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On the Possibility of a Static Universe

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Abstract: After a century of cosmological observations, we have a solid standard model of cosmology. However, from a theoretical viewpoint, it is a compelling question if the cosmological data inevitably require an expanding universe independently of the theoretical framework. The possibility of obtaining a viable cosmological model with a constant scale-factor is discussed in the context of the Brans–Dicke class of scalar–tensor theories. It is shown that a flat spatial section requires the presence of a stiff matter fluid. However, some kinematical properties of the standard cosmological model can be reproduced. A realistic scenario may require a more complex class of scalar–tensor theories.

Keywords: cosmology; expanding universe; static universe; scalar-tensor theories

1. Introduction

The publication of the article of Alexander Friedmann in 1922 proposing the possibility of a dynamical universe [1] was one of the most important revolutions in our view of the cosmos. For the first time, to our knowledge, in the history of science, the universe was considered as an evolving system. Until Friedmann, the known universe was described essentially as a static system. Even, the first cosmological model constructed from the recently proposed new theory of gravity, General Relativity, was static and inevitably unstable due to the attractive character of the gravitational interaction [2,3]. In spite of the unstable nature of any static cosmological system, the Friedmann proposal of a dynamical universe initially received some opposition. It must be remembered that the concept of galaxies distributed in the universe emerged after a long debate and, only after the Friedmann article, that the measuring of the spectra of galaxies was obtained showing the systematic redshift of the spectral lines, an indication of cosmic expansion. Unfortunately, Friedmann did not live enough to watch the triumph of his speculations about a dynamical universe with the formulation of the law for the recessions of the galaxies made, mainly, by Hubble and Lemaître.

Is there any reason to consider the possibility of a static universe? The answer most probably is no, for two main reasons. First, to explain the redshift of the spectral lines of distant objects is not simple without a dynamical cosmos. The hypothesis of the tired light [4,5], for example, has difficulties in incorporating a hot phase and the consequent primordial nucleosynthesis and the spectrum of the Cosmic Microwave Background Radiation (CMB), besides structure formation. Second, due to the attractive character of gravity, any static universe would be unstable. This is a feature difficult to circumvent and it is hard to conceive a model within General Relativity that can change this picture.

Notwithstanding, the previous remarks refer to a *completely* static universe. It is possible to conceive of a universe with a constant scale factor, but with some other possible dynamical quantity [6–10]. This is the case for scalar–tensor theories where gravity is

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coupled to a scalar field: there are specific configurations for which the scale factor is constant but the scalar field is dynamical. Since in many scalar–tensor theories the scalar field is connected with the gravitational coupling, a time-dependent scalar field may imply a cosmical dynamics in spite of a constant scale factor. There are simple examples where a stable configuration can be obtained with a constant scale factor and a dynamical gravitational coupling, but they are very particular. This will be discussed in next section. It is far from obvious how this particular configuration can be generalized in order to have a realistic cosmological scenario, incorporating the different phases of the evolution of the universe.

A cosmological scenario without expansion containing at least one static phase has been discussed by Wetterich, leading to a viable model [11,12]. The Wetterich model also contains contracting phases. The effects typically identified as being due to the expansion of the universe (redshift for example) are transferred to a time-dependent mass of the elementary particles [13–15]. In this model, the cosmic initial singularity can be avoided. The model is formulated by using a scalar field non-minimally coupled with the geometry and with the matter sector. A realistic scenario for all the phases of the universe can be achieved.

The possibility of a cosmological model with a constant scale factor in all cosmological phases has been qualitatively evoked in Ref. [16]. The static model should be constructed in the minimal coupled frame, connected with the non-minimal coupled frame through a conformal transformation. Note that the present approach is substantially different from that used in refs. [11,12], which considered a non-minimal coupled frame between a scalar field and geometry, besides a non-trivial coupling with the scalar field in the matter sector. Evidently, the present approach, formulated in the minimal coupled frame, may be seen as with less freedom than refs. [11,12]. Nevertheless, it allows a connection to some traditional theoretical frameworks, like the Jordan–Wagoner–Brans–Dicke theory, as the resulting theory presented in this paper contains a scalar field minimally coupled to gravity, but non-trivially coupled to the matter sector. Furthermore, the coupling parameter ω of the kinetic term of the scalar field is not constant. Our approach also differs from Wetterich's previous work [13,14] inasmuch as he works with the original Brans–Dicke (BD) theory [17], where ω is a constant, and also includes a potential.

