_2)], \tag{81}$$

for which the limits of vanishing k_1 and k_2 are well posed (Figure 1).

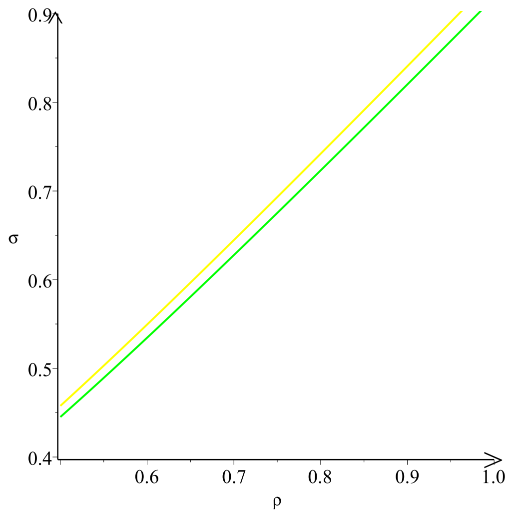

Figure 1. The $\varphi = 0$ sections of the embedding surfaces σ for the Schwarzschild spacetime (yellow–light gray) and for the spacetime $k_1 = 1/4r_s$, $k_2 = 0$ (green–gray) after Equation (81) in units $M = 1$; the difference in the two $R_{1212}(\sigma)$s is appreciated. The role of the linear term in the modification of the Schwarzschild radius is therefore outlined.

7.2. Weak-Field Limit

For static spherically symmetric spacetimes, the weak-field limit depends on the weak-field limit (w-fl) of the Christoffel symbol Γ^r_{00} in the linearized regime as for from the weak-field-limit of generalized potential $\Phi(r)$ defined after $g_{tt} = 1 + \phi$ as $\Gamma^r_{00} \mid_{w-fl} = -\Phi(r) \mid_{w-fl,r}$, $\Phi(r)$ being a gravitational potential; the Schwarzschild terms lead to the Newtonian potential, the k_2 term is higher orders, while the k_1 term is kept and accounted, i.e., for helping the galaxy rotation curve.

More in detail, the weak-field limit is considered for the Christoffel symbol from the spherically symmetric metric

$$g_{tt} \equiv 1 - \frac{r_s}{r} + \psi(r) \tag{82}$$

as defined and specified in Section 3

$$\Gamma^r{}_{00} \equiv \frac{1}{2}(1 - \frac{r_s}{r^2} + \psi(r))(\frac{r_s}{r} + \frac{d\psi}{dr}). \tag{83}$$

When specified for the here-considered generalizations of the Schwarzschild spacetimes, i.e., $g_{tt} = 1 - \frac{r_s}{r} - k_1 r - k_2 r^2$, one obtains that the Newtonian gravitational potential $\Phi(r)$ descends from the r_s addends, the nonnegligible modification terms descend from the k_1 addends, and the addends containing the k_2 are negligible. The analysis of the addend containing $k_1 \cdot k_2$ is achieved.

8. Prospective Studies

The understanding of the mechanisms underlying the appearance of the linear term and of the cosmological-constant term can be tentatively drawn from a host of disciplines, aimed at fixing the pertinent possible values of these term for the generalizations of the Schwarzschild spacetimes.

8.1. Experimental Validation: Data Analyses

The study of the allowed values of the linear term and of that of the cosmological constant term have widely been very recently from a data analysis point of view throughout the literature.

The features of the terms corresponding to the fluid, which are worked out from the linear-term effects, have been scrutinized within the experimental framework of COBE-Planck in [47] for the deSitter case.

The potentiality of the determinations of the values of the linear term and of that of the cosmological constant have been recently envisaged in the observational analyses of CMB, gravitational waves, dark matter, and dark radiation from string cosmology [48].

An attempt to constrain the values of the (positive) cosmological constant and that of the linear term for a generalized Schwarzschild–deSitter spacetime was proposed in [49] from a data analysis point of view.

8.2. Experimental Validation: Geometry Methods

The null geodesics in an cosmological-constant-driven expanding universe were studied only in [50], in which the black hole shadows could be envisaged.

The effects of the dilaton in reducing the size of innermost stable circular time-like orbits with respect to the Schwarzschild scheme were compared in [51] within the framework of the generalization of the flat-curve-condition spacetime with the tools of the Einstein field equations with respect to gravitational lensing.

From [44], broader investigations about the strong gravitational lensing of the Kiselev blackhole are available.

The phenomenon of gravitational lensing from quasi-periodic flares of star–accretion disc collisions was studied in the Schwarzschild case and in the Kerr case in [52].

Time-like geodesics of generalized Schwarzschild black holes were studied in [53–55] also in comparison with other types of black holes, from a quantum viewpoint, and in comparison with the naked singularity.

Schwarzschild–deSitter spacetimes, Reissner–Nordstrom–de Sitter spacetimes, and Kerr–deSitter spacetimes were compared in [56] as far as the influence of the duty of the cosmological constant is concerned regarding the properties of accretion discs orbiting black holes for quasars and for the active galactic nuclei.

The time-like geodesics and the geodesic deviation within the framework of a two-dimensional "stringy" black hole spacetime in the Schwarzschild gauge were studied in [55], from which the low-energy limit should be derived for a comparison with the present result of the paper.

8.3. Quantum Implementation

The quantum features of generalized spacetimes can be investigated, as indicated from [9].

The Gibbons–Hawking temperature is defined as $T_{GH}(r^*) \equiv \frac{1}{4\pi}\left[\frac{dg_{tt}}{dr}\right]_{r=r^*}$, i.e., to be evaluated ate the coordinate point $r = r^*$.

