Janis–Newman Algorithm: Generating Rotating and NUT Charged Black Holes

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Abstract: In this review we present the most general form of the Janis–Newman algorithm. This extension allows generating configurations which contain all bosonic fields with spin less than or equal to two (real and complex scalar fields, gauge fields, metric field) and with five of the six parameters of the Plebański–Demiański metric (mass, electric charge, magnetic charge, NUT charge and angular momentum). Several examples are included to illustrate the algorithm. We also discuss the extension of the algorithm to other dimensions.

Keywords: Janis-Newman algorithm; rotating black hole; Taub-NUT metric; Kerr-Newman metric; supergravity

1. Introduction

1.1. Motivations

General relativity is the theory of gravitational phenomena. It describes the dynamical evolution of spacetime through the Einstein–Hilbert action that leads to Einstein equations. The latter are highly non-linear differential equations and finding exact solutions is a notoriously difficult problem.

There are different types of solutions, but this review will cover only black-hole-like solutions (type-D in the Petrov classification) which can be described as particle-like objects that carry some charges, such as a mass or an electric charge.

Black holes are important objects in any theory of gravity for the insight they provide into the quantum gravity realm. For this reason it is a key step, in any theory, to obtain all possible black holes solutions. Rotating black holes are the most relevant subcases for astrophysics as it is believed that most astrophysical black holes are rotating. These solutions may also provide exterior metric for rotating stars.

The most general solution of this type in pure Einstein–Maxwell gravity with a cosmological constant Λ is the Plebański–Demiański metric [1,2]: it possesses six charges: mass m, NUT charge n, electric charge q, magnetic charge p, spin a and acceleration α. A challenging work is to generalize this solution to more complex Lagrangians, involving scalar fields and other gauge fields with non-minimal interactions, as is typically the case in supergravity. As the complexity of the equations of motion increase, it is harder to find exact analytical solutions, and one often consider specific types of solutions (extremal, BPS), truncations (some fields are constant, equal or vanishing) or solutions with restricted number of charges. For this reason it is interesting to find solution generating algorithms—procedures which transform a seed configuration to another configuration with a greater complexity (for example with a higher number of charges).

An *on-shell* algorithm is very precious because one is sure to obtain a solution if one starts with a seed configuration which solves the equations of motion. On the other hand, off-shell algorithms do not necessarily preserve the equations of motion but they are nonetheless very useful: they provide a motivated ansatz, and it is always easier to check if an ansatz satisfy the equations than

solving them from scratch. Even if in practice this kind of solution generating technique does not provide so many new solutions, it can help to understand better the underlying theory (which can be general relativity, modified gravities or even supergravity) and it may shed light on the structure of gravitational solutions.

1.2. The Janis–Newman Algorithm

The Janis–Newman (JN) algorithm is one of these (off-shell) solution generating techniques, which—in its original formulation—generates rotating metrics from static ones. It was found by Janis and Newman as an alternative derivation of the Kerr metric [3], while shortly after it has been used again to discover the Kerr–Newman metric [4].

This algorithm provides a way to generate axisymmetric metrics from a spherically symmetric seed metric through a particular complexification of radial and (null) time coordinates, followed by a complex coordinate transformation. Often one performs eventually a change of coordinates to write the result in Boyer–Lindquist coordinates.

The original prescription uses the Newman–Penrose tetrad formalism, which appears to be very tedious since it requires to invert the metric, to find a null tetrad basis where the transformation can be applied, and lastly to invert again the metric. In [5] Giampieri introduced another formulation of the JN algorithm which avoids gymnastics with null tetrads and which appears to be very useful for extending the procedure to more complicated solutions (such as higher dimensional ones). However it has been so far totally ignored in the literature. We stress that *all* results are totally equivalent in both approaches, and every computation that can be done with the Giampieri prescription can be done with the other. Finally [6] provides an alternative view on the algorithm.

In order for the metric to be still real, the radial functions inside the metric must be transformed such that reality is preserved.[1] Despite that there is *no* rigorous statement concerning the possible complexification of these functions, some general features have been worked out in the last decades and a set of rules has been established. Note that this step is the same in both prescriptions. In particular these rules can be obtained by solving the equations of motion for some examples and by identifying the terms in the solution [7]. Another approach consists in expressing the metric functions in terms of the Boyer–Lindquist functions—that appear in the change of coordinates and which are real—, the latter being then determined from the equations of motion [8,9].

It is widely believed that the JN algorithm is just a trick without any physical or mathematical basis, which is not accurate. Indeed it was proved by Talbot [10] shortly after its discovery why this transformation was well-defined, and he characterizes under which conditions the algorithm is on-shell for a subclass of Kerr–Schild (KS) metrics (see also [11]).[2] KS metrics admit a very natural formulation in terms of complex functions for which (some) complex change of coordinates can be defined. Note that KS metrics are physically interesting as they contain solutions of Petrov type II and D. Another way to understand this algorithm has been provided by Schiffer et al. [12] who showed that some KS metrics can be written in terms of a unique complex generating function, from which other solutions can be obtained through a complex change of coordinates. In various papers, Newman shows that the imaginary part of complex coordinates may be interpreted as an angular momentum, and there are similar correspondences for other charges (magnetic. . .) [13–15]. More recently Ferraro shed a new light on the JN algorithm using Cartan formalism [16]. Uniqueness results for the case of pure Einstein theory have been derived in [8]. A recent account on these different points can be found in [17].

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[1]　For simplifying, we will say that we complexify the functions inside the metric when we perform this transformation, even if in practice we "realify" them.

[2]　It has not been proved that the KS condition is necessary, but all known examples seem to fit in this category.

In its current form the algorithm is independent of the gravity theory under consideration since it operates independently at the level of each field in order to generate an ansatz, and the equations of motion are introduced only at the end to check if the configuration is a genuine solution. We believe that a better understanding of the algorithm would lead to an on-shell formulation where the algorithm would be interpreted as some kind of symmetry or geometric property. One intuition is that every configuration found with the JN algorithm and solving the equations of motion is derived from a seed that also solves the equations of motion (in particular no useful ansatz has been generated from an off-shell seed configuration).

Other solution generating algorithms rely on a complex formulation of general relativity which allows complex changes of coordinates. This is the case of the Ernst potential formulation [18,19] or of Quevedo's formalism who decomposes the Riemann tensor in irreducible representations of $SO(3,\mathbb{C}) \sim SO(3,1)$ and then uses the symmetry group to generate new solutions [20,21].

Despite its long history the Janis–Newman algorithm has not produced any new rotating solution for non-fluid configurations (which excludes radiating and interior solutions) beside the Kerr–Newman metric [4], and very few known examples have been reproduced [3,22–25]. Generically the application the Janis–Newman algorithm to interior and radiating systems [9,26–30] consist in deriving a configuration that do not solve the equations of motion by itself and to interpret the mismatch as a fluid (whose properties can be studied)—in this review we will not be interested by this kind of applications. Moreover the only solutions that have been fully derived using the algorithm are the original Kerr metric [3], the $d = 3$ BTZ black hole [23,24] and the d-dimensional Myers–Perry metric with a single angular momentum [22]: only the metric was found in the other cases [4,25] and the other fields had to be obtained using the equations of motion.

A first explanation is that there is no real understanding of the algorithm in its most general form (as reviewed above it is understood in some cases): there is no geometric or symmetry-related interpretation. Another reason is that the algorithm has been defined only for the metric (and real scalar fields) and no extension to the other types of fields was known until recently. It has also been understood that the algorithm could not be applied in the presence of a cosmological constant [7]: in particular the (a)dS–Kerr(–Newman) metrics [31] (see also [1,2,32,33]) cannot be derived in this way despite various erroneous claims [29,34]. Finally many works [35–43] (to cite only few) are (at least partly) incorrect or not reliable because they do not check the equations of motion or they perform non-integrable Boyer–Lindquist changes of coordinates [30,44,45].

The algorithm was later shown to be generalizable by Demiański and Newman who demonstrated by writing a general ansatz and solving the equations of motion that other parameters can be added [7,46], even in the presence of a cosmological constant. While one parameter corresponds to the NUT charge, the other one did not receive any interpretation until now.[3] Unfortunately Demiański did not express his result to a concrete algorithm (the normal prescription fails in the presence of the NUT charge and of the cosmological constant) which may explain why this work did not receive any further attention. Note that the algorithm also failed in the presence of magnetic charges.

A way to avoid problems in defining the changes of coordinates to the Boyer–Lindquist system and to find the complexification of the functions has been proposed in [8] and extended in [30]: the method consists of writing the unknown complexified function in terms of the functions of the coordinate transformation. This philosophy is particularly well-suited for providing an ansatz which does not rely on a static seed solution.

More recently it has been investigated whether the JN algorithm can be applied in modified theories of gravity. Pirogov put forward that rotating metrics obtained from the JN algorithm in Brans–Dicke theory are not solutions if $\alpha \neq 1$ [50]. Similarly Hansen and Yunes have shown a similar

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3　　Demiański's metric has been generalized in [47–49].

result in quadratic modified gravity (which includes Gauss–Bonnet) [51].[4] These do not include Sen's dilaton–axion black hole for which $\alpha = 1$ (Section 5.4), nor the BBMB black hole from conformal gravity (Section 5.2). Finally it was proved in [53] that it does not work either for Einstein–Born–Infled theories.[5] We note that all these no-go theorem have been found by assuming a transformation with only rotation.

Previous reviews of the JN algorithm can be found in [8,17,37,54] (see also [55]).

1.3. Overview

The goal of the current work is to review a series of recent papers [56–59] in which the JN algorithm has been extended in several directions, opening the doors to many new applications. This review evolved from the thesis of the author [60], which presented the material from a slightly different perspective, and from lectures given at HRI (Allahabad, India).

As explained in the previous section, the JN algorithm was formulated only for the metric and all other fields had to be found using the equations of motion (with or without using an ansatz). For example neither the Kerr–Newman gauge field or its associated field strength could be derived in [4]. The solution to this problem is to perform a gauge transformation in order to remove the radial component of the gauge field in null coordinates [56]. It is then straightforward to apply the JN algorithm in either prescription.[6] Another problem was exemplified by the derivation of Sen's axion–dilaton rotating black hole [62] by Yazadjiev [25], who could find the metric and the dilaton, but not the axion (nor the gauge field). The reason is that while the JN algorithm applies directly to real scalar fields, it does not for complex scalar fields (or for a pair of real fields that can naturally be gathered into a complex scalar). Then it is necessary to consider the complex scalar as a unique object and to perform the transformation without trying to keep it real [59]. Hence this completes the JN algorithm for all bosonic fields with spin less than or equal to two.