In the present text this possibility is investigated in more detail. It is shown that, at least in the context of BD theory with a dynamical BD parameter ω, as suggested in ref. [16], it is possible to obtain at most a kinematical consistent background description of the cosmic evolution. However, it appears important differences concerning how to obtain equivalent kinematical descriptions between the static model developed here and the dynamical Friedmann models. In the Friedmann models, the behavior of the scale factor is essentially dictated by the equation of state of the matter component. In the corresponding static universe with a dynamical gravitational coupling, a given description of the cosmic evolution is determined essentially by an appropriate choice of the non-trivial coupling function of the scalar field kinetic term, denoted by $\omega(\phi)$. In both cases, expanding or static universes, a similar cosmic red-shift relation can be obtained by choosing appropriately $\omega(\phi)$. In spite of this kinematical equivalence, important phenomena present in the cosmic history demanding a perturbative analysis, such as the CMB and structure formation, very probably can not be incorporated in a static scenario, even if a more detailed analysis is required. In this sense, the analysis to be made here can be seen as a kind of no-go theorem for a universe with a constant scale factor (but possibly with a dynamical gravitational coupling) in all its phases. Of course, it is not excluded that different classes of extensions of the GR theory may change the conclusions presented here.

In the next section we review the Einstein static universe and its instability and the particular cases of the BD theory with some possible static, stable configurations. From now on we use the term "static universe" to denote a universe with constant scale factor even if the gravitational coupling is varying. In Section 3, the Wagoner–Brans–Dicke–Jordan scalar-tensor theory is discussed both in the Jordan and Einstein frames. In Section 4, it is shown how the variation of the mass of elementary particles can lead to a shift in the spectral lines of the hydrogen atom. In Section 5, it is shown how a static universe in the Einstein frame can lead to a scenario where the standard cosmological model is reproduced in the Jordan frame from the kinetic point of view. In Section 6, we conclude with some final remarks.

2. Stability of Static Models in GR and BD Theories

The static model in the GR and BD theories are briefly revised in what follows. The GR equations in presence of a cosmological constant Λ are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} + g_{\mu\nu}\Lambda, \tag{1}$$

$$T^{\mu\nu}{}_{;\mu} = 0. \tag{2}$$

For a static metric, with a spatial curvature k (which can be positive, negative or zero), a pressureless fluid, a cosmological constant and fixing the constant scale factor equal to unity, the equations reduce to,

$$3k = 8\pi G\rho + \Lambda, \tag{3}$$

$$k = \Lambda, \tag{4}$$

$$\dot{\rho} = 0. \tag{5}$$

For $k = 0$, the universe turns out to be completely empty, while for k negative (a pseudo-sphere), the energy density becomes negative. Only for positive k do we have a consistent scenario with

$$4\pi G\rho = \Lambda > 0. \tag{6}$$

However, this solution is unstable. The perturbative equations in the synchronous coordinate condition $h_{\mu 0} = 0$ for a given fluid with density ρ and pressure p are given by [18],

$$\ddot{h} + 2H\dot{h} = 8\pi G\rho\delta, \tag{7}$$

$$\dot{\delta} + (1+\alpha)\left(\theta - \frac{\dot{h}}{2}\right) = 0, \tag{8}$$

$$(1+\alpha)\left[\delta\theta + (2-3\alpha)H\dot{\theta}\right] = \frac{v_s^2}{a^2}\delta, \tag{9}$$

