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Review

Monopole-Based Scenarios of Confinement and Deconfinement in 3D and 4D

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Abstract: This review discusses confinement, as well as the topological and critical phenomena, in the gauge theories which provide the condensation of magnetic monopoles. These theories include the 3D SU(N) Georgi-Glashow model, the 4D $[\mathrm{U}(1)]^{N-1}$-invariant compact QED , and the $[\mathrm{U}(1)]^{N-1}$-invariant dual Abelian Higgs model. After a general introduction to the string models of confinement, an analytic description of this penomenon is provided at the example of the 3D SU(N) Georgi-Glashow model, with a special emphasis placed on the so-called Casimir scaling of k-string tensions in that model. We further discuss the string representation of the 3D $[\mathrm{U}(1)]^{N-1}$-invariant compact QED, as well as of its 4D generalization with the inclusion of the Θ-term. We compare topological effects, which appear in the latter case, with those that take place in the 3D QED extended by the Chern-Simons term. We further discuss the string representation of the 't Hooft-loop average in the $[\mathrm{U}(1)]^{N-1}$-invariant dual Abelian Higgs model extended by the Θ-term, along with the topological effects caused by this term. These topological effects are compared with those occurring in the 3D dual Abelian Higgs model (i.e., the dual Landau-Ginzburg theory) extended by the Chern-Simons term. In the second part of the review, we discuss critical properties of the weakly-coupled 3D confining theories. These theories include the 3D compact QED, along with its fermionic extension, and the 3D Georgi-Glashow model.

Keywords: magnetic monopoles; Abelian-type models of confinement; string representation of the Wilson- and the 't Hooft-loop averages; Aharonov-Bohm and other topological effects; Casimir scaling of k-string tensions; critical properties of the 3D weakly coupled confining theories

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1. Topological Effects in the Abelian-Type Confining Theories

1.1. Introduction

In this review, we discuss various non-perturbative phenomena that take place in the Abelian-type confining gauge theories. We start this discussion with recalling some basic facts about confinement, the large-distance static quark-antiquark potential associated with it, and the related models of the confining string. As is well known, because of confinement in QCD, quarks and gluons do not exist as individual particles, but appear only in the form of bound states (for recent reviews, see [1–3]). The latter include mesons, baryons, glueballs, and the so-called hybrids consisting of a quark, an antiquark, and one or several gluons. Confining interactions that take place between the constituents of the bound states, can occur through string-like Euclidean configurations of the Yang-Mills field. Such effective strings can be viewed as the microscopic tubes that carry fluxes of the gauge field from one constituent to another, which is the reason for calling them "the QCD flux tubes" [4–7]. Similar flux tubes, called Abrikosov vortices [8,9] (for a relativistic generalization, see [10]), exist in type-II superconductors, in which case they represent stable cylindrically-symmetric solutions to the classical equations of motion. This observation inspired 't Hooft and Mandelstam [11–13] to put forward their famous scenario of confinement as a dual superconductor. To describe this scenario, we recall that the vacuum of the usual superconductor contains electron-electron Cooper pairs, whose condensation is modeled by an electrically charged Abelian Higgs field. Once two external monopoles of the opposite magnetic charge are immersed into the superconductor, they get confined through an Abrikosov vortex extending between them. Close to the center of the vortex, namely inside its so-called core, the magnetic field created by the monopole-antimonopole pair, partially destroys the condensate of Cooper pairs. Accordingly, the confinement scenario of Refs. [11–13] suggests the QCD vacuum to be of the type of a *dual* superconductor, which can be characterized by a magnetically-charged dual-Higgs condensate [1]. The insertion into such a magnetically charged medium of a static pair of the mutually opposite electric charges leads to the formation between them of a dual Abrikosov vortex, which represents a tube of the electric flux. In the case of the group SU(N), the corresponding dual Abelian Higgs model allows one to describe confinement of particles which are charged with respect to the maximal Abelian $[\mathrm{U}(1)]^{N-1}$-subgroup of SU(N) [2]. The dedicated lattice simulations [2,3,6,7,16–18] indicate that the transverse-distance dependence of the chromo-electric field in the QCD flux tube is indeed very similar to that of the magnetic field in Abrikosov vortices, which is known from the theory of type-II superconductors [8–10].

In reality, however, static sources of the (chromo-)electric field do not exist, and even heavy quarks are always dynamical. The dynamics of the quark-antiquark bound states can be described in terms of the gauge-invariant amplitudes of the vacuum-to-vacuum transition. For illustrative purposes, let us disregard quark spin degrees of freedom, and take into account only quark electromagnetic interactions. The corresponding Euclidean Lagrangian is that of a complex-valued scalar field coupled to an Abelian gauge field, namely $\mathcal{L} = \frac{1}{4}F_{\mu\nu}^2 + |D_\mu\varphi|^2$. Here, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the strength tensor of the gauge field, and $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative, with e being the electric charge. We consider further the simplest amplitude, which describes the propagation of the corresponding Coulomb bound state from the point x to the point y. Such an amplitude, given by the Green function of this bound state, has the form

$$G(x,y) \equiv \int \mathcal{D}A_\mu \, \mathrm{e}^{-\frac{1}{4}\int d^4x\, F_{\mu\nu}^2} \langle x|\, \frac{1}{D_\mu^2}\, |y\rangle \langle y|\, \frac{1}{D_\mu^2}\, |x\rangle \,. \tag{1}$$

With the use of the world-line representation for $\langle x|\, \frac{1}{D_\mu^2}\, |y\rangle$ (see e.g., [19]), Equation (1) can further be explicitly written as the following integral over trajectories $z_\mu(\tau)$ and $\bar{z}_\mu(\tau)$ of the quark and the antiquark [20,21]:

$$G(x,y) = \int\limits_0^\infty ds \int\limits_0^\infty d\bar{s} \int\limits_{z(0)=x}^{z(s)=y} \mathcal{D}z_\mu \int\limits_{\bar{z}(0)=y}^{\bar{z}(\bar{s})=x} \mathcal{D}\bar{z}_\mu \, \mathrm{e}^{-\frac{1}{4}\int_0^s d\tau\, \dot{z}_\mu^2 - \frac{1}{4}\int_0^{\bar{s}} d\tau\, \dot{\bar{z}}_\mu^2} \left\langle \exp\left[ie\left(\int\limits_x^y dz_\mu\, A_\mu(z) + \int\limits_y^x d\bar{z}_\mu\, A_\mu(\bar{z}) \right)\right]\right\rangle . \tag{2}$$

Here the average over the gauge field is defined as

$$\langle \cdots \rangle \equiv \int \mathcal{D}A_\mu \, \mathrm{e}^{-\frac{1}{4}\int d^4x\, F_{\mu\nu}^2} (\cdots), \tag{3}$$

and the dot denotes the derivative with respect to the proper time τ. As we see, the gauge field enters Equation (2) only through the exponential $\exp[\cdots]$. This exponential, representing the phase factor taken along a closed contour C, which is formed by the trajectories $z_\mu(\tau)$ and $\bar{z}_\mu(\tau)$, is called the Wilson loop [22]. One can parameterize the entire contour C by some vector-function $x_\mu(\tau)$, and consider the corresponding Wilson-loop average $\langle W(C)\rangle = \left\langle \exp\left[ie \oint_C dx_\mu\, A_\mu(x)\right]\right\rangle$. In the case of only

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1 Note that the dedicated lattice simulations [14,15] have confirmed this scenario of confinement with a very high accuracy.

2 Note, however, that, unlike the Abelian Higgs model, the Yang-Mills theory does not possess any string-like solutions to the classical equations of motion. That is, no indications exist that, in the absence of external static sources of the gauge field, the Yang-Mills vacuum can contain stable classical string-like field configurations.

electromagnetic interactions at issue, the average (3) is Gaussian, and the Wilson-loop average can be readily calculated. The corresponding leading result has the form [23,24]

$$\langle W(C)\rangle \simeq \mathrm{e}^{-\frac{e^2}{8\pi}\cdot\frac{L(C)}{a}}, \tag{4}$$

where $L(C)$ is the length of the contour C, and a is the inverse ultraviolet cut-off. The latter appears due to the fact that the leading contribution to the Wilson-loop average in electrodynamics stems from the two-point interactions mediated by the photon propagator, being therefore ultraviolet-divergent. The exponential fall-off of $\langle W(C)\rangle$ with $L(C)$, given by Equation (4), is called the perimeter law.

Let us further mention another important example of a gauge-invariant vacuum-to-vacuum transition amplitude. To this end, we integrate over the fields φ and φ^* in the partition function

$$\mathcal{Z} = \int \mathcal{D}A_\mu\, \mathcal{D}\varphi\, \mathcal{D}\varphi^*\, \mathrm{e}^{-\int d^4x\, \mathcal{L}}.$$

This integration yields $\mathcal{Z} = \langle \exp[-\mathrm{tr}\,\ln(-D_\mu^2)]\rangle$, where "tr" in the Abelian case at issue stands only for the functional trace over the space-time coordinates. Retaining in the cumulant expansion of this mean value only the first term, one calculates the partition function in the one-loop approximation. This approximation accounts for an infinite set of diagrams containing one loop of the φ-field and a certain number of external lines of the A_μ-field. The corresponding expression for the partition function reads $\mathcal{Z} \simeq \exp(-\langle\Gamma[A_\mu]\rangle)$, where $\Gamma[A_\mu]$ is the one-loop effective action, which can be written as

$$\Gamma[A_\mu] = -\int_0^\infty \frac{ds}{s}\,\langle x|\,\mathrm{e}^{sD_\mu^2}\,|x\rangle = -\Omega\int_0^\infty \frac{ds}{s} \oint\limits_{x_\mu(0)=x_\mu(s)} \mathcal{D}x_\mu\, \mathrm{e}^{-\frac{1}{4}\int_0^s d\tau\, \dot{x}_\mu^2}\, W(C). \tag{5}$$

Here Ω is the four-dimensional volume occupied by the system, and we have disregarded an inessential additive A_μ-independent constant. Thus, the free-energy density of the φ-field, given in the one-loop approximation by $\langle\Gamma[A_\mu]\rangle$, is completely expressed in this approximation through the Wilson-loop average.

Instead of the perimeter law (4) for the Wilson-loop average, which holds for both small and large contours in non-confining gauge theories, in confining theories the so-called area law holds for sufficiently large contours [22]. As follows from its name, the area law corresponds to an exponential fall-off of the Wilson-loop average with the area Σ_{min} of the minimal surface bounded by the contour C, namely $\langle W(C)\rangle \to \mathrm{e}^{-\sigma\Sigma_{\mathrm{min}}}$ for $\Sigma_{\mathrm{min}} \gtrsim \frac{1}{\sigma}$, where the coefficient σ has the dimensionality of $(\mathrm{mass})^2$. This coefficient is called the string tension, since it represents the energy-per-unit-length of the above-discussed confining string, which is formed between a quark and an antiquark at their separations $\gtrsim \frac{1}{\sqrt{\sigma}}$. In general, the string tension depends on the representation of the gauge group under which the confined particles transform. This dependence will be discussed in Section 1.3 below. In QCD, for quarks transforming under the fundamental representation of the SU(3) group, the numerical value of the string tension, $\sigma \simeq (440\,\mathrm{MeV})^2$, can be obtained from the Regge phenomenology. Notice also that in QCD, as well as in other non-Abelian gauge theories, the vector-potential is matrix-valued, so that the exponential in the definition of the Wilson loop should be path-ordered and traced. That is,

$$W(C) = \frac{1}{N}\,\mathrm{tr}\,\mathcal{P}\,\exp\left[ig\oint_C dx_\mu\, A_\mu(x)\right], \tag{6}$$

where "tr" stands for the trace over color indices, $\mathcal{P}$ denotes the path ordering, g is the gauge coupling, $A_\mu \equiv T^a A_\mu^a$ with T^a being a generator of a given representation of SU(N) under which the quark transforms, and $a = 1, \ldots, N^2 - 1$.

For a flat contour C, whose extension T in the 4-th direction is much larger than its spatial extension R, the Wilson-loop average can be used for a derivation of the static quark-antiquark potential $V(R)$ through the formula

$$V(R) = -\frac{1}{T}\ln\langle W(C)\rangle\Big|_{T\to\infty}. \tag{7}$$

Since $\Sigma_{\rm min} = RT$ for such contours, one has $V(R) = \sigma R$. That is, in the Yang-Mills and other gauge theories where the Wilson-loop average exhibits the area law, the static potential at large distances is the linear confining one [3]. Regardless of the shape of the contour C, this contour unambiguously defines the corresponding minimal surface $S_{\rm min}$. This surface, being the world sheet of the maximally stretched string, should appear as a saddle point in the representation of the Wilson-loop average in the form of a functional-integral sum over all surfaces bounded by the contour C. For the case of $\Sigma_{\rm min} \gtrsim \frac{1}{\sigma}$ at issue, such a sum can be formally written as

$$\langle W(C)\rangle \simeq {\rm e}^{-\sigma\Sigma_{\rm min}} = \sum_S {\rm e}^{-\mathcal{A}[S]}. \tag{8}$$

Here $\Sigma_{\rm min}$ is the area of $S_{\rm min}$, and $\mathcal{A}$ is some action associated with the surface S. Thus, the problem of string representation of a certain confining gauge theory implies the derivation from that theory of both the action $\mathcal{A}$ and the measure in the functional-integral sum $\sum\limits_S$.

Clearly, Equation (8) resembles the known representation of the partition function of a point particle in terms of the integral over all possible trajectories of that particle [cf. Equation (5)]. Within this analogy, $S_{\rm min}$ corresponds to the classical trajectory of a particle, while all other surfaces S correspond to quantum trajectories. However, while the measure in the sum over paths of a particle is known [4], the measure in the functional-integral sum $\sum\limits_S$ is unknown. Fortunately, at least the string action $\mathcal{A}[S_{\rm min}]$ can be explicitly derived in the certain limits of Abelian gauge theories with confinement, such as the 3D Georgi-Glashow model (26) or the 4D dual Abelian Higgs model (75) [5]. These models, along with the corresponding string representations, will be discussed in full detail in Sections 1.3 and 1.6 below. The resulting action $\mathcal{A}[S_{\rm min}]$ turns out to have a non-local form of the interaction of two infinitesimal world-sheet elements $d\sigma_{\mu\nu}$, which is mediated by the propagator D_m of a dual vector boson of mass m, namely

$$\mathcal{A}[S_{\rm min}] \propto m^2 \int_{S_{\rm min}} d\sigma_{\mu\nu}(x) \int_{S_{\rm min}} d\sigma_{\mu\nu}(x')\, D_m(x-x'). \tag{9}$$

The way in which this mass is generated depends on a particular confining gauge theory. For instance, in the case of the 3D Georgi-Glashow model, a non-vanishing value of the mass m is provided by the Debye screening of the dual vector boson in the monopole-antimonopole plasma, while in the dual Abelian Higgs model a non-vanishing mass appears owing to the Higgs mechanism.

