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Elucidating the Dark Energy and Dark Matter Phenomena Within the Scale-Invariant Vacuum (SIV) Paradigm

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Abstract: The enigmatic phenomenon of dark energy (DE) is the elusive entity driving the accelerated expansion of our Universe. A plausible candidate for DE is the non-zero Einstein Cosmological Constant Λ_E manifested as a constant energy density of the vacuum, yet it seemingly defies gravitational effects. In this work, we interpret the non-zero Λ_E through the lens of scale-invariant cosmology. We revisit the conformal scale factor λ and its defining equations within the Scale-Invariant Vacuum (SIV) paradigm. Furthermore, we address the profound problem of the missing mass across galactic and extragalactic scales by deriving an MOND-like relation, $g \sim \sqrt{a_0 \, g_N}$, within the SIV context. Remarkably, the values obtained for Λ_E and the MOND fundamental acceleration, a_0, align with observed magnitudes, specifically, $a_0 \approx 10^{-10}\,\mathrm{m\,s^{-2}}$ and $\Lambda_E \approx 1.8 \times 10^{-52}\,\mathrm{m^{-2}}$. Moreover, we propose a novel early dark energy term, $\tilde{T}_{\mu\nu} \sim \kappa H$, within the SIV paradigm, which holds potential relevance for addressing the Hubble tension.

Keywords: cosmology; theory; dark energy; dark matter; MOND; Weyl integrable geometry

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1. Introduction

Modern physics faces a tantalizing situation wherein the two theories describing most phenomena, Quantum Field Theory (QFT) and General Relativity (GR), have been successfully tested to high precision on Earth and via solar system observations. However, the models of phenomena on galactic, inter-galactic, and cosmic scales have suggested that normal matter barely accounts for ≈5% of the energy content of the Universe. Meanwhile, ≈70% of the energy content is related to the expansion of the Universe due to dark energy (DE), and the other component is ≈25% due to dark matter (DM) [1].

Following the conventional wisdom, there is no shortage of proposals as to what DE and DM could be: either possible new fields, new particles, or modifications of the Einstein GR (EGR) [2–4]. While one is often very successful in continuing an ongoing trajectory, the usual explanation of DE and DM has faced a detection deficit for over 40 years while accumulating many tensions as a working paradigm [5].

Here, we would like to provide a possible solution to the DE and DM puzzles via a symmetry extension of the EGR. This extension was already proposed by Weyl in 1918 [6,7], but was rejected, for good reason, by Einstein [8]. The initial concern was resolved within the Weyl integrable geometry (WIG) reformulation [9,10]. The idea of scale invariance was then advocated by Dirac [10], and it was later the cornerstone of the scale-covariant cosmology of Canuto et al. [11]. The initial works [10,11] used the Large

Numbers Hypothesis of Dirac [12], which was faced with skepticism. However, in 2016, the idea of a Scale-Invariant Vacuum (SIV) was proposed in three papers posted to the arXiv preprint server in 2016 and published in 2017 by Maeder [13]. Since then, the idea has been stress tested on various phenomena. For a short, recent overview, see [14]. The foundations of the framework have been revisited [15,16], and the potential link between the SIV and DM and DE was stated in 2020 by Maeder and Gueorguiev [17]. Here, we provide our current understanding of the phenomena of dark energy via the Einstein Cosmological Constant and dark matter via the MOND-like acceleration relation within the SIV paradigm, along with the numerical values of the relevant expressions.

2. Framework for Scale-Invariant Cosmology

As was already pointed out, the Weyl geometry as the foundation of a scale-invariant cosmology was discussed by Dirac [10] and Canuto et al. [11], and possible astronomical applications have been considered by Bouvier and Maeder [18] and Maeder and Bouvier [19]. More modern but very abstract mathematical formulations have been reviewed [20]; however, nature-based considerations of phenomena with potential observational validations are rare [14,19,21], and even fewer have had a particle physics focus [22].

The scale-covariant cosmology equations were first introduced in 1977 by Canuto et al. [11] in the following form:

$$\frac{8\,\pi G\varrho}{3} = \frac{k}{a^2} + \frac{\dot{a}^2}{a^2} + 2\,\frac{\dot{\lambda}\,\dot{a}}{\lambda\,a} + \frac{\dot{\lambda}^2}{\lambda^2} - \frac{\Lambda_{\mathrm{E}}\lambda^2}{3}\,, \tag{1}$$

$$-8\,\pi G p = \frac{k}{a^2} + 2\frac{\ddot{a}}{a} + 2\frac{\ddot{\lambda}}{\lambda} + \frac{\dot{a}^2}{a^2} + 4\frac{\dot{a}\,\dot{\lambda}}{a\,\lambda} - \frac{\dot{\lambda}^2}{\lambda^2} - \Lambda_{\mathrm{E}}\,\lambda^2\,. \tag{2}$$