These equations are valid even in presence of a cosmological constant. In these expressions, we have introduced the following definitions:

$$H = \frac{\dot{a}}{a}\,, \qquad h = \frac{h_{kk}}{a^2}\,, \qquad \delta = \frac{\delta\rho}{\rho}\,, \tag{10}$$

$$\theta = \partial_k \delta v^k\,, \qquad \alpha = \frac{p}{\rho}\,, \qquad v_s^2 = \frac{\partial p}{\partial\rho}\,. \tag{11}$$

H is the Hubble function, h_{kk} is the trace of metric fluctuations, δ is the density contrast, θ is related with the velocity perturbation, α is the equation of state parameter and v_s^2 is the sound speed. For a static universe with pressureless fluid ($\alpha = v_s^2 = 0$), the perturbed equations reduce to

$$\ddot{\delta} - 4\pi G\delta = 0 \quad , \qquad \dot{\theta} = 0\,. \tag{12}$$

Consequently, the matter perturbation, expressed by the density contrast δ, grows exponentially, characterizing the instability due to the attractive nature of the gravitational interaction.

The perturbative analysis of the Brans-Dicke cosmological models has been carried out in ref. [19]. The inflationary case will be considered just as a simple example. The background solutions for an equation of state $p = -\rho$ are

$$a(t) \propto t^{\omega+1/2}, \quad (13)$$
$$\phi(t) \propto t^2. \quad (14)$$

The universe is static if $\omega = -1/2$. For this case, the perturbations behave as

$$\frac{\delta\phi}{\phi} \equiv \lambda = \frac{1}{t}\int\left\{c_1 J_{3/2}(nt) + c_2 J_{-3/2}(nt)\right\}, \quad (15)$$

n denoting the wavenumber of the perturbations. The solution displays a growing mode and a decreasing mode. Asymptotically, the growing mode behaves as,

$$\lambda \propto t^2. \quad (16)$$

The growing mode is not exponential as in GR case. It presents a mild instability that is necessary, after all, to induce the formation of structures observed in the universe.

The same properties are verified for a matter dominated universe in a Brans–Dicke cosmology but with $\omega = -1$.

3. Field Equations in the Jordan and Einstein Frames

The example discussed in the previous section shows that it is possible to have a static universe in scalar-tensor theories without exponential instabilities. However, it is not clear how to describe the different expansion phases of the standard model in the static frame. Our proposal is to consider a dynamical parameter $\omega(\phi)$. Following the qualitative discussion presented in ref. [16], our starting point is the Bergmann–Wagoner–Brans–Dicke theory whose Lagrangian in the original Jordan frame is [17,20]

$$\mathcal{L} = \sqrt{-g}\left\{\phi R - \omega(\phi)\frac{\phi_{;\rho}\phi^{;\rho}}{\phi}\right\} + \mathcal{L}_m(g_{\mu\nu}, \Psi), \quad (17)$$

with the matter Lagrangian given by $\mathcal{L}_m(g_{\mu\nu}, \Psi)$, Ψ representing the matter fields. The gravitational coupling is dynamical and related to the inverse of the scalar field ϕ.

The field equations are the following:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi}{\phi}T_{\mu\nu} + \frac{\omega(\phi)}{\phi^2}\left(\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi_{;\rho}\phi^{;\rho}\right) + \frac{1}{\phi}\left(\phi_{;\mu;\nu} - g_{\mu\nu}\Box\phi\right), \quad (18)$$
$$\Box\phi = \frac{8\pi T}{3+2\omega(\phi)} - \frac{\omega_\phi}{3+2\omega(\phi)}\phi_{;\rho}\phi^{;\rho}, \quad (19)$$
$$T^{\mu\nu}{}_{;\mu} = 0. \quad (20)$$

Now, we perform a conformal transformation, with $g_{\mu\nu} = \phi^{-1}\tilde{g}_{\mu\nu}$, as indicated in the Appendix A. The new equations are:

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} = 8\pi G\tilde{T}_{\mu\nu} + \frac{(\omega(\phi)+3/2)}{\phi^2}\left(\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\phi_{;\rho}\phi^{;\rho}\right), \quad (21)$$
$$\tilde{\Box}\phi = \frac{8\pi G\phi\tilde{T}}{3+2\omega(\phi)} - \left(\frac{\phi\omega_\phi}{3+2\omega(\phi)} - 1\right)\frac{\phi_{;\rho}\phi^{;\rho}}{\phi}, \quad (22)$$
$$\tilde{T}^{\mu\nu}_{;\mu} = -\frac{\tilde{g}^{\nu\mu}\phi_{;\mu}\tilde{T}}{2\phi}. \quad (23)$$

In writing these equations, we have made the redefinition,

$$G\tilde{\rho} = \frac{\rho}{\phi^2}, \quad G\tilde{p} = \frac{p}{\phi^2}, \tag{24}$$

G being the present value of the cosmological coupling.

4. The Redshift Relation

In the static universe, the mass of the particles must vary with time in order to obtain a change in the spectral lines, as observed. The mechanism to generate the observed redshift will now be described.

In the Einstein frame, the energy conservation law for any perfect fluid satisfying $\tilde{p} = \alpha\tilde{\rho}$ in a homogeneous and isotropic spacetime reads,

$$\tilde{\rho}' + 3\tilde{H}(1+\alpha)\tilde{\rho} = -\frac{(1-3\alpha)}{2}\frac{\phi'}{\phi}\tilde{\rho}, \tag{25}$$

where the primes mean derivative with respect to τ, and $\tilde{H} = b'/b$ is the Einstein Hubble function, which is considered to be zero. Integrating this equation for the case of a fluid composed of massive non-relativistic particles ($\alpha = 0$), we obtain

$$\tilde{\rho} = \tilde{\rho}_0\left(\frac{\phi_0}{\phi}\right)^{1/2} = n\tilde{m}, \tag{26}$$

where n is the particle number density, which is a constant in a static universe, and $\tilde{\rho}_0$ is a constant. Hence, the relation between the constant mass m in the expanding universe and the varying mass $\tilde{m}$ in the static universe is given by,

$$\tilde{m} \propto m\phi^{-1/2}. \tag{27}$$

Assuming that ϕ is positive and it decreases in time, namely $\infty > \phi > 0$, the masses increase with time, meaning that the wavelength of the emitted radiation will decrease with time. In other words, the electronic transition occurred in the past will have a wavelength greater than observed today in the laboratory. We remark that a decreasing ϕ implies a growing gravitational coupling and a decreasing Planck's mass. Also, the Planck length grows with time. Our approach is purely classical but it suggests that the quantum gravity regime may be achieved at much smaller energy scales as time goes on and the gravitational interaction becomes stronger.

The relation (27) is equivalent to the invariance of the test particle's Lagrangian under the conformal transformation

$$l = -\int m ds = -\int m\phi^{-1/2}d\tilde{s} = -\int \tilde{m}d\tilde{s}. \tag{28}$$

The spectral lines of the hydrogen atom in the static universe are given by

$$\frac{1}{\lambda} = \frac{\Delta E}{hc} = \frac{\tilde{m}Z^2e^4}{4\pi c\hbar^3}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right), \tag{29}$$

with n_i and n_f designating the initial and final electronic principal quantum numbers. Hence, the wavelength of the emitted photon varies as

$$\lambda \propto \frac{1}{\tilde{m}} \propto \phi^{1/2}. \tag{30}$$

As a consequence, the photons emitted in the past have a higher wavelength compared with the emissions in the laboratory today.

The discussion above is an example of how the Jordan and Einstein frames may describe the same phenomena in a complete different way. For the discussion on the equivalence of both frames, see ref. [21] and references therein.

5. A Static Universe: General Relations

We will try now to construct a static universe in the Einstein frame. The difficulty lies in that the conservation law in the Jordan frame implies that each matter component depends only on the scale factor, making hard to obtain a transition from a cosmic phase to another. This can be achieved in the Einstein frame due to the second term in (23) and the presence of the arbitrary function $\omega(\phi)$.

