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A Small Cosmological Constant from a Large Number of Extra Dimensions

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Abstract: In this article, we consider the $4 + n$ dimensional spacetimes among which one is the four dimensional physical Universe and the other is an n-dimensional sphere with constant radius in the framework of Lanczos-Lovelock gravity. We find that the curvature of extra dimensional sphere contributes a huge but negative energy density provided that its radius is sufficiently small, such as the scale of Planck length. Therefore, the huge positive vacuum energy, i.e., the large positive cosmological constant is exactly cancelled out by the curvature of extra sphere. In the mean time the higher order of Lanczos-Lovelock term contributes an observations-allowed small cosmological constant if the number of extra dimensions is sufficiently large, such as $n \approx 69$.

Keywords: Lanczos-Lovelock gravity; cosmological constant problem; dark energy

1. Introduction

The first ideas of existence for new spatial dimensions beyond our four dimensional sapcetime can be dated back to the early works of Kaluza and Klein [1,2]. They tried to unify Maxwell theory with Einstein gravity by proposing a model with a compact fifth dimension where the Maxwell theory was originated. However, this theory is mired with problems, some of which are explored by Wesson in [3]. On the other hand, as one of the most important and fascinating theory in modern theoretical physics, string theory [4,5] suggests that our universe must be higher dimensional and the compact extra dimensions may be as small as the Planck length and therefore nearly impossible to detect. To this day, the idea of extra dimensions has lodged itself in the scientific theories [6].

Lanczos-Lovelock (Thereafter, we use LL to represent it) gravity [7–10] represents a natural and elegant generalization of Einstein's theory of gravity from four dimensions to higher dimensions. It is characterized by the fact that the field equations always contain up to second derivatives of the metric even though the action functional can be a quadratic or very higher degree polynomial in the Riemann curvature tensor. This is very different from the Einstein gravity where only a linear term of Riemann curvature is in the presence. It is found the resulting LL tensor (generalized Einstein tensor) contains the Einstein tensor as a particular case, and it reduces to the latter when the spacetime goes back to four dimensions. The subtlety of second-order differential equations of motion lies in that the linearized LL gravity (around Minkowski space) is ghost-free [11,12]. Furthermore, the number of gravitational degrees of freedom in LL gravity turns out to be exactly that in Einstein gravity [13]. In higher dimensions than four, the LL gravity differs from Einstein gravity and brings us an even richer physical landscapes [14–17].

For cosmology, Deruelle and Fariña-Busto constructed systematic cosmological solutions in the LL gravity. These solutions cover the maximally symmetric space-times, the Robertson-Walker universes and the product manifolds of symmetric subspaces [18]. Mena Marugan [19] considered a class of higher dimensional LL gravity in the presence of a positive cosmological constant whose metric is given by the product of the four-dimensional physical universe and an n-dimensional maximally symmetric space. It was shown that when all the LL coefficients are non-negative, the theory admits classical solutions with a constant scale extra space, and a four-dimensional Einstein gravity with a non-negative cosmological constant. Kirnos and Makarenko [20] studied the accelerating cosmologies in LL gravity with dilaton field. The metric is also assumed to be the product of a four-dimensional universe plus an n-dimensional maximally symmetric extra space. It is found that there are stationary, power-law, exponential and exponent-of-exponent forms of cosmological solutions. Paul and Mukherjee [21] investigated a 10-dimensional cosmology with the Gauss-Bonnet term. By this way, they proposed a scenario which includes a spontaneous compactification of extra space and an inflationary four-dimensional universe followed by a radiation-dominated epoch. Furthermore, the cosmological constant problem is argued to be solved by the dynamics of compact space. Mena Marugan [22] developed the perturbative formalism for the LL gravity in higher dimensional spacetime with the induced metric given by the product of two maximally symmetric spaces. A good feature of this formalism is that it admits the study of multidimensional cosmological models of physical interest, in which all except four dimensions could become compactified. Pavluchenko [23] derived the equations of motion for Bianchi-I cosmological models in the LL gravity. The equations derived apply to any number of spatial dimensions and any order of the LL corrections. Müller-Hoissen [24] studied the LL gravity up to the third order in the Riemann curvature. The equations of motion are elaborated in the Kaluza-Klein cosmology ansatz and solved in the special case where the extra dimensions appear as an n-dimensional sphere with constant radius.