In the physically interesting case of the Yang-Mills theory, a non-local action of the form (9) appears within the Gaussian approximation to the so-called Stochastic Vacuum Model [26–28] (for reviews, see [29–31]). There, $m^2 D_m(x-x')$ becomes replaced by a certain function $D(x-x')$, which is regular at $x = x'$. This function turns out to be proportional to the Green function of a thought bound state, called a 2-gluon gluelump, which is formed by two gluons together with a static source of the gauge field

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3 In full QCD, the quark-antiquark string breaks at a certain distance by converting its energy into a production of a meson constituting of a light quark and the corresponding antiquark. These light quark and antiquark further recombine with the initial static quark-antiquark pair to form two heavy-light mesons. This leads to the screening of the color charge of the static quarks and to the flattening of the linearly rising static potential at large distances. Accordingly, the Wilson-loop average associated with a static quark-antiquark pair, ceases to be an order parameter for confinement in full QCD.

4 Namely [25], $\mathcal{D}x_\mu = \lim\limits_{n\to\infty}\frac{1}{(4\pi\varepsilon)^{D(n+1)/2}}\prod\limits_{k=1}^{n}\int d^D x^{(k)}$, where $\varepsilon \equiv \frac{s}{n}$, and D is the dimensionality of the Euclidean space-time.

5 In this review, we do not discuss string representation of supersymmetric gauge theories, which is based on the so-called AdS/CFT-correspondence. For a recent review of this topic see Ref. [1].

transforming under the adjoint representation of the group SU(N). The distance at which this Green function exponentially falls off, defines the correlation length λ of confining stochastic background Yang-Mills fields (cf. Refs. [32–35]). Such fields, whose typical momenta are smaller than $1/\lambda$, lead to the formation of the confining string which sweeps out the surface $S_{\rm min}$. Accordingly, one can expect that gluons with momenta larger than $1/\lambda$ can lead to the formation of the so-called gluon chain [4,36,37], in which several such gluons are interconnected by strings. Since these high-momentum gluons possess their own degrees of freedom, they can produce various types of excitations of the gluon chain, which would quantify the functional-integral sum $\sum\limits_{S}$ in Equation (8). Nevertheless, the dynamics of such a many-body bound state of massless relativistic particles, with linear interactions between the nearest neighbours, appears quite complicated (cf. Ref. [38]), hindering the construction of an explicit analytic formula for the functional-integral sum $\sum\limits_{S}$. Still, the approximate result which can be considered reliable in the Yang-Mills theory, is the above-mentioned phenomenological action of the form (9), which describes the string that sweeps out the minimal surface $S_{\rm min}$. Thus, given the exponential fall-off of both $D_m(x)$ and $D(x)$ at large $|x|$, one can say that, when proceeding from confining Abelian gauge theories to the Yang-Mills theory, the mass of a dual vector boson in the string action essentially becomes substituted by the mass of a 2-gluon gluelump.

1.2. The Large-Distance Static Quark-Antiquark Potential and the Models of the Confining String

As discussed above, the static potential $V(R)$ resulting from the area law for the Wilson-loop average $\langle W(C)\rangle$ rises linearly at large distances. Indeed, applying Equation (7) to the case where the contour C has the form of a rectangle, $C = R\times T$, one obtains $V(R) = \sigma R$. Thus, for quark-antiquark separations $R \gtrsim \frac{1}{\sqrt{\sigma}}$, the leading term in the static potential is the linear one. This term represents the free energy of a straight-line confining string of length R. In reality, however, the string is a dynamical object, so that the straight-line string can only appear in the semi-classical approximation to the functional-integral sum (8) over string world sheets. Depending on the dynamics of the confining string, which is defined primarily by the action $\mathcal{A}[S]$, one can obtain various corrections to the linear potential.

In general, the length of the confining string significantly exceeds its thickness. Therefore, in the leading approximation, the effects produced by the string thickness can be disregarded altogether. This leads to the so-called Nambu-Goto string action [39–41], $\mathcal{A}[S] = \sigma\Sigma$, where Σ is the area of the surface S. Explicitly, this action has the form

$$\mathcal{A}[S] = \sigma \int d^2\xi \sqrt{\det g_{ab}}. \tag{10}$$

Here $g_{ab} = \partial_a x_\mu \cdot \partial_b x_\mu$ is the tensor of the induced metric corresponding to the vector-function $x_\mu = x_\mu(\vec{\xi})$ which parameterizes the string world sheet S. Henceforth, the indices a and b take the values 1 and 2, while $\mu = 1,\ldots,D$, where D is the dimensionality of the embedding Euclidean space-time, and $\vec{\xi} = (\xi_1, \xi_2)$.

In order to obtain the leading correction to the linear potential, which is produced by small fluctuations of the Nambu-Goto string about the flat surface lying in the $(1,4)$-plane, one parameterizes the string world sheet by the vector-function $x_\mu = (x_1, x_i, x_4)$, where $x_1 = R\xi_1$, $x_4 = T\xi_2$, with $\xi_1 \in [0,1]$, $\xi_2 \in [0,1]$. Since fluctuations of the string occur in the directions perpendicular to the $(1,4)$-plane, they are described by the components $x_i = (x_2, x_3)$ of the vector-function x_μ. Using the explicit form of the components of the induced-metric tensor, $g_{11} = R^2 + (\partial_1 x_i)^2$, $g_{22} = T^2 + (\partial_2 x_i)^2$, $g_{12} = g_{21} = \partial_1 x_i \cdot \partial_2 x_i$, one has $\det g_{ab} = (TR)^2\left[1 + \left(\frac{\partial x_i}{\partial x_1}\right)^2 + \left(\frac{\partial x_i}{\partial x_4}\right)^2\right]$, where the $O(|x_i|^4)$-terms have been disregarded. This yields for the Nambu-Goto action, $\mathcal{A}[S] = \sigma\int_0^1 d\xi_1 \int_0^1 d\xi_2 \sqrt{\det g_{ab}}$, the following expression: $\mathcal{A}[S] \simeq \sigma RT + \frac{\sigma}{2}\int_0^R dx_1 \int_0^T dx_4 \left[\left(\frac{\partial x_i}{\partial x_1}\right)^2 + \left(\frac{\partial x_i}{\partial x_4}\right)^2\right]$.

Accordingly, in the D-dimensional Euclidean space, where the index i acquires $D-2$ values, the Wilson-loop average (8) has the form

$$\langle W(C)\rangle = \int \mathcal{D}x_i \, \mathrm{e}^{-\mathcal{A}[S]} \simeq \mathrm{e}^{-\sigma RT - \frac{D-2}{2}\cdot \mathrm{tr}\, \ln\left[-(\partial/\partial x_1)^2-(\partial/\partial x_4)^2\right]}.$$

Using further the representation of the logarithm in the form $\ln x = -\frac{\partial x^{-\beta}}{\partial\beta}\Big|_{\beta\to 0}$, and sending T to infinity, one has

$$\mathrm{tr}\,\ln\left[-(\partial/\partial x_1)^2 - (\partial/\partial x_4)^2\right] = -T\frac{\partial}{\partial\beta}\sum_{n=1}^{\infty}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\frac{1}{\left[\omega^2+\left(\frac{\pi n}{R}\right)^2\right]^\beta}\Bigg|_{\beta\to 0}.$$

The ω-integration in this formula can further be carried out by exponentiating the denominator, which yields [42]

$$\mathrm{tr}\,\ln\left[-(\partial/\partial x_1)^2 - (\partial/\partial x_4)^2\right] = \frac{\pi T}{R}\,\zeta(-1) = -\frac{\pi T}{12R}.$$

Here, the value $\zeta(-1) = -\frac{1}{12}$ of the Riemann ζ-function, $\zeta(z) = \sum\limits_{n=1}^{\infty}\frac{1}{n^z}$, obtained via the analytic continuation, has been used. Thus, integrating over string fluctuations x_i's, and regularizing the so-emerging functional determinant via the ζ-function, one obtains the following static potential:

$$V(R) \simeq \sigma R - \frac{\pi(D-2)}{24R}. \tag{11}$$

The obtained correction to the linear potential, $-\frac{\pi(D-2)}{24R}$, is called the Lüscher term [43]. Being produced by the fluctuating confining string, this term is developed at the distances $R \gtrsim \frac{1}{\sqrt{\sigma}}$. This feature of the Lüscher term distinguishes it from the Coulomb term in the static quark-antiquark potential,

$$V_{\mathrm{Coul}}(R) = -\frac{g^2 C_r}{4\pi R}, \tag{12}$$

which is also $\propto \frac{1}{R}$. In fact, the Coulomb term dominates the full quark-antiquark potential at the distances $R \lesssim \frac{1}{\sqrt{\sigma}}$, whereas the Lüscher term is only a correction to the linear potential at the distances $R \gtrsim \frac{1}{\sqrt{\sigma}}$. In Equation (12), g stands for the Yang-Mills coupling, and C_r is the quadratic Casimir operator of the representation r of the group SU(N) under which the quark and the antiquark are transformed. The proportionality of the Coulomb potential to the quadratic Casimir operator, $V(R) \propto C_r$, means that the Coulomb quark-antiquark potential respects the so-called Casimir scaling [44]. Clearly, the Lüscher term does not respect this scaling, which is one more feature distinguishing it from the Coulomb potential in QCD. Furthermore, unlike the Coulomb potential, the Lüscher term is coupling-independent altogether, i.e., "universal". Yet another distinguishing feature of the Lüscher term is that it depends on the space-time dimensionality only via the factor $(D-2)$, being otherwise $\propto \frac{1}{R}$ in any number of dimensions. On the contrary, the R-dependence of the Coulomb potential changes with the Euclidean space-time dimensionality D in a non-trivial way, namely as $V_{\mathrm{Coul}}(R) \propto \frac{1}{R^{D-2}}$ for $D > 2$.

Furthermore, owing to the same fluctuations of the confining string that yield the Lüscher term, the thickness of the string increases (albeit only logarithmically) with its length R. Specifically, at $R \gg \frac{1}{\sqrt{\sigma}}$, the following expression holds for the mean squared transverse size of the string [43]: $\langle x_\perp^2\rangle \propto \alpha'\cdot\ln\frac{R^2}{\alpha'}$, where $\alpha' \equiv \frac{1}{2\pi\sigma}$ is the slope of linear hadronic Regge trajectories. This slope appears in the asymptotic Regge behavior of the high-energy scattering amplitudes, $A \to \mathrm{e}^{\alpha' t\ln(s/t)}$, at $s \gg t$, where s and t are the Mandelstam variables. The Regge behavior with a linear trajectory $\alpha(t) = \alpha' t$ of zero intercept, $\alpha(0) = 0$, is associated with the classical string. Once fluctuations of the string are taken into account, a non-vanishing Reggeon intercept appears. Since these fluctuations are the same ones as those which yield the Lüscher term, the intercept turns out to be related to the coefficient of the

Lüscher term as [45–47] $\alpha(0) = \frac{D-2}{24}$. Although this result has been rigorously obtained in the large-D limit, there exist convincing arguments [47] that it can be valid for $D \sim 1$ as well. Note also that the bosonic string can be consistently quantized only in $D = 26$ dimensions [48] [6], in which case $\alpha(0) = 1$, while the above-quoted result can be written as $\alpha(0) = 1 + \frac{D-26}{24}$.

Owing to the negative sign of the Lüscher term, the potential (11) respects the inequalities [49]

$$\frac{dV}{dR} > 0 \quad \text{and} \quad \frac{d^2V}{dR^2} \leq 0,$$

which should be respected by any confining potential [7]. The same is true for the full static potential produced by the Nambu-Goto string, which can be obtained in the limit of $D \gg 1$ [42] [8]. Such a full potential has the form

$$V(R) = \sigma\sqrt{R^2 - R_c^2}, \tag{13}$$

where $R_c = \sqrt{\frac{\pi(D-2)}{12\sigma}}$ is the minimum quark-antiquark separation for which it is still legitimate to consider small fluctuations of the string. Clearly, the large-R limit of the above-quoted potential (13) recovers both the linear quark-antiquark potential and the Lüscher term, namely $V(R) = \sigma R - \frac{\pi(D-2)}{24R} + \mathcal{O}\left(\frac{1}{\sigma R^3}\right)$.

Equation (13) can be represented in the equivalent form $V(R) = \sigma(R)R$, with the effective R-dependent string tension

$$\sigma(R) \equiv \sigma\sqrt{1 - (R_c/R)^2}. \tag{14}$$

By virtue of this formula, one can obtain the critical behavior of the string tension at temperatures close to the deconfinement one [50]. Indeed, the static potential at finite temperature T is related to the connected two-point correlation function of the so-called Polyakov loops as

$$\langle P^\dagger(\vec{\mathrm{R}})P(\vec{0})\rangle_{\mathrm{conn}} = \mathrm{e}^{-\beta V(R)}. \tag{15}$$

Here β is the inverse temperature, and $P(\vec{\mathrm{R}}) \equiv \frac{1}{N}\,\mathrm{tr}\,\mathcal{T}\exp\left[ig\int_0^\beta dx_4\,A_4(\vec{\mathrm{R}}, x_4)\right]$ is the Polyakov loop [51] [9]. In this formula, $\mathcal{T}$ stands for time ordering, and we have assumed for concreteness that quarks are transformed under the fundamental representation of the group SU(N), so that $A_4 \equiv A_4^a T^a$, where T^a is a generator of that representation. Considering the correlation function (15) in the limit of $R \gg \beta$, and comparing it with the zero-temperature formula (7), we observe that the Euclidean time T at zero temperature corresponds to the quark-antiquark separation R at finite temperatures, while R at zero temperature corresponds to β. Accordingly, the Wilson-loop average $\langle W(C)\rangle \simeq \mathrm{e}^{-\sigma(R)R\cdot T}$ at zero temperature corresponds to the two-point correlation function of Polyakov loops $\langle P^\dagger(\vec{\mathrm{R}})P(\vec{0})\rangle_{\mathrm{conn}} \simeq \mathrm{e}^{-\sigma(\beta)\beta\cdot R}$ at finite temperatures, and $\sigma(\beta)$ is given by Equation (14) with R replaced by β. Thus, the Nambu-Goto model of the confining string yields the following critical behavior of the string tension:

$$\sigma(\mathrm{T}) \to \sigma\sqrt{1 - (\mathrm{T}/\mathrm{T}_c)} \;\; \text{at} \;\; \mathrm{T} \to \mathrm{T}_c, \tag{16}$$

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[6] Cf. also the discussion around Equation (21) below.

[7] These inequalities are, however, not respected by the potential $V(R) \propto R^{1+\alpha}$ with $\alpha > 0$. In particular, they are not respected by the harmonic-oscillator potential, which means that this potential cannot describe confinement.

[8] Note, however, that strings can unlikely provide confinement of point particles in the spaces of dimensionality $D > 4$. This issue is discussed in more detail at the end of Section 1.5 below.

[9] This operator is also called the Polyakov or the Wilson line.

where

$$T_c = \sqrt{\frac{3\sigma}{\pi(D-2)}} \tag{17}$$

is the critical temperature of the deconfinement phase transition. As we see, the model at issue leads to the deconfinement phase transition owing to the negative sign of the Lüscher term, i.e., owing to the fact that the force exerted by this term on the quark and the antiquark, is attractive.