A second aspect for which the original form of the algorithm was deficient is configuration with magnetic and NUT charges and in presence of a cosmological constant. The issue corresponds to finding how one should complexify the functions: the usual rules do not work and if there were no way to obtain the functions by complexification then the JN algorithm would be of limited interest as it could not be exported to other cases (except if one is willing to solve equations of motion, which is not the goal of a solution generating technique). We have found that to reproduce Demiański's result [7] it is necessary to complexify also the mass and to consider the complex parameter $m + in$ [58,59] and to shift the curvature of the spherical horizon. Similarly for configurations with magnetic charges one needs to consider the complex charge $q + ip$ [59]. Such complex combinations are quite natural from the point of view of the Plebański–Demiański solution [1,2] described previously. It is to notice that the appearance of complex coordinate transformations mixed with complex parameter transformations was a feature of Quevedo's solution generating technique [20,21], yet it is unclear what the link with our approach really is. Hence the final metric obtained from the JN algorithm may contain (for vanishing cosmological constant) five of the six Plebański–Demiański parameters [1,2] along with Demiański's parameter.

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[4]   There are some errors in the introduction of [51]: they report incorrectly that the result from [50] implies that Sen's black hole cannot be derived from the JN algorithm, as was done by Yazadjiev [25]. But this black hole corresponds to $\alpha = 1$ and as reported above there is no problem in this case (see [52] for comparison). Moreover they argue that several works published before 2013 did not take into account the results of Pirogov [50], published in 2013.

[5]   It may be possible to circumvent the result of [53] by using the results described in this review since several tools were not known by its author.

[6]   Another solution has been proposed by Keane [61] but it is applicable only to the Newman–Penrose coefficients of the field strength. Our proposal requires less computations and yields directly the gauge field from which all relevant quantities can easily be derived.

An interesting fact is that the previous argument works in the presence of the cosmological constant only if one considers the possibility of having a generic topological horizons (flat, hyperbolic or spherical) and for this reason we have provided an extension of the formalism to this case [58].

We also propose a generalization of the algorithm to any dimension [57], but while new examples could be found for $d = 5$ the program could not be carried to the end for $d > 5$.

All these results provide a complete framework for most of the theories of gravity that are commonly used. As a conclusion we summarize the features of our new results:

- all bosonic fields with spin ≤ 2;
- topological horizons;
- charges m, n, q, p, a (with a only for $\Lambda = 0$);
- extend to $d = 3, 5$ dimensions (and proposal for higher).

We have written a general *Mathematica* package for the JN algorithm in Einstein–Maxwell theory.[7] Here is a list of new examples that have been completely derived using the previous results (all in $4d$ except when said explicitly):

- Kerr–Newman–NUT;
- dyonic Kerr–Newman;
- Yang–Mills Kerr–Newman black hole [63];
- adS–NUT Schwarzschild;
- Demiański's solution [7];
- ungauged $N = 2$ BPS solutions [64];
- non-extremal solution in T^3 model [62] (partly derived in [25]);
- SWIP solutions [65];
- (a)dS–charged Taub–NUT–BBMB [66];
- $5d$ Myers–Perry [67];
- $5d$ BMPV [68];
- NUT charged black hole[8] in gauged $N = 2$ sugra with $F = -i X^0 X^1$ [69].

We also found a more direct derivation of the rotating BTZ black hole (derived in another way by Kim [23,24]).

1.4. Outlook

A major playground for this modified Janis–Newman (JN) algorithm is (gauged) supergravity—where many interesting solutions remain to be discovered—since all the necessary ingredients are now present. Moreover important solutions are still missing in higher-dimensional Einstein–Maxwell (in particular the charged Myers–Perry solution) and one can hope that understanding the JN algorithm in higher dimensions would shed light on this problem. Another open case is whether black rings can also be derived using the algorithm.

A major question about the JN algorithm is whether it is possible to include rotation for non-vanishing cosmological constant. A possible related problem concerns the addition of acceleration α, which is the only missing parameter when $\Lambda = 0$. It is indeed puzzling that one could get all Plebański–Demiański parameters but the acceleration, which appears in the combination $a + i\alpha$. Both problems are linked to the fact that the JN algorithm – in its current form – does not take into account various couplings between the parameters (such as the spin with the cosmological constant or the acceleration with the mass in the simplest cases). On the other hand it does not mean that it is impossible to find a generalization of the algorithm: philosophically the problem is identical to the ones of adding NUT and magnetic charges.

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[7]   Available at http://www.lpthe.jussieu.fr/~erbin/.
[8]   Derived by D. Klemm and M. Rabbiosi, unpublished work.

In any case the meaning and a rigorous derivation of the JN algorithm—perhaps elevating it to the status of a true solution generating algorithm—are still to be found. It is also interesting to note that almost all of the examples quoted in the previous section can be embedded into $N = 2$ supergravity. This calls for a possible interpretation of the algorithm in terms of some hidden symmetry of supergravity, or even of string theory.

We hope that our new extension of the algorithm will help to bring it out of the shadows where it stayed since its creation, and to establish it as a standard tool for deriving new solutions in the various theories of gravity.

1.5. Summary

In Section 2 we review the original Janis–Newman algorithm and its alternative form due to Giampieri before illustrating it with some examples. Section 3 shows how to extend the algorithm to more complicated set of fields (complex scalars, gauge fields) and parameters (magnetic and NUT charges, topological horizon). Then Section 4 provides a general description of the algorithm in its most general form. Section 5 describes several examples.

Appendix A reviews briefly the main properties of $N = 2$ supergravity.

In our conventions the spacetime signature is mostly plus.

2. Algorithm: Main Ideas

In this section we summarize the original algorithm together with its extension to gauge fields. We will see that the algorithm involves the transformations of two different objects (the tensor structure and the coordinate-dependent functions of the fields) which can be taken care of separately. The transformation of the tensor structure is simple and no new idea (for $d = 4$) will be needed after this section since we will be dealing with the two most general tensor structures for bosonic fields of spin less than or equal to two (the metric and vector fields). On the other hand, the transformation of the functions is more involved and we will introduce new concepts through simple examples in the next section before giving the most general formulation in Section 4. We review the two different prescriptions for the transformation and we illustrate the algorithm with two basic examples: the flat space and the Kerr–Newman metrics.

2.1. Summary

The general procedure for the Janis–Newman algorithm can be summarized as follows:

1. Perform a change of coordinates (t, r) to (u, r) and a gauge transformation such that $g_{rr} = 0$ and $A_r = 0$.
2. Take $u, r \in \mathbb{C}$ and replace the functions $f_i(r)$ inside the real fields by new real-valued functions $\tilde{f}_i(r, \bar{r})$ (there is a set of "empirical" rules).
3. Perform a complex change of coordinates and transform accordingly:

   (a) the tensor structure, i.e., the dx^μ (two prescriptions: Janis–Newman [3] and Giampieri [5]);
   (b) the functions $\tilde{f}_i(r, \bar{r})$.

4. Perform a change of coordinates to simplify the metric (for example to Boyer–Lindquist system). If the transformation is infinitesimal then one should check that it is a valid diffeomorphism, i.e., that it is integrable.

Note that in the last point the operations (a) and (b) are independent. In practice one is performing the algorithm for a generic class of configurations with unspecified $f_i(r)$ in order to obtain general formulas. One leaves point 2 and (3b) implicit since the other steps are independent of the form of the functions. Then given a specific configuration one can perform 2 and (3b).

Throughout the review we will not be interested in showing that the examples discussed are indeed solutions but merely to explain how to extend the algorithm. All examples we are discussing

have been shown to be solutions of the theory under concerned and we refer the reader to the original literature for more details. For this reason we will rarely mention the action or the equations of motion and just discussed the fields and their expressions.

One could add a fifth point to the list: checking the equations of motion. We stress again that the algorithm is *off-shell* and there is no guarantee (except in some specific cases [17]) that a solution is mapped to a solution.

2.2. Algorithm

We present the algorithm for a metric $g_{\mu\nu}$ and a gauge field A_μ associated with a U(1) gauge symmetry. This simple case is sufficient to illustrate the main features of the algorithm.

As already mentioned in the introduction, the authors of [4] failed to derive the field strength of the Kerr–Newman black hole from the Reissner–Nordström one. In the null tetrad formalism it is natural to write the field strength in terms of its Newman–Penrose coefficients, but a problem arises when one tries to generate the rotating solution since one of the coefficients is zero in the case of Reissner–Nordström, but non-zero for Kerr–Newman. Three different prescriptions have been proposed: two works in the Newman–Penrose formalism—one with the field strength [61] and one with the gauge field [56]—while the third extends Giampieri's approach to the gauge field [56]. Since the proposals from [56] fit more directly (and parallel each other) inside the prescriptions of Janis–Newman and Giampieri, we will focus on them. It is also more convenient to work with the gauge fields since any other quantity can be easily computed from them.

2.2.1. Seed Metric and Gauge Fields

The seed metric and gauge field take the form

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + r^2 d\Omega^2, \qquad d\Omega^2 = d\theta^2 + H(\theta)^2\,d\phi^2, \tag{1a}$$

$$A = f_A(r)\,dt. \tag{1b}$$

The normalized curvature of the (θ, ϕ) sections (or equivalently of the horizon) is denoted by κ

$$\kappa = \begin{cases} +1 & S^2, \\ -1 & H^2 \end{cases} \tag{2}$$

where S^2 and H^2 are respectively the sphere and the hyperboloid,[9] and one has

$$H(\theta) = \begin{cases} \sin\theta & \kappa = 1, \\ \sinh\theta & \kappa = -1. \end{cases} \tag{3}$$

In all this section we will consider the case of spherical horizon with $\kappa = 1$.
Introduce Eddington–Finkelstein coordinates (u, r)

$$du = dt - f^{-1}dr \tag{4}$$

in order to remove the g_{rr} term of the metric [3]. Under this transformation the gauge field becomes

$$A = f_A\,(du + f^{-1}dr). \tag{5}$$

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[9]   We leave aside the case of the plane $\mathbb{R}^2$ with $\kappa = 0$. The formulas can easily be extended to this case.

The changes of coordinate has introduced an A_r component but since it depends only on the radial coordinate $A_r = A_r(r)$ it can be removed by a gauge transformation.

At the end the metric and gauge fields are

$$ds^2 = -f\,dt^2 + 2du\,dr + r^2 d\Omega^2, \tag{6a}$$

$$A = f_A\,du. \tag{6b}$$

This last step was missing in [4] and explains why they could not derive the full solution from the algorithm. The lesson to draw is that the validity of the algorithm depends a lot on the coordinate basis and of the parametrization of the fields, although guiding principle founded on all known examples seems that one needs to have

$$g_{rr} = 0, \qquad A_r = 0. \tag{7}$$

2.2.2. Janis–Newman Prescription: Newman–Penrose Formalism

The Janis–Newman prescription for transforming the tensor structure relies on the Newman–Penrose formalism [3,4,17].