The paper is also devoted to the study of higher dimensional cosmologies in the framework of LL gravity. Staring from the metric of four dimensional Robertson-Walker Universe plus an n-dimensional sphere with constant radius, we derive the equations of motion in LL gravity. Here we take into account three terms, a large cosmological constant, the Einstein-Hilbert term and a higher order LL term except for the matter one in the action. We find that the curvature of extra space plays an important role because it brings us a huge but negative energy density if its radius is in the order of Planck length. Therefore, the large cosmological constant can be exactly balanced. In other words, the huge vacuum energy bends the extra dimensional space into a Planck-length sized sphere while leaving the four dimensional physical spacetime flat. On the other hand, the higher order LL term results in a small cosmological constant which is consistent with the cosmic observations. In order that the resulting cosmological constant is small enough to explain the cosmic speed-up, the number of extra dimensions should be very large, such as $n \approx 69$.

We point out that the effect of extra spatial curvature in the evolution of universe has been explored in the many references. We briefly describe as follows. Wetterich [25] showed that in order to obtain a zero four-dimensional cosmological constant, one must require the existence of compact extra space. In the presence of non-compact extra space, one can envisage dynamical mechanisms leading to a vanishing or extremely small cosmological constant. Rubakov and Shaposhniko [26] discussed the possibility that the cosmological constant problem is solved by raising the number of spatial dimensions in Einstein gravity but with the Kaluza-Klein metric ansatz. In 6-dimensional spacetime, it is found that Einstein gravity with the cosmological constant admits classical solutions with vanishing cosmological constant and the two extra dimensions are compacted. However, there also

exist solutions with the non-vanishing cosmological constant. A vanishing cosmological constant is not preferred at the classical level. Canfora et al. studied the cosmological dynamics in LL gravity (with the Gauss-Bonnet term) in arbitrary dimensions [27]. It is shown that the four dimensional Einstein gravity with small cosmological constant and small extra space can be obtained by spontaneous compactification of Lovelock gravity in vacuum. On the other hand, Chirkov and Toporensky [28] studied the stability problem of extra dimensions with the cubic term of LL term. They showed that the extra space is always stable in the case of negative spatial curvature. On the contrary, for positive spatial curvature, the stability depends on the coupling constants of LL terms. In [29], they numerically checked the stabilization of extra space for negative spatial curvature in the presence of 3-d LL term. Finally, Pavluchenko [30] performed a stability analysis for exponential solutions in LL gravity with the Gauss-Bonnet term and the cubic term, respectively. It is found that the isotropic universe is always stable provided that the Hubble parameter is positive. Other studies on the stability of compactified extra space with both negative and positive curvature are included in [31–33].

Throughout this paper, we adopt the system of units in which $G = c = \hbar = 1$ and the metric signature $(-, +, +, +)$.

2. The Large and Small Cosmological Constants: Cancellation and Generation

Following Deruelle and Fariña-Busto's research work [18], we consider the line element

$$ds^2 = -dt^2 + a(t)^2 d\sigma_3^2 + b^2 d\sigma_n^2 \,, \tag{1}$$

where $d\sigma_3^2$ denotes our three dimensional Euclid space and $d\sigma_n^2$ the n-dimensional sphere for extra space. The scale factor b of extra space is a constant because it has been compactified. The size of b is commonly assumed to be several orders of magnitude of Planck length, l_p,

$$l_p = \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \cdot 10^{-35} \text{ m} \,, \tag{2}$$

in order that it is consistent with gravitational and astronomical observations. $a(t)$ is the scale factor of our four dimensional Universe. We note that the scale factor b is in general also the function of cosmic time t. However, there are no problems with the assumption that it has been compactified as a constant which is frequently adopted in literature, for example, in Refs. [24,34,35]. Our model has the following action

$$S = \int d^{4+n}x \sqrt{-g} \left[\frac{1}{16\pi} \left(-\frac{2\alpha_0}{l_p^2} + \alpha_1 L_1 + \frac{\alpha_N l_p^{2N-2}}{n_{2N-1}} L_N \right) + L_m \right] \,. \tag{3}$$