Notice also the factor of $\sqrt{3}$ in T_c, instead of $\sqrt{12}$, where the latter would correspond to the naive use of the above expression for $1/R_c$. The value of this coefficient is determined by the fact that the confining string is in general described by a two-dimensional conformal field theory, which is characterized by the so-called conformal-anomaly number c (also called the central charge). In particular, for the bosonic string at issue, one has $c = 1$. In terms of such a conformal field theory, the Lüscher term, $-\frac{c(D-2)\pi}{24R}$, represents the zero-point energy of the corresponding two-dimensional system of the spatial extension R, which is subject to the Dirichlet boundary conditions. Rather, the correlation function (15) yields the zero-point energy of the system which is confined in a long cylinder of circumference β. This zero-point energy reads [52,53] $-\frac{c(D-2)\pi}{6}T$. Accordingly, while the Lüscher term stems from the large-R expansion of the effective string tension (14), $\sigma(R) = \sigma - \frac{c(D-2)\pi}{24}R^{-2} + O(R^{-4})$, an analogous large-$\beta$ expansion of the finite-temperature string tension has the form $\sigma(T) = \sigma - \frac{c(D-2)\pi}{6}T^2 + O(T^4)$.

Notice further that, on the purely theoretical grounds, one cannot exclude the possibility for the confining string to be described by some fermionic extension of the bosonic Nambu-Goto string theory. In such a case, massless fermionic modes propagating over the string world sheet, change the central charge, so that its value becomes $1/4$ for the fermionic string and $3/2$ for the so-called Neveu-Schwarz string [54–58]. Since in both cases the central charge remains positive-definite, the Lüscher term in these string theories has the same negative sign as in the Nambu-Goto case, i.e., the presence of fermionic string modes does not affect the existence of the deconfinement phase transition. This is, however, no longer the case for the so-called supersymmetric string, for which the central charge is equal to zero (cf. Ref. [54–58]). Anyway, regardless of these theoretical possibilities, the coefficient of the Lüscher term obtained in the lattice measurements [59] corresponds to the purely bosonic case. We notice that the accuracy of these lattice measurements is high enough as to safely exclude all possible fermionic extensions of the Nambu-Goto string from the list of potential candidates of the confining string in QCD.

The square-root fall-off (16) of $\sigma(T)$ at $T \to T_c$ means that the critical index ν characterizing the corresponding deconfinement phase transition, is equal to $1/2$, i.e., the phase transition in the Nambu-Goto model of the confining string is second-order and of the mean-field universality class. Clearly, this result does not depend on the number of colors N, which contradicts the so-called Svetitsky-Yaffe conjecture [60]. According to that conjecture, the deconfinement phase transition in the D-dimensional SU(N) Yang-Mills theory should be of the same universality class as the deconfinement phase transition in the $(D-1)$-dimensional N-state Potts model. The conjecture is based on the observation [51] that the deconfinement phase transition in the SU(N) Yang-Mills theory corresponds to the spontaneous breaking of the center-subgroup symmetry of SU(N). The center subgroup, which consists of those elements of the group SU(N) that commute with all the elements, is the same discrete Z_N group as the one that characterizes the N-state Potts model. Hence, according to the Svetitsky-Yaffe conjecture, the deconfinement phase transition in the four-dimensional SU(2) Yang-Mills theory should be second order, with the universality class of the three-dimensional Ising model, which corresponds to $\nu \simeq 0.63$ (cf. Ref. [61]). For $N = 3$, one has the so-called weak first-order phase transition, in which case it is still possible to formally attribute to the critical exponent ν the value of $\frac{1}{D-1} = \frac{1}{3}$ (cf. Refs. [62,63]). For $N > 3$, the phase transition is first order, so that it can no longer be characterized by the critical exponents. Thus, only for $N = 2$ is the deconfinement phase transition in the SU(N) Yang-Mills theory second order. Even in that case, the value of $\nu \simeq 0.63$ corresponding to the universality class of the three-dimensional Ising model, exceeds the above-obtained value of $\nu = 1/2$, which follows from the

Nambu-Goto string model and corresponds to the mean-field universality class. One can also compare these values of ν with the value of $\nu = 1$, which can be obtained within the deconfinement scenario based on the condensation of long closed strings [51]. In this scenario, the linear fall-off of $\sigma(\mathrm{T})$ at $\mathrm{T} \to \mathrm{T}_c$ comes out as a mere consequence of the formula $\sigma(\mathrm{T}) = \sigma - \frac{\mathrm{T}S}{L}$, since the entropy S of a closed string is proportional to its length L. Indeed, the entropy S in this case is given by the logarithm of the number of possibilities to realize on the lattice a closed trajectory of length L. In particular, for a hypercubic lattice of spacing h, it reads $S = \frac{L}{h} \cdot \ln(2D-1)$. Thus, the free energy of a closed string, $F = \sigma L - \mathrm{T}S$, vanishes linearly at $\mathrm{T} \to \mathrm{T}_c$, where T_c is given by the formula $\mathrm{T}_c = \frac{\sigma h}{\ln(2D-1)}$. Since the value of $\nu = 1$ implies the universality class of the *two-dimensional* Ising model, the corresponding phase transition cannot take place in the 4D Yang-Mills theory.

The static potential (13) can be identified with the ground-state energy E_0 in the representation of the Wilson-loop average as a partition function of the Nambu-Goto string. That is, one represents $\langle W(C)\rangle$ as the following sum over the string states of definite energies: $\langle W(C)\rangle = \sum\limits_{n=0}^{\infty} w_n \,\mathrm{e}^{-E_n T}$. The contour C here has a rectangular shape, with the temporal extension T being much larger than the spatial extension R. Furthermore, the "eigenenergies" E_n's are R-dependent functions, while the coefficients w_n's are just integers. The canonical quantization of the Nambu-Goto string with fixed ends yields then the following energy spectrum [64]:

$$E_n(R) = \sigma R\,\sqrt{1 + \frac{2\pi}{\sigma R^2}\left(n - \frac{D-2}{24}\right)} = \sigma R + \frac{\pi}{R}\left(n - \frac{D-2}{24}\right) + O\left(\frac{1}{\sigma R^3}\right). \tag{18}$$

The coefficients w_n's, which account for level multiplicities, read [65–69]: $w_0 = 1$, $w_1 = D-2$, $w_2 = \frac{1}{2}(D-2)(D+1)$, etc. In the particular case of $D = 3$, w_n is just the number of partitions of n, so that [70]

$$w_n \simeq \frac{1}{4\sqrt{3}n}\,\mathrm{e}^{\pi\sqrt{2n/3}} \tag{19}$$

for $n \gg 1$. Together with Equation (18), this formula yields the following lower bound for the temporal extension of the contour C: $T_{\rm min} = \sqrt{\frac{\pi}{3\sigma}}$. We note that, at finite temperatures, where the Euclidean time becomes periodic with the period β, this expression leads to an upper bound for the temperature: $\mathrm{T}_{\rm max} = \frac{1}{\beta_{\rm min}} = \sqrt{\frac{3\sigma}{\pi}}$. Remarkably, this expression coincides with Equation (17) at $D = 3$, which was obtained without recourse to the asymptotic formula (19).

The Nambu-Goto string action (10) is semiclassically equivalent to the so-called Polyakov string action [48]

$$\mathcal{A}_{\rm P} = \frac{\sigma}{2}\int d^2\xi\,\sqrt{\gamma}\gamma^{ab}g_{ab},$$

where γ_{ab} is an auxiliary metric, and $\gamma \equiv \det\gamma_{ab}$ [10]. Owing to this equivalence, one can perform the string quantization by integrating over x_μ, which yields the string partition function in the form of a functional integral over γ_{ab}. Furthermore, since the Polyakov action is invariant under the reparametrizations of the surface, a certain gauge in the group of reparametrizations should be fixed, which yields an additional integration over the ghost fields. It is convenient to use the so-called conformal gauge, in which the metric γ_{ab} is diagonal, namely $\gamma_{ab} = \sqrt{\gamma}\delta_{ab}$. In this gauge, one has

$$\mathcal{A}_{\rm P} = \frac{\sigma}{2}\int d^2\xi\,(\partial_a x_\mu)^2, \tag{20}$$

[10] This equivalence can be proved by using the formula $\delta\gamma = -\gamma\cdot\gamma_{ab}\cdot\delta\gamma^{ab}$, which yields $\delta\mathcal{A}_{\rm P} = \frac{\sigma}{2}\int d^2\xi\,\sqrt{\gamma}\cdot\delta\gamma^{ab}\cdot T_{ab}$, where $T_{ab} = g_{ab} - \frac{1}{2}g_{cd}\gamma^{cd}\gamma_{ab}$ is the energy-momentum tensor. The corresponding classical equation of motion, $T_{ab} = 0$, defines the stationary point in the functional integral over auxiliary metrics. This equation has a solution $\gamma_{ab} = g_{ab}$, for which $\mathcal{A}_{\rm P}$ is indeed equal to the Nambu-Goto action (10).

which represents the theory of a free massless bosonic field x_μ. Integrating over x_μ, one arrives at the so-called Liouville theory:

$$\int \mathcal{D}x_\mu \, \mathrm{e}^{-\mathcal{A}_\mathrm{P}} = \mathrm{e}^{\frac{D}{48\pi}\int d^2\xi \left[\frac{1}{2}(\partial_a\varphi)^2+\mu^2\mathrm{e}^\varphi\right]},$$

where $\varphi = \frac{1}{2}\ln\gamma$ and $\mu^2 = \frac{48\pi\sigma}{26-D}$. The subsequent integration over the ghosts yields for the corresponding Faddeev-Popov determinant a parametrically similar result, namely $\mathrm{e}^{-\frac{26}{48\pi}\int d^2\xi\left[\frac{1}{2}(\partial_a\varphi)^2+\mu^2\mathrm{e}^\varphi\right]}$. Combining these two expressions together, one obtains the sought string partition function in the form [48]

$$\mathcal{Z} = \int \mathcal{D}\varphi \, \mathrm{e}^{-\frac{26-D}{48\pi}\int d^2\xi\left[\frac{1}{2}(\partial_a\varphi)^2+\mu^2\mathrm{e}^\varphi\right]}. \tag{21}$$

This result clearly indicates that the conformal anomaly cancels only for $D = 26$, which means that the bosonic string can be selfconsistently quantized only in 26 dimensions. The existence of such a unique critical value for the space-time dimensionality makes the Nambu-Goto model of the bosonic string radically different from the field-theoretical models of point particles, which can be quantized in the space-time of any dimensionality where their renormalizability is provided. Accordingly, when used as a model of the hadronic string, the Nambu-Goto string can at most be treated semiclassically, as it was done above in the present Section, but not at the fully quantum level.

An attempt to make the Nambu-Goto string quantizable in the physically important case of $D = 4$ can be based on the observation that the partition function (21) can be equivalently rewritten as

$$\mathcal{Z} = \int \mathcal{D}\varphi \, \mathrm{e}^{-\sigma\int d^2\xi\,\sqrt{\gamma}-\frac{26-D}{96\pi}\int d^2\xi d^2\xi'\,\sqrt{\gamma}R\left(-\frac{1}{\Delta}\right)_{\xi,\xi'}\sqrt{\gamma'}R'}.$$

Here, $\left(\frac{1}{\Delta}\right)_{\xi,\xi'}$ is the Green function of the Laplacian $\Delta = \frac{1}{\sqrt{\gamma}}\partial_a\sqrt{\gamma}\gamma^{ab}\partial_b$, $R = -\mathrm{e}^{-\varphi}\partial^2\varphi$ is the conformal-gauge expression for the scalar curvature of the world sheet, and $\gamma' = \gamma(\vec{\xi}')$, $R' = R(\vec{\xi}')$. Given the semiclassical equality $\gamma_{ab} = g_{ab}$ (cf. Footnote 10), one can say that, starting from an extension of the Nambu-Goto model by the non-local term [11] $\kappa\int d^2\xi d^2\xi'\,\sqrt{\gamma}R\left(-\frac{1}{\Delta}\right)_{\xi,\xi'}\sqrt{\gamma'}R'$, one can hope to have the situation with $\kappa = \frac{4-26}{96\pi}$, which would make such a string model quantizable at $D = 4$. Clearly, the mentioned non-local term can naturally emerge from the integration over some scalar field $\Phi(\vec{\xi})$ coupled to the world sheet of the Nambu-Goto string as $\int d^2\xi\,\Phi\,\sqrt{g}R[g_{ab}]$. However, in the Yang-Mills theory of interest, a possible origin of such a scalar field "living" on the string world sheet, is unknown.

The Polyakov string action suggests a yet another model of the deconfinement phase transition [72]. It is based on the idea that, due to the compactification of the Euclidean time at finite temperature, the confining string should also be compactified on the corresponding cylinder of the radius $R = (2\pi\mathrm{T})^{-1}$. Upon such a compactification, the action (20) takes the form $\mathcal{A}_P = \frac{\sigma R^2}{2}\int d^2\xi(\partial_a\phi)^2$, where $\phi \in [0, 2\pi)$. Thus, one arrives at the 2D XY model, which contains vortices that exist in the molecular phase at low temperatures and in the plasma phase at high temperatures. The two phases are separated from each other by the Kosterlitz-Thouless phase transition, which takes place at the temperature $\mathrm{T}_c = \frac{1}{2}\sqrt{\frac{\sigma}{2\pi}}$. One can compare this critical temperature with the critical temperature (17) of the Hagedorn phase transition, which occurs due to the exponentially growing number of string states of a given mass. As has been noticed in [72], the two critical temperatures become equal to each other precisely at $D = 26$. In spite of this remarkable coincidence, such a model based on the compactified bosonic string, is unlikely to be a realistic model of the deconfinement phase transition in the 4D Yang-Mills theory. Indeed, the Kosterlitz-Thouless phase transition predicted by this model is of infinite order (i.e., an arbitrary-order temperature derivative of the free energy is continuous across

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[11] This term can be viewed as a non-trivial 2D theory of gravity (cf. Footnote 26 below). Once written in the quasi-local form $-\kappa\int d^2\xi(\partial_a\ln\sqrt{\gamma})^2$, it is sometimes called the Polchinski-Strominger term [71].

the critical temperature), while the deconfinement phase transition in the Yang-Mills theory is either first or second order, as discussed above.

Accounting for the thickness of the string, one obtains corrections to the Nambu-Goto action. Of those, the leading one is the so-called rigidity term, which corresponds to the bending energy of a rigid stick [73]. Its action has the form

$$\mathcal{A}_{\rm rid} = \frac{1}{2\alpha}\int d^2\xi\,\sqrt{g}\,(\Delta x_\mu)^2, \tag{22}$$

where $g \equiv \det g_{ab}$ is the determinant of the induced-metric tensor, $\Delta = \frac{1}{\sqrt{g}}\partial_a\sqrt{g}g^{ab}\partial_b$ is the Laplacian associated with the induced metric, and α is a dimensionless coupling constant. The static potential stemming from the full string action,

$$\mathcal{A} = \mathcal{A}_{\rm P} + \mathcal{A}_{\rm rid}, \tag{23}$$

recovers Equation (13) in the limit of $\alpha \to \infty$, while going over to the strictly linear potential $V(R) = \sigma\cdot(R - R_c)$ in the opposite limit of $\alpha \to 0$, where the rigidity of the string fully suppresses its fluctuations [74–76].