First one needs to obtain the contravariant expressions of the metric and of the gauge field

$$\frac{\partial^2}{\partial s^2} = g^{\mu\nu}\partial_\mu\partial_\nu = f\,\partial r^2 - 2\,\partial u\partial r + \frac{1}{r^2}\left(\partial_\theta^2 + \frac{\partial_\phi^2}{\sin^2\theta}\right), \tag{8a}$$

$$A = -f_A\,\partial_r. \tag{8b}$$

Then one introduces null complex tetrads

$$Z_a^\mu = \{\ell^\mu, n^\mu, m^\mu, \bar{m}^\mu\} \tag{9}$$

with flat metric

$$\eta^{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{10}$$

such that

$$g^{\mu\nu} = \eta^{ab}Z_a^\mu Z_b^\nu = -\ell^\mu n^\nu - \ell^\nu n^\mu + m^\mu \bar{m}^\nu + m^\nu \bar{m}^\mu. \tag{11}$$

The explicit tetrad expressions are

$$\ell^\mu = \delta_r^\mu, \qquad n^\mu = \delta_u^\mu - \frac{f}{2}\,\delta_r^\mu, \qquad m^\mu = \frac{1}{\sqrt{2}\bar{r}}\left(\delta_\theta^\mu + \frac{i}{\sin\theta}\,\delta_\phi^\mu\right) \tag{12}$$

and the gauge field is

$$A^\mu = -f_A\,\ell^\mu. \tag{13}$$

Note that without the gauge transformation there would be an additional term and the expression of A^μ in terms of the tetrads would be ambiguous.

At this point u and r are allowed to take complex values but keeping ℓ^μ and n^μ real and $(m^\mu)^* = \bar{m}^\mu$ and replacing

$$f(r) \longrightarrow \tilde{f}(r,\bar{r}) \in \mathbb{R}, \qquad f_A(r) \longrightarrow \tilde{f}_A(r,\bar{r}) \in \mathbb{R}. \tag{14}$$

Consistency implies that one recovers the seed for $\bar{r} = r$ and $\bar{u} = u$.

Finally one can perform a complex change of coordinates

$$u = u' + ia\cos\theta, \qquad r = r' - ia\cos\theta \tag{15}$$

where a is a parameter (to be interpreted as the angular momentum per unit of mass) and $r', u' \in \mathbb{R}$. While this transformation seems arbitrary, general consistency limits severely the possibilities [58]. The tetrads transform as vectors

$$Z_a'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} Z_a^\nu \tag{16}$$

and this lead to the expressions

$$\ell'^\mu = \delta_r^\mu, \qquad n'^\mu = \delta_u^\mu - \frac{\tilde{f}}{2}\delta_r^\mu,$$

$$m'^\mu = \frac{1}{\sqrt{2}(r' + ia\cos\theta)}\left(\delta_\theta^\mu + \frac{i}{\sin\theta}\delta_\phi^\mu - ia\sin\theta\,(\delta_u^\mu - \delta_r^\mu)\right). \tag{17}$$

After inverting the contravariant form of the metric and the gauge field one is lead to the final expressions

$$ds'^2 = -\tilde{f}\,(du' - a\sin^2\theta\,d\phi)^2 - 2\,(du' - a\sin^2\theta\,d\phi)(dr' + a\sin^2\theta\,d\phi) + \rho^2 d\Omega^2, \tag{18a}$$

$$A' = \tilde{f}_A\,(du' - a\sin^2\theta\,d\phi). \tag{18b}$$

where

$$\rho^2 = |r|^2 = r'^2 + a^2\cos^2\theta. \tag{19}$$

The coordinate dependence of the functions can be written as

$$\tilde{f} = \tilde{f}(r, \bar{r}) = \tilde{f}(r', \theta) \tag{20}$$

in the new coordinates (and similarly for $\tilde{f}_A$), but note that the θ dependence is not arbitrary and comes solely from $\text{Im}\,r$.

2.2.3. Giampieri Prescription

The net effect of the transformation (15) on the tensor structure amounts to the replacements

$$du \longrightarrow du' - a\sin^2\theta\,d\phi, \qquad dr \longrightarrow dr' + a\sin^2\theta\,d\phi \tag{21}$$

by comparing (6) and (18), up to the $r^2 \to \rho^2$ in front of $d\Omega^2$. Is it possible to obtain the same effect by avoiding the Newman–Penrose formalism and all the computations associated to changing from covariant to contravariant expressions? Inspecting the infinitesimal form of (15)

$$du = du' - ia\sin\theta\,d\theta, \qquad dr = dr' + ia\sin\theta\,d\theta, \tag{22}$$

one sees that (21) can be recovered if one sets [5]

$$id\theta = \sin\theta\,d\phi. \tag{23}$$

Note that it should be done only in the infinitesimal transformation and not elsewhere in the metric. Although some authors [16,29] mentioned the equivalence between the tetrad computation and (21), it is surprising that this direction has not been followed further.

While this new prescription is not rigorous and there is no known way to derive (23), it continues to hold for the most general seed (Section 4) and it gives systematically the same results as the Janis–Newman prescription, as can be seen by simple inspection. In particular this approach is not

adding nor removing any of the ambiguities due to the function transformations that are already present and well-known in JN algorithm. Since this prescription is much simpler we will continue to use it throughout the rest of this review (we will show in Section 4 how it is modified for topological horizons).

Finally the comparison of the two prescriptions clearly shows that the r^2 factor in front of $d\Omega^2$ should be considered as a function instead of a part of the tensor structure: the replacement $r^2 \to \rho^2$ is dictated by the rules given in the next section. We did not want to enter into these subtleties here but this will become evident in Section 4.

2.2.4. Transforming the Functions

The transformation of the functions is common to both the Janis–Newman and Giampieri prescriptions since they are independent of the tensor structure. This step is the main weakness of the Janis–Newman algorithm because there is no unique way to perform the replacement and for this reason the final result contains some part of arbitrariness. This provides another incentive for checking systematically if the equations of motion are satisfied. Nonetheless examples have provided a small set of rules [3,4,8,56]

$$r \longrightarrow \frac{1}{2}(r + \bar{r}) = \operatorname{Re} r, \tag{24a}$$

$$\frac{1}{r} \longrightarrow \frac{1}{2}\left(\frac{1}{r} + \frac{1}{\bar{r}}\right) = \frac{\operatorname{Re} r}{|r|^2}, \tag{24b}$$

$$r^2 \longrightarrow |r|^2. \tag{24c}$$

The idea is to use geometric or arithmetic means. All other functions can be reduced to a combination of them, for example $1/r^2$ is complexified as $1/|r|^2$.

Every known configuration which does not involve a magnetic, a NUT charge, complex scalar fields or powers higher of r than quadratic can be derived with these rules (these cases will be dealt with in Sections 3 and 4). Hence despite the fact that there is some arbitrariness, it is ultimately quite limited and very few options are possible in most cases.

2.2.5. Boyer–Lindquist Coordinates

Boyer–Lindquist coordinates are defined to be those with the minimal number of non-zero off-diagonal components in the metric. Performing the transformation (the primes in (18) are now omitted)

$$du' = dt' - g(r)dr', \qquad d\phi = d\phi' - h(r)dr, \tag{25}$$

the conditions $g_{tr} = g_{r\phi'} = 0$ are solved for

$$g(r) = \frac{r^2 + a^2}{\Delta}, \qquad h(r) = \frac{a}{\Delta} \tag{26}$$

where we have defined

$$\Delta(r) = \tilde{f}\rho^2 + a^2 \sin^2 \theta. \tag{27}$$

As indicated by the r-dependence this change of variables is integrable provided that g and h are functions of r only. However Δ as given in (27) for a generic configuration contains a θ dependence: one should check that this dependence cancels once restricted to the example of interest. Otherwise one is not allowed to perform this change of coordinates (but other systems may still be found).

Given (26) one gets the metric and gauge fields (deleting the prime)

$$ds^2 = -\tilde{f}\,dt^2 + \frac{\rho^2}{\Delta}\,dr^2 + \rho^2 d\theta^2 + \frac{\Sigma^2}{\rho^2}\sin^2\theta\,d\phi^2 + 2a(\tilde{f}-1)\sin^2\theta\,dtd\phi, \tag{28a}$$

$$A = \tilde{f}_A\left(dt - \frac{\rho^2}{\Delta}\,dr - a\sin^2\theta\,d\phi\right) \tag{28b}$$

with

$$\frac{\Sigma^2}{\rho^2} = r^2 + a^2 + ag_{t\phi}. \tag{29}$$

The *rr*-term has been computed from

$$g - a\sin^2\theta\,h = \frac{\rho^2}{\Delta}. \tag{30}$$

Generically the radial component of the gauge field depends only on radial coordinate $A_r = A_r(r)$ (θ-dependence of the function $\tilde{f}_A$ sits in a factor $1/\rho^2$ which cancels the one in front of dr) and one can perform a gauge transformation in order to set it to zero, leaving

$$A = \tilde{f}_A\left(dt - a\sin^2\theta\,d\phi\right). \tag{31}$$

2.3. Examples

2.3.1. Flat Space

It is straightforward to check that the algorithm applied to the Minkowski metric—which has $f = 1$, leading to $\tilde{f} = 1$—in spherical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right) \tag{32}$$

gives again the Minkowski metric but in spheroidal coordinates (after a Boyer–Lindquist transformation)

$$ds^2 = -dt^2 + \frac{\rho^2}{r^2 + a^2}\,dr^2 + \rho^2 d\theta^2 + (r^2 + a^2)\sin^2\theta\,d\phi^2, \tag{33}$$

recalling that $\rho^2 = r^2 + a^2\cos^2\theta$. The metric is exactly diagonal because $g_{t\phi} = 0$ for $\tilde{f} = 1$ from (28a).

Hence for flat space the JN algorithm reduces to a change of coordinates, from spherical to (oblate) spheroidal coordinates: the 2-spheres foliating the space in the radial direction are deformed to ellipses with semi-major axis a.

This fact is an important consistency check and it has been useful for extending the algorithm to higher dimensions [57] or to other coordinate systems (such as one with direction cosines). Moreover in this case one can forget about the time direction and consider only the transformation of the radial coordinate.

2.3.2. Kerr–Newman

The seed function is the Reissner–Nordström for which the metric and gauge field are

$$f(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2}, \qquad f_A = \frac{q}{r}. \tag{34}$$

Applications of the rules (24) leads to

$$\tilde{f} = 1 - \frac{2m\,\mathrm{Re}\,r}{|r|^2} + \frac{q^2}{|r|^2} = 1 + \frac{q^2 - 2mr'}{\rho^2}, \tag{35a}$$

$$\tilde{f}_A = \frac{q\,\mathrm{Re}\,r}{|r|^2} = \frac{qr'}{\rho^2}. \tag{35b}$$

These functions together with (28) describe correctly the Kerr–Newman solution [17,70]. For completeness we spell out the expressions of the quantities appearing in the metric

$$\frac{\Sigma^2}{\rho^2} = r^2 + a^2 - \frac{q^2 - 2mr}{\rho^2}\,a^2 \sin^2\theta, \tag{36a}$$

$$\Delta = r^2 - 2mr + a^2 + q^2. \tag{36b}$$

In particular Δ does not contain any θ dependence and the BL transformation is well defined. Moreover the radial component of the gauge field is

$$A_r = -\frac{\tilde{f}_A \rho^2}{\Delta} = \frac{qr}{\Delta} \tag{37}$$

and it is independent of θ.