Here L_1 and L_N are defined by

$$L_1 = R \,, \quad L_N = 2^{-N} \delta_{j_1 j_2 \cdots j_{2N}}^{i_1 i_2 \cdots i_{2N}} R_{i_1 i_2}^{j_1 j_2} \cdots R_{i_{2N-1} i_{2N}}^{j_{2N-1} j_{2N}} \,. \tag{4}$$

We see L_1 and L_N represent the Ricci scalar and the N order LL invariant, respectively. They have the dimension of L^{-2} and L^{-2N}, respectively, because the Riemann curvature has the dimension of L^{-2} (L denotes the length). $\delta_{j_1 j_2 \cdots j_{2N}}^{i_1 i_2 \cdots i_{2N}}$ is the Kronecker symbol of order $2N$ and R_{ef}^{ij} is the $(n+4)$-dimensional Riemann tensor. n is the dimension of extra space and α_i are dimensionless constants. The term $-\alpha_0/l_p^2$ corresponds to the Einstein cosmological constant and it has the dimension of l_p^{-2}. L_1 term is the contribution of Einstein-Hilbert term and the L_N term comes from the higher order LL term. L_m is the Lagrangian of matters.

We note that different from the convention of Deruelle and Fariña-Busto in reference [18], a quotient coefficient of n_{2N-1} arises in the front of L_N. Its subtlety lies in the fact that it helps create an effective small cosmological constant. General Relativity predicts $\alpha_1 = 1$. N is a positive integer and is constrained by the dimension of spacetime as follows

$$2 < N \leq \frac{4+n}{2} \,, \tag{5}$$

when n is an even number, and

$$2 < N \leq \frac{3+n}{2} \,, \tag{6}$$

when is odd. According to the convention of Deruelle and Fariña-Busto in reference [18], n_1 and n_{2N-1} are defined by

$$n_1 \equiv n(n-1) \,, \quad n_{2N-1} \equiv n(n-1)(n-2) \cdots [n-(2N-1)] \,. \tag{7}$$

Variation of action, Equation (3), with respect to the metric, one obtain the LL gravitational equations

$$-\frac{\alpha_0}{l_p^2} g_{\mu\nu} + \alpha_1 G_{\mu\nu} + \frac{\alpha_N l_p^{2N-2}}{n_{2N-1}} G_{\mu\nu}^{(N)} = 8\pi T_{\mu\nu} \,, \tag{8}$$

with

$$G_{\mu\nu}^{(N)} = -2^{-N-1} g_{\mu\rho} \delta_{\nu j_1 j_2 \cdots j_{2p}}^{\rho i_1 i_2 \cdots i_{2p}} R_{i_1 i_2}^{j_1 j_2} \cdots R_{i_{2p-1} i_{2p}}^{j_{2p-1} j_{2p}} \,. \tag{9}$$

$T_{\mu\nu}$ is the energy-momentum tensor in $n+4$ dimensional spacetime. For the cosmic fluid, it can be written as

$$T_{\mu\nu} = [\rho \,, -p g_{ij} \,, -q g_{mn}] \,, \tag{10}$$

where ρ and p are the energy density and pressure in 3-dimensional physical space and q is the pressure in n-dimensional extra space. g_{ij} and g_{mn} denote the metric of physical space and extra space, respectively.

Ref. [18] gives the equations of motion

$$-\frac{2\alpha_0}{l_p^2} + \alpha_1 \left(6H^2 + \frac{n_1}{b^2}\right) + \alpha_N \left[\left(\frac{l_p}{b}\right)^{2N} \cdot \frac{1}{l_p^2} + \frac{6N n_{2N-3}}{n_{2N-1}} \cdot H^2 \cdot \left(\frac{l_p}{b}\right)^{2N-2}\right] = 16\pi\rho \,, \tag{11}$$

$$-\frac{2\alpha_0}{l_p^2} + \alpha_1 \left(2H^2 + \frac{n_1}{b^2}\right) + \alpha_N \left[\left(\frac{l_p}{b}\right)^{2N} \cdot \frac{1}{l_p^2} + \frac{2N n_{2N-3}}{n_{2N-1}} \cdot H^2 \cdot \left(\frac{l_p}{b}\right)^{2N-2}\right]$$
$$+4\frac{\ddot{a}}{a}\left[\alpha_1 + \frac{N\alpha_N n_{2N-3}}{n_{2N-1}} \cdot \left(\frac{l_p}{b}\right)^{2N-2}\right] = -16\pi p \,, \tag{12}$$