Furthermore, the action (23) can be used to determine the scale dependence of the coupling α. To this end, following the usual renormalization-group procedure, one splits $x_\mu(\vec{\xi})$ into a low-energy part and a high-energy fluctuation, and integrates over such fluctuations in the Gaussian approximation. This yields for α the following one-loop expression [48]:

$$\alpha(p) = \frac{\alpha_0}{1 - \frac{\alpha_0 D}{4\pi}\ln\frac{\Lambda}{p}}, \tag{24}$$

where $\alpha_0 = \alpha(\Lambda)$, and $p \equiv |\vec{p}|$. Thus, the running coupling $\alpha(p)$ in the theory (23) is asymptotically free, which makes this theory similar to the two-dimensional O(N) sigma-model. The dimensional transmutation, which takes place because of the asymptotic freedom, leads to the appearance of a mass parameter $m = \Lambda\,{\rm e}^{-c/\alpha}$, where Λ is an ultra-violet cut-off, and c is some positive dimensionless constant. Accordingly, it is natural to expect that two unit vectors orthogonal to the string world sheet get correlated at the distances $\sim \frac{1}{m}$, i.e., the correlation length in the theory (23) is as small as $\frac{1}{\Lambda}$. This is the essence of the so-called problem of crumpling of the string world sheet. Had the running coupling $\alpha(p)$ possessed an infra-red stable fixed point α_* such that $\alpha_* < c$, the problem of crumpling at α's close to α_* would be solved, since the correlation length, estimated as $\frac{1}{m} = \frac{1}{\Lambda}\,{\rm e}^{c/\alpha_*}$, could be exponentially larger than $\frac{1}{\Lambda}$. However, no indications of possible existence of an infra-red stable fixed point for the rigid-string running coupling have been found [77,78], which forces one to seek alternative solutions to the problem of crumpling [12].

One such possibility can be based on the so-called string θ-term [48], which can be introduced in the physically relevant case of $D = 4$, and reads θn, where

$$n = \frac{1}{2\pi}\int d^2\xi\,\sqrt{g}\,g^{ab}(\partial_a t_{\mu\nu})(\partial_b \tilde{t}_{\mu\nu}) \tag{25}$$

is the number of self-intersections of the string world sheet. In Equation (25), $t_{\mu\nu} = \varepsilon^{ab}(\partial_a x_\mu)(\partial_b x_\nu)/\sqrt{g}$ is the so-called extrinsic-curvature tensor of the world sheet, and a tilde denotes the dual tensor. Thus, for the case of $\theta = \pi$, the contribution of the θ-term to the string partition function, given by the factor ${\rm e}^{i\theta n}$, becomes equal to $(-1)^n$. Accordingly, the sum of contributions to the partition function which are produced by some two world sheets with nearly the same string actions, but with n's differing

[12] Such a non-trivial infra-red fixed point exists though in the extension of the rigid-string theory by a Wess-Zumino term $\propto {\rm tr}\,(\Omega^{-1}{\rm d}\Omega)^3$, where the elements of matrix Ω are pairwise products of tangent and/or normal vectors to the world sheet [79].

from each other by 1, vanishes. Thus, the string θ-term at $\theta = \pi$ can provide a mechanism of mutual cancellations among contributions to the string partition function which are produced by highly crumpled world sheets.

Of course, it looks desirable not just to add the rigidity- and the θ-term to the Nambu-Goto action, but to derive them from an underlying confining gauge theory. It turns out that, for a variety of confining theories, the resulting string action has a non-local form of Equation (9). Furthermore, the subsequent derivative expansion of this non-local action leads to the appearance of the rigidity term with a negative coupling α [13]. An advantageous feature of the negative sign of the rigidity coupling is that it allows one to consistently define the correlation length between the unit vectors orthogonal to the string world sheet. To illustrate this, one represents $x_\mu(\vec{\xi})$ as a sum of a low-energy part, which yields a surface with a constant induced metric g_{ab}, and a fluctuation $y_\mu(\vec{\xi})$ about this surface. The correlation length between the unit vectors orthogonal to the world sheet with the constant induced metric, is defined then by the infra-red behavior of the correlation function $\langle y_\mu(\vec{\xi})y_\nu(\vec{0})\rangle$. To obtain this correlation function, one can use the conformal gauge, $g_{ab} = \sqrt{g}\delta_{ab}$, in which the action (22) yields the following $O(y_\mu^2)$-term: $\frac{1}{2\alpha}\int d^2\xi \frac{1}{\sqrt{g}}(\partial^2 y_\mu)^2$. Then, owing to the fact that $g = \text{const}$, one obtains from the action (23) in the case of $\alpha < 0$ the following correlation function:

$$\langle y_\mu(\vec{\xi})y_\nu(\vec{0})\rangle = \delta_{\mu\nu}\int \frac{d^2p}{(2\pi)^2}\frac{\mathrm{e}^{i\vec{p}\vec{\xi}}}{\frac{1}{\alpha\sqrt{g}}p^4 - \sigma p^2} = \frac{\delta_{\mu\nu}}{2\pi\sigma}\left[K_0(m|\vec{\xi}|) - \ln\frac{L}{|\vec{\xi}|}\right].$$

Here $m^2 = |\alpha|\sqrt{g}\sigma$, and $\vec{\xi}$ has been attributed the dimensionality of [length], so that $L = |\vec{\xi}|_{\max}$ is the inverse infra-red cut-off. Using the known behavior of the Macdonald function $K_0(x)$ at $x \ll 1$ and $x \gg 1$, one can readily see that the correlation function $\langle y_\mu(\vec{\xi})y_\nu(\vec{0})\rangle$ stays finite at $|\vec{\xi}| \to 0$, while the corresponding correlation length, equal to $1/m$, is indeed well defined for $\alpha < 0$ [14]. An analogy with the rigid-string running coupling (24) suggests further that $|\alpha(p)|$ in confining gauge theories can also grow in the infra-red limit. Accordingly, the problem of crumpling of large world sheets in these theories is likely to persist, necessitating for its solution the above-discussed string θ-term, which should be derived within the same theories. This issue will be addressed in detail in Section 1.5 below.

1.3. SU(N) Georgi-Glashow Model: Area Law and k-String Tensions in 3D

The Yang-Mills theory extended by the Higgs field Φ^a, which transforms under the adjoint representation of the gauge group SU(N), is called the Georgi-Glashow model. In the (2+1)-dimensional Euclidean space-time, classical equations of motion in such a model possess a non-perturbative solution, called the 't Hooft-Polyakov monopole [83,84]. In the limit of a sufficiently small electric coupling, monopoles together with antimonopoles form a dilute quantum plasma, which provides the Debye screening of a test magnetically charged particle immersed into it, along with the generation of the Debye mass M_D of the dual-photon field [84]. Accordingly, the appearance of a finite vacuum correlation length $1/M_D$ leads to confinement. We discuss first the generation of the Debye mass in the simplest case of the SU(2) 3D Georgi-Glashow model, proceeding further to the quantitative description of confinement in the general SU(N) case.

The Euclidean action of the SU(2) 3D Georgi-Glashow model has the form

$$S = \int d^3x\left[\frac{1}{4g^2}\left(F^a_{\mu\nu}\right)^2 + \frac{1}{2}\left(D_\mu\Phi^a\right)^2 + \frac{\lambda}{4}\left((\Phi^a)^2 - \eta^2\right)^2\right], \tag{26}$$

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[13] Cf. Refs. [30,80,81].

[14] At the same time, one can show [82] that the one-loop contribution to the zero-point energy of the static quark-antiquark pair is negative for any sign of α. For this reason, the negative sign of α is not associated with the formation of any metastable quark-antiquark state.

where the covariant derivative acts on the Higgs field according to the formula $D_\mu\Phi^a = \partial_\mu\Phi^a + \varepsilon^{abc}A^b_\mu\Phi^c$. The actual reason for which confinement in the model (26) allows for an analytic description is the fact that it holds in the so-called weak-coupling regime of $g \ll \eta$, which will be assumed henceforth. Clearly, this limit parallels the requirement of having sufficiently large η's in order to ensure the spontaneous SU(2)→U(1) symmetry breaking. Expanding then the action (26) around its minimum, which is provided by the Higgs configuration $\Phi^a = \delta^{a3}\eta$, one obtains the perturbative spectrum of the resulting U(1)-invariant model. This spectrum consists of a massless photon A^3_μ, as well as the vector bosons $W^\pm_\mu = (A^1_\mu \mp A^2_\mu)/\sqrt{2}$ with the masses $m = g\eta$, and a scalar particle $\sigma = \Phi^3 - \eta$ with the mass $M = \eta\sqrt{2\lambda}$. While the photon and the σ-particle and neutral with respect to the U(1)-group, the W-bosons are charged with respect to that group, with the corresponding electric charges equal to ±1 (in the units of g).

To obtain the 't Hooft-Polyakov monopole solution to the classical equations of motion, one assumes that the Higgs part of this solution is directed along the 3-rd axis, i.e., $\Phi^a = \delta^{a3}u(r)$, where $r \equiv |\vec{x}|$. The equations of motion then yield $u(0) = 0$ and $u(r) \to \eta - \frac{e^{-Mr}}{gr}$ at $r \to \infty$. For the off-diagonal components of the vector-field part of the monopole solution, the equations of motion yield an exponential fall-off with the distance, namely $A^1_\mu \sim A^2_\mu = \mathcal{O}(e^{-mr})$. Instead, the diagonal component A^3_μ of the monopole solution falls off as $\mathcal{O}(1/r)$, yielding the long-range Coulomb interactions in the monopole-antimonopole plasma. Furthermore, the action of a single monopole has the form

$$S_0 = 4\pi \cdot \frac{m}{g^2} \cdot \varepsilon(M/m). \tag{27}$$

Here $4\pi \cdot \frac{m}{g^2}$ is the classical monopole action, while the correcting function ε is produced by quantum fluctuations in the monopole background. This function turns out to be monotonic and very slowly varying, so that [85,86] $\varepsilon(0) = 1$, while at $M \gg m$ one gets numerically a very close value [87] $\varepsilon(\infty) \simeq 1.8$.

In general, interactions in the monopole-antimonopole ensemble are mediated not only by the vector-field part A^3_μ of the monopole solution but also by the Higgs-field part $u(r)$. In what follows, however, we will be interested in the limit of $M \gg m$, where the interaction mediated by $u(r)$ is exponentially suppressed with respect to the interaction mediated by A^3_μ, so that the action of a configuration consisting of n monopoles and antimonopoles reads

$$S_n \simeq nS_0 + \frac{g_m^2}{8\pi}\sum_{\substack{a,b=1\\a\neq b}}^{n}\frac{q_a q_b}{|\vec{z}_a - \vec{z}_b|}. \tag{28}$$

In Equation (28), q_a's are the monopole charges in the units of the magnetic coupling g_m. The energy of a given configuration of monopoles is proportional to the square of the magnetic-field flux they produce, so that the energy of some $k > 1$ monopoles of a unit charge is lower than the energy of one monopole of charge k. For this reason, all monopoles carrying magnetic charges $k > 1$ dissociate into monopoles of charge 1. Accordingly, when constructing the grand canonical monopole-antimonopole partition function, it suffices to perform the summation over monopoles and antimonopoles with magnetic charges $q_a = \pm 1$, and to disregard all those with $|q_a| > 1$. Hence, the partition function of such a grand canonical ensemble reads [15]

$$\mathcal{Z}_{\rm mon} = 1 + \sum_{n=1}^{\infty}\frac{1}{n!}\left(\alpha \cdot \frac{m^{7/2}}{g}\, e^{-S_0}\right)^n \prod_{a=1}^{n}\int d^3z_a \sum_{q_a=\pm 1} e^{-\frac{g_m^2}{8\pi}\sum\limits_{\substack{a,b=1\\a\neq b}}^{n}\frac{q_a q_b}{|\vec{z}_a-\vec{z}_b|}}. \tag{29}$$

15 Notice also that the superposition of n (anti-)monopoles interacting with each other Coulomb-like in the limit of $M \gg m$, whose action is given by Equation (28), can be shown to be an approximate solution to the classical equations of motion, the corrections to which are suppressed at least as $\mathcal{O}(e^{-M/m})$ [84].

The dimensionless function $\alpha = \alpha(M/m)$ in this formula stems from quantum fluctuations around the monopole solution.

To calculate the partition function (29), it is convenient to represent Coulomb interactions between (anti-)monopoles by means of an auxiliary scalar field $\chi(\vec{x})$ as follows:

$$\mathrm{e}^{-\frac{g_m^2}{8\pi}\sum\limits_{\substack{a,b=1\\ a\neq b}}^{n}\frac{q_a q_b}{|\vec{z}_a-\vec{z}_b|}} = \int \mathcal{D}\chi\, \mathrm{e}^{-\int d^3x\left[\frac{1}{2}(\partial_\mu\chi)^2+i\chi\rho\right]},$$

where $\rho(\vec{x}) = g_m \sum\limits_{a=1}^{n} q_a\delta(\vec{x}-\vec{z}_a)$ is the density of magnetic charge. This representation readily yields an expression for the grand canonical monopole-antimonopole ensemble in the form of a three-dimensional sine-Gordon model [16]

$$\mathcal{Z}_{\mathrm{mon}} = \int \mathcal{D}\chi\, \mathrm{e}^{-\int d^3x\left[\frac{1}{2}(\partial_\mu\chi)^2-2\zeta\cos(g_m\chi)\right]}. \tag{30}$$

Here, we have introduced the monopole fugacity $\zeta = \alpha\cdot\frac{m^{7/2}}{g}\mathrm{e}^{-S_0}$, which has the dimensionality of (mass)^3. Clearly, in the weak-coupling regime of $g \ll \eta$, this quantity possesses an exponential smallness. Notice that this smallness takes place regardless of whether $M \gg m$ or not. Namely, as was shown in Ref. [88], although the function $\alpha(M/m)$ increases for $M \lesssim m$, this increase is slower than $\mathcal{O}(\mathrm{e}^{S_0})$.