The metric in the minimal coupled frame is given by,

$$d\tilde{s}^2 = d\tau^2 - \frac{b^2}{1+k\frac{r^2}{4}}(dx^2+dy^2+dz^2), \tag{31}$$

with $r^2 = x^2+y^2+z^2$ We shall assume a barotropic equation of state $\tilde{p} = \alpha\tilde{\rho}$ with α constant. The Equations (21)–(23) become:

$$3\tilde{H}^2 + 3\frac{k}{b} = 8\pi G\tilde{\rho} + \left(\frac{3+2\omega}{4}\right)\left(\frac{\phi'}{\phi}\right)^2, \tag{32}$$

$$2\tilde{H}' + 3\tilde{H}^2 + \frac{k}{b} = -8\pi G\tilde{p} - \left(\frac{3+2\omega}{4}\right)\left(\frac{\phi'}{\phi}\right)^2, \tag{33}$$

$$\frac{\phi''}{\phi} + 3\tilde{H}\frac{\phi'}{\phi} = \frac{8\pi G}{3+2\omega}(1-3\alpha)\tilde{\rho} - \left(\frac{\phi\omega_\phi}{3+2\omega(\phi)} - 1\right)\left(\frac{\phi'}{\phi}\right)^2, \tag{34}$$

$$\tilde{\rho}' + 3\tilde{H}(1+\alpha)\tilde{\rho} = -\frac{(1-3\alpha)}{2}\frac{\phi'}{\phi}\tilde{\rho}. \tag{35}$$

The primes mean derivative with respect to τ, and $\tilde{H} = b'/b$. A universe without expansion in this frame means $\tilde{H} = 0$, and without loss of generality we fix $b = 1$. The previous equations reduce to

$$3k = 8\pi G\tilde{\rho} + \left(\frac{3+2\omega}{4}\right)\left(\frac{\phi'}{\phi}\right)^2, \tag{36}$$

$$k = -8\pi G\alpha\tilde{\rho} - \left(\frac{3+2\omega}{4}\right)\left(\frac{\phi'}{\phi}\right)^2, \tag{37}$$

$$\frac{\phi''}{\phi} = \frac{8\pi G}{3+2\omega}(1-3\alpha)\tilde{\rho} - \left(\frac{\phi\omega_\phi}{3+2\omega(\phi)} - 1\right)\left(\frac{\phi'}{\phi}\right)^2, \tag{38}$$

$$\tilde{\rho}' = -\frac{(1-3\alpha)}{2}\frac{\phi'}{\phi}\tilde{\rho}. \tag{39}$$

The last equation can be easily integrated as

$$\tilde{\rho} = \tilde{\rho}_0\phi^{-\frac{(1-3\alpha)}{2}}. \tag{40}$$

Adding (36) and (37), we obtain

$$4k = 8\pi G(1-\alpha)\tilde{\rho}. \tag{41}$$

Subtracting (36) and 3× (37),

$$0 = 8\pi G(1+3\alpha)\tilde{\rho} + (3+2\omega)\left(\frac{\phi'}{\phi}\right)^2 \tag{42}$$

Three conclusions can be obtained from these relations:

$$\alpha = 1 \quad \rightarrow \quad k = 0; \tag{43}$$

$$\alpha \neq 1, \frac{1}{3} \quad \rightarrow \quad \tilde{\rho} = \text{constant} \rightarrow \phi = \text{constant}, \quad k > 0: \tag{44}$$

$$\alpha = \frac{1}{3} \quad \rightarrow \quad \tilde{\rho} = \text{constant} \rightarrow \phi = \text{constant or dynamical}, \quad k > 0. \tag{45}$$

The Einstein static model can be obtained from the previous relation by fixing ϕ constant, and a two fluid model, one a matter component ($\alpha = 0$) and a cosmological constant ($\alpha = -1$), leading to

$$k = 4\pi G\tilde{\rho}_m, \tag{46}$$

$$\tilde{\rho}_\Lambda = \frac{\tilde{\rho}_m}{2}. \tag{47}$$

6. An Example of a Static Universe

Let us consider specifically the case $\alpha = 1$, leading to $k = 0$, and evaluate the distance-redshift relation in this situation.