$$6\frac{\ddot{a}}{a}\left\{\alpha_1 + N\alpha_N\left[\frac{(n-1)_{2N-2}}{n_{2N-1}}\cdot\left(\frac{l_p}{b}\right)^{2N-2} + \frac{2(N-1)(n-1)_{2N-4}H^2l_p^2}{n_{2N-1}}\cdot\left(\frac{l_p}{b}\right)^{2N-4}\right]\right\}$$

$$-\frac{2\alpha_0}{l_p^2} + \alpha_N\left[\frac{(n-1)_{2N}}{n_{2N-1}}\cdot\left(\frac{l_p}{b}\right)^{2N}\cdot\frac{1}{l_p^2} + \frac{6N(n-1)_{2N-2}}{n_{2N-1}}\cdot H^{2N-2}\cdot\left(\frac{l_p}{b}\right)^2\right]$$

$$+\alpha_1\left[6H^2 + \frac{(n-1)_2}{b^2}\right] = -16\pi q \ , \tag{13}$$

where the dot denotes the derivative with respect to cosmic time, t. H is the Hubble parameter which has the dimension of L^{-1}. Now we have four variables, a, ρ, p and q, but three in-dependant equations. The fourth in-dependent equation is needed. It is the equation of state for matters,

$$p = F(\rho, q) \ . \tag{14}$$

If the scale factor b of extra space is also time-dependant, the energy conservation equation

$$T^{\mu\nu}_{;\nu} = 0 \ , \tag{15}$$

turns out to be

$$\frac{d\rho}{dt} + 3\frac{\dot{a}}{a}(\rho + p) + n\frac{\dot{b}}{b}(\rho + q) = 0 \ . \tag{16}$$

Now the scale factor b is a constant, we have

$$\frac{d\rho}{dt} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \ . \tag{17}$$

In this case, Equation (12) can be replaced with Equation (17). As a result, Equations (11), (13) and (17) plus the equation of state

$$p = F(\rho) \ , \tag{18}$$

constitute the closed system of equations of motion.

Now let's turn to the Friedmann equation Equation (11) which can be rewritten as

$$\alpha_1 3H^2 = 8\pi\rho + \frac{\alpha_0}{l_p^2} - \alpha_1\frac{n_1}{2b^2} - \frac{1}{2}\alpha_N\left[\left(\frac{l_p}{b}\right)^{2N}\cdot\frac{1}{l_p^2} + \frac{6Nn_{2N-3}H^2}{n_{2N-1}}\cdot\left(\frac{l_p}{b}\right)^{2N-2}\right] \ . \tag{19}$$

In order that it restores to the Friedmann equation in General Relativity, we must require

$$\alpha_1 = 1 \ . \tag{20}$$

Now let's make an analysis of the terms on the right hand side of the equation. The first term is cosmic energy density (or mass density) contributed mainly by dark matter and relativistic matter. For the present-day Universe, it is roughly

$$\rho \longrightarrow \frac{\rho}{\rho_p}\cdot\frac{1}{l_p^2} = 10^{-124}\cdot\frac{1}{l_p^2} \ , \tag{21}$$

in the unit of inverse of length squared. ρ_p is called the Planck mass density. If the dimensionless constant α_0 is in the order of unit one $\alpha_0 \approx 1$, we find the ratio of first term

to second term is about 10^{-124}, an extremely small number. It is also the case for the third term if we assume α_1 is in the order of unit one, n_1 not a large number and b a few order of Planck length. The second term and the third term have different signs. This inspires us the second term contributes the huge vacuum energy and the third term is used to counteract it. As for the fourth and fifth term in square bracket, they can be in the order of 10^{-124} provided that N is sufficiently large. This allows us to interpret the two terms as dark energy. We shall explain it in the following manner.