Expanding the cosine in Equation (30), we obtain from the leading term of this expansion (equal to unity) the mean density of the monopole-antimonopole plasma. It follows from the standard formula $\rho_{\mathrm{mean}} = \frac{1}{\Omega}\frac{\partial \ln \mathcal{Z}_{\mathrm{mon}}}{\partial \ln\zeta}$, where Ω is the 3-dimensional volume occupied by the system, and has the form $\rho_{\mathrm{mean}} \simeq 2\zeta$. The approximate equality in this expression corresponds to the neglection of Coulomb-exchange corrections, and the factor of 2 stems from the fact that the mean density of either monopoles or antimonopoles is equal to ζ. The second term in the expansion of the cosine in Equation (30) yields the magnetic Debye mass M_D of the field χ, which reads $M_D = g_m\sqrt{2\zeta}$. The corresponding Debye radius, $1/M_D$, defines the distance at which the Coulomb field of a test magnetic charge immersed into the monopole-antimonopole plasma, becomes Debye screened by the random fields produced by monopoles and antimonopoles. Accordingly, the correlation length in the monopole-antimonopole plasma is also equal to $1/M_D$, being therefore exponentially larger than the mean distance between the constituents of the monopole-antimonopole plasma. Indeed, the correlation length is of the order of $\mathcal{O}(\zeta^{-1/2})$, while the mean distance between the constituents of the plasma is $\sim \rho_{\mathrm{mean}}^{-1/3}$, i.e., it has the order of $\mathcal{O}(\zeta^{-1/3})$. Notice also that the Debye volume $\frac{4\pi}{3M_D^3}$ contains $\frac{4\pi}{3M_D^3}\cdot\rho_{\mathrm{mean}} \propto \frac{1}{g_m^3\sqrt{\zeta}}$ monopoles and antimonopoles. Owing to the exponential largeness of this number, fluctuations of individual (anti-)monopoles can be safely disregarded. This fact fully justifies the adopted mean-field description of the plasma in terms of the field χ.

We proceed now to the general case of the SU(N) Georgi-Glashow model. Similarly to the SU(2)$\to$U(1) symmetry-breaking pattern considered above, the SU(N) symmetry in that model is spontaneously broken down to the maximal Abelian subgroup $[\mathrm{U}(1)]^{N-1}$ of the group SU(N). The generators of this so-called Cartan group $[\mathrm{U}(1)]^{N-1}$ are given by the mutually commuting diagonal generators of the group SU(N), which form an $(N-1)$-dimensional (matrix) vector $\vec{\mathrm{H}}$. The remaining off-diagonal generators of SU(N) can be grouped pairwise into certain linear combinations which, in analogy with the SU(2)-case, are called step (rising and lowering) generators $E_{\pm i}$, where $i = 1,\ldots,\frac{N^2-N}{2}$. The algebra of SU($N$)-generators resulting from such a decomposition reads $\left[\vec{\mathrm{H}}, E_{\pm i}\right] = \vec{\mathrm{q}}_{\pm i}E_{\pm i}$, $[E_i, E_{-i}] = \vec{\mathrm{q}}_i\vec{\mathrm{H}}$, where the vectors $\vec{\mathrm{q}}_i$ and $\vec{\mathrm{q}}_{-i}$ are called respectively positive

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[16] The path-integral measure $\mathcal{D}\chi$ in Equation (30) is normalized by the condition $\int\mathcal{D}\chi\,\mathrm{e}^{-\frac{1}{2}\int d^3x\,(\partial_\mu\chi)^2} = 1$.

and negative root vectors of the group SU(N). Accordingly, one can represent the entire matrix-valued vector potential $A^a_\mu T^a$ (where $a = 1, \ldots, N^2 - 1$) as a sum of the off-diagonal and the diagonal parts, $A^a_\mu T^a = \left(W^+_\mu\right)^i E_{-i} + \left(W^-_\mu\right)^i E_i + \vec{\rm A}_\mu \vec{\rm H}$. That is, W-bosons in this decomposition are charged with respect to the unbroken $[{\rm U}(1)]^{N-1}$ symmetry group, whereas "photon" fields $\vec{\rm A}_\mu$ are neutral with respect to this group. Similarly to the case of the SU(2) Georgi-Glashow model, the SU(N)$\to[{\rm U}(1)]^{N-1}$ symmetry breaking keeps photon fields massless, while giving masses to W-bosons. Due to the latter fact, W-bosons are unable to mediate long-range interactions in the monopole-antimonopole plasma. Noticing also that the magnetic charges of monopoles are $g_m\vec{\rm q}_i$, while the magnetic charges of antimonopoles are $g_m\vec{\rm q}_{-i}$, and using the fact that $\vec{\rm q}_{-i} = -\vec{\rm q}_i$, one obtains the following generalization of the partition function (30) (cf. Ref. [89]):

$$\mathcal{Z}^N_{\rm mon} = \int \mathcal{D}\vec{\chi}\, \exp\left\{-\int d^3x \left[\frac{1}{2}\left(\partial_\mu\vec{\chi}\right)^2 - 2\zeta \sum_{i=1}^{(N^2-N)/2} \cos\left(g_m \vec{\rm q}_i \vec{\chi}\right)\right]\right\}. \tag{31}$$

The Debye mass of the $(N-1)$-component "dual-photon" field $\vec{\chi}$ can be obtained by using the formula [90] $\sum\limits_{i=1}^{(N^2-N)/2} q^m_i q^n_i = \frac{N}{2}\delta^{mn}$ (where the indices m and n run from 1 to $N-1$), which complies with the normalization of the root vectors $|\vec{\rm q}_i| = 1 \;\; \forall i$. This Debye mass reads

$$M_D = g_m \sqrt{N\zeta}. \tag{32}$$

Furthermore, similarly to the SU(2)-case, one can use the formula $\rho_{\rm mean} = \frac{1}{\Omega}\frac{\partial \ln \mathcal{Z}^N_{\rm mon}}{\partial \ln \zeta}$ to calculate the mean density of the monopole-antimonopole plasma. This mean density reads

$$\rho_{\rm mean} \simeq (N^2 - N)\zeta, \tag{33}$$

in agreement with the number of species of monopoles and antimonopoles, equal to $N^2 - N$. The corresponding number of monopoles and antimonopoles contained in the Debye volume $\frac{4\pi}{3M_D^3}$ appears proportional to $\frac{N-1}{g_m^3\sqrt{N\zeta}}$. The exponential largeness of this quantity, which is provided by the factor of $\frac{1}{\sqrt{\zeta}}$, ensures the validity of the mean-field approximation.

We proceed now to the quantitative description of confinement of the static quark-antiquark pair in the SU(N) Georgi-Glashow model. To this end, we notice that the charges which the quarks possess with respect to the maximal Abelian $[{\rm U}(1)]^{N-1}$ subgroup of SU(N), have the form $g\vec{\mu}_\alpha$, where the $(N-1)$-component vectors $\vec{\mu}_\alpha$'s are called the weight vectors of the group SU(N), and $\alpha = 1, \ldots, N$. By virtue of the relation ${\rm tr}\, {\rm e}^{i\vec{\rm Q}\vec{\rm H}} = \sum\limits_{\alpha=1}^{N} {\rm e}^{i\vec{\rm Q}\vec{\mu}_\alpha}$, which holds for an arbitrary $(N-1)$-dimensional vector $\vec{\rm Q}$, one can calculate the contribution produced to the Wilson-loop average by a configuration consisting of some $n > 0$ monopoles and/or antimonopoles. To do so, we consider the magnetic-charge density corresponding to this configuration. It has the form $g_m\vec{\rho}_n(\vec{\rm x})$, where

$$\vec{\rho}_n(\vec{\rm x}) = \sum_{k=1}^{n} \vec{\rm q}_{i_k}\delta(\vec{\rm x} - \vec{\rm z}_k), \tag{34}$$

and $\vec{\rm z}_k$'s are the positions of (anti-)monopoles [17]. One can further introduce a strength tensor $\vec{\rm F}^{(n)}_{\mu\nu}$ which violates the Bianchi identities in a way yielding the magnetic-charge density $g_m\vec{\rho}_n(\vec{\rm x})$, namely as $\frac{1}{2}\varepsilon_{\mu\nu\lambda}\partial_\mu\vec{\rm F}^{(n)}_{\nu\lambda} = g_m\vec{\rho}_n(\vec{\rm x})$. Noticing that the strength tensor $\vec{\rm F}^{(n)}_{\mu\nu}$ is produced entirely by monopoles, so

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[17] Clearly, monopoles correspond to those k's for which $i_k > 0$, while antimonopoles correspond to k's with $i_k < 0$.

that it does not contain any contribution of free "photons", one can readily write down a solution to the latter equation: $\vec{\mathrm{F}}^{(n)}_{\mu\nu}(\vec{\mathrm{x}}) = -g_m \varepsilon_{\mu\nu\lambda}\partial^x_\lambda \int d^3y\, D_0(\vec{\mathrm{x}}-\vec{\mathrm{y}})\vec{\rho}_n(\vec{\mathrm{y}})$, where $D_0(\vec{\mathrm{x}}) = 1/(4\pi|\vec{\mathrm{x}}|)$ is the 3D Coulomb propagator. In analogy with the Stokes' theorem for the electromagnetic field, one can further write down the corresponding contribution to the Wilson-loop average:

$$W^{(n)}(C) = \frac{1}{N}\,\mathrm{tr}\,\exp\left(\frac{ig}{2}\vec{\mathrm{H}}\int_S d\sigma_{\mu\nu}\vec{\mathrm{F}}^{(n)}_{\mu\nu}\right) = \frac{1}{N}\sum_{\alpha=1}^{N} W^{(n)}_\alpha(C), \tag{35}$$

where $W^{(n)}_\alpha(C) \equiv \exp\left(\frac{ig}{2}\vec{\mu}_\alpha \int_S d\sigma_{\mu\nu}\vec{\mathrm{F}}^{(n)}_{\mu\nu}\right)$, and S is some surface bounded by the contour C. Using further the above expression for $\vec{\mathrm{F}}^{(n)}_{\mu\nu}$, along with the quantization condition $gg_m = 4\pi$ [18], we can represent $W^{(n)}_\alpha(C)$ in the form

$$W^{(n)}_\alpha(C) = \exp\left(i\vec{\mu}_\alpha \int d^3x\, \vec{\rho}_n \eta\right),$$

where $\eta(\vec{\mathrm{x}}, C) = \int_S d\sigma_\mu(\vec{\mathrm{y}})\,\partial^x_\mu \frac{1}{|\vec{\mathrm{x}}-\vec{\mathrm{y}}|}$ is the solid angle under which the contour C is seen from the point $\vec{\mathrm{x}}$, and $d\sigma_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda}d\sigma_{\nu\lambda}$ [19]. We can now prove that $W^{(n)}_\alpha(C)$ does not depend on a particular choice of the surface S. To this end, one can consider the ratio of two $W^{(n)}_\alpha(C)$'s which are defined at some two different surfaces, S_1 and S_2, bounded by the same contour C. Using the explicit form of $\vec{\rho}_n$, Equation (34), one obtains for this ratio the following expression:

$$\prod_{k=1}^{n} \exp\left(-i\vec{\mu}_\alpha \vec{\mathrm{q}}_{i_k} \int\limits_{S_1\cup S_2} d\sigma_\mu(\vec{\mathrm{x}})\partial^x_\mu \frac{1}{\left|\vec{\mathrm{z}}_k - \vec{\mathrm{x}}\right|}\right). \tag{36}$$

According to the Gauss' theorem, the integral in this expression can only be non-vanishing (and equal to -4π) for those points $\vec{\mathrm{z}}_k$'s that lie inside the volume bounded by the surface $S_1 \cup S_2$. In order to find the scalar product $\vec{\mu}_\alpha \vec{\mathrm{q}}_{i_k}$, it suffices to notice that

$$\vec{\mu}_\alpha \vec{\mu}_\beta = \frac{1}{2}\left(\delta_{\alpha\beta} - \frac{1}{N}\right), \tag{37}$$

and that every root vector is a difference of two weight vectors (cf. e.g., Ref. [91]). Therefore, instead of labelling a root vector $\vec{\mathrm{q}}_i$ with the index i, one can label it with a pair of indices α and β as

$$\vec{\mathrm{q}}_{\alpha\beta} = \vec{\mu}_\alpha - \vec{\mu}_\beta. \tag{38}$$

Consequently, as can be seen from Equation (37), the only non-vanishing values of the scalar product $\vec{\mu}_\alpha \vec{\mathrm{q}}_{i_k}$ are equal to $\pm\frac{1}{2}$. Therefore, those exponentials in Equation (36) which are not equal to unity for the trivial reason of having vanishing arguments, are nevertheless still equal to unity as $\mathrm{e}^{\pm 2\pi i}$. Thus, we conclude that $W^{(n)}_\alpha(C)$ is indeed independent of a particular choice of the surface S.

The summation over the grand canonical ensemble of monopoles and antimonopoles promotes Equation (35) to a complete expression for the Wilson-loop average, $W(C) = \frac{1}{N}\sum\limits_{\alpha=1}^{N} W_\alpha(C)$, where [92]

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18 Here, the factor of 4π stems from the fact that the Georgi-Glashow model allows for the inclusion of fields transforming under the fundamental representation of the group SU(N). The electric charge acquired by such fields upon the SU(N)$\to$[U(1)]$^{N-1}$ symmetry-breaking, would be $g/2$, so that the Dirac quantization condition for this minimal admissible electric charge becomes $g_m \cdot \frac{g}{2} = 2\pi$.

19 In particular, expanding the contour C and then shrinking it so as to form a closed surface surrounding the point $\vec{\mathrm{x}}$, one can readily prove through the Gauss' theorem that the full solid angle corresponding to the above formula reproduces the expected value of 4π.

$$W_\alpha(C) = \frac{1}{\mathcal{Z}_{\rm mon}^N} \int \mathcal{D}\vec{\rho}\mathcal{D}\vec{\chi} \exp\left\{ -\frac{g_m^2}{2} \int d^3x d^3y\, \vec{\rho}(\vec{x}) D_0(\vec{x}-\vec{y})\vec{\rho}(\vec{y}) + \right.$$
$$\left. \int d^3x \left[i g_m \vec{\chi}\vec{\rho} + 2\zeta \sum_{i=1}^{(N^2-N)/2} \cos\left(g_m \vec{q}_i \vec{\chi}\right) + i\vec{\mu}_\alpha \vec{\rho}\,\eta \right] \right\}. \quad (39)$$

In this formula, $\vec{\rho}$ is a dynamical monopole density, the integration over which in the case of $\eta = 0$ recovers the grand canonical partition function (31), so that the normalization $W_\alpha(0) = 1$ is respected. Alternatively, one can express the Wilson-loop average in terms of the magnetic field, whose divergence yields this dynamical monopole density, i.e., $\partial_\mu \vec{B}_\mu = \vec{\rho}$. The corresponding expression reads

$$W_\alpha(C) = \frac{1}{\mathcal{Z}_{\rm mon}^N} \int \mathcal{D}\vec{B}_\mu\, \delta\left(\varepsilon_{\mu\nu\lambda}\partial_\nu \vec{B}_\lambda\right) \int \mathcal{D}\vec{\chi}\, \exp\left\{ \int d^3x \left[-\frac{g_m^2}{2}\vec{B}_\mu^2 + \right.\right.$$
$$\left.\left. i g_m \vec{\chi}\partial_\mu \vec{B}_\mu + 2\zeta \sum_{i=1}^{(N^2-N)/2} \cos\left(g_m \vec{q}_i \vec{\chi}\right) \right] + 4\pi i \vec{\mu}_\alpha \int_S d\sigma_\mu \vec{B}_\mu \right\}, \quad (40)$$

where we have taken into account that the field $\vec{B}_\mu$ obeys the Maxwell equation $\varepsilon_{\mu\nu\lambda}\partial_\nu \vec{B}_\lambda = 0$, since monopoles do not produce any electric fields.