With the result for the matter density (40), and remembering that with $\alpha = 1$ and $k = 0$, Equation (36) becomes identical to Equation (37), while Equation (38) is just the derivative of those equations. Hence, there is just one equation to be integrated,

$$8\pi G\tilde{\rho} + \left(\frac{3+2\omega}{4}\right)\left(\frac{\phi'}{\phi}\right)^2 = 0. \tag{48}$$

Let us write $w(\phi) =: -3/2 - f(\phi)$. Then, as for $\alpha = 1$ one obrains $\tilde{\rho}/\tilde{\rho}_0 = \phi/\phi_0$, Equation (48) can be written as

$$\frac{f^{1/2}(\phi)}{\phi^{3/2}} \mathrm{d}\phi = \sqrt{C_1}\, c\, \mathrm{d}\tau, \tag{49}$$

where $C_1 = 16\pi G\tilde{\rho}_0/(\phi_0 c^2)$, remembering that the 0 subscript denotes quantities evaluated today, and c is the speed of light. With the integration from proper time τ_1 when a source emitted light towards an observer that receives it at τ_0, we obtain

$$I(\phi_0) - I(\phi_1) = \sqrt{C_1} d, \tag{50}$$

where d is the distance between the source and the observer, which in a flat static universe is just $d = c(\tau_0 - \tau_1)$, and $I(\phi)$ is the function resulting from the integral in ϕ.

Using Equation (50) and writing $\phi_1 = (1+z)^2\phi_0$ we obtain the exact distance–redshift relation,

$$d = \frac{I(\phi_0) - I(\phi_0(1+z)^2)}{\sqrt{C_1}}, \tag{51}$$

For small z we obtain

$$d = \frac{c f^{1/2}(\phi_0)}{2\sqrt{G\tilde{\rho}_0}} z + \dots, \tag{52}$$

where the constant in front of z (without c) plays the role of the inverse of the Hubble constant in this scenario.

As an example, let us take a power law functional form for f, yielding,

$$\omega(\phi) = -\frac{3}{2} - \kappa\phi^n. \tag{53}$$

The parameter κ can be fixed equal to unity by absorbing it in the expression for ϕ. Equation (48) can be easily integrated, leading to

$$\phi = \phi_0(\pm\tau)^{\frac{2}{n-1}}. \tag{54}$$

In order to have $\infty > \phi > 0$ during time evolution, when the exponent is negative the plus sign must be chosen, implying $0 \leq \tau < \infty$, and vice-versa.

In the case of a spatially flat expanding universe dominated by stiff matter and described by standard GR, we have that $1 + z \propto a^{-1}(t) \propto t^{-1/3}$, where t is the proper time. In case of static universes in the Einstein frame of a generalized Brans–Dicke theory as described above, one has $1 + z \propto \phi^{1/2}(\tau) \propto \tau^{1/(n-1)}$ with respect to the proper time in the Einstein frame. Hence, in order to have the same proper time dependence, one must have $n = -2$.

Some observational tests of the unperturbed universe depend essentially on the behavior of the scale factor. It is possible to choose the functional form of $\omega(\phi)$ in order to mimic the scale factor of the standard cosmological model in its different phases by translating the results in the static Einstein frame to the Jordan frame where the scale factor is a function of time. First we remember that, with $b = 1$,

$$a = \phi^{-1/2}, \tag{55}$$

$$dt = \pm\phi^{-1/2} d\tau. \tag{56}$$

Hence, using Equation (54), the scale factor in the Jordan frame is given, in terms of the cosmic time in the same frame, as

$$a = a_0(t)^{\frac{1}{2-n}}, \tag{57}$$

with $0 \leq t < \infty$ for $n < 2$. The convenient choice of n may lead to the same kinematical behavior of the standard cosmological model: $n = 0$ for the radiative phase, $n = 1/2$ for the matter dominated phase and $1 < n \leq \infty$ for the dark energy phase ($n = 2$ corresponds to a de Sitter phase and $n > 2$ to a phantom dark energy phase). In this way, the main phases of the expanding universe can be mapped in the corresponding static models by choosing conveniently the function $\omega(\phi)$.