As a scale-invariant cosmology, any λ could be used. Thus, one has to make a "gauge" choice for λ to proceed with comparison to observations[1]. Some possible choices have already been discussed by Canuto et al. [11] based on the Large Numbers Hypothesis by Dirac [12]. Subsequent studies, e.g., [19], noticed that, for the vacuum solutions of the GR equations, the conformal equivalence of de Sitter space to Minkowski space can be achieved explicitly by fixing λ to satisfy

$$3\lambda^{-2}/(c^2t^2\Lambda_E) = 1. \tag{3}$$

This property of the vacuum, that *the empty space is scale invariant*, was further formalized in [13], and, in subsequent works, was emphasized as the Scale-Invariant Vacuum hypothesis for imposing the SIV gauging condition that fixes λ. Therefore, if one is to choose an λ that does not depend on the matter behavior explicitly, and by setting $\Lambda = \Lambda_E\lambda^2$, then one can obtain the following set of relationships given first in [13] and further re-derived from an action in [16], while the observational consequences have been summarized in [14]:

$$3\,\frac{\dot{\lambda}^2}{\lambda^2} = \Lambda\,, \quad \text{and} \quad 2\frac{\ddot{\lambda}}{\lambda} - \frac{\dot{\lambda}^2}{\lambda^2} = \Lambda\,, \tag{4}$$

$$\text{or} \quad \frac{\ddot{\lambda}}{\lambda} = 2\,\frac{\dot{\lambda}^2}{\lambda^2}\,, \quad \text{and} \quad \frac{\ddot{\lambda}}{\lambda} - \frac{\dot{\lambda}^2}{\lambda^2} = \frac{\Lambda}{3}\,. \tag{5}$$

Upon the use of the SIV choice (4) first introduced in 2017 by Maeder [13] or its equivalent form (5), one observes that *the cosmological constant disappears* from (1) and (2). In doing so, one recovers the scale invariance of the vacuum for flat cosmology ($k = 0$), which is broken only by the presence of source fields characterized by the energy density of matter and its pressure:

$$\frac{8\pi G\varrho}{3} = \frac{k}{a^2} + \frac{\dot{a}^2}{a^2} + 2\,\frac{\dot{a}\dot{\lambda}}{a\lambda}\,, \tag{6}$$

$$-8\pi G p = \frac{k}{a^2} + 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + 4\frac{\dot{a}\dot{\lambda}}{a\lambda}\,. \tag{7}$$

Making sense of this choice of λ, called the Scale-Invariant Vacuum (SIV) choice, and linking it to the observed phenomena in nature are the purposes of this paper. For this purpose, we go back to the origin of the equations and highlight the key properties of the Ricci tensor and scalar related to understanding our viewpoint. After a Weyl transformation [18,19], one has the following Ricci tensor and Ricci scalar expressions:

$$g_{\mu\nu} \to \lambda^2 g_{\mu\nu} \Rightarrow R_{\mu\nu} \to R_{\mu\nu} + K_{\mu\nu}, \tag{8}$$

where $K_{\mu\nu}$ is given by

$$K_{\mu\nu} = g_{\mu\nu}\kappa^{\rho}\kappa_{\rho} + 2\kappa_{\mu}\kappa_{\nu} + \kappa_{\mu;\nu} + \kappa_{\nu;\mu} - 2g_{\mu\nu}\kappa^{\rho}_{;\rho}. \tag{9}$$

Such expressions were first discussed in Equation (89.2) by Eddington [9] and later by Dirac [10] and Parker [22] as well. Upon contracting (8), one sees that the Ricci scalar becomes $R \to (R+K)/\lambda^2$, where

$$K = 6\kappa^{\rho}\kappa_{\rho} - 6\kappa^{\rho}_{;\rho}. \tag{10}$$

Here, $\kappa_{\mu} = -\partial_{\mu}\ln\lambda$ is the WIG connexion vector.

3. The Various Faces of the Cosmological Term Λ

3.1. The Einstein Cosmological Constant Λ_E

The conventional Einstein equation with cosmological constant Λ_E within general relativity is as follows:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda_E g_{\mu\nu} = \varkappa T_{\mu\nu}. \tag{11}$$

This formulation utilizes a metric-compatible connection where the first two terms form the Einstein tensor with zero-divergence while on the right-hand side. When on the left-hand side, one has the stress–energy tensor satisfying similar zero divergence, leading to the relevant covariant conservation laws. All of this is consistent as long as $\varkappa = 8\pi G/c^4$ and the cosmological constant Λ_E are constants; otherwise, one would have $\Lambda_{E,\mu} = \varkappa_{,\nu} T^{\nu}_{\mu}$. Thus, the constancy of G and c imply the constancy of the Einstein Cosmological Constant Λ_E.

3.2. Cosmological Constant or Dark Energy

One usually expects that an appropriate averaging over the matter distribution would result in a stress–energy tensor $T_{\mu\nu} = \rho g_{\mu\nu} + O(\delta T)$ where the energy density ρ will be related to the zero-point/vacuum energy of the matter fields. Since, in a co-moving frame, for an observer at infinity, the metric is expected to be Minkowski-like, one has $T_{00} \propto \rho$ and $T_{ii} \propto p$; therefore, one naturally considers ρ as a dark energy contribution to the stress–energy tensor with $p = -\rho$ due to $\eta_{00} = -\eta_{ii}$. Thus, we can move the term $\Lambda_E\, g_{\mu\nu}$ from the LHS to the RHS of (11) and consider Λ_E to be related to the zero-point/vacuum energy of the matter fields. Unfortunately, this leads to the cosmological constant problem manifested in the enormous discrepancy between the estimated value, based on QFT arguments, and the observed/measured actual value [1,23–25]. Another issue comes from the parallels between Newtonian Gravity, where a homogeneous and isotropic mass distribution has no gravitational effect, and GR, where energy is on the RHS of (11) and therefore has an influence on the metric. Thus, the question of "Should or shouldn't the vacuum gravitate if it has a non-zero energy density?" comes to mind.