Let us further choose C to be a circular contour located in the (1,2)-plane. Since the monopole contribution to the Wilson-loop average has been proven independent of a particular choice of the surface S, we take for S a planar surface bounded by the contour C. We can now proceed to the saddle-point integrations over the fields $\vec{\chi}$ and $\vec{B}_\mu$. Clearly, a non-trivial solution to the saddle-point equations exists only for those points $\vec{x}$ for which $\sqrt{x_1^2 + x_2^2}$ is smaller than the radius of the circular contour C. For such points $\vec{x}$, the solution to the saddle-point equations is expected to depend only on the distance to the (1,2)-plane, i.e., it can be sought in the form $\vec{B}_\mu = \delta_{\mu 3}\vec{B}(z)$, $\vec{\chi} = \vec{\chi}(z)$, where $z \equiv x_3$. This ansatz leads to the following saddle-point equations:

$$i g_m \vec{\chi}' + g_m^2 \vec{B} = 4\pi i \vec{\mu}_\alpha \delta(z), \qquad i\vec{B}' - 2\zeta \sum_{i=1}^{(N^2-N)/2} \vec{q}_i \sin\left(g_m \vec{q}_i \vec{\chi}\right) = 0, \quad (41)$$

where $' \equiv d/dz$. Noticing further the distinguished role played in these equations by the vector $\vec{\mu}_\alpha$, we seek $\vec{B}$ and $\vec{\chi}$ in the form $\vec{B}(z) = \vec{\mu}_\alpha B(z)$ and $\vec{\chi} = \vec{\mu}_\alpha \chi(z)$. Multiplying then the second of the two saddle-point Equations (41) by $\vec{\mu}_\alpha$, and using Equations (37) and (38), we cast the saddle-point equations to the form

$$2i\phi' + g_m^2 B = 4\pi i \delta(z), \qquad B' + 2i\zeta N \sin\phi = 0, \quad (42)$$

where $\phi \equiv g_m \chi/2$. A solution to this system of equations has the form

$$B(z) = i\,\frac{8M_D}{g_m^2}\,\frac{e^{-M_D|z|}}{1+e^{-2M_D|z|}}, \qquad \phi(z) = 4\,\mathrm{sgn}\,z \cdot \arctan\left(e^{-M_D|z|}\right), \quad (43)$$

where the Debye mass M_D is given by Equation (32). In particular, we see that the function $\phi(z)$ jumps from the value of π to the value of $-\pi$ when z changes from $+0$ to -0, while the magnetic field $B(z)$ exponentially falls off above and below the (1,2)-plane at the distance $1/M_D$, which is equal to the vacuum correlation length. Thus, we explicitly see that the radius of the confining string in the 3D SU(N) Georgi-Glashow model is equal to the vacuum correlation length in that model.

The value of the string tension σ in the fundamental representation can be determined up to an overall numerical factor, which depends on whether one defines σ through $B(0)$ or through the mean value of the magnetic field that can be obtained by averaging $B(z)$ over the interval $-\frac{1}{M_D} < z < \frac{1}{M_D}$.

Apart from this factor, the string tension reads $\sigma \propto \vec{\mu}_\alpha^2 |B(0)|$, so that its dependence on the parameters of the model has the form

$$\sigma \propto \frac{N-1}{\sqrt{N}} \cdot \frac{\sqrt{\zeta}}{g_m}. \tag{44}$$

Notice that this result depends on g as $\sigma \propto g \cdot \mathrm{e}^{-\mathrm{const}\cdot\eta/g}$, where the exponential cannot be expanded in a Taylor series, since we work in the weak-coupling regime of $g \ll \eta$. For this reason, the dependence of σ on g appears non-analytic, i.e., the obtained string tension is manifestly non-perturbative. This result resembles the one for the string tension in the 4D Yang-Mills theory. There, the string tension is proportional to the square of the only dimensionful parameter of the theory, called the QCD scale parameter, i.e., $\sigma \propto \Lambda_{\mathrm{QCD}}^2$. Since Λ_{QCD} appears as a consequence of the dimensional transmutation, it depends on the Yang-Mills coupling constant g as $\mathrm{e}^{-\mathrm{const}/g^2}$. Thus, although the string tension in both the 3D SU(N) Georgi-Glashow model and the 4D Yang-Mills theory has a non-analytic coupling-dependence, the origin of this non-analyticity in these two theories is different. Namely, in the first case the non-analyticity stems from the monopole fugacity ζ, while in the second case it stems from the dimensional transmutation, which itself is a consequence of the asymptotic freedom that holds in the Yang-Mills theory.

We proceed now to the calculation of the so-called k-string tensions in the 3D SU(N) Georgi-Glashow model with $N \geq 3$. Here, $k = 1, \ldots, N-1$ denotes the so-called $\mathcal{N}$-ality of a given representation of SU(N). It is defined as the modulo-N difference between the number of quark and antiquark fields which constitute an object transforming under a certain higher representation of the group SU(N). Accordingly, representations with $\mathcal{N}$-alities $k \leq N/2$ and $N-k$ are related to each other via the complex conjugation, which corresponds to the replacement of quarks by antiquarks and vice versa, so that confining strings associated with these representations have equal tensions. The representations relevant for confinement are given by rank-k antisymmetric tensors, while all other representations are contained in a tensor product of some number of adjoint representations, and have zero $\mathcal{N}$-ality. These representations are irrelevant for confinement since an $\mathcal{N}$-ality-zero static object gets screened by gluons, so that its Wilson-loop average exhibits only the perimeter law for sufficiently large contours. In general, any representation of a non-zero $\mathcal{N}$-ality is contained in a direct product of a certain rank-k antisymmetric representation and some number of adjoint representations. Accordingly, for all possible representations of the color source, there exist only N string tensions σ_k's which characterize confinement. Of those, σ_1 is the string tension corresponding to the fundamental representation, while $\sigma_N = 0$. The quantity σ_k can be interpreted as a tension of a k-string, i.e., a confining string which interconnects k quarks with k antiquarks. As mentioned, the equality $\sigma_k = \sigma_{N-k}$ takes place, owing to which only $[N/2]$ of all string tensions σ_k's are mutually independent. Thus, the full information about confinement is encoded in these $[N/2]$ numbers.

Clearly, a k-string can only be stable provided the inequality $\sigma_k < k\sigma_1$ holds. In the large-N limit, interactions between strings composing a k-string are suppressed, so that $\sigma_k \to k\sigma_1$ at $N \to \infty$ and a fixed k. In the 2D Yang-Mills theory, where confinement stems just from the one-gluon exchange between the sources of the gauge field, one has $\sigma_k \propto C_2^{(k)}$, where $C_2^{(k)} = \frac{k(N-k)(N+1)}{2N}$ is the eigenvalue of the quadratic Casimir operator of a rank-k antisymmetric representation. For this reason, the ratio $\frac{\sigma_k}{\sigma_1}$ in the 2D Yang-Mills theory obeys exactly the so-called Casimir-scaling formula [93], $\frac{\sigma_k}{\sigma_1} = \frac{k(N-k)}{N-1}$, i.e., indeed $\frac{\sigma_k}{\sigma_1} < k$. We demonstrate now that the Casimir scaling of k-string tensions holds also in the 3D SU(N) Georgi-Glashow model. To this end, we consider a generalization of Equation (40) to the case of a k-string. Such a generalization is given by the k-th power of Equation (40), and reads

$$W_k(C) = \sum_{\alpha_1=1}^{N} \cdots \sum_{\alpha_k=1}^{N} W_{\alpha_1,\ldots,\alpha_k}(C). \tag{45}$$

Here $W_{\alpha_1,\ldots,\alpha_k}(C)$ is given by Equation (40) with $\vec{\mu}_\alpha$ replaced by the sum $\vec{\mathrm{M}}_k = \sum_{i=1}^{k} \vec{\mu}_{\alpha_i}$, in which some of the vectors $\vec{\mu}_{\alpha_i}$ can be the same. Accordingly, the vector $\vec{\mathrm{M}}_k$ substitutes the vector $\vec{\mu}_\alpha$ in the first of the two saddle-point equations (41), so that $\vec{\mathrm{B}} \propto \vec{\mathrm{M}}_k$. For this reason, one gets the area law of the form $W_{\alpha_1,\ldots,\alpha_k}(C) \simeq \mathrm{e}^{-(\vec{\mathrm{M}}_k)^2\sigma'\Sigma}$ with some k-independent string tension σ'. Clearly, this law holds for sufficiently large areas Σ of the planar surface bounded by the contour C. Hence, the nested sum (45) consists of exponentials of the form [20]

$$\mathrm{e}^{-\left(n_1\vec{\mu}_{a_1}+\cdots+n_p\vec{\mu}_{a_p}+\sum\limits_{j=1}^{n}\vec{\mu}_{a_j}\right)^2\sigma'\Sigma}, \quad \text{where} \quad n_1+\cdots+n_p+n=k. \tag{46}$$

That is, every group of mutually coinciding weight vectors is characterized by some integer n_i, where $i=1,\ldots,p$. Instead, all the vectors $\vec{\mu}_{a_j}$'s with $j=1,\ldots,n$ are mutually different. In Appendix A, we calculate the square of the sum entering the exponential (46),

$$S \equiv \left(\sum_{i=1}^{p} n_i\vec{\mu}_{a_i} + \sum_{j=1}^{n}\vec{\mu}_{a_j}\right)^2, \tag{47}$$

which yields

$$S = \frac{1}{2}\sum_{i=1}^{p} n_i^2 + \frac{n}{2} - \frac{(k-n)^2+kn}{2N}. \tag{48}$$

In order to identify the exponential that yields the dominant contribution to the sum, we should find the value of n and the set $\{n_1,\ldots,n_p\}$ that minimize S. We notice first of all that the sum $\sum_{i=1}^{p} n_i$ is a fixed number for a given n. Therefore, n_i's which minimize the sum $\sum_{i=1}^{p} n_i^2$, should all be equal to each other, i.e. $n_i = \frac{k-n}{p}$ $\forall i=1,\ldots,p$. Indeed, let us assume the opposite, namely that for a certain index j, $n_j = \frac{k-n}{p} + a$ with some $a \neq 0$. This means that some other index l exists, such that $n_l = \frac{k-n}{p} - a$. Then $\sum_{i=1}^{p} n_i^2 = \frac{(k-n)^2}{p} + 2a^2$, which is larger than the value $\frac{(k-n)^2}{p}$ of this sum in the case where all n_i's are equal to each other.

Furthermore, the number of possibilities to represent the integer $(k-n)$ as a sum of p equal integers varies from 2 to $(k-n)$, i.e., $2 \leq p \leq (k-n)$. Therefore, the value of the sum $\sum_{i=1}^{p} n_i^2 = \frac{(k-n)^2}{p}$ varies from $(k-n)$ to $\frac{(k-n)^2}{2}$, so that

$$S_{\min} = \frac{k-n}{2} + \frac{n}{2} - \frac{(k-n)^2+kn}{2N} = \frac{1}{2}\left(k - \frac{n^2-kn+k^2}{N}\right).$$

The maximum of the function $S_{\min}(n)$ is achieved at $n=\frac{k}{2}$, while its minimum is achieved at $n=k$, and reads [21]

$$S_{\min}(k) = \frac{k(N-k)}{2N}.$$

Hence, the minimum of $S_{\min}$, and therefore of S, is achieved in the case where *all* k weight vectors in Equation (46) are mutually different [94,95]. For this value of n, the exponential reads $\mathrm{e}^{-\frac{k(N-k)}{2N}\cdot\sigma'\Sigma}$.

We can further calculate the number of occurrences of the term (46) in the nested sum (45). To this end, we notice that $C_k^{n_1} \equiv \frac{k!}{n_1!(k-n_1)!}$ possibilities exist to choose out of k weight vectors n_1 coinciding

[20] The corresponding pre-exponentials are also calculated below.

[21] The value of $S_{\min}(1)$ is larger than that of $S_{\min}(k)$, namely $S_{\min}(1) = S_{\min}(k) + \frac{k-1}{2N}$.

ones, whose index can acquire any value from 1 to N. Once these vectors are chosen and their index is fixed, $C^{n_2}_{k-n_1}$ possibilities exist to choose out of the remaining $(k-n_1)$ vectors n_2 coinciding ones, whose index can acquire any of the remaining $(N-1)$ values, and so on. At the last step, $C^{n_p}_{k-n_1-\cdots-n_{p-1}}$ possibilities exist to choose n_p vectors, and their index can acquire any of the remaining $(N-(p-1))$ values. After that, $k-n_1-\cdots-n_p = n$ mutually different weight vectors remain. The number of possibilities to choose one of them is equal to n, and the index of that vector can acquire any of the remaining $(N-p)$ values. Once this vector is chosen and its index is fixed, $(n-1)$ possibilities exist to choose the next vector, whose index can acquire one of $(N-p-1)$ possible values. Finally, the last vector out of this group of n mutually different vectors can acquire $(N-p-(n-1))$ values. Altogether, we obtain for the sought number of occurrences the following expression:

$$C^{n_1}_k N * C^{n_2}_{k-n_1}(N-1) * \cdots * C^{n_p}_{k-n_1-\cdots-n_{p-1}}(N-p+1) *$$
$$n(N-p) * (n-1)(N-p-1) * \cdots * 1(N-p-n+1).$$

Explicitly, this product reads

$$\frac{k!}{n_1!(k-n_1)!} \cdot \frac{(k-n_1)!}{n_2!(k-n_1-n_2)!} \cdots \frac{(k-n_1-\cdots-n_{p-1})!}{n_p!(k-n_1-\cdots-n_p)!} \cdot \frac{N!}{(N-p-n)!} \cdot n!$$
$$= \frac{k!N!}{n_1!\cdots n_p!\,(N-p-n)!}.$$

In the above-discussed case of $n=k$, this expression takes the form $\frac{k!N!}{(N-k)!}$. In the particular case of $k \sim N$, this "entropy factor" grows as strongly as $O(k!)$, so that the Stirling's formula yields for the full exponential: $e^{k\ln k - \frac{k(N-k)}{2N}\cdot\sigma'\Sigma}$. Consequently, for a given $N \gg 1$, the area Σ should be at least as large as $\Sigma > O(\frac{\ln N}{\sigma'})$ in order to ensure the stability of k strings even for $k \sim N$.