3. Extension through Simple Examples

In this section we motivate through simple examples modifications to the original prescription for the transformation of the functions.

3.1. Magnetic Charges: Dyonic Kerr–Newman

The dyonic Reissner–Nordström metric is obtained from the electric one (34) by the replacement [71]

$$q^2 \longrightarrow |Z|^2 = q^2 + p^2 \tag{38}$$

where Z corresponds to the central charge

$$Z = q + ip. \tag{39}$$

Then the metric function reads

$$f = 1 - \frac{2m}{r} + \frac{|Z|^2}{r^2}. \tag{40}$$

The gauge field receives a new ϕ-component

$$A = f_A\,\mathrm{d}t - p\cos\theta\,\mathrm{d}\phi = f_A\,\mathrm{d}u - p\cos\theta\,\mathrm{d}\phi \tag{41}$$

(the last equality being valid after a gauge transformation) and

$$f_A = \frac{q}{r}. \tag{42}$$

The transformation of the function f under (15) is straightforward and yields

$$\tilde{f} = 1 - \frac{2mr' - |Z|^2}{\rho^2}. \tag{43}$$

On the other hand transforming directly the r inside f_A according to (24) does not yield the correct result. Instead one needs to first rewrite the gauge field function as

$$f_A = \text{Re}\left(\frac{Z}{r}\right) \tag{44}$$

from which the transformation proceeds to

$$\tilde{f}_A = \frac{\text{Re}(Z\bar{r})}{|r|^2} = \frac{qr' - pa\cos\theta}{\rho^2}. \tag{45}$$

Note that it not useful to replace p by $\text{Im}\, Z$ in (41) since it is not accompanied by any r dependence. Moreover it is natural that the factor $|Z|^2$ appears in the metric and this explains why the charges there do not mix with the coordinates.

The gauge field in BL coordinates is finally

$$A = \frac{qr - pa\cos\theta}{\rho^2}\,\mathrm{d}t + \left(-\frac{qr}{\rho^2}a\sin^2\theta + \frac{p(r^2 + a^2)}{\rho^2}\cos\theta\right)\mathrm{d}\phi \tag{46a}$$

$$= \frac{qr}{\rho^2}(\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi) + \frac{p\cos\theta}{\rho^2}\left(a\,\mathrm{d}t + (r^2 + a^2)\,\mathrm{d}\phi\right). \tag{46b}$$

The radial component has been removed thanks to a gauge transformation since it depends only on r

$$\Delta \times A_r = -\frac{qr - pa\cos\theta}{\rho^2}\rho^2 - pa\cos\theta = -qr. \tag{47}$$

There is a coupling between the parameters a and p which can be interpreted from the fact that a rotating magnetic charge has an electric quadrupole moment. This coupling is taken into account from the product of the imaginary parts which yield a real term. In view of the form of the algorithm such contribution could not arise from any other place. Moreover the combination $Z = q + ip$ appears naturally in the Plebański–Demiański solution [1,2].

The Yang–Mills Kerr–Newman black hole found by Perry [63] can also be derived in this way, starting from the seed

$$A^I = \frac{q^I}{r}\,\mathrm{d}t + p^I\cos\theta\,\mathrm{d}\phi, \qquad |Z|^2 = q^I q^I + p^I p^I \tag{48}$$

where q^I and p^I are constant elements of the Lie algebra.

3.2. NUT Charge and Cosmological Constant and Topological Horizon: (Anti-)de Sitter Schwarzschild–NUT

In this subsection we consider general topological horizons

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + H(\theta)^2\,\mathrm{d}\phi^2, \qquad H(\theta) = \begin{cases} \sin\theta & \kappa = 1 \quad (S^2), \\ \sinh\theta & \kappa = -1 \quad (H^2). \end{cases} \tag{49}$$

The cosmological constant is denoted by Λ. We give only the main formulas to motivate the modification of the algorithm, leaving the details of the transformation for Section 4.

The complex transformation that adds a NUT charge is

$$u = u' - 2\kappa \ln H(\theta), \qquad r = r' + in, \tag{50a}$$

$$m = m' + i\kappa n, \qquad \kappa = \kappa' - \frac{4\Lambda}{3}n^2. \tag{50b}$$

Note that it is κ and not κ' that appears in m. After having shown

The metric derived from the seed (1a) is

$$ds^2 = -\tilde{f}\,(dt - 2\kappa n H'(\theta)\,d\phi)^2 + \tilde{f}^{-1}\,dr^2 + \rho^2\,d\Omega^2, \tag{51}$$

see (97), where

$$\rho^2 = r'^2 + n^2. \tag{52}$$

The function corresponding to the (a)dS–Schwarzschild metric is

$$f = \kappa - \frac{2m}{r} - \frac{\Lambda}{3}\,r^2 = \kappa - 2\,\mathrm{Re}\left(\frac{m}{r}\right) - \frac{\Lambda}{3}\,r^2. \tag{53}$$

The transformation is

$$\tilde{f} = \kappa - \frac{2\,\mathrm{Re}(m\bar{r})}{|r|^2} - \frac{\Lambda}{3}\,|r|^2 = \kappa' - \frac{4\Lambda}{3}\,n^2 - \frac{2\left[m'r' + \left(\kappa' - \frac{4\Lambda}{3}\,n^2\right)n^2\right]}{\rho^2} - \frac{\Lambda}{3}\,\rho^2 \tag{54}$$

which after simplification gives

$$\tilde{f} = \kappa' - \frac{2m'r' + 2\kappa'n^2}{\rho^2} - \frac{\Lambda}{3}(r'^2 + 5n^2) + \frac{8\Lambda}{3}\,\frac{n^4}{\rho^2} \tag{55}$$

which corresponds correctly to the function of (a)dS–Schwarzschild–NUT [72].

Note that it is necessary to consider the general case of massive black hole with topological horizon (if $\Lambda \neq 0$ for the latter) even if one is ultimately interested in the $m = 0$ or $\kappa = 1$ cases.

The transformation (50) can be interpreted as follows. In similarity with the case of the magnetic charge, writing the mass as a complex parameter is needed to take into account some couplings between the parameters that would not be found otherwise. Moreover the shift of κ is required because the curvature of the (θ, ϕ) section should be normalized to $\kappa = \pm 1$ but the coupling of the NUT charge with the cosmological constant modifies the curvature: the new shift is necessary to balance this effect and to normalize the (θ, ϕ) curvature to $\kappa' = \pm 1$ in the new metric. The NUT charge in the Plebański–Demiański solution [1,2] is

$$\ell = n\left(1 - \frac{4\Lambda}{3}\,n^2\right) \tag{56}$$

so the natural complex combination is $m + i\ell$ and not $m + i\kappa n$ from this point of view, and similarly for the curvature [73] (Section 5.3) (such relations appear when taking limit of the Plebański–Demiański solution to recover subcases).

Finally we conclude this section with two remarks to quote different contexts where the above expression appear naturally :

- Embedding Einstein–Maxwell into $N = 2$ supergravity with a negative cosmological constant $\Lambda = -3g^2$, the solution is BPS if [72]

$$\kappa' = -1, \qquad n = \pm\frac{1}{2g}, \tag{57}$$

  in which case $\kappa' = \kappa$.
- The Euclidean NUT solution is obtained from the Wick rotation

$$t = -i\tau, \qquad n = i\nu. \tag{58}$$

The condition for regularity is [74,75]

$$m = m' - v \left(\kappa + \frac{4\Lambda}{3} v^2 \right) = 0. \tag{59}$$

3.3. Complex Scalar Fields

For a complex scalar field, or any pair of real fields that can be naturally gathered as a complex field, one should treat the full field as a single entity instead of looking at the real and imaginary parts independently. In particular one should not impose any reality condition. A typical case of such system is the axion–dilaton pair

$$\tau = e^{-2\phi} + i\sigma. \tag{60}$$

In order to demonstrate this principle consider the seed (for a complete example see Section 5.4)

$$\tau = 1 + \frac{\mu}{r} \tag{61}$$

where only the dilaton is non-zero.

Then the transformation (15) gives

$$\tau' = 1 + \frac{\mu}{r} = 1 + \frac{\mu}{r' - ia\cos\theta} = 1 + \frac{\mu r'}{\rho^2} + i\,\frac{\mu a\cos\theta}{\rho^2}. \tag{62}$$

The transformation generates an imaginary part which cannot be obtained if $\mathrm{Im}\,\tau$ is treated separately: the algorithm does not change fields that vanish except if they are components of a larger field. Note that both τ and τ' are harmonic functions.

4. Complete Algorithm

In this section we gather all the facts on the Janis–Newman algorithm and we explain how to apply it to a general setting. We write the formulas corresponding to the most general configurations that can be obtained. We insist again on the fact that all these results can also be derived from the tetrad formalism.

4.1. Seed Configuration

We consider a general configuration with a metric $g_{\mu\nu}$, gauge fields A^I_μ, complex scalar fields τ^i and real scalar fields q^u. The initial parameters of the seed configuration are the mass m, electric charges q^I, magnetic charges p^i and some other parameters λ^A (such as the scalar charges). The electric and magnetic charges are grouped in complex parameters

$$Z^I = q^I + ip^I. \tag{63}$$

All indices run over some arbitrary ranges.

The seed configuration is spherically symmetric and in particular all the fields depend only on the radial direction r

$$ds^2 = -f_t(r)\,dt^2 + f_r(r)\,dr^2 + f_\Omega(r)\,d\Omega^2, \tag{64a}$$
$$A^I = f^I(r)\,dt + p^I H'(\theta)\,d\phi, \tag{64b}$$
$$\tau^i = \tau^i(r), \qquad q^u = q^u(r) \tag{64c}$$

where

$$d\Omega^2 = d\theta^2 + H(\theta)^2\,d\phi^2, \qquad H(\theta) = \begin{cases} \sin\theta & \kappa = 1 \quad (S^2), \\ \sinh\theta & \kappa = -1 \quad (H^2). \end{cases} \tag{65}$$

Note that only two functions in the metric are relevant since the last one can be fixed through a diffeomorphism. All the real functions are denoted collectively by

$$f_i = \{f_t, f_r, f_\Omega, f^I, q^u\}. \tag{66}$$

The transformation to null coordinates is

$$\mathrm{d}t = \mathrm{d}u - \sqrt{\frac{f_r}{f_t}}\,\mathrm{d}r \tag{67}$$

and yields

$$\mathrm{d}s^2 = -f_t\,\mathrm{d}u^2 - 2\sqrt{f_t f_r}\,\mathrm{d}r^2 + f_\Omega\,\mathrm{d}\Omega^2, \tag{68a}$$
$$A^I = f^I\,\mathrm{d}u + p^I H'\,\mathrm{d}\phi \tag{68b}$$

where the radial component of the gauge field

$$A_r^I = f^I \sqrt{\frac{f_r}{f_t}} \tag{69}$$

has been set to zero through a gauge transformation.