If the cosmological constant is understood as the mass density of vacuum, the estimation of quantum field theory gives the following magnitude for dimensionless constant α_0

$$\alpha_0(\text{theory}) \approx 1 . \tag{22}$$

When the second term is transformed to the unit of mass density, we find

$$\frac{\alpha_0(\text{theory})}{l_p^2} = \frac{1}{l_p^2} \rightarrow \frac{l_p^2}{l_p^2} \cdot \rho_p = \frac{c^5}{\hbar G^2} \approx 5.2 \cdot 10^{96} \text{ kg} \cdot \text{m}^{-3} . \tag{23}$$

Apart from numerical factors this density is the sum of zero-point energies of oscillation from the electromagnetic field. However, assuming that dark energy in the present-day Universe is contributed by the cosmological constant, we are led to the following value

$$\alpha_0(\text{observations}) \approx 10^{-124} , \tag{24}$$

by various observations. In the unit of mass density, it is

$$\frac{\alpha_0(\text{observations})}{l_p^2} = \frac{10^{-124}}{l_p^2} \rightarrow \rho_X = 10^{-124} \cdot \frac{l_p^2}{l_p^2} \cdot \rho_p = 10^{-124} \frac{c^5}{\hbar G^2} \approx 5.2 \cdot 10^{-28} \text{ kg} \cdot \text{m}^{-3} . \tag{25}$$

Thus we see that the theoretical result $\alpha_0 \approx 1$ is the order of 10^{124} of the observational value $\alpha_0 \approx 10^{-124}$. This is the famous cosmological constant problem. We note that the term n_1 has the dimension of L^{-2}. If the scale of extra space is in the order of Planck length, the n_1 term would contribute a huge but negative energy density, i.e.,

$$-\alpha_1 \frac{n_1}{2b^2} = -\frac{n_1}{2l_p^2} \rightarrow -\frac{n_1}{2} \cdot \frac{l_p^2}{l_p^2} \cdot \rho_p = -\frac{n_1}{2} \rho_p . \tag{26}$$

This tells us the huge vacuum energy can be interacted by the curvature of extra sphere. If we let

$$\frac{\alpha_0(\text{theory})}{l_p^2} - \frac{n_1}{2b^2} \rightarrow 10^{-124} \cdot \frac{1}{l_p^2} , \tag{27}$$

this means that we are required to adjust the counter term of $\frac{n_1}{2b^2}$ to at least one part in 10^{124}. This amounts to the accuracy that we require a weight difference of 10^{-114} kg between two mountains. Thus an extremely fine-tuning problem appears. Therefore, we demand

$$\frac{\alpha_0(\text{theory})}{l_p^2} - \frac{n_1}{2b^2} = 0 . \tag{28}$$

Now the huge positive cosmological constant is exactly neutralized by negative energy of curvature of extra space. We then obtain

$$b = \sqrt{\frac{n_1}{2\alpha_0(\text{theory})}} l_p \, . \tag{29}$$

This reveals that in order to counteract the huge vacuum energy by the curvature of extra space, the size of extra space could be several orders of Planck length. The scale of extra space b is proportional to $\sqrt{n_1}$. This means the radius b increases with the increasing of dimension n. If we assume the scale of extra space is in the order of from current accelerator experiments on Kaluza-Klein theories given by

$$b < 10^{-9} \, \text{m} \, , \tag{30}$$

we find

$$n < 10^{27} \, . \tag{31}$$

Namely, the dimension of extra space can not be arbitrarily large. In Table 1 we list some values of dimension and scale for extra space. The minimum of dimension and scale for extra space is two and one Planck length, respectively. String theory predicts the extra space is seven dimensional. If this is the case, the corresponding scale for extra space is in the order of five Planck length.

Table 1. The dimension and scale of extra space.

n	2	4	6	7	10^{10}	10^{20}	10^{27}
$b(l_p)$	1	2.4	3.9	4.6	10^{10}	10^{20}	10^{27}

Substituting the scale of extra space, Equation (29) into Equation (19), we obtain the density of dark energy

$$\rho_X = -\frac{1}{16\pi}\alpha_N\left[\left(\frac{2\alpha_0}{n_1}\right)^N \cdot \rho_p + \frac{6Nn_{2N-3}H^2}{n_{2N-1}} \cdot \left(\frac{2\alpha_0}{n_1}\right)^{N-1}\right] \, . \tag{32}$$