Hence, we restrict ourselves to $\vec{\mathrm{M}}_k$'s consisting of mutually different vectors $\vec{\mu}_{\alpha_i}$'s, and replace $\vec{\mu}_\alpha$ by $\vec{\mathrm{M}}_k$ in the saddle-point equations (41). Setting further $\vec{\mathrm{B}} = \vec{\mathrm{M}}_k B(z)$ and $\vec{\chi} = \vec{\mathrm{M}}_k \chi(z)$, we see that the first of equations (41) takes the same form as in the case of the fundamental representation. To simplify the second saddle-point equation, we represent positive root vectors entering that equation by using the relation (38). This yields

$$\sum_{\beta<\alpha} \vec{\mathrm{M}}_k(\vec{\mu}_\alpha - \vec{\mu}_\beta) \cdot \sin[g_m \vec{\mathrm{M}}_k(\vec{\mu}_\alpha - \vec{\mu}_\beta)\chi] = \frac{i}{2\zeta} C_k B', \tag{49}$$

where the square of the vector $\vec{\mathrm{M}}_k$ consisting of mutually different weight vectors $\vec{\mu}_{\alpha_i}$'s reads

$$C_k \equiv \vec{\mathrm{M}}_k^2 = \frac{k(N-k)}{2N}. \tag{50}$$

Using further the $\alpha \leftrightarrow \beta$ symmetry of the expression standing under the sum in Equation (49), we can rewrite the left-hand side of Equation (49) as

$$\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} \vec{\mathrm{M}}_k\vec{\mu}_\alpha \left[\sin(g_m\vec{\mathrm{M}}_k\vec{\mu}_\alpha\chi)\cos(g_m\vec{\mathrm{M}}_k\vec{\mu}_\beta\chi) - \cos(g_m\vec{\mathrm{M}}_k\vec{\mu}_\alpha\chi)\sin(g_m\vec{\mathrm{M}}_k\vec{\mu}_\beta\chi)\right].$$

We should now calculate the four sums in this expression. Starting with the first one, $\sum\limits_{\alpha=1}^{N} \vec{\mathrm{M}}_k\vec{\mu}_\alpha \sin(g_m\vec{\mathrm{M}}_k\vec{\mu}_\alpha\chi)$, we notice that this sum contains k terms for which $\vec{\mu}_\alpha$ coincides with some of the k weight vectors entering $\vec{\mathrm{M}}_k$. Using Equation (37), we have in the case of every such term: $\vec{\mathrm{M}}_k\vec{\mu}_\alpha = \frac{N-1}{2N} - (k-1)\frac{1}{2N} = \frac{N-k}{2N}$. For the remaining $N-k$ terms in the sum, the vector $\vec{\mu}_\alpha$ does not coincide with any of the weight vectors entering $\vec{\mathrm{M}}_k$, so that in the case of every such term we have: $\vec{\mathrm{M}}_k\vec{\mu}_\alpha = -\frac{k}{2N}$. Altogether, the sum reads

$$\sum_{\alpha=1}^{N}\vec{\mathrm{M}}_k\vec{\mu}_\alpha \sin(g_m\vec{\mathrm{M}}_k\vec{\mu}_\alpha\chi) = k\cdot\tfrac{N-k}{2N}\sin\left(g_m\tfrac{N-k}{2N}\chi\right) + \\ (N-k)\cdot\tfrac{k}{2N}\sin\left(g_m\tfrac{k}{2N}\chi\right) = C_k\cdot\left[\sin\left(g_m\tfrac{k}{2N}\chi\right)+\sin\left(g_m\tfrac{N-k}{2N}\chi\right)\right]. \tag{51}$$

In the same way, we obtain for the three other sums the following expressions:

$$\sum_{\beta=1}^{N}\cos(g_m\vec{\mathrm{M}}_k\vec{\mu}_\beta\chi) = k\cdot\cos\left(g_m\frac{N-k}{2N}\chi\right)+(N-k)\cdot\cos\left(g_m\frac{k}{2N}\chi\right), \tag{52}$$

$$\sum_{\alpha=1}^{N}\vec{\mathrm{M}}_k\vec{\mu}_\alpha\cos(g_m\vec{\mathrm{M}}_k\vec{\mu}_\alpha\chi) = C_k\cdot\left[\cos\left(g_m\frac{N-k}{2N}\chi\right)-\cos\left(g_m\frac{k}{2N}\chi\right)\right], \tag{53}$$

$$\sum_{\beta=1}^{N}\sin(g_m\vec{\mathrm{M}}_k\vec{\mu}_\beta\chi) = k\cdot\sin\left(g_m\frac{N-k}{2N}\chi\right)-(N-k)\cdot\sin\left(g_m\frac{k}{2N}\chi\right). \tag{54}$$

We notice that each of the sums (51)–(54) is invariant under the interchange of quarks and antiquarks that are confined by the k-string, which corresponds to the replacement $k \leftrightarrow (N-k)$. Bringing these sums together, we find that the left-hand side of Equation (49) is equal to $C_kN\sin\frac{g_m\chi}{2}$, so that Equation (49) coincides with the second of Equations (42). Accordingly, the saddle-point fields $B(z)$ and $\phi(z)$ are given by Equation (43), so that the resulting string tension $\tilde{\sigma} \propto |B(0)|$ is manifestly k-independent. This yields $W_k(C) \simeq \mathrm{e}^{-\frac{k(N-k)}{2N}\tilde{\sigma}\Sigma}$, and therefore $\frac{\sigma_k}{\sigma_1} = \frac{k(N-k)}{N-1}$, i.e., the Casimir-scaling ratio. Thus, for the case of a flat contour C, we have demonstrated Casimir scaling in the 3D SU(N) Georgi-Glashow model.

Nevertheless, in the 4D SU(N) Yang-Mills theory with $N=4$ and $N=6$, lattice data [59,96,97] on the $\frac{\sigma_k}{\sigma_1}$-ratio show that corrections to the Casimir scaling are of the order of 10%, while corrections to the so-called Sine scaling [98–101], $\frac{\sigma_k}{\sigma_1} = \frac{\sin\frac{\pi k}{N}}{\sin\frac{\pi}{N}}$, amount to only a few percent. The Sine-scaling ratio has been found analytically in supersymmetric gauge theories [98], as well as through a possible duality between such gauge theories and string theories [99–102], but not in the 4D Yang-Mills theory itself. The principal difference of the Sine scaling from the Casimir scaling is reflected in the N-dependence of the leading correction to the large-N limit $\frac{\sigma_k}{\sigma_1} \to k$. Namely, for the Sine scaling, this correction reads $\frac{\pi^2}{6}\frac{k(1-k^2)}{N^2}$, being therefore $O(\frac{1}{N^2})$, while in the case of the Casimir scaling it reads $\frac{k(1-k)}{N}$, thereby behaving with N as $O(\frac{1}{N})$. Physicswise, this correction yields the strength of pairwise attractive interactions between the k strings that constitute a k-string. This is the reason as to why the parametric N-dependence of such a leading correction to the large-N limit of the k-string tension is important. However, the current level of accuracy of lattice simulations does not allow one to unambiguously decide in favor of either the $O(\frac{1}{N^2})$- or the $O(\frac{1}{N})$-behavior of this correction. On the theory side too, there is no reason to expect either the Sine or the Casimir scaling to be an exact result in the 4D non-supersymmetric Yang-Mills theory. Yet, as has been shown in Ref. [95], the Casimir scaling takes place as the leading result in the realistic 4D $[\mathrm{U}(1)]^{N-1}$-invariant dual Abelian Higgs model of confinement.

1.4. Confining Strings in the 3D [U(1)]$^{N-1}$-Invariant Compact QED

As we have seen at the example of the 3D SU(N) Georgi-Glashow model, if the SU(N) gauge symmetry is spontaneously broken down to the $[\mathrm{U}(1)]^{N-1}$ symmetry, the resulting group $[\mathrm{U}(1)]^{N-1}$ appears compact, thereby allowing for the existence of magnetic monopoles. The corresponding gauge theory with the compact group $[\mathrm{U}(1)]^{N-1}$ can be naturally called a $[\mathrm{U}(1)]^{N-1}$-invariant compact QED. In addition to magnetic monopoles, it also contains an $(N-1)$-component free-photon field $\vec{\mathrm{A}}_\mu$. Similarly to the U(1)-invariant compact QED [84,103], the monopole and the photon contributions to the vacuum expectation value of any gauge-invariant operator in the $[\mathrm{U}(1)]^{N-1}$-invariant compact QED get factorized. This fact makes compact QED for any $N \geq 1$ similar to the 2D XY model, since disorder in both these theories is produced by topological defects, which are monopoles in the first case and vortices in the second case. On the contrary, free photons in compact QED cannot produce the degree of disorder sufficient for confinement of external electrically charged particles, yielding for the corresponding Wilson-loop average only the perimeter, rather than the area, law. The counterpart of free photons in the 2D XY model is the spin waves, which can only produce disorder in the spin-spin correlation functions at short distances.

One should, however, mention the following difference between the 2D XY model and the 3D compact QED. In the 2D XY model, free vortices and antivortices exist only at temperatures higher than a certain critical one, whereas below that temperature vortices form bound states with antivortices [104–106]. In other words, vortices and antivortices in the 2D XY model exist in a plasma phase (i.e., the system is disordered) only at sufficiently high temperatures, whereas at low temperatures they exist in a molecular phase. Instead, in the zero-temperature 3D compact QED, where the strength of the monopole-antimonopole Coulomb interaction is defined by the coupling constant g_m, monopoles and antimonopoles exist in the plasma phase for all values of g_m. This is, however, no longer the case in the 4D compact QED [103], where monopoles are not point-like objects as in 3D, but are closed loops. The action of such a monopole loop of length L is proportional to $\frac{L}{g^2}$, while its entropy also increases linearly with L. Consequently, monopole loops can only condense provided g is larger than a certain critical value, whereas for small g's only small-length monopole loops survive, so that the degree of disorder produced by such loops is not sufficient to provide confinement of external electric charges. Thus, in the 4D compact QED, confinement takes place only in the strong-coupling regime.

Let us now consider the Wilson-loop average (40) defined at some contour C, which is not necessarily flat, and whose mean size, which can be estimated as $\sqrt{\Sigma_{\mathrm{min}}}$, is much larger than the vacuum correlation length $1/M_D$. It turns out that monopoles and free photons can be described in a unified way, namely through one and the same antisymmetric tensor field $\vec{\mathrm{h}}_{\mu\nu}$ related to the magnetic field $\vec{\mathrm{B}}_\mu$ as $\vec{\mathrm{B}}_\mu = \frac{1}{2g_m}\varepsilon_{\mu\nu\lambda}\vec{\mathrm{h}}_{\nu\lambda}$. The coefficient on the right-hand side of this relation has been chosen in such a way as to reproduce the Coulomb interaction of monopole densities from Equation (39). Namely, the following equalities hold:

$$\frac{1}{4}\int d^3x\,\vec{\mathrm{h}}_{\mu\nu}^2 = \frac{g_m^2}{2}\int d^3x\,\vec{\mathrm{B}}_\mu^2 = \frac{g_m^2}{2}\int d^3x d^3y\,\vec{\rho}(\vec{\mathrm{x}})D_0(\vec{\mathrm{x}}-\vec{\mathrm{y}})\vec{\rho}(\vec{\mathrm{y}}).$$

Furthermore, free photons can be taken into account in the functional integral (40) by relaxing the constraint imposed through the functional δ-function $\delta\left(\varepsilon_{\mu\nu\lambda}\partial_\nu\vec{\mathrm{B}}_\lambda\right)$. Indeed, in terms of the field $\vec{\mathrm{h}}_{\mu\nu}$, this constraint reads $\partial_\mu\vec{\mathrm{h}}_{\mu\nu} = 0$, which corresponds to the absence of the free photons $\vec{\mathrm{A}}_\mu$ in the following general formula for an antisymmetric rank-2 tensor:

$$\vec{\mathrm{h}}_{\mu\nu} = \partial_\mu\vec{\mathrm{A}}_\nu - \partial_\nu\vec{\mathrm{A}}_\mu + \varepsilon_{\mu\nu\lambda}\partial_\lambda\vec{\varphi}. \tag{55}$$

Relaxing the said constraint, we arrive at the following expression for the Wilson-loop average (40) in terms of the antisymmetric tensor field $\vec{\mathrm{h}}_{\mu\nu}$:

$$\begin{gathered} W_\alpha = \frac{1}{\mathcal{Z}_{\mathrm{mon}}^N}\int \mathcal{D}\vec{\mathrm{h}}_{\mu\nu}\int\mathcal{D}\vec{\chi}\exp\Bigg\{\int d^3x\Bigg[-\frac{1}{4}\vec{\mathrm{h}}_{\mu\nu}^2 + \\ \frac{i}{2}\vec{\chi}\varepsilon_{\mu\nu\lambda}\partial_\mu\vec{\mathrm{h}}_{\nu\lambda} + 2\zeta\sum_{i=1}^{(N^2-N)/2}\cos\left(g_m\vec{\mathrm{q}}_i\vec{\chi}\right)\Bigg] + \frac{ig}{2}\vec{\mu}_\alpha\int_S d\sigma_{\mu\nu}\vec{\mathrm{h}}_{\mu\nu}\Bigg\}. \end{gathered} \tag{56}$$

Since the exponential in Equation (56) does not contain a kinetic term of the field $\vec{\chi}$, it is legitimate to perform the functional integration over this field in the saddle-point approximation. The corresponding saddle-point equation can be readily solved by using the ansatz $\vec{\mathrm{h}}_{\mu\nu} = \vec{\mu}_\alpha h_{\mu\nu}$. Solving the saddle-point equation, we arrive at the replacement of $\int d^3x[\frac{i}{2}\vec{\chi}\varepsilon_{\mu\nu\lambda}\partial_\mu\vec{\mathrm{h}}_{\nu\lambda} + 2\zeta \sum_{i=1}^{(N^2-N)/2} \cos\left(g_m\vec{\mathrm{q}}_i\vec{\chi}\right)]$ in Equation (56) by $-V[\vec{\mathrm{h}}_{\mu\nu}]$, where $V[\vec{\mathrm{h}}_{\mu\nu}]$ is the following multi-valued potential of the antisymmetric tensor field $\vec{\mathrm{h}}_{\mu\nu}$:

$$V[\vec{\mathrm{h}}_{\mu\nu}] \equiv 2(N-1)\zeta \int d^3x \left[H_\alpha \operatorname{arcsinh} H_\alpha - \sqrt{1+H_\alpha^2}\right], \tag{57}$$

and

$$H_\alpha \equiv \frac{\vec{\mu}_\alpha}{2g_m(N-1)\zeta}\varepsilon_{\mu\nu\lambda}\partial_\mu\vec{\mathrm{h}}_{\nu\lambda}. \tag{58}$$

As was first noticed in Ref. [107] for the case of 3D compact QED, corresponding to $N = 2$, the summation over branches of the potential V can provide the sought mechanism of the summation over string world sheets in Equation (8). Furthermore, owing to the relation $\varepsilon_{\mu\nu\lambda}\partial_\mu\vec{\mathrm{h}}_{\nu\lambda} = 2g_m\vec{\rho}$, the potential V can be viewed as a potential of monopole densities $\vec{\rho}$'s. By means of the Cauchy-Schwarz inequality, we obtain from Equation (58):

$$|H_\alpha| \le \frac{|\vec{\mu}_\alpha|\,|\vec{\rho}|}{(N-1)\zeta} = \frac{|\vec{\rho}|}{\sqrt{2N(N-1)}\,\zeta}.$$