4.2. Janis–Newman Algorithm

4.2.1. Complex Transformation

One performs the complex change of coordinates

$$r = r' + i\,F(\theta), \qquad u = u' + i\,G(\theta). \tag{70}$$

In the case of topological horizons the Giampieri ansatz (23) generalizes to

$$i\,\mathrm{d}\theta = H(\theta)\,\mathrm{d}\phi \tag{71}$$

leading to the differentials

$$\mathrm{d}r = \mathrm{d}r' + F'(\theta)H(\theta)\,\mathrm{d}\phi, \qquad \mathrm{d}u = \mathrm{d}u' + G'(\theta)H(\theta)\,\mathrm{d}\phi. \tag{72}$$

The ansatz (71) is a direct consequence of the fact that the 2-dimensional slice (θ, ϕ) is given by $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + H(\theta)^2\,\mathrm{d}\phi^2$, such that the function in the RHS of (71) corresponds to $\sqrt{g_{\phi\phi}^\Omega}$ (where g is the static metric), as can be seen by doing the computation with $i\,\mathrm{d}\theta = \mathcal{H}(\theta)\mathrm{d}\phi$ and identifying $\mathcal{H} = H$ at the end.

The most general known transformation is

$$F(\theta) = n - a\,H'(\theta) + c\left(1 + H'(\theta)\ln\frac{H(\theta/2)}{H'(\theta/2)}\right), \tag{73a}$$

$$G(\theta) = \kappa a\,H'(\theta) - 2\kappa n \ln H(\theta) - \kappa c\,H'(\theta)\ln\frac{H(\theta/2)}{H'(\theta/2)}, \tag{73b}$$

$$m = m' + i\kappa n, \tag{73c}$$

$$\kappa = \kappa' - \frac{4\Lambda}{3}\,n^2, \tag{73d}$$

where $a, c \neq 0$ only if $\Lambda = 0$. The mass that is transformed is the physical mass: even if it written in terms of other parameters one should identify it and transform it.

The parameters a and n correspond respectively to the angular momentum and to the NUT charge. On the other hand the constant c did not receive any clear interpretation (see for example [7,17,76]). It can be noted that the solution is of type II in Petrov classification (and thus the JN algorithm *can* change the Petrov type) and it corresponds to a wire singularity on the rotation axis. Moreover the BL transformation is not well-defined.

4.2.2. Function Transformation

All the real functions $f_i = f_i(r)$ must be modified to be kept real once $r \in \mathbb{C}$

$$\tilde{f}_i = \tilde{f}_i(r, \bar{r}) = \tilde{f}_i(r', F(\theta)) \in \mathbb{R}. \tag{74}$$

The last equality means that $\tilde{f}_i$ can depend on θ only through $\operatorname{Im} r = F(\theta)$. The condition that one recovers the seed for $\bar{r} = r = r'$ is

$$\tilde{f}_i(r', 0) = f_i(r'). \tag{75}$$

If all magnetic charges are vanishing or in terms without electromagnetic charges the rules for finding the $\tilde{f}_i$ are

$$r \longrightarrow \frac{1}{2}(r + \bar{r}) = \operatorname{Re} r, \tag{76a}$$

$$\frac{1}{r} \longrightarrow \frac{1}{2}\left(\frac{1}{r} + \frac{1}{\bar{r}}\right) = \frac{\operatorname{Re} r}{|r|^2}, \tag{76b}$$

$$r^2 \longrightarrow |r|^2. \tag{76c}$$

Up to quadratic powers of r and r^{-1} these rules determine almost uniquely the result. This is not anymore the case when the configurations involve higher power. These can be dealt with by splitting it in lower powers: generically one should try to factorize the expression into at most quadratic pieces. Some examples of this with natural guesses are

$$r^4 - b^2 = (r^2 + b)(r^2 - b), \qquad r^4 + b = r^2\left(r^2 + \frac{b}{r^2}\right). \tag{77}$$

Moreover the same power of r can be transformed differently, for example

$$\frac{1}{r^n} \longrightarrow \frac{1}{r^{n-2}} \frac{1}{|r|^2}. \tag{78}$$

Denoting by $Q(r)$ and $P(r)$ collectively all functions that multiply q^I and p^I respectively, all such terms should be rewritten as

$$\left(q^I Q(r), p^I P(r)\right) = \left(\operatorname{Re}\left(Z^I Q(r)\right), \operatorname{Im}\left(Z^I P(r)\right)\right) \tag{79}$$

before performing the transformation (70). Note that in this case one does not use the rules (76).

Finally the transformed complex scalars are obtained by simply plugging (70)

$$\tau'^i(r', \theta) = \tau^i\left(r + iF(\theta)\right). \tag{80}$$

4.2.3. Null Coordinates

Plugging the transformation (70) inside the seed metric and gauge fields (68) leads to[10]

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[10] We stress that at this stage these formula do not satisfy Einstein equations, they are just proxies to simplify later computations.

$$ds^2 = -\tilde{f}_t \left(du' + \alpha\, dr' + \omega H\, d\phi \right)^2 + 2\beta\, dr'd\phi + \tilde{f}_\Omega \left(d\theta^2 + \sigma^2 H^2\, d\phi^2 \right), \tag{81a}$$

$$A^I = \tilde{f}^I \left(du' + G'H\, d\phi \right) + p^I H'\, d\phi \tag{81b}$$

where (one should not confuse the primes to indicate derivatives from the primes on the coordinates)

$$\omega = G' + \sqrt{\frac{\tilde{f}_r}{\tilde{f}_t}}\, F', \qquad \sigma^2 = 1 + \frac{\tilde{f}_r}{\tilde{f}_\Omega}\, F'^2, \qquad \alpha = \sqrt{\frac{\tilde{f}_r}{\tilde{f}_t}}, \qquad \beta = \tilde{f}_r\, F'H. \tag{82}$$

4.2.4. Boyer–Lindquist Coordinates

The Boyer–Lindquist transformation

$$du' = dt' - g(r')dr', \qquad d\phi = d\phi' - h(r')dr', \tag{83}$$

can be used to remove the off-diagonal tr and $r\phi$ components of the metric

$$g_{t'r'} = g_{r'\phi'} = 0. \tag{84}$$

The solution to these equations is

$$g(r') = \frac{\sqrt{(\tilde{f}_t \tilde{f}_r)^{-1}}\, \tilde{f}_\Omega - F'G'}{\Delta}, \qquad h(r') = \frac{F'}{H\Delta} \tag{85}$$

where

$$\Delta = \frac{\tilde{f}_\Omega}{\tilde{f}_r}\, \sigma^2 = \frac{\tilde{f}_\Omega}{\tilde{f}_r} + F'^2. \tag{86}$$

Remember that the changes of coordinate is valid only if g and h are functions of r' only. Inserting (85) into (81) yields

$$ds^2 = -\tilde{f}_t \left(dt' + \omega H\, d\phi' \right)^2 + \frac{\tilde{f}_\Omega}{\Delta}\, dr'^2 + \tilde{f}_\Omega \left(d\theta^2 + \sigma^2 H^2\, d\phi'^2 \right), \tag{87a}$$

$$A^I = \tilde{f}^I \left(dt' - \frac{\tilde{f}_\Omega}{\Delta\sqrt{\tilde{f}_t \tilde{f}_r}}\, dr' + G'H\, d\phi' \right) + p^I H'\, d\phi' \tag{87b}$$

where we recall that

$$\omega = G' + \sqrt{\frac{\tilde{f}_r}{\tilde{f}_t}}\, F', \qquad \sigma^2 = 1 + \frac{\tilde{f}_r}{\tilde{f}_\Omega}\, F'^2. \tag{88}$$

Generically one finds $A_r = A_r(r)$ which can be set to zero thanks to a gauge transformation.

Before closing this section we simplify the above formulas for few simple cases that are often used.

Degenerate Schwarzschild Seed

A degenerate seed (one unknown function) in Schwarzschild coordinates has

$$f_r = f_t^{-1}, \qquad f_\Omega = r^2. \tag{89}$$

In this case the above formulas reduced to

$$ds^2 = -\tilde{f}_t \left(dt + \omega H \, d\phi\right)^2 + \frac{\rho^2}{\Delta} \, dr^2 + \rho^2 \left(d\theta^2 + \sigma^2 H^2 \, d\phi^2\right), \tag{90a}$$

$$A = \tilde{f}_A \left(dt - \frac{\rho^2}{\Delta} \, dr + G' H \, d\phi\right). \tag{90b}$$

where

$$\omega = G' + \tilde{f}^{-1} F', \quad \sigma^2 = 1 + \frac{F'^2}{\tilde{f}\rho^2}, \quad \Delta = \tilde{f}\rho^2 \sigma^2. \tag{91}$$

Degenerate Isotropic Seed

A degenerate seed in isotropic coordinates has

$$f_t = f^{-1}, \quad f_r = f, \quad f_\Omega = r^2 f. \tag{92}$$

In this case the above formulas reduced to

$$ds^2 = -\tilde{f}^{-1} \left(dt + \omega H \, d\phi\right)^2 + \tilde{f}\rho^2 \left(\frac{dr^2}{\Delta} + d\theta^2 + \sigma^2 H^2 \, d\phi^2\right), \tag{93a}$$

$$A^I = \tilde{f}^I \left(dt - \frac{\tilde{f}\rho^2}{\Delta} \, dr + G' H \, d\phi\right) + p^I H' \, d\phi \tag{93b}$$

where we recall that

$$\omega = G' + \tilde{f} F', \quad \sigma^2 = 1 + \frac{F'^2}{\rho^2}, \quad \Delta = \tilde{f}\rho^2 + F'^2. \tag{94}$$

Constant *F*

The expressions simplify greatly if $F' = 0$ (for example when $\Lambda \neq 0$). First all functions depend only on r since $F(\theta) = \text{cst}$

$$\tilde{f}_i(r, \theta) = \tilde{f}_i(r, 0). \tag{95}$$

As a consequence the Boyer–Lindquist transformation (85)

$$g(r') = \sqrt{\frac{\tilde{f}_r}{\tilde{f}_t}}, \quad h(r') = 0 \tag{96}$$

is always well-defined.