In principle, a more general choice for $\omega(\phi)$ may lead to smooth transitions between these phases. For example,

$$\omega(\phi) = -\frac{3}{2} - \frac{\kappa_1\phi^2}{\sqrt{\kappa_2 + \kappa_3\phi^3 + \kappa_4\phi^4}}, \tag{58}$$

interpolates smoothly the radiative phase ($\phi \to \infty$) and a de Sitter phase ($\phi \to 0$) passing in between by a matter dominated phase. The parameters κ_i are constants. Since $\infty > \phi > 0$, all three main phases of the standard cosmological model would be generated by this functional form, with a smooth transition between them. The necessary duration of each phase may be achieved by choosing conveniently the values of the parameters κ_i. However, the explicit dependence of ϕ on τ can not be obtained in terms of elementary functions, requiring a numerical integration. Below we present a numerical calculation for the evolution of $\phi(\tau)$ in two scenarios: one with a continuous transition following (58), with $\kappa_2 = 0$ and $\kappa_1 = \kappa_3 = \kappa_4 = 1$, and the other with a piecewise transition at $\phi = 1$ from $\kappa_2 = \kappa_3 = 0$ and $\kappa_4 = 1$ (radiation) to $\kappa_2 = \kappa_4 = 0$ and $\kappa_3 = 1$ (matter), see Figure 1. In both cases, $\kappa_1 = 1$. We note that time runs from left to right (from larger to smaller values of τ).

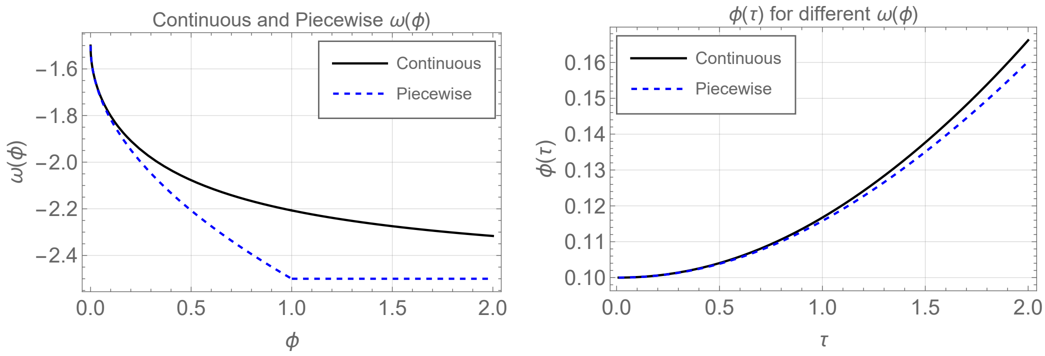

Figure 1. (**Left**) Comparison between $\omega(\phi)$ for the continuous case, (58), where $\kappa_1 = \kappa_3 = \kappa_4 = 1$ and $\kappa_2 = 0$, and the case where there is a sharp transition from $\kappa_2 = \kappa_3 = 0$ and $\kappa_1 = \kappa_4 = 1$ (radiation) to $\kappa_2 = \kappa_4 = 0$ and $\kappa_1 = \kappa_3 = 1$ (matter). (**Right**) Numerical evolution of the scalar field for the two aforementioned cases.

Among the possible limitations of the scenario sketched above, there is one that is particularly relevant; namely, the relation with the standard cosmological model established in the Jordan frame is purely kinematical. Even if we can satisfy some observational tests connected with Hubble–Lemaître law, a detailed perturbative study must be carried out in order to verify if, for example, the structure formation and CMB anisotropies are reproduced, at least in their general lines. This will be object of a separate study.