Since $\alpha_0(\text{theory}) \approx 1$, we conjecture the dimensionless constant α_N is also in the order of unit one, i.e., $\alpha_N \approx -1$ and we let

$$\frac{1}{16\pi}\left(\frac{2}{n_1}\right)^N = 10^{-124} \, . \tag{33}$$

As indicated earlier, the maximum of N is $\frac{n+4}{2}$ or $\frac{n+3}{2}$. Considering this situation, we have

$$\frac{1}{16\pi}\left[\frac{2}{n(n-1)}\right]^{\frac{n+4}{2}} = 10^{-124} \, , \quad or \quad \frac{1}{16\pi}\left[\frac{2}{n(n-1)}\right]^{\frac{n+3}{2}} = 10^{-124} \, . \tag{34}$$

Therefore, we conclude the number of extra spatial dimensions is

$$n \approx 69 \, , \tag{35}$$

in order that the density of dark energy is in the order

$$\rho_X \approx 10^{-124}\rho_p \, . \tag{36}$$

For convenience, the symbol $\alpha_0(\text{theory})$ would be be abbreviated to α_0 in the following. One can check that in the present-day Universe, the second term is very much smaller than the first one by making an examination on their ratio

$$\mathcal{R} \equiv \frac{\frac{6Nn_{2N-3}H_0^2}{n_{2N-1}} \cdot \left(\frac{2\alpha_0}{n_1}\right)^{N-1}}{\left(\frac{2\alpha_0}{n_1}\right)^N \cdot \rho_p} = \frac{8\pi Nn(n-1)}{\alpha_0(n-2N+1)(n-2N+2)} \cdot \frac{\rho_0}{\rho_p} , \tag{37}$$

where

$$\rho_0 = \frac{8\pi H_0^2}{3} , \tag{38}$$

is the present-day cosmic mass density. H_0 is the Hubble parameter today. Substituting

$$n = 69 , \quad N = 36 \text{ (or } 36.5) , \quad \alpha_0 = 1 , \quad \rho_0 = 10^{-28} \text{kg} \cdot \text{m}^{-3} , \tag{39}$$

into above equation, we obtain

$$\mathcal{R} = 10^{-119} \ll 1 . \tag{40}$$

Therefore, the second term can be safely neglected. Eventually, we obtain the expression for a small cosmological constant

$$\rho_X = -\frac{1}{16\pi}\alpha_N \left(\frac{2\alpha_0}{n_1}\right)^N \rho_p . \tag{41}$$

It is apparent α_n is required to be negative in order for the energy density to be positive. We expect α_N is the order of unit one. We note that, in order to get an observations allowed dark energy density, we may not select the maximum of N if the dimension of extra space is sufficiently large, for example, the pairs of $n = 10^7$ with $N = 9$, $n = 10^9$ with $N = 7$, $n = 10^{12}$ with $N = 5$, and so on.

3. Conclusions and Discussion

To sum up, we have proposed a mechanism that cancels out the huge cosmological constant and a small cosmological constant is produced. Our strategy is to bend the extra space into an n-dimensional sphere with a constant radius of order of Planck length. To achieve this, a huge positive energy density is needed. This is precisely the role of vacuum energy. To put it in another way, the n-dimensional sphere contains huge but negative energy density and it exactly neutralize the huge and positive vacuum energy. In the mean time an observational allowed small cosmological constant is produced. In this instance, a higher order LL term is needed. If the number of extra dimensions is sufficiently large, such as $n \approx 69$, the scale of extra space would be approximately $6l_p$. Up to this point our limit on the number of extra dimensions is based on the LL gravity. In contrast, Lin et al. [36], in the framework of Einstein gravity, provide another novel way to limit the number of extra dimensions in the braneworld model by using the observations of gravitational waves.

Why can the huge vacuum energy density be cancelled out by extra space? From the perspective of dimensional analysis, we see the vacuum energy density is proportional to l_p^{-2}. It is subtly that the extra sphere contains a minus energy tensity which is similarly proportional to b^{-2}. If the scale of extra space, b, is the order of Planck length, the two energy densities can be precisely offset. On the other hand, why can a small cosmological constant be produced by a large number of extra dimensions? Also from the perspective of dimensional analysis, we see the density of dark energy is proportional to $(\frac{l_p}{b})^{2N}$ or b^{-2N} from Equation (19). This is exactly the dimension of $N - $th order LL invariant. Since $b > l_p$, we see the larger magnitude the index N, the smaller magnitude the energy density. Only with a sufficiently large number of extra dimensions can we obtain a sufficiently large N. So a large number of extra dimensions are required in order to get a small cosmological constant.