For the same reason it is always possible to perform a gauge transformation. Finally the metric and gauge fields (87) becomes

$$ds^2 = -\tilde{f}_t \left(dt + G' H \, d\phi\right)^2 + \tilde{f}_r \, dr^2 + \tilde{f}_\Omega \, d\Omega^2, \tag{97a}$$

$$A^I = \tilde{f}^I \left(dt' + G' H \, d\phi'\right) + p^I H' \, d\phi'. \tag{97b}$$

4.3. Open Questions

The algorithm we have described help to work with five (four if $\Lambda \neq 0$) of the six parameters of the Plebański–Demiański (PD) solution. It is tempting to conjecture that it can be extended to the full set of parameters by generalizing the ideas described in Section 3.2 (shifting κ, writing $a + i\alpha$...). Indeed we have found that these operations were quite natural in the context of the PD solution and inspiration could be found in [73].

5. Examples

In this section we list several examples that can be derived from the JN algorithm described in Section 4. Other examples were described previously: Kerr–Newman in Section 2.3.2, dyonic Kerr–Newman and Yang–Mills Kerr–Newman in Section 3.1. For simplicity we will always consider the case $\kappa = 1$ except when $\Lambda \neq 0$.

The first two examples are the Kerr–Newmann–NUT solution and the charged (a)dS–BBMB–NUT solution in conformal gravity. We will also give examples from ungauged $N = 2$ supergravity coupled to $n_v = 0, 1, 3$ vector multiplets (pure supergravity, T^3 model and STU model): this theory is reviewed in Appendix A.

5.1. Kerr–Newman–NUT

The Reissner–Nordström metric and gauge fields are given by

$$ds^2 = -f\,dt^2 + f^{-1}\,dr^2 + r^2 d\Omega^2, \qquad f = 1 - \frac{2m}{r} + \frac{q^2}{r^2}, \tag{98a}$$

$$A = f_A\,dt, \qquad f_A = \frac{q}{r}, \tag{98b}$$

m and q being the mass and the electric charge.

The two functions are complexified as

$$\tilde{f} = 1 - \frac{2\,\mathrm{Re}(m\bar{r}) + q^2}{|r|^2}, \qquad \tilde{f}_A = \frac{q\,\mathrm{Re}\,r}{|r|^2}. \tag{99}$$

Performing the transformation

$$u = u' + \big(a\cos\theta - 2n\ln\sin\theta\big), \qquad r = r' + i\big(n - a\cos\theta\big), \qquad m = m' + in \tag{100}$$

gives (omitting the primes)

$$\tilde{f} = 1 - \frac{2mr + 2n(n - a\cos\theta) - q^2}{\rho^2}, \qquad \rho^2 = r^2 + (n - a\cos\theta)^2. \tag{101}$$

The metric and the gauge fields in BL coordinates are

$$ds^2 = -\tilde{f}\,(dt + \Omega\,d\phi)^2 + \frac{\rho^2}{\Delta}\,dr^2 + \rho^2(d\theta^2 + \sigma^2 H^2 d\phi^2), \tag{102a}$$

$$A = \frac{qr}{\rho^2}\left(dt - (a\sin^2\theta + 2n\cos\theta)d\phi\right) + A_r\,dr \tag{102b}$$

where

$$\Omega = -2n\cos\theta - (1 - \tilde{f}^{-1})\,a\sin^2\theta, \qquad \sigma^2 = \frac{\Delta}{\tilde{f}\rho^2}, \tag{103}$$

$$\Delta = r^2 - 2mr + a^2 + q^2 - n^2.$$

This corresponds to the Kerr–Newman–NUT solution [72].

One can check that A_r is a function of r only

$$A_r = -\frac{qr}{\Delta} \tag{104}$$

and it can be removed by a gauge transformation.

5.2. Charged (a)dS–BBMB–NUT

The action of Einstein–Maxwell theory with cosmological constant conformally coupled to a scalar field is [66]

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left(R - 2\Lambda - \frac{1}{6} R\phi^2 - (\partial\phi)^2 - 2\alpha\phi^4 - F^2 \right), \tag{105}$$

where α is a coupling constant, and we have set $8\pi G = 1$.

For $F, \alpha, \Lambda = 0$, the Bocharova–Bronnikov–Melnikov–Bekenstein (BBMB) solution [77,78] is static and spherically symmetric – it can be seen as the equivalent of the Schwarzschild black hole in conformal gravity.

The general static charged solution with cosmological constant and quartic coupling reads

$$ds^2 = -f \, dt^2 + f^{-1} \, dr^2 + r^2 \, d\Omega^2, \tag{106a}$$

$$A = \frac{q}{r} \, dt, \qquad \phi = \sqrt{-\frac{\Lambda}{6\alpha}} \, \frac{m}{r-m}, \tag{106b}$$

$$f = -\frac{\Lambda}{3} r^2 + \kappa \frac{(r-m)^2}{r^2}, \tag{106c}$$

where the horizon can be spherical or hyperbolic. There is one constraint among the parameters

$$q^2 = \kappa m^2 \left(1 + \frac{\Lambda}{36\alpha} \right) \tag{107}$$

and one has $\alpha\Lambda < 0$ in order for ϕ to be real.

In order to add a NUT charge one performs the JN transformation[11]

$$u = u' - 2n \ln H(\theta), \qquad r = r' + in, \qquad m = m' + in, \qquad \kappa = \kappa' - \frac{4\Lambda}{3} n^2. \tag{108}$$

One obtains the metric (omitting the primes)

$$ds^2 = -\tilde{f} \left(dt - 2nH' \, d\phi \right)^2 + \tilde{f}^{-1} \, dr^2 + (r^2 + n^2) \, d\Omega^2 \tag{109}$$

where the function $\tilde{f}$ is

$$\tilde{f} = -\frac{\Lambda}{3} (r^2 + n^2) + \left(\kappa - \frac{4\Lambda}{3} n^2 \right) \frac{(r-m)^2}{r^2 + n^2}. \tag{110}$$

Note that the term $(r - m)$ is invariant. Similarly one obtains the scalar field

$$\phi = \sqrt{-\frac{\Lambda}{6\alpha}} \, \frac{\sqrt{m^2 + n^2}}{r - m} \tag{111}$$

where the m in the numerator as been complexified as $|m|$.

Finally it is trivial to find the gauge field

$$A = \frac{q}{r^2 + n^2} \left(dt - 2n \cos\theta \, d\phi \right) \tag{112}$$

and the constraint (107) becomes

$$q^2 = \left(\kappa - \frac{4\Lambda}{3} n^2 \right) (m^2 + n^2) \left(1 + \frac{\Lambda}{36\alpha} \right). \tag{113}$$

---

[11] Due to the convention of [66] there is no κ in the transformations.

An interesting point is that the radial coordinate is redefined in [66] when obtaining the stationary solution from the static one.

Note that the BBMB solution and its NUT version are obtained from the limit

$$\Lambda, \alpha \longrightarrow 0, \qquad \text{with} \qquad -\frac{\Lambda}{36\alpha} \longrightarrow 1, \tag{114}$$

which also implies $q = 0$ from the constraint (107). Since no other modifications are needed, the derivation from the JN algorithm also holds in this case.

5.3. Ungauged $N = 2$ BPS Solutions

A BPS solution is a classical solution which preserves a part of the supersymmetry. The BPS equations are obtained by setting to zero the variations of the fermionic partners under a supersymmetric transformation. These equations are first order and under some conditions their solutions also solve the equations of motion.

In [64] (Section 3.1) (see also [79] (Section 2.2) for a summary), Behrndt, Lüst and Sabra obtained the most general stationary BPS solution for $N = 2$ ungauged supergravity. The metric for this class of solutions reads

$$ds^2 = f^{-1}(dt + \Omega \, d\phi)^2 + f \, d\Sigma^2, \tag{115}$$

with the 3-dimensional spatial metric given in spherical or spheroidal coordinates

$$d\Sigma^2 = h_{ij} \, dx^i dx^j = dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \tag{116a}$$

$$= \frac{\rho^2}{r^2 + a^2} \, dr^2 + \rho^2 d\theta^2 + (r^2 + a^2)\sin^2\theta \, d\phi^2, \tag{116b}$$

where i, j, k are flat spatial indices (which should not be confused with the indices of the scalar fields). The functions f and Ω depend on r and θ only.

Then the solution is entirely given in terms of two sets of (real) harmonic functions[12] $\{H^\Lambda, H_\Lambda\}$

$$f = e^{-K} = i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda), \tag{117a}$$

$$\varepsilon_{ijk}\partial_j\Omega_k = 2\,e^{-K}\mathcal{A}_i = (H_\Lambda\partial_i H^\Lambda - H^\Lambda\partial_i H_\Lambda), \tag{117b}$$

$$F^\Lambda_{ij} = \frac{1}{2}\varepsilon_{ijk}\partial_k H^\Lambda, \qquad G_{\Lambda\,ij} = \frac{1}{2}\varepsilon_{ijk}\partial_k H_\Lambda, \tag{117c}$$

$$i(X^\Lambda - \bar{X}^\Lambda) = H^\Lambda, \qquad i(F_\Lambda - \bar{F}_\Lambda) = H_\Lambda. \tag{117d}$$

The object Ω_i is the connection of the line bundle corresponding to the fibration of time over the spatial manifold (its curl is related to the Kähler connection). Its only non-vanishing component is $\Omega_\phi \equiv \Omega = \omega H$.

Starting from the metric (115) in spherical coordinates with $\Omega = 0$, one can use the JN algorithm of Section 4 with

$$f_t = f^{-1}, \qquad f_r = f, \qquad f_\Omega = r^2 f, \tag{118}$$

leading to the formula (93). The function Ω reads

$$\Omega = \omega H = a(1 - \tilde{f})\sin^2\theta + 2n\cos\theta. \tag{119}$$

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12   We omit the tilde that is present in [64] to avoid the confusion with the quantities that are transformed by the JNA. No confusion is possible since the index position will always indicate which function we are using.

Then one needs only to find the complexification of f and to check that it gives the correct ω, as would be found from the Equation (117). However it appears that one cannot complexify directly f since it should be viewed as a composite object made of complex functions. Therefore one needs to complexify first the harmonic functions H_Λ and H^Λ (or equivalently X^Λ), and then to reconstruct the other quantities. Nonetheless, Equation (117) ensure that finding the correct harmonic functions gives a solution, thus it is not necessary to check these equations for all the other quantities.

In the next subsections we provide two examples,[13] one for pure supergravity as an appetizer, and then one with $n_v = 3$ multiplets (STU model).