The entropy of the Schwarzschild–deSitter black holes was calculated in [57]; the thermodynamics of asymptotically anti-deSitter black holes was determined in [58]. The thermodynamical features of Schwarzschild–anti-deSitter black holes in various spacetime dimensions was defined in [59].

The chosen generalized Schwarzschild spacetimes can be demonstrated to be a low-energy limit of other structures.

The Schwarzschild–deSitter black holes and the Schwarzschild–anti-deSitter ones are studied in the loop-quantum-gravitational regime [60].

The Schwarzschild–deSitter black holes and the Schwarzschild–anti-deSitter ones are established to constitute the low-energy limit of a supersymmetric scheme in [61].

The understanding of the quantum features of generalized Schwarzschild spacetimes might lead to a better insight if its modes spectrum analysis. In [62], the analysis was performed regarding generalized Schwarzschild spacetimes with a cosmological constant.

The quasinormal modes and the Hawking radiation of a Schwarzschild spacetime were studied after assuming Bardeen-inspired terms as the quantum corrections in [63].

The perturbation methods can also be used to investigate the features of generalized Schwarzschild spacetimes, and the results should be recast to be used as far as the classical description is concerned. As an example, in [2], generalized Schwarzschild deSitter spacetimes and generalized Schwarzschild–anti-deSitter spacetimes were investigated with perturbative methods within the Regge–Wheeler formalism for the use of the Schroedinger equation. The Gibbons–Hawking coordinate system was analyzed. The results learned from the perturbative point of view are that there exist regions of the parameter space of the models, for which the Kruskal coordinates are regular, from Equation (A10) from [2].

The study of the parameter space for the regularity of the Kruskal coordinates opens the way to the study of the further generalizations of the Schwarzschild spacetimes with the perturbative method of the Schroedinger equation in the quantum regime and comparison with the results found in the present work.

More in detail, the study of the tasks of the linear term from the quantum-perturbative methods point of view and the low-energy limit of the opportune Christoffel symbol should be compared.

9. Remarks

The aim of this paper was to study some of the possible generalizations of the Schwarzschild spacetimes, which find agreement with the confrontation of phenomenological evidence.

In particular, three instances of those were newly investigated from an analytical point of view: generalized Schwarzschild spacetime, endowed with a Schwarzschild solid-angle element, with a cosmological-constant term; generalized Schwarzschild spacetime, endowed with a Schwarzschild solid-angle element, with a linear term; and generalized Schwarzschild spacetime, endowed with a Schwarzschild solid-angle element, with a linear term and with a cosmological-constant term. From the analytical point of view, the new analytical expressions of the analytical radii were newly found and discussed, i.e., such that the parameter spaces of the models were newly delineated as far as the allowed sets of values of the cosmological-constant term and of those of the linear term are concerned. From the analytical expression of the radii, the new conditions to write down the analytical expressions of the new physical horizons were newly worked out: the new apt conditions further frame the parameter space.

In the case of generalized Schwarzschild spacetime, endowed with a Schwarzschild solid-angle element, with a cosmological-constant term, the analytical radii were found, and the conditions to obtain the physical horizons were framed. In [2], the relation between the Schwarzschild radius and the positive value of the cosmological-constant term in the deSitter case only were hinted at. Within the present new analysis, the results of [2] are newly exhaustively completed; furthermore, the new exploration of the context is extended newly to the negative values of the cosmological-constant term in the anti-deSitter case. As a result, the new generalized Schwarzschild–deSitter spacetimes and the new generalized Schwarzschild–anti-deSitter spacetimes were newly demonstrated to be characterized in a different manner after the presence of the cosmological-constant term; more in detail, the presence of the cosmological constant was newly proven to be allowed to modify the role of the Schwarzschild radius only slightly in the Schwarzschild–deSitter case and in the Schwarzschild–anti-deSitter case; moreover, the modification was newly proven to happen in a different manner in the two cases. The Nariai coordinate-singularity-avoiding coordinates extensions was newly proven to hold also in the anti-deSitter case: indeed, the calculations of the orders of the remainders were newly proven, not modified, after the sign of the cosmological-constant term.

Generalized Schwarzschild spacetimes, endowed with a Schwarzschild solid-angle element, modeled after the presence of a linear term, were newly examined. The analytical expressions of the analytical radii and those of the physical horizons were newly constrained in order to set the newly-discovered qualities of the parameter space of the model. The new coordinate-singularity-avoiding coordinates extensions were found; the new initial values of the new variables were newly found; the new role of the linear term in the modification of the Schwarzschild radius was newly focused; the new constraints on the parameter space of the model were newly assessed.

Generalized Schwarzschild spacetimes, endowed with a Schwarzschild solid-angle element, with a cosmological constant and a linear term, were newly analyzed. The new analytical expressions of the analytical radii were newly written, from whose well-posedness the parameter space was newly constrained; the analytical expressions of the physical horizons were newly worked out, from which the parameter space of the scheme was newly investigated and newly constrained. The new coordinate-singularity avoiding coordinates extensions were newly provided; the new role of the linear term and those of the cosmological term in generalized deSitter case and in generalized anti-deSitter case were newly outlined as modifying the qualities of the Schwarzschild radius; the new initial values of the new coordinates were newly calculated. The remainders were newly calculated, in which the sign of the value of the cosmological constant was newly discovered not to alter the orders of the remainders; the expansions of the integrand functions were newly found to hold in both the generalized deSitter case and the generalized anti-deSitter case.

The results are of interest for the comparison with Lemaitre–Tolman–Bondi spacetimes features [64,65].

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