As mentioned earlier, the scale factor b is generally the function of cosmic time t. Its asymptotic value after compactification is not arbitrary, but determined by the coupling constants of the theory. How to get a desirable asymptotic value for b? This is a fundamental problem. In this situation, we would have three Einstein equations but five unknown functions which include two scale factors a and b, the energy density ρ, two pressures p and q. Therefore, the other two equations of state for ρ, p and q are needed in order that the equations of motion are closed. In this sense, the stability and asymptotic value of b are determined by the two equations of state that were chosen. This was a practice sanctified by tradition. However, it is found that, without any simplifying assumptions on $b(t)$, the system of differential equations is extremely difficult to solve.

Therefore, the simplifying hypothesis which is frequently adopted in this type of problem, is to assume that the scale factor of internal space is a constant, namely, its asymptotic value. The assumption has not any problem in theory. How to obtain this value? This is a much deeper issue. Assuming b being a constant is the first step of research. As the second step, we could presume the behavior of b, for example, $b = b_0 \frac{\sinh t/t_0}{\cosh t/t_0}$ with the desirable asymptotic value b_0, instead of assuming one of the equation of state for ρ, p and q. Then the system of equations are also closed. This is another option which is different from the traditional one for pursuing a consistent asymptotic value.

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References

1. Kaluza, T. Zum Unitätsproblem der Physik. In *Sitzungsberichte der Koniglich Preußischen Akademie der Wissenschaften (Mathematical Physics)*; Akademie der Wissenschaften: Berlin, Germany, 1921; Volume 131, pp. 966–972.
2. Klein, O. Quantum theory and five-dimensional relativity theory. *Z. Phys.* **1926**, *37*, 895. [CrossRef]
3. Wesson, P.S. *Space-Time-Matter: Modern Kaluza-Klein Theory*; World Scientific: Singapore, 1998.
4. Witten, E. String Theory Dynamics in Various Dimensions. *Nucl. Phys. B* **1995**, *443*, 85. [CrossRef]
5. Vafa, C. Evidence for F-Theory. *Nucl. Phys. B* **1996**, *469*, 403–415. [CrossRef]
6. Green, M.B.; Schwarz, J.H.; Witten, E. *Superstring Theory*; Cambridge University Press: Cambridge, UK, 1987.
7. Lanczos, C. Electromagnetism as a natural feature of Riemann's geometry. *Z. Phys.* **1932**, *73*, 147. [CrossRef]
8. Lanczos, C. A Remarkable property of the Riemann-Christoffel tensor in four dimensions. *Ann. Math.* **1938**, *39*, 842. [CrossRef]
9. Lovelock, D. Divergence-free tensorial concomitants. *Aequationes Math.* **1970**, *4*, 127. [CrossRef]
10. Lovelock, D. The Einstein tensor and its generalizations. *J. Math. Phys.* **1971**, *12*, 498. [CrossRef]
11. Zumino, B. Gravity Theories in More Than Four-Dimensions. *Phys. Rep.* **1986**, *137*, 109. [CrossRef]
12. Zwiebach, B. Curvature Squared Terms and String Theories. *Phys. Lett. B* **1985**, *156*, 315. [CrossRef]
13. Buchbinder, I.L.; Karataeva, I.Y.; Lyakhovich, S.L. Multidimensional R**2 gravity: The Structure of constraints and canonical quantization. *Class. Quantum Grav.* **1991**, *8*, 1113. [CrossRef]
14. Padmanabhan, T.; Kothawala, D. Lanczos-Lovelock models of gravity. *Phys. Rep.* **2013**, *531*, 115. [CrossRef]
15. Charmousis, C. Higher order gravity theories and their black hole solutions. *Lect. Notes Phys.* **2009**, *769*, 299.
16. Myers, R.C.; Simon, J.Z. Black Hole Thermodynamics in Lovelock Gravity. *Phys. Rev. D* **1988**, *38*, 2434. [CrossRef] [PubMed]
17. Paranjape, A.; Sarkar, S.; Padmanabhan, T. Thermodynamic route to field equations in Lancos-Lovelock gravity. *Phys. Rev. D* **2006**, *74*, 104015. [CrossRef]
18. Deruelle, N.; Fari, L. ña-Busto, The Lovelock Gravitational Field Equations in Cosmology. *Phys. Rev. D* **1990**, *41*, 3696. [CrossRef] [PubMed]
19. Mena Marugan, G.A. Dynamically generated four-dimensional models in Lovelock cosmology. *Phys. Rev. D* **1992**, *46*, 4340. [CrossRef] [PubMed]
20. Kirnos, I.V.; Makarenko, A.N. Accelerating cosmologies in Lovelock gravity with dilaton. *Open Astron. J.* **2010**, *3*, 37. [CrossRef]