7. Final Remarks

The possibility to have a static universe compatible with the observational data has been discussed in this text. By static universe, it is understood here as a universe with a constant scale factor but with possible other dynamical fields, like a scalar field related to the gravitational coupling. The Brans–Dicke theory, with a variable ω, was used as an example. In this case, it has been shown that a spatially flat static universe is possible only if the content of the universe is given by a stiff matter fluid. A two-fluid model is also possible, including radiation, but only if there is positive spatial curvature. This can be verified by generalizing Equation (41) including radiation and stiff matter.

In spite of this deceiving limitation, it is possible that some more complex scalar–tensor theories may allow us to surmount the limitations of the static model discussed here. Appealing to other classes of Horndeski theories [22,23] may circumvent the restrictions given specially by relations (41) and (42). In this case, it is maybe possible to mimic different phases of the universe in the behavior of the dynamical scale factor by choosing a convenient function $\omega(\phi)$. This property may lead to a kinematical equivalence between the static universe and the standard mode at the background level. Otherwise, the static universe can be connected only to a given phase in the cosmic history, as in the models discussed in refs. [11,12], which contain, beside a static phase, contracting universes in other phases. However, it seems hard to maintain this equivalence at the perturbative level. This issue must be addressed in a separate analysis.

It is important to remember that the Brans–Dicke theory with a stiff matter fluid has many peculiarities as discussed in ref. [24]. It must be stressed that we have exploited here a conformal transformation in order to rewrite the theory formulated in the Jordan frame in the Einstein frame. The use of disformal transformations my bring other possibilities as discussed in ref. [25], where a particular attention has been given to the case of a stiff matter fluid. The use of the unimodular constraint can also lead to a well-posed scenario [26].

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Appendix A. Conformal Transformation

Under the conformal transformation

$$g_{\mu\nu} = \phi^{-1}\tilde{g}_{\mu\nu}, \quad g^{\mu\nu} = \phi\tilde{g}^{\mu\nu}, \tag{A1}$$

the connection transforms as

$$\Gamma^{\rho}_{\mu\nu} = \tilde{\Gamma}^{\rho}_{\mu\nu} - \frac{1}{2}\left(\delta^{\rho}_{\mu}\frac{\partial_\nu\phi}{\phi} + \delta^{\rho}_{\nu}\frac{\partial_\mu\phi}{\phi} - \tilde{g}_{\mu\nu}\tilde{g}^{\rho\sigma}\frac{\partial_\sigma\phi}{\phi}\right). \tag{A2}$$

The Ricci tensor and the Ricci scalar takes the form

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} + \frac{\phi_{;\mu;\nu}}{\phi} - \frac{1}{2}\frac{\phi_{;\mu}\phi_{;\nu}}{\phi^2} + \frac{1}{2}\tilde{g}_{\mu\nu}\left(\frac{\tilde{\Box}\phi}{\phi} - 2\frac{\phi_{;\rho}\phi^{;\rho}}{\phi^2}\right), \tag{A3}$$

$$R = \phi\left\{\tilde{R} + 3\frac{\tilde{\Box}\phi}{\phi} - \frac{9}{2}\frac{\phi_{;\rho}\phi^{;\rho}}{\phi}\right\}, \tag{A4}$$

The energy momentum–tensor becomes

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu} = \phi\left\{(\rho + p)\tilde{u}^\mu\tilde{u}^\nu - p\tilde{g}^{\mu\nu}\right\} \tag{A5}$$

$$= \phi^3\left\{(\tilde{\rho} + \tilde{p})u^\mu u^\nu - \tilde{p}\tilde{g}^{\mu\nu}\right\}, \tag{A6}$$

with

$$\tilde{\rho} = \rho\phi^{-2}, \quad \tilde{p} = p\phi^{-2}. \tag{A7}$$

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