3.3. Connecting the Dots Within the SIV Paradigm

The observed value for Λ_E is well within the order of magnitude estimate based on the relevant parameters for such a system [26], that is, using the values of c, G, R_H where the Hubble radius is $R_H = c/H_0$ with H_0 being the Hubble constant. However, if one considers a non-zero positive constant energy density for a homogeneous and isotropic universe, then one concludes that, at a sufficiently large distance R_S, such a universe should possess a black hole event horizon. For example, if the constant energy density is due to the zero-point energy of the familiar matter and radiation fields, then one expects $\rho_0 = const \gtrsim 0$. Now, consider a ball of radius r. Such a constant value results in an effective mass $M = \frac{4\pi}{3} r^3 \rho_0$, in geometric units ($c = 1, G = 1$), and its Schwarzschild radius will be $R_S = 2M$. If the mass distribution is within a radius $r < R_S$, then one has a black hole. No matter how small the positive constant energy density $\rho_0 \gtrsim 0$ is, there is always a sufficiently big ball of radius r_b, so that beyond $r > r_b$, one has a black hole with an event horizon at $r_b = \frac{8\pi}{3} r_b^3 \rho_0 \Rightarrow r_b = \sqrt{\frac{3}{8\pi\rho_0}}$. Of course, this situation does not apply if the constant energy density is negative. Therefore, one can inevitably conclude that we are inside a black hole, which may be the case as argued in [27,28]. However, if it is so, then one would expect a contracting flow of matter towards the center of such a black hole, located somewhere in the past, where there should be the biggest concentration of matter; however, we observe an expanding Hubble flow towards the future event horizon, which would indicate that the constant energy density should be negative. Such a negative energy density will result in negative effective mass, which is at odds with positive probabilities in quantum mechanics [29].

An alternative viewpoint is to consider the observed age of the Universe $\tau_0 \approx 1/H_0$ instead. Then again, c, G, τ_0 will give us the ballpark estimate for Λ_E but with a different understanding when viewed within the SIV paradigm. For this purpose, we shall consider the Weyl transformation (8). That is, the metric in the Einstein GR (EGR) frame $g'_{\mu\nu}$ will be related to a metric $g_{\mu\nu}$ within a Weyl integrable geometry (WIG) via the factor λ, that is, $g'_{\mu\nu} = \lambda^2 g_{\mu\nu}$. From now on, we will denote the EGR frame quantities with primes and no primes for the more general WIG quantities. Upon utilizing (8), (9), and (10) within (11), such a Weyl transformation expresses (11) into the more general WIG framework and the Einstein equation becomes

$$R_{\mu\nu} + K_{\mu\nu} - \frac{1}{2}\lambda^{-2}(R + K)\lambda^2 g_{\mu\nu} + \Lambda_E \lambda^2 g_{\mu\nu} = \varkappa T_{\mu\nu}. \quad (12)$$

The above Equation (12) should be viewed as a rewriting of the Einstein GR equations with a cosmological constant (11) for the metric $g'_{\mu\nu}$ into extended equations within the WIG framework where now the objects on the LHS and RHS of (12) depend on λ and $g_{\mu\nu}$. If λ is chosen to be a constant, then (12) reverts to (11) because the connexion vector κ_μ becomes zero and so is $K_{\mu\nu}$, as seen from (10), but for an appropriate new choice of units that is reflected in the appropriate rescaling of Λ_E, one will have the extended Equation (12). In what follows, we are interested in a non-trivial choice for λ that will depend on time only. To specify the functional form of this non-trivial λ, one can make the following considerations: To understand the forthcoming expressions for λ, the new Equation (12) can now be split into two equations, one containing $\Lambda = \lambda^2 \Lambda_E$ along with λ and its derivatives (via k_μ), and another equation that does not have the Λ term:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \varkappa T_{\mu\nu} - \tilde{T}_{\mu\nu}, \tag{13}$$

$$K_{\mu\nu} - \frac{1}{2} K g_{\mu\nu} + \Lambda g_{\mu\nu} = \tilde{T}_{\mu\nu}. \tag{14}$$