21. Paul, B.C.; Mukherjee, S. Higher-dimensional cosmology with Gauss-Bonnet terms and the cosmological-constant problem. *Phys. Rev. D* **1990**, *42*, 2595. [CrossRef]

22. Mena Marugan, G.A. Perturbative formalism of Lovelock gravity. *Phys. Rev. D* **1992**, *46*, 4320. [CrossRef] [PubMed]

23. Pavluchenko, S.A. General features of Bianchi-I cosmological models in Lovelock gravity. *Phys. Rev. D* **2009**, *80*, 107501. [CrossRef]

24. Müller-Hoissen, F. Spontaneous compactification with quadratic and cubic curvature terms. *Class. Quantum Grav.* **1986**, *3*, 665.

25. Wetterich, C. The cosmological constant and non-compact internal spaces in Kaluza-Klein theories. *Nucl. Phys. B* **1985**, *255*, 480. [CrossRef]

26. Rubakov, V.A.; Shaposhnikov, M.E. Extra spacetime dimensions: Towards a solution to the cosmological constant problem. *Phys. Lett. B* **1983**, *125*, 139. [CrossRef]

27. Canfora, F.; Giacomini, A.; Troncoso, R.; Willison, S. General relativity with small cosmological constant from spontaneous compactification of Lovelock theory in vacuum. *Phys. Rev. D* **2009**, *80*, 044029. [CrossRef]

28. Chirkov, D.; Toporensky, A. Stability analysis of compactification in 3-d order Lovelock gravity. *Gravit. Cosmol.* **2023**, *29*, 262–268. [CrossRef]

29. Chirkov, D.; Giacomini, A.; Toporensky, A. Dynamic compactification with stabilized extra dimensions in cubic Lovelock gravity. *Gen. Rel. Grav.* **2018**, *50*, 98. [CrossRef]

30. Pavluchenko, S.A. Stability analysis of exponential solutions in Lovelock cosmologies. *Phys. Rev. D* **2015**, *92*, 104017. [CrossRef]

31. Ivashchuk, V.D. On stability of exponential cosmological solutions with non-static volume factor in the Einstein–Gauss–Bonnet model. *Eur. Phys. J. C* **2016**, *76*, 431. [CrossRef]

32. Ivashchuk, V.D. On stable exponential solutions in the Einstein–Gauss–Bonnet cosmology with zero variation of G. *Gravit. Cosmol.* **2016**, *22*, 329. [CrossRef]

33. Ernazarov, K.K.; Ivashchuk, V.D. Stable exponential cosmological solutions with zero variation of G and three different Hubble-like parameters in the Einstein–Gauss–Bonnet model with a Λ-term. *Eur. Phys. J. C* **2017**, *77*, 402. [CrossRef]

34. Demaret, J.; Caprasse, H.; Moussiaux, A.; Tombal, P.; Papadopoulos, D. Ten-dimensional Lovelock type space-times. *Phys. Rev. D* **1990**, *41*, 1163. [CrossRef] [PubMed]

35. Arkani-Hamed, N.; Dimopoulos, S.; Dvali, G. The Hierarchy Problem and New Dimensions at a Millimeter. *Phys. Lett. B* **1998**, *429*, 263. [CrossRef]

36. Lin, Z.C.; Yu, H.; Gong, Y. Limiting the number of extra dimensions with shortcuts. *Phys. Rev. D* **2024**, *109*, 104015. [CrossRef]