Noticing then that the mean density $\rho_{\rm mean}$ of the monopole-antimonopole plasma is given by Equation (33), we see that the weak-field limit, $|H_\alpha| \ll 1$, corresponds to monopole densities whose absolute values $|\vec{\rho}|$'s are smaller than $\rho_{\rm mean}$ by a factor of $O(N)$ in the large-N limit. Thus, the weak-field limit corresponds to low monopole densities (cf. Ref. [108]). In this limit, the integrand in Equation (57) becomes a quadratic function of H_α, and the summation over branches of the potential (57) gets lost. For this reason, the Wilson-loop average in the weak-field limit acquires an explicit S-dependence, and takes the form

$$W_\alpha \simeq \frac{1}{\mathcal{Z}_{\rm mon}^N}\int \mathcal{D}\vec{\mathrm{h}}_{\mu\nu} \exp\left[-\int d^3x\left(\frac{1}{12M_D^2}\vec{\mathrm{H}}_{\mu\nu\lambda}^2 + \frac{1}{4}\vec{\mathrm{h}}_{\mu\nu}^2\right) + \frac{ig}{2}\vec{\mu}_\alpha\int_S d\sigma_{\mu\nu}\vec{\mathrm{h}}_{\mu\nu}\right]. \tag{59}$$

Here $\vec{\mathrm{H}}_{\mu\nu\lambda} = \partial_\mu\vec{\mathrm{h}}_{\nu\lambda} + \partial_\lambda\vec{\mathrm{h}}_{\mu\nu} + \partial_\nu\vec{\mathrm{h}}_{\lambda\mu}$ is the strength tensor of the antisymmetric-tensor field $\vec{\mathrm{h}}_{\mu\nu}$, and the Debye mass M_D of the dual photon is given by Equation (32). The Gaussian integration over the field $\vec{\mathrm{h}}_{\mu\nu}$, whose details are presented in Appendix B, yields for the Wilson-loop average the following expression:

$$W_\alpha \simeq \exp\left\{-\frac{(g\vec{\mu}_\alpha)^2}{2}\left[\frac{M_D^2}{2}\int_S d\sigma_{\mu\nu}(\vec{\mathrm{x}})\int_S d\sigma_{\mu\nu}(\vec{\mathrm{y}}) + \oint_C dx_\mu \oint_C dy_\mu\right] D_{M_D}(\vec{\mathrm{x}}-\vec{\mathrm{y}})\right\}, \tag{60}$$

where $D_M(\vec{\mathrm{x}}) = \mathrm{e}^{-M|\vec{\mathrm{x}}|}/(4\pi|\vec{\mathrm{x}}|)$ is the Yukawa propagator. Note that, in the formal limit of $M_D \to 0$, where monopoles are suppressed, Equation (60) recovers the standard expression for the Wilson-loop average in the non-compact $[\mathrm{U}(1)]^{N-1}$-invariant QED, which is provided just by the free photons, and reads [22] $W_\alpha^{\rm phot} = \exp\left[-\frac{(g\vec{\mu}_\alpha)^2}{2}\oint_C dx_\mu \oint_C dy_\mu D_0(\vec{\mathrm{x}}-\vec{\mathrm{y}})\right]$. In the general case of a non-vanishing M_D, where monopoles are present in addition to the free photons, the $\vec{\mathrm{h}}_{\mu\nu}$-field can be represented in the form (55), which leads to the factorization of the photon and the monopole contributions to the Wilson-loop average. Namely, one has $W_\alpha = W_\alpha^{\rm phot}W_\alpha^{\rm mon}$, which can serve as a definition of the monopole contribution $W_\alpha^{\rm mon}$. The obtained factorization of the Wilson-loop average illustrates the

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[22] This result can be seen directly from Equation (59), where, in the limit of $M_D \to 0$, the dominant contribution to the $\vec{\mathrm{h}}_{\mu\nu}$-integral is produced by the fields for which $\vec{\mathrm{H}}_{\mu\nu\lambda} = 0$. Such fields can be represented as $\vec{\mathrm{h}}_{\mu\nu} = \partial_\mu\vec{\mathrm{A}}_\nu - \partial_\nu\vec{\mathrm{A}}_\mu$, which is nothing but the strength tensor of the free-photon field.

general principle mentioned at the beginning of this Section, which states that the photon and the monopole contributions to the vacuum expectation value of any gauge-invariant operator in compact QED get factorized. It is remarkable that the photon contribution gets eventually cancelled by the massless part of the monopole contribution. As a consequence, both the surface-surface and the contour-contour interactions in the resulting Equation (60) are mediated entirely by the massive dual photon.

Equation (60) yields a non-local string action

$$\mathcal{A} = \frac{(g\vec{\mu}_\alpha M_D)^2}{4} \int_S d\sigma_{\mu\nu}(\vec{x}) \int_S d\sigma_{\mu\nu}(\vec{y}) D_{M_D}(\vec{x}-\vec{y}), \tag{61}$$

which has the form of Equation (9). As has been mentioned at the end of Section 1.2, the two leading terms in the derivative expansion of this action are the Nambu-Goto term and the rigidity term with a negative coupling. If the surface S in Equation (60) is the minimal surface for a given contour C, e.g., a flat surface in the case of a flat contour, then only the Nambu-Goto term survives in the derivative expansion of the non-local action, while the rigidity and all the higher-derivative terms vanish. In particular, for the rigidity term, this can be seen directly from the corresponding expression (22) by noticing that the minimal surface is defined through the 2D Laplace equation [23] $\Delta x_\mu = 0$.

Let us first consider the Nambu-Goto term, which yields the string tension. As follows from Equation (60), in the weak-field limit at issue, one readily obtains Casimir scaling of k-string tensions even for a non-flat surface S [94]. Indeed, the corresponding Wilson-loop average is given by Equation (45), where the expression for $W_{\alpha_1,\ldots,\alpha_k}(C)$ follows from Equation (60) upon the replacement of $\vec{\mu}_\alpha$ by the sum $\vec{\mathrm{M}}_k = \sum_{i=1}^{k} \vec{\mu}_{\alpha_i}$. As has been shown in Section 1.3, the dominant contribution to the sum (45) stems from those terms where $\vec{\mathrm{M}}_k$ consists of mutually different vectors $\vec{\mu}_{\alpha_i}$'s. Then, owing to Equation (50) for the square of such vectors $\vec{\mathrm{M}}_k$'s, we obtain Casimir scaling of k-string tensions.

One can further perform the derivative expansion of the non-local string action (61) defined at some non-minimal surface S. As a result, one obtains the values of the string tension and of the rigidity-term coupling. Instead of the action (61), one can consider a general action of this type, namely

$$\mathcal{A}_{\mathrm{str}}[S] = \int_S d\sigma_{\mu\nu}(x) \int_S d\sigma_{\mu\nu}(x')\, \mathcal{D}\left[\left(\frac{x-x'}{\lambda}\right)^2\right], \tag{62}$$

where the vector $x_\mu(\vec{\xi})$ parameterizing the surface S, can correspond to the Euclidean space-time of any dimension. The function $\mathcal{D}(x)$, which has the dimensionality of [mass]^4, falls off as $O(\mathrm{e}^{-|x|/\lambda})$ at $|x| \gtrsim \lambda$, where λ is the vacuum correlation length in a given confining gauge theory. Since the derivatives with respect to the world-sheet coordinates have the order of $O(1/\sqrt{\Sigma_{\mathrm{min}}})$, the derivative expansion converges provided

$$\lambda < \sqrt{\Sigma_{\mathrm{min}}}. \tag{63}$$

Physicswise, this inequality means that confinement in a certain gauge theory takes place and allows for an effective string description provided confined particles are separated from each other by the distances which are larger than the vacuum correlation length in that theory. More specifically, it turns out that the terms of the derivative expansion are proportional to the even-order integral moments $\int d^2z\, |\vec{z}|^{2n}\, \mathcal{D}(z^2)$ of the function $\mathcal{D}$, so that the actual parameter of the expansion is $\lambda^2/\Sigma_{\mathrm{min}}$.

The details of the derivative expansion can be found in Ref. [30]. An important relation used in the course of this derivation is the so-called Gauss-Weingarten formula $D_a D_b x_\mu = K^i_{ab} n^i_\mu$ for the covariant

[23] Note that this equation is actually highly non-linear, since the metric g_{ab} entering the Laplacian $\Delta = \frac{1}{\sqrt{g}}\partial_a \sqrt{g} g^{ab} \partial_b$, is induced by the same vector-function $x_\mu(\vec{\xi})$, i.e., $g_{ab} = \partial_a x_\mu \cdot \partial_b x_\mu$.

derivative $D_a D_b x_\mu \equiv D_a \partial_b x_\mu = \partial_a \partial_b x_\mu - \Gamma^c_{ab} \partial_c x_\mu$. This formula allows one to replace the products of ordinary derivatives $\partial_a \partial_b x_\mu$ by the products of covariant derivatives $D_a D_b x_\mu$. In the above relations, Γ^c_{ab} is a Christoffel symbol defined with respect to the induced metric g_{ab}, n^i_μ's are the unit normals to the world sheet, which are labeled by the index $i = 1, \ldots, D-2$, and K^i_{ab} is the second fundamental form of the world sheet. The normals n^i_μ's obey the condition $n^i_\mu \cdot \partial_a x_\mu = 0$, which yields the following orthogonality relation: $D_a D_b x_\mu \cdot \partial_c x_\mu = 0$. In particular, by virtue of this relation, one can prove a complete mutual cancellation of the $O(\lambda^4)$-terms proportional to $\int d^2\xi (\partial_a \ln \sqrt{g})^2$, in $\mathcal{A}_{\rm str}[S]$ [24]. At the final step of the calculation, one converts the so-emerging products of the covariant derivatives [25], $D_a D_b x_\mu$, into the products of the second fundamental form, by using, e.g., the formula

$$(g^{ab}g^{cd} + g^{ad}g^{bc} + g^{ac}g^{bd}) D_a D_b x_\mu \cdot D_c D_d x_\mu = 3(\partial_a \partial^a x_\mu)^2 + 2(K^i_{ab} K^{i,ab} - K^{i\,a}_a K^{i\,b}_b),$$

which can be proved through the orthonormality relation $n^i_\mu n^j_\mu = \delta^{ij}$. One can further make use of the relation $K^i_{ab} K^{i,ab} - K^{i\,a}_a K^{i\,b}_b = -R$, where R is the scalar curvature of the world sheet. In the conformal gauge, it has the form $R = -\frac{\partial^2 \ln \sqrt{g}}{\sqrt{g}}$, so that $\sqrt{g}(K^i_{ab} K^{i,ab} - K^{i\,a}_a K^{i\,b}_b)$ yields a full derivative, which does not contribute to the string action [26]. Altogether, up to the irrelevant full derivatives, one obtains the following result for the two leading terms of the derivative expansion of the non-local string action:

$$\mathcal{A}_{\rm str}[S] \simeq \sigma \int d^2\xi \sqrt{g} + \frac{1}{2\alpha} \int d^2\xi \sqrt{g}\, (\Delta x_\mu)^2, \tag{64}$$

where

$$\sigma = 2\lambda^2 \int d^2 z\, \mathcal{D}(\vec{z}^{\,2}) \quad \text{and} \quad \frac{1}{\alpha} = -\frac{\lambda^4}{4} \int d^2 z\, \vec{z}^{\,2}\, \mathcal{D}(\vec{z}^{\,2}) \tag{65}$$

are the string tension and the rigidity-term coupling, respectively. In particular, we see that α comes out negative, as was mentioned at the end of Section 1.2. Among the terms that have been omitted in Equation (64), the leading ones have the coefficients proportional to the next even-order integral moment of the function $\mathcal{D}$, i.e., these coefficients have the order of $\lambda^8 \int d^2 z\, |\vec{z}|^4\, \mathcal{D}(\vec{z}^{\,2})$.

As follows from Equations (61) and (62), one has $\mathcal{D} = \frac{(g\vec{\mu}_\alpha M_D)^2}{4} D_{M_D}$ for the case of the 3D $[{\rm U}(1)]^{N-1}$-invariant compact QED under discussion. Equations (65) with this function $\mathcal{D}$ yield [92]

$$\sigma = 2\pi^2 \cdot \frac{N-1}{\sqrt{N}} \cdot \frac{\sqrt{\zeta}}{g_m} \quad \text{and} \quad \frac{1}{\alpha} = -\frac{\pi^2}{4} \cdot \frac{N-1}{N^{3/2}} \cdot \frac{1}{g_m^3 \sqrt{\zeta}}.$$

The obtained expression for σ provides a particular value of the overall numerical coefficient in Equation (44), which applies to the limiting case where the monopole density $|\vec{\rho}|$ is much smaller than the mean density (33). On the other hand, the above-obtained expression for σ has an advantage over Equation (44) of being applicable to an arbitrarily shaped, and not only flat, surface S.

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[24] This means that the non-local term $\int d^2\xi d^2\xi' \sqrt{g} R \left(-\frac{1}{\Delta}\right)_{\xi,\xi'} \sqrt{g'} R'$ discussed in Section 1.2, does not appear in the course of the curvature expansion, i.e., this expansion does not provide the possibility to cancel the conformal anomaly for some value of D other than 26.

[25] These products are only non-trivial for $a \neq b$, since $\Gamma^b_{aa} = 0$.

[26] Note that this is precisely the reason why the Einstein equations (in vacuum) are identically satisfied in the 2D case, i.e., General Relativity is trivial in 2D. In particular, according to the so-called Gauss-Bonnet theorem, the Einstein-Hilbert action in 2D reads $\int d^2\xi \sqrt{g}\, R = 4\pi\chi$, where χ is the Euler characteristic of the world sheet. And the other way around, one can say that General Relativity in vacuum, as described by the 4D Einstein-Hilbert action, is nothing but a generalization of the Euler characteristic to the 4D case, with the replacement of the induced metric $g_{ab} = \partial_a x_\mu \cdot \partial_b x_\mu$ by a certain 4D metric $g_{\mu\nu}(x)$.

1.5. Self-Intersections of the Confining-String World Sheet Due to the Θ*-term*

In this Section, we will consider an example of a confining gauge theory, where the derivative expansion of the resulting non-local string action yields a string θ-term proportional to the number of string self-intersections (25). Namely, the string θ-term turns out to appear in the 4D $[\mathrm{U}(1)]^{N-1}$-invariant compact QED extended by the field-theoretical Θ-term. The derivative expansion of the non-local string action yields then the string coupling θ expressed through the vacuum angle Θ. As a result, Θ's corresponding to the critical value of $\theta = \pi$, which was discussed beneath Equation (25), will be expressed in terms of the gauge coupling g and the number of colors N.

Prior to the start of this analysis, let us discuss similar topological phenomena which take place in the lower-dimensional spaces. In 2D, the world-line representation for the propagator of a free fermion yields the number of self-intersections of fermionic trajectories. It stems from the fermion's spin factor, which is proportional to the commutator of γ-matrices. In 2D, this commutator yields the totally antisymmetric tensor $\varepsilon_{\mu\nu}$. If one parameterizes the trajectory through a vector-function $x_\mu(\tau)$, where the parameter