To evaluate $\tilde{T}_{\mu\nu}$, we look at the LHS of (14) and use (9) and (10): $\tilde{T}_{\mu\nu} = 2\kappa_\mu \kappa_\nu + \kappa_{\mu;\nu} + \kappa_{\nu;\mu} + (\Lambda - 2\kappa^\rho \kappa_\rho + \kappa^\rho_{;\rho}) g_{\mu\nu}$. Therefore, imposing $\kappa_{\mu;\nu} = \kappa_{\nu;\mu} = -\kappa_\mu \kappa_\nu$ along with $\Lambda = 3\kappa_\mu \kappa^\mu$ will guarantee $\tilde{T}_{\mu\nu} = 0$. By looking at $\kappa_{\mu;\nu} = \kappa_{\mu,\nu} + \Gamma^\rho_{\mu\nu} \kappa_\rho$, the first condition is readily satisfied within the WIG ($\kappa_{\mu;\nu} = \kappa_{\nu;\mu}$) of Canuto et al. [11], while the second condition implies the relationship $\kappa_{\mu,\nu} + \Gamma^\rho_{\mu\nu} \kappa_\rho + \kappa_\mu \kappa_\nu = 0$. For the SIV "gauge" choice $\lambda \propto 1/t$, the only non-zero component is $\kappa_0 = 1/t$, which will require $\Gamma^0_{\mu\nu} = 0$. In particular, $\Gamma^0_{00} = 0$ implies a time-independent g_{00} for constructing the metric-compatible covariant derivative. Thus, for a general metric, the LHS of (14) may result in non-zero $\tilde{T}_{\mu\nu} = \Gamma^0_{\mu\nu} \kappa_0$ within the SIV paradigm. Therefore, (14) defines $\tilde{T}_{\mu\nu}$ once the choice of λ is made. In this respect, $\tilde{T}_{\mu\nu}$ can bring early dark energy effects into (13) due to the time dependence of κ_0. The contemporary view on the resolution of the Hubble tension is the possibility for early dark energy [30], which could be supplied by $\tilde{T}_{\mu\nu}$. For the case of SIV theory, the appearance of a non-zero $\tilde{T}_{\mu\nu}$ and its dependence on $\Gamma^0_{\mu\nu}$ and κ result in the coupling of the Hubble parameter $H = \dot{a}/a$ to $\kappa = -\dot{\lambda}/\lambda$, as seen by the last terms in (6) and (7). This gives an explicit new model for early dark energy based on $\tilde{T}_{\mu\nu} \sim \kappa H$. From what follows, this split of (12) is viewed as a foresight[2]. When investigating how to choose the "gauge" factor λ, the second Equation (14) defines $\tilde{T}_{\mu\nu}$ once the choice of λ is made, and one is left only with the first Equation (13).

Now, consider $\kappa^\rho_{;\rho} = -\kappa^\rho \kappa_\rho$ along with $\Lambda = 2\kappa^\rho \kappa_\rho + \kappa^\rho_{;\rho} = 3\kappa^\rho \kappa_\rho$, which implies $\Lambda = K/4$ and $\tilde{T} = 0$. In the special co-moving frame where the time-covariant derivative is given by the partial derivative, with λ dependent only on time and $\kappa = \kappa_0 = -\dot{\lambda}/\lambda$, one obtains a key SIV equation:

$$\Lambda = \frac{3}{2}\left(\kappa^2 - \dot{\kappa}\right) \Leftrightarrow \Lambda = 3\kappa^2 = 3\left(\frac{\dot{\lambda}}{\lambda}\right)^2 : \text{iff } \dot{\kappa} = -\kappa^2 \tag{15}$$

The above equations are those given by the first expressions in (4) and (5). By taking the time derivative of κ/λ, we see that it will be equal to $(\dot{\kappa} + \kappa^2)/\lambda$, which will vanish if $\dot{\kappa} = -\kappa^2$; thus, Λ/λ^2 will be a constant[3]. Therefore, we will denote this constant also by Λ_E since these two constants will coincide eventually. Thus, for the case $\dot{\kappa} = -\kappa^2$, the solution for $\lambda(t)$ is very simple and one can use the constant $\Lambda_E = \Lambda/\lambda^2 = 3\dot{\lambda}^2/\lambda^4$ to characterize it:

$$\varepsilon(t_0 - t)\sqrt{\Lambda_E/3} = 1/\lambda - 1/\lambda_0. \tag{16}$$

That is, $\kappa^2 = \dot{\lambda}^2/\lambda^2 = \Lambda/3 = (\Lambda_E/3)\lambda^2$, along with

$$\lambda = \lambda_0 / \left(1 + \lambda_0 \varepsilon (t_0 - t)\sqrt{\Lambda_E/3}\right), \tag{17}$$

and $\kappa = -\dot{\lambda}/\lambda = -\varepsilon\lambda\sqrt{\Lambda_E/3}$ are key SIV expressions; therefore, $\dot{\kappa} \sim \dot{\lambda} = \varepsilon\lambda^2\sqrt{\Lambda_E/3}$ implies $\dot{\kappa} = -\varepsilon^2(\Lambda_E/3)\lambda^2 = -\kappa^2$ as required. Here, $\varepsilon = \pm 1$, and therefore $\varepsilon^2 = 1$. By setting $\lambda_0\sqrt{\Lambda_E/3} = 1/t_0$, we have $\lambda = \lambda_0 t_0/t$ for $\varepsilon = -1$; thus, we have recovered (3), and therefore $\kappa = -\dot{\lambda}/\lambda = 1/t$ with $t \in [t_{\text{in}}, t_0]$, where t_{in} is the moment of the Big Bang when $a(t_{\text{in}}) = 0$.

It is often convenient to choose $\lambda_0 = 1$ along with SIV time units such that $t_0 = 1$. Thus, one is led to the same expression of the scale factor λ as obtained by the fundamental

SIV hypothesis, according to which the macroscopic empty space is scale invariant, homogeneous, and isotropic [13,16].