5.3.1. Pure Supergravity

As a first example we consider pure (or minimal) supergravity, i.e., $n_v = 0$ [64] (Section 4.2). The prepotential reads

$$F = -\frac{i}{4}(X^0)^2. \tag{120}$$

The function H_0 and H^0 are related to the real and imaginary parts of the scalar X^0

$$H_0 = \frac{1}{2}(X^0 + \bar{X}^0) = \mathrm{Re}\, X^0, \qquad \bar{H}^0 = i(X^0 - \bar{X}^0) = -2\,\mathrm{Im}\, X^0, \tag{121}$$

while the Kähler potential is given by

$$f = \mathrm{e}^{-K} = X^0 \bar{X}^0. \tag{122}$$

The static solution corresponds to

$$H_0 = X^0 = 1 + \frac{m}{r} \tag{123}$$

Performing the JN transformation for the angular momentum gives

$$\tilde{X}^0 = 1 + \frac{m(r + ia\cos\theta)}{\rho^2}. \tag{124}$$

This corresponds to the second solution of which is stationary with

$$\omega = \frac{m(2r + m)}{\rho^2}\, a \sin^2\theta. \tag{125}$$

Alternatively one can use the JN algorithm to add a NUT charge. In this case using the rule

$$r \longrightarrow \frac{1}{2}(r + \bar{r}) = \mathrm{Re}\, r = r' \tag{126}$$

must be use for transforming f and r^2 (in front of $d\Omega$), leading to

$$X^0 = 1 + \frac{m + in}{r}. \tag{127}$$

Note that it gives

$$\tilde{f} = \left(1 + \frac{m}{r}\right)^2 + \frac{n^2}{r^2}. \tag{128}$$

---

[13] They correspond to singular solutions, but we are not concerned with regularity here.

It is slightly puzzling that the above rule should be used instead of the two others in (76). One possible explanation is the following: in the seed solution shift the radial coordinate such that $r = R - m$ and apply the JN transformation in this coordinate system. It is clear that every function of r is left unchanged while the tensor structure transforms identically since $dr = dR$. After the transformation one can undo the coordinate transformation. As we mentioned earlier the algorithm is very sensible to the coordinate system and to the parametrization (but it is still not clear why the R-coordinate is the natural one). This kind of difficulty will reappear in the SWIP solution (Section 5.5).

5.3.2. STU Model

We now consider the STU model $n_v = 3$ with prepotential [64] (Section 3)

$$F = -\frac{X^1 X^2 X^3}{X^0}. \tag{129}$$

The expressions for the Kähler potential and the scalar fields in terms of the harmonic functions are complicated and will not be needed (see [64] (Section 3) for the expressions). Various choices for the functions will give different solutions.

A class of static black hole-like solutions are given by the harmonic functions [64] (Section 4.4)

$$H_0 = h_0 + \frac{q_0}{r}, \qquad H^i = h^i + \frac{p^i}{r}, \qquad H^0 = H_i = 0. \tag{130}$$

These solutions carry three magnetic p^i and one electric q_0 charges.

Let us form the complex harmonic functions

$$\mathcal{H}_0 = H_0 + i\, H^0, \qquad \mathcal{H}_i = H^i + i\, H_i. \tag{131}$$

Then the rule for complex function leads to

$$\mathcal{H}_0 = h_0 + \frac{q_0(r + ia\cos\theta)}{\rho^2}, \qquad \mathcal{H}_i = h^i + \frac{p^i(r + ia\cos\theta)}{\rho^2}, \tag{132}$$

for which the various harmonic functions read explicitly

$$H_0 = h_0 + \frac{q_0 r}{\rho^2}, \qquad H^i = h^i + \frac{p^i r}{\rho^2}, \qquad H^0 = \frac{q_0 a\cos\theta}{\rho^2}, \qquad H_i = \frac{p^i a\cos\theta}{\rho^2}. \tag{133}$$

This set of functions corresponds to the stationary solution of [64] (Section 4.4) where the magnetic and electric dipole momenta are not independent parameters but obtained from the magnetic and electric charges instead.

5.4. Non-Extremal Rotating Solution in T^3 Model

The T^3 model under consideration corresponds to Einstein–Maxwell gravity coupled to an axion σ and a dilaton ϕ (with specific coupling constants) and the action is given by (147) with $M = 1$. This model can be embedded in $N = 2$ ungauged supergravity with $n_v = 1$, equal gauge fields $A \equiv A^0 = A^1$ and prepotential[14]

$$F = -i\, X^0 X^1, \tag{134}$$

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[14]　This model can be obtained from the STU model by setting the sections pairwise equal $X^2 = X^0$ and $X^3 = X^1$ [80]. It is also a truncation of pure $N = 4$ supergravity.

The dilaton and the axion corresponds to the complex scalar field

$$\tau = e^{-2\phi} + i\,\sigma. \tag{135}$$

Sen derived the rotating black hole for this theory using the fact that it can be embedded in heterotic string theory [62].

The static metric, gauge field and the complex field read respectively

$$ds^2 = -\frac{f_1}{f_2}\,dt^2 + f_2\left(f_1^{-1}\,dr^2 + r^2\,d\Omega^2\right), \tag{136a}$$

$$A = \frac{f_A}{f_2}\,dt, \tag{136b}$$

$$\tau = e^{-2\phi} = f_2 \tag{136c}$$

where

$$f_1 = 1 - \frac{r_1}{r}, \qquad f_2 = 1 + \frac{r_2}{r}, \qquad f_A = \frac{q}{r}. \tag{137}$$

The radii r_1 and r_2 are related to the mass m and the charge q by

$$r_1 + r_2 = 2m, \qquad r_2 = \frac{q^2}{m}. \tag{138}$$

Applying the Janis–Newman algorithm with rotation, the two functions f_1 and f_2 are complexified as

$$\tilde{f}_1 = 1 - \frac{r_1 r}{\rho^2}, \qquad \tilde{f}_2 = 1 + \frac{r_2 r}{\rho^2}. \tag{139}$$

The final metric in BL coordinates is given by

$$ds^2 = -\frac{\tilde{f}_1}{\tilde{f}_2}\left[dt - a\left(1 - \frac{\tilde{f}_2}{\tilde{f}_1}\right)\sin^2\theta\,d\phi\right]^2 + \tilde{f}_2\left(\frac{\rho^2 dr^2}{\Delta} + \rho^2 d\theta^2 + \frac{\Delta}{\tilde{f}_1}\sin^2\theta\,d\phi^2\right) \tag{140}$$

for which the BL functions are

$$g(r) = \frac{\hat{\Delta}}{\Delta}, \qquad h(r) = \frac{a}{\Delta} \tag{141}$$

with

$$\Delta = \tilde{f}_1\rho^2 + a^2\sin^2\theta, \qquad \hat{\Delta} = \tilde{f}_2\rho^2 + a^2\sin^2\theta. \tag{142}$$

Once f_A has been complexified as

$$\tilde{f}_A = \frac{qr}{\rho^2} \tag{143}$$

the transformation of the gauge field is straightforward

$$A = \frac{\tilde{f}_A}{\tilde{f}_2}\left(dt - a\sin^2\theta\,d\phi\right) - \frac{qr}{\Delta}\,dr. \tag{144}$$

The A_r depending solely on r can again be removed thanks to a gauge transformation.

Finally the scalar field is complex and is transformed as

$$\tau = 1 + \frac{r_2\bar{r}}{\rho^2}. \tag{145}$$

The explicit values for the dilaton and axion are then

$$e^{-2\phi} = \tilde{f}_2, \qquad \sigma = \frac{r_2 a\cos\theta}{\rho^2}. \tag{146}$$

This reproduces Sen's solution and it completes the computation from [25] which could not derive the gauge field nor the axion. It is interesting to note that for another value of the dilaton coupling we cannot use the transformation [50,52].[15]

5.5. SWIP Solutions

Let us consider the action [65,81]

$$S = \frac{1}{16\pi} \int \mathrm{d}^4x \sqrt{|g|} \left(R - 2(\partial\phi)^2 - \frac{1}{2}\,\mathrm{e}^{4\phi}\,(\partial\sigma)^2 - \mathrm{e}^{-2\phi} F^i_{\mu\nu} F^{i\mu\nu} + \sigma\,F^i_{\mu\nu} \tilde{F}^{i\mu\nu} \right) \tag{147}$$

where $i = 1, \ldots, M$. When $M = 2$ and $M = 6$ this action corresponds respectively to $N = 2$ supergravity with one vector multiplet and to $N = 4$ pure supergravity, but we keep M arbitrary. The axion σ and the dilaton ϕ are naturally paired into a complex scalar

$$\tau = \sigma + i\,\mathrm{e}^{-2\phi}. \tag{148}$$

In order to avoid redundancy we first provide the general metric with $a, n \neq 0$, and we explain how to find it from the restricted case $a = n = 0$. The stationary Israel–Wilson–Perjés (SWIP) solutions correspond to

$$\mathrm{d}s^2 = -\,\mathrm{e}^{2U} W(\mathrm{d}t + A_\phi\,\mathrm{d}\phi)^2 + \mathrm{e}^{-2U} W^{-1}\mathrm{d}\Sigma^2, \tag{149a}$$

$$A^i_t = 2\,\mathrm{e}^{2U}\,\mathrm{Re}(k^i H_2), \qquad \tilde{A}^i_t = 2\,\mathrm{e}^{2U}\,\mathrm{Re}(k^i H_1), \qquad \tau = \frac{H_1}{H_2}, \tag{149b}$$

$$A_\phi = 2n\cos\theta - a\sin^2\theta\,(\,\mathrm{e}^{-2U} W^{-1} - 1), \tag{149c}$$

$$\mathrm{e}^{-2U} = 2\,\mathrm{Im}(H_1\bar{H}_2), \qquad W = 1 - \frac{r_0^2}{\rho^2}. \tag{149d}$$

This solution is entirely determined by the two harmonic functions

$$H_1 = \frac{1}{\sqrt{2}}\,\mathrm{e}^{\phi_0}\left(\tau_0 + \frac{\tau_0\mathcal{M} + \bar{\tau}_0 \mathrm{Y}}{r - ia\cos\theta} \right), \qquad H_2 = \frac{1}{\sqrt{2}}\,\mathrm{e}^{\phi_0}\left(1 + \frac{\mathcal{M} + \mathrm{Y}}{r - ia\cos\theta} \right). \tag{150}$$

The spatial 3-dimensional metric $\mathrm{d}\Sigma^2$ reads

$$\mathrm{d}\Sigma^2 = h_{ij}\,\mathrm{d}x^i\mathrm{d}x^j = \frac{\rho^2 - r_0^2}{r^2 + a^2 - r_0^2}\,\mathrm{d}r^2 + (\rho^2 - r_0^2)\mathrm{d}\theta^2 + (r^2 + a^2 - r_0^2)\sin^2\theta\,\mathrm{d}\phi^2. \tag{151}$$

Finally, r_0 corresponds to

$$r_0^2 = |\mathcal{M}|^2 + |\mathrm{Y}|^2 - \sum_i \left|\Gamma^i\right|^2 \tag{152}$$

where the complex parameters are

$$\mathcal{M} = m + in, \qquad \Gamma^i = q^i + ip^i, \tag{153}$$

m being the mass, n the NUT charge, q^i the electric charges and p^i the magnetic charges, while the axion–dilaton charge Y takes the form

$$\mathrm{Y} = -\frac{1}{2}\sum_i \frac{(\bar{\Gamma}^i)^2}{\mathcal{M}}. \tag{154}$$

---

[15] The authors of [51] report incorrectly that [50] is excluding all dilatonic solutions.