3.4. Interpretation of the Cosmological Constant Within the SIV Framework

The presence of a non-zero cosmological term Λ_E leads to some severe problems: (1) a mismatch of the observed value with the zero-point energy estimates based on QFT, and (2) the puzzling conclusion that we may be inside a black hole. The Quantum Field Theory (QFT) predicts an enormous value of vacuum energy when viewed as the zero-point energy of the matter fields (i.e., $c^7/\hbar/G^2 \sim 10^{114}$ erg/cm^2), while anthropic considerations à la Weinberg and even a simple dimensional estimate using the relevant physical constants (i.e., $H_0^2 c^2/G \sim 10^{-8}$ erg/cm^2) seem to arrive at the correct order of magnitude for the vacuum energy related to the cosmological constant. Thus, one can conclude that quantum effects involving Planck's constant $\hbar$ have nothing to do with the observed Λ_E. Therefore, quantum vacuum fluctuations are just that: fluctuations whose mean value is zero at large cosmic scales. Within the SIV, this is reflected in removing the Λ_E from the Freedman equations, as seen in (13), in favor of an early dark energy term defined by (14) that involves the conformal factor λ. It can be interpreted as a choice of parameterization that brings the GR equations into the true co-moving frame with no cosmological constant Λ_E as extra energy density. It is similar to what happens when identifying the co-moving frame such that the kinetic energy of a system is zero and therefore there is no relative special motion. However, we do leave in 4D spacetime, which brings up the question of relative time parameterizations; that is, what if the coordinate time of the observer is different from the proper time of the system under study? It seems the relative time parametrization controlled by λ also controls the amount of extra energy that there could be.

Another way to understand the situation is to recognize that the positive cosmological constant Λ_E on the LHS of (11) indicates extra energy density as part of the RHS (11). The presence of Λ_E explicitly breaks the global rescaling symmetry along with the ρ, p, and k/a^2 terms in the Freedman equations. The breaking is still there even for the macroscopic vacuum, characterized by $\rho = p = k = 0$, if Λ_E is non-zero. This can be viewed as a manifestation of *unproper* time parametrization, since, for proper time parametrization, one expects zero energy density instead. To correct the time parametrization, one can apply global conformal transformation $\lambda(t)$ instead of the commonly discussed local conformal gauge $\lambda(x)$. The use of $\lambda(x)$ would imply the presence of a physical field whose excitations should manifest as particles, which is not permissible [22]. Thus, the idea of using $\lambda(t)$ is well justified in order to preserve isotropy and homogeneity of space. It is aligned with the idea about the role of time parametrization. Therefore, the existence of $\lambda(t)$ as defined by (17) removes Λ_E from the Freedman equations and results in (6) and (7), which are clearly scale invariant when $\rho = p = k = 0$. This demonstrates the relationship between the scale-breaking term Λ_E and its relation to the symmetry-restoring WIG frame defined by $\lambda(t)$ given by (17).

Therefore, the "gauge" symmetry of the SIV theory is not like the usual local gauge symmetry, which we are familiar with from particle physics. As such, one can circumvent the earlier mentioned problems by showing that Λ_E is an actual constant within the SIV. As mentioned earlier, the Einstein Cosmological Constant Λ_E must be a constant if one views the Newton Gravitational Constant G as a true constant (see Section 3.1). This can be used to construct a Weyl transformation that removes the cosmological term Λ. It implies that the extra energy density due to the cosmological constant can be viewed as an observer effect, just as the kinetic energy of a system depends on the relative motion of the two systems; however, in the case of the cosmological constant, this seems to be about the difference in time parameterizations. That is, the metric $g_{\mu\nu}$ provides Λ-free EGR equations, as seen in (13), while the presence of a non-zero constant Λ_E term is due to the choice of the EGR metric tensor $g'_{\mu\nu} = \lambda^2 g_{\mu\nu}$, where the factor $\lambda = t_0/t$ is defined via $\sqrt{\Lambda_E/3} = 1/t_0$ (17), in agreement with the early observation by [13,19] that one can obtain the Minkowski line element based on (3). Thus, this relates the value of the cosmological constant Λ_E to the age of the Universe t_0, which is consistent with the simple dimensional estimates [26]. In the usual SI units where the age of the Universe is $\tau_0 = 13.8$ billion years and the speed of light is $c = 3 \times 10^8$ m/s, one obtains

$$\Lambda_E = 3/(c\tau_0)^2 \approx 1.8 \times 10^{-52}\ \text{m}^{-2}, \tag{18}$$

which is reasonably close to the measured value [1,26].

In the considerations above, we have settled that the time dependence of λ only is justifiable based on assumptions of homogeneous and isotropic space at cosmological scales. This is a cornerstone of the SIV paradigm [13,16]. An alternative justification of a time-dependent-only λ is based on re-parametrization invariance, which has been fruitful in justifying the known classical long-range forces and some of the key properties of physical systems [29]. Either way, one arrives at the above arguments and concludes that the Einstein Cosmological Constant Λ_E is a manifestation of the choice of parametrization where the age of the Universe is a natural parameter that measures how much stuff has become causally connected within the observed Universe. Thus, upon a proper choice of λ and metric $g_{\mu\nu}$, the Λ_E term disappears, and one has only (13) without the observer-related cosmological constant Λ_E given by (18).

4. The Missing Mass Problem

At astronomical and cosmological scales, it has been observed that motions in galaxies and their clusters exhibit behavior that cannot be explained by the observed known matter and its laws of motion. The idea of an invisible dark matter has been proposed to explain observations (see the next section for more details). However, its absence from laboratory tests for the past 40 years has become a puzzle. That is, the absence of any laboratory detection of new dark matter particles raises questions as to the validity of the idea of the existence of dark matter. On the other hand, these observations could be addressed by an alternative to the dark matter hypothesis known as Modified Newtonian Dynamics (MOND) [31]. It has been gaining supporters due to its successes in fitting observational data about galaxies [32]. Furthermore, it has been shown that MOND could be viewed as a particular manifestation of the SIV paradigm [33]. Thus, the presence of the scale factor λ within the SIV paradigm can be utilized to address this missing matter observational puzzle by its connection to MOND.

4.1. The Dark Matter Option

There is unambiguous observational evidence for deviation away from the Keplerian fall-off $v^2 \sim M/r$ when observing the motion of stars in the outer layers of galaxies. The observations point to flat rotational curves $v \sim const$ as one moves further away from the central bulge of the visible part of a galaxy. This observation has been addressed as a continuation of the matter paradigm by assuming the presence of extra non-luminous matter named dark matter, which forms halos around galaxies and extends far beyond the luminous part of the host galaxy [34]. This is a natural initial guess as to what may be causing the deviation from the Keplerian fall-off and into the presence of flat rotational curves. There is a large variety of proposed dark matter candidates that are yet to be observed if the idea is correct. In addition to the fact that such dark matter is a no-show in labs, there are many additional issues with the various proposed dark matter solutions.

4.2. The MOND Option

The Modified Newtonian Dynamics (MOND) resolution to the observed flat rotational curves does not consider a new dark matter component(s) but assumes that the dynamics are changed once the Newtonian acceleration falls below a certain cut-off value a_0. Since the FLRW scale factor at the current epoch is often chosen to be 1, this ambiguity in notation with the MOND acceleration a_0 should presumably be absent.

For the very low acceleration $g \ll a_0$, the initial MOND suggest that $g \sim \sqrt{g_N a_0}$, where $g = v^2/r$ is the observed acceleration, while $g_N = GM/r^2$ is the Newtonian gravitational acceleration [31]. For systems with acceleration $g \gg a_0$, the dynamics are reduced to the standard dynamics $g = g_N$ (limit of $a_0 \to 0$), while, for accelerations $g \ll a_0$, the system is in the Deep-MOND (DMOND) regime, where it also should exhibit scale-invariant space–time dynamics $(t,r) \to \lambda \times (t,r)$ [32,35]. Furthermore, in the scale-invariant DMOND regime, $g \sim \sqrt{g_N a_0}$, where the limit could be viewed as $a_0 \to \infty$ along with $G \to 0$ but the product $Ga_0 \to \mathcal{A}_0$ stays constant. In this DMOND scale-invariant regime, λ is an overall constant scale factor, where $(t,r) \to \lambda \times (t,r)$ could be viewed as rescaling the coordinate functions but keeping the units the same; thus, G and a_0 are constants that are invariant upon this rescaling within the DMOND regime. An alternative viewpoint is to consider a change of units that removes the λ scaling of the coordinates while inducing change in the values of the dimension-full constants q, say with units $[q] = [l]^a[t]^b[m]^c$, then $q \to \lambda^{-(a+b)}q$ [32]. From this viewpoint, $a_0 \to \lambda a_0$ and $G \to G/\lambda$ and the limits are obtained as $\lambda \to 0$ or ∞. Notice that mass-related quantities are not affected within MOND. The value of the MOND acceleration a_0 is expected to satisfy $a_0 \approx cH_0/2\pi$ [32].

4.3. Deriving MOND-like Acceleration Within SIV

In the SIV paradigm, the scale invariance is the primary idea, and the form of the scale factor λ can be deduced, as discussed earlier. Furthermore, the equations of motion given by the equations of the geodesics within GR are generalized via Dirac co-calculus to include an extra velocity-dependent term as part of the scale-covariant Newtonian equation of motion [16,18,19,36,37]:

$$\frac{d^2\overrightarrow{r}}{dt^2} = -\frac{G_t M(t)}{r^2}\frac{\overrightarrow{r}}{r} + \kappa(t)\frac{d\overrightarrow{r}}{dt}, \tag{19}$$

Here, $\kappa(t) = -\dot{\lambda}/\lambda$ and G_t is Newton's Gravitational Constant in the system of units related to the choice of time parametrization t within the SIV, where G_t is viewed as a true constant but the mass is expected to have relevant time dependence. For the above equation to exhibit scale invariance, one has to assume that $G_t M(t) \propto \lambda$, and if G_t is kept constant, then $M(t) = M_0\lambda(t)$. The variation of mass is demanded by the conservation law associated with the scale-invariant equation [13]. In the subsequent considerations, we will drop the time label for G_t and $M(t)$ to simplify the notation.

To arrive at an expression for an MOND-like acceleration a_0 within the SIV, one considers the ratio of the Newtonian acceleration $g_N = GM/r^2$ to the additional dynamic acceleration $\kappa(t)v$ (magnitudes):

$$x = \frac{\kappa v r^2}{GM}. \tag{20}$$

Next, we will use the relation given by the instantaneous radial acceleration $v^2/r = GM/r^2$ to eliminate the speed v. Then, by using $g_N = GM/r^2$, we arrive at

$$x = \frac{\kappa v r^2}{GM} = \kappa\sqrt{\frac{r^3}{GM}} = \kappa\sqrt{\frac{r}{g_N}}.$$

The quantity x has been discussed previously [15,37], and is finally utilized in connecting the SIV to MOND in [33]. Here, we are re-deriving the relevant expressions with a focus on keeping the time component $\kappa = \kappa_0 = -\dot{\lambda}/\lambda$ of the connexion vector within the WIG explicit. Thus, when the dynamic acceleration dominates over the Newtonian ($x \gg 1$), one has

$$g = g_N + x g_N \approx x g_N = \kappa \sqrt{r g_N}.$$

Therefore, we have arrived at the DMOND-type relation $g \sim \sqrt{a_0 g_N}$, from which we can deduce an expression for a_0 within the SIV:

$$a_0 \approx \kappa^2 r.$$

The upper bound on a_0 corresponds to utilizing the Hubble horizon $r \to r_H = c/H_0$. Therefore,

$$a_0 \approx \kappa^2 r_H = \kappa^2 c/H_0. \tag{21}$$

In SIV units ($1 = \lambda_0 = t_0 = c$), one has $\kappa = 1/t$, and, for the matter-dominated epoch, using the scale factor $a(t) = \left((t^3 - \Omega_\mathrm{m})/(1 - \Omega_\mathrm{m})\right)^{2/3}$ [38], one obtains $H = 2t^2/(t^3 - \Omega_\mathrm{m})$; therefore, $a_0 \approx \kappa^2 c/H_0 = (1 - \Omega_\mathrm{m})/2 = \Omega_\lambda/2$, where, within the SIV, $\Omega_\lambda = 2/(H\,t)$—assuming a flat Universe ($\Omega_\mathrm{k} = 0$). Therefore, $\Omega_\lambda + \Omega_\mathrm{m} = 1$ is trivially true within the SIV. SIV models with non-zero curvature k are also possible [13]. Thus, a non-zero κ term in (19) gives rise to non-zero DMOND-like acceleration within the SIV:

$$a_0 \approx \Omega_\lambda/2. \tag{22}$$

Next, we look at the DMOND-like acceleration a_0, as expressed in the usual SI units where the age of the Universe is $\tau_0 = 13.8$ billion years, the Hubble constant is $H_0 = 68$ km/s/Mp, and the speed of light is $c = 3 \times 10^8$ m/s. Using $\kappa \approx 1/\tau_0$, one can immediately estimate the value to be $a_0 \approx c/\tau_0$ since $H_0\tau_0 \approx 1$. However, to be more precise, one has to take into account that $\kappa = -\dot{\lambda}/\lambda$ and therefore $\kappa(\tau) = (dt/d\tau)\kappa(t)$. To find $dt/d\tau$, we consider

$$\frac{t - t_\mathrm{in}}{t_0 - t_\mathrm{in}} = \frac{\tau - \tau_\mathrm{in}}{\tau_0 - \tau_\mathrm{in}}$$

for $\tau_\mathrm{in} = 0$ and $t_\mathrm{in} = \Omega_\mathrm{m}^{1/3} \Rightarrow t = \Omega_\mathrm{m}^{1/3} + \frac{\tau}{\tau_0}(1 - \Omega_\mathrm{m}^{1/3})$,

$$\frac{dt}{d\tau} = \frac{t_0 - t_\mathrm{in}}{\tau_0} = \frac{(1 - \Omega_\mathrm{m}^{1/3})}{\tau_0}.$$

Thus, there is a correction factor to our use of $\kappa \approx 1/\tau_0$ above; that is, $\kappa(\tau_0) = (1 - \Omega_\mathrm{m}^{1/3})/\tau_0$ results in

$$a_0 \approx (1 - \Omega_\mathrm{m}^{1/3})^2 c/\tau_0 = (1 - \Omega_\mathrm{m}^{1/3})^2 c H_0/\xi,$$

where $H_0\tau_0 = \xi \approx 1$. For $\Omega_\mathrm{m} = 5\%$, this gives $a_0 \approx 2.75 \times 10^{-10}$ m/s^2. Here, the estimate is based on the ΛCDM model; its fit to observational data results in $\Omega_\mathrm{m} = 5\%$ for baryonic matter. Within the SIV, we do not have such parameter determination yet; however, a value of Ω_m could be estimated based on the self-consistency requirement about the age of the Universe and the value of the Hubble constant. That is, assuming $H_0\tau_0 = \xi \approx 1$, one obtains $2(1 - \Omega_\mathrm{m}^{1/3})/(1 - \Omega_\mathrm{m}) \approx 1$, which gives $\Omega_\mathrm{m} \approx 23.6\%$ within the SIV. Thus, $a_0 \approx 10^{-10}$ m/s^2. Therefore, the two estimates above for a_0 result in about the same order of magnitude values $a_0 \approx 10^{-10}$ m/s^2. Notice that this value for $\Omega_\mathrm{m} = 23.6\%$ is closer to the total matter content within the ΛCDM model.

The SIV paradigm suggests that the MOND-like acceleration a_0 may be epoch dependent and could have different values depending on the redshift of the system under observation. This could be used to differentiate and test the SIV and MOND paradigms [39].

5. Conclusion

To summarize, we have shown that the Einstein Cosmological Constant Λ_E is a true constant within the framework of scale-invariant cosmology. In particular, within the SIV paradigm, one can derive an expression (17) for the scale factor $\lambda(t)$ based on (14), which results in the condition (15), while the metric tensor satisfies the usual Einstein equation without cosmological constant (13). That is, without a cosmological constant term of the form $\Lambda_E g_{\mu\nu}$, but with the potential introduction of time-dependent dark energy term $\tilde{T}_{\mu\nu}$, that could be significant in the early universe and then diminish later. The contemporary view on the resolution of the Hubble tension is the possibility for early dark energy [30], which could be supplied by $\tilde{T}_{\mu\nu}$. This explicit new model for early dark energy has the form $\tilde{T}_{\mu\nu} \sim \kappa H$; thus, it could be used to test the SIV theory and its impact on the Hubble tension. However, one first will have to determine the model parameter Ω_m and the validity of the SIV paradigm. This could be performed by determination of the relevant SIV Ω_m from the cosmological parameters such as deceleration, jerk, and snap, which were recently constrained using a model-independent kinematic cosmographic study utilizing three different data sets and their combinations [40,41]. Our approach to Λ_E avoids the puzzling conclusion that we may be inside of a black hole if Λ_E is associated with the zero-point energy of the vacuum—labeled as dark energy that does not gravitate. Furthermore, we avoid the puzzling observation that there is a disagreement of 123 orders of magnitude with the QFT estimates of the zero-point vacuum energy since it does not contribute to $T_{\mu\nu}$. In our approach, the presence of a non-zero Λ_E within ΛCDM is due to the choice of time parameterization. That is, as there is extra kinetic energy in the case of spatial motion, there is also extra energy due to differences in the time parametrization since this is a relative temporal motion.

Next, we have shown how to derive an expression for the MOND-like acceleration a_0 (21) that controls the transition to DMOND where scale invariance is expected. In this respect, we may have explained the dark matter problem via an MOND-like paradigm. In our case, however, a_0 is a consequence of the scale-invariant equations of motion (19), while the usual MOND limits are controlled by the parameter x (20). Thus, there is a small tangency of the SIV and MOND over a limited interval of low gravities and timescales!

Our estimates of the fundamental MOND acceleration $a_0 \approx 10^{-10}$ m/s^2 and the Einstein Cosmological Constant $\Lambda_E \approx 10^{-52}$ m^{-2} show the correct order of magnitude for these two important constants. Both values are controlled by the age of the Universe, while a_0 is also related to the matter content of the Universe.

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Notes

1 As it stands, one can fit observations and deduce the model parameters, but the choice of λ has to make sense from a physics viewpoint.

2 The need to fix λ has been anticipated by Dirac [10], Canuto et al. [11], but they used the Large Numbers Hypothesis Dirac [12]. Here, we present the SIV approach, which seems to be relevant for understanding the cosmological constant and the dark matter phenomena. Another "gauge" fixing is the λ constant that is the EGR frame. There could be other "gauge" choices within the WIG that will correspond to specific WIG frameworks. The significance of these frameworks is something to be understood in the future. In particular, the more correct expression for Λ in (15) contains a linear term in κ, of the form $\Gamma^{\rho}_{\rho 0}\kappa_0$, that comes from $\kappa^{\rho}_{;\rho}$. This term, along with other terms that result in an overall non-zero value for the LHS of (14), can be part of the stress–energy tensor $\tilde{T}_{\mu\nu}$ determining $g_{\mu\nu}$ via (13). These extra terms to $T_{\mu\nu}$ could be viewed as dark energy that are not directly related to the cosmological constant. For example, another metric-specific term is $\Gamma^{0}_{0i}\kappa_0$, which is not balanced in general when considering (9), (10), and (14). Remarkably, all the terms with an explicit $g_{\mu\nu}$ will cancel out upon using the more general expression $\kappa^{\rho}_{\rho;} = -\kappa^{\rho}\kappa_{\rho}$ instead of $\dot{\kappa} = -\kappa^2$, as given in (15), but $\dot{\kappa} = -\kappa^2$ is important since it guarantees a constant value for Λ/λ^2 and therefore constancy of Λ_E. In this respect, the unique choice for λ that follows from (15), which is equivalent to (4) and (5), is an equivalent definition of the main SIV equations within a special co-moving frame. Furthermore, the SIV theory associated with the unique "gauge" choice defined by Equation (4) and/or the equivalent set (5) is also supported by the unique scale-invariant action principle discussed recently in [16].

3 The SIV equations for λ have been redirived from an action principle [16], but were first introduced and studied, since 2017, by Maeder [13], within the scale-invariant cosmology by Dirac [10], Canuto et al. [11]. Thus, the property of $\dot{\kappa} = -\kappa^2$ has been noticed before and in particular the result within the SIV that $\dot{\lambda}^2/\lambda^4$ is constant. Here, we turn this observations into a reasonable choice for determining the functional form of λ that results in Λ_E being a constant according to the SIV.

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