The latter together with the asymptotic values τ_0 are defined by

$$\tau \sim \tau_0 - i\,e^{-2\phi_0}\frac{2Y}{r}. \tag{155}$$

The complex constant k^i are determined by

$$k^i = -\frac{1}{\sqrt{2}}\frac{\mathcal{M}\Gamma^i + \bar{Y}\bar{\Gamma}^i}{|\mathcal{M}|^2 - |Y|^2}. \tag{156}$$

As discussed in the previous section, the transformation of scalar fields is different depending on one is turning on a NUT charge or an angular momentum. For this reason, starting from the case $a = n = 0$, one needs to perform the two successive transformations

$$u = u' - 2in\ln\sin\theta, \qquad r = r' + in, \qquad m = m' + in, \tag{157a}$$
$$u = u' + ia\cos\theta, \qquad r = r' - ia\cos\theta, \tag{157b}$$

the order being irrelevant (for definiteness we choose to add the NUT charge first), the reason being that the transformations of the functions are different in both cases (as in Section 5.3.1). The group properties of the JN algorithm ensure that the metric will be transformed as if only one transformation was performed. Then the metric and the gauge fields are directly obtained, which ensures that the general form of the solution (149) is correct. For that one needs to shift r^2 by r_0^2 in order to bring the metric (151) to the form (116). This modifies the function but one does not need this fact to obtain the general form. Then one can shift by $-r_0^2$ before dealing with the complexification of the functions. See [65] (p. 17) and Section 5.3.1 for discussions about the changes of coordinates. Since all the functions and the parameters depend only on $\mathcal{M}$, H_1 and H_2, it is sufficient to explain their complexification.

The function W is transformed as a real function. On the other hand H_1 and H_2 are complex harmonic functions and should be transformed accordingly. For the NUT charge one should use the rule

$$r \longrightarrow \mathrm{Re}\,r. \tag{158}$$

Then one can perform the second transformation (157b) in order to add the angular momentum by applying the usual rules (76). On can see that it yields the correct result.

Finally let us note that it seems possible to also start from $p^i = 0$ and to turn them on using the transformation

$$q^i = q'^i = q^i + ip^i, \tag{159}$$

using different rules for complexifying the various terms (depending whether one is dealing with a real or a complex function/parameter).

5.6. Gauged N = 2 Non-Extremal Solution

The simplest deformation of $N = 2$ supergravity with n_v vector multiplets consists in the so-called Fayet–Iliopoulos (FI) gauging. It amounts to gauging $(n_v + 1)$ times the diagonal U(1) group of the SU(2) part of the R-symmetry group (automorphism of the supersymmetry algebra). The potential can be entirely written in terms of the quantities defined in Appendix A and of the $(n_v + 1)$ coupling constants g_I, where $I = 0, \ldots, n_v$.

We consider the model with prepotential (see also Section 5.4)

$$F = -i\,X^0 X^1. \tag{160}$$

for which the potential generated by the FI gauging is

$$V(\tau, \bar{\tau}) = -\frac{4}{\tau + \bar{\tau}} \left(g_0^2 + g_0 g_1 (\tau + \bar{\tau}) + g_1^2 |\tau|^2 \right). \tag{161}$$

The goal of this section is to derive the NUT charged black hole from [69] using the JN algorithm.[16] The seed solution is taken to be Equation (4.22) from [69] with $j = N = 0$

$$f_t = \kappa - \frac{2mr - 2\ell^2 \sum_I g_I |Z^I|^2}{f_\Omega} + \frac{f_\Omega}{\ell^2}, \tag{162a}$$

$$f_\Omega = r^2 - \Delta^2 - \delta^2, \tag{162b}$$

$$f^I = \frac{(r - \Delta)Q^I - \delta\, P^I}{f_\Omega}, \tag{162c}$$

$$\tau = \frac{g_0}{g_1} \frac{r + \Delta - i\delta}{r - \Delta + i\delta}. \tag{162d}$$

where the following quantities have been defined

$$m = \frac{\ell^2 P^0}{\Delta} \frac{g_1^2 \left[-(P^1)^2 P^0 + (Q^1)^2 P^0 - 2Q^0 Q^1 P^1 \right] + g_0^2 P^0 |Z^0|^2}{|Z^0|^2}, \tag{163a}$$

$$\delta = -\Delta \frac{Q^0}{P^0}. \tag{163b}$$

The independent parameters are given by Q^I (electric charges), P^I (magnetic charges), g_Λ (FI gaugings), Δ (scalar charge) and $\Lambda = -3/\ell^2$ (the cosmological constant).

In order to perform the complexification the functions are first rewritten as

$$f_t = \kappa - \frac{2\,\mathrm{Re}(m\bar{r}) - 2\ell^2 \sum_I g_I |Z^I|^2}{f_\Omega} + \frac{f_\Omega}{\ell^2}, \tag{164a}$$

$$f_\Omega = |r|^2 - \Delta^2 - \delta^2 = |r|^2 - \frac{\Delta^2 |Z^1|^2}{\mathrm{Im}(Z^1)^2}, \tag{164b}$$

$$f^I = \frac{\mathrm{Re}(Q^I \bar{r})\,\mathrm{Im}\,Z^1 - \Delta\,\mathrm{Im}(Z^I Z^1)}{\mathrm{Im}\,Z^1\, f_\Omega}, \tag{164c}$$

$$\tau = \frac{g_0}{g_1} \frac{\bar{r} + \Delta - i\delta}{\bar{r} - \Delta + i\delta}. \tag{164d}$$

Applying the transformations (70) with (73a) gives (omitting the primes)

$$\tilde{f}_t = \kappa + \frac{4n^2}{\ell^2} - \frac{2mr + 2\left(\kappa + 4n^2/\ell^2\right)n^2 - 2\ell^2 \sum_I g_I |Z^I|^2}{\tilde{f}_\Omega} + \frac{\tilde{f}_\Omega}{\ell^2}, \tag{165a}$$

$$\tilde{f}_\Omega = r^2 + n^2 - \Delta^2 - \delta^2, \tag{165b}$$

$$\tilde{f}^I = \frac{(Q^I r + P^I n)\,\mathrm{Im}\,Z^1 - \Delta\,\mathrm{Im}(Z^I Z^1)}{\mathrm{Im}\,Z^1\, \tilde{f}_\Omega}, \tag{165c}$$

$$\tilde{\tau} = \frac{g_0}{g_1} \frac{r + \Delta - i(\delta + n)}{r - \Delta + i(\delta - n)}. \tag{165d}$$

The last step is to simplify these expressions

$$\tilde{f}_t = \kappa + \frac{4n^2}{\ell^2} - \frac{2mr + 2\kappa n^2 + 8n^4/\ell^2 - 2\ell^2 \sum_I g_I |Z^I|^2}{\tilde{f}_\Omega} + \frac{\tilde{f}_\Omega}{\ell^2}, \tag{166a}$$

$$\tilde{f}_\Omega = r^2 + n^2 - \Delta^2 - \delta^2, \tag{166b}$$

$$\tilde{f}^I = \frac{Q^I(r - \Delta) + P^I(n - \delta)}{\tilde{f}_\Omega}, \tag{166c}$$

$$\tilde{\tau} = \frac{g_0}{g_1} \frac{r + \Delta - i(\delta + n)}{r - \Delta + i(\delta - n)}. \tag{166d}$$

It is straightforward to check that the form of the metric and gauge fields are correctly reproduced by the algorithm given in Section 4 for the tensor structure. In total this reproduces the Equation (4.22) and formulas below in [69] with $j = 0$.

An important thing that we learn here is that the mass parameter needs to be transformed as if it was not composed of other parameters.

Appendix A. Review of $N = 2$ Ungauged Supergravity

In order for this review to be self-contained we recall the basic elements of $N = 2$ supergravity without hypermultiplets – we refer the reader to the standard references for more details [82–84].

The gravity multiplet contains the metric and the graviphoton

$$\{g_{\mu\nu}, A^0\} \tag{A1}$$

while each of the vector multiplets contains a gauge field and a complex scalar field

$$\{A^i, \tau^i\}, \qquad i = 1, \dots, n_v. \tag{A2}$$

The scalar fields τ^i (the conjugate fields $(\tau^i)^*$ are denoted by $\bar{\tau}^{\bar{i}}$) parametrize a special Kähler manifold with metric $g_{i\bar{j}}$. This manifold is uniquely determined by an holomorphic function called the prepotential F. The latter is better defined using the homogeneous (or projective) coordinates X^Λ such that

$$\tau^i = \frac{X^i}{X^0}. \tag{A3}$$

The first derivative of the prepotential with respect to X^Λ is denoted by

$$F_\Lambda = \frac{\partial F}{\partial X^\Lambda}. \tag{A4}$$

Finally it makes sense to regroup the gauge fields into one single vector

$$A^\Lambda = (A^0, A^i). \tag{A5}$$

One needs to introduce two more quantities, respectively the Kähler potential and the Kähler connection

$$K = -\ln i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda), \qquad \mathcal{A}_\mu = -\frac{i}{2}(\partial_i K \, \partial_\mu \tau^i - \partial_{\bar{i}} K \, \partial_\mu \bar{\tau}^{\bar{i}}). \tag{A6}$$

The Lagrangian for the theory without gauge group is given by

$$\mathcal{L} = -\frac{R}{2} + g_{i\bar{j}}(\tau, \bar{\tau})\, \partial_\mu \tau^i \partial^\nu \bar{\tau}^{\bar{i}} + \mathcal{I}_{\Lambda\Sigma}(\tau, \bar{\tau})\, F^\Lambda_{\mu\nu} F^{\Sigma\,\mu\nu} - \mathcal{R}_{\Lambda\Sigma}(\tau, \bar{\tau})\, F^\Lambda_{\mu\nu} \star F^{\Sigma\,\mu\nu} \tag{A7}$$

where R is the Ricci scalar and $\star F^\Lambda$ is the Hodge dual of F^Λ. The matrix

$$\mathcal{N} = \mathcal{R} + i\mathcal{I} \tag{A8}$$

can be expressed in terms of F. From this Lagrangian one can introduce the symplectic dual of F^Λ

$$G_\Lambda = \frac{\delta\mathcal{L}}{\delta F^\Lambda} = \mathcal{R}_{\Lambda\Sigma} F^\Sigma - \mathcal{I}_{\Lambda\Sigma} \star F^\Sigma. \tag{A9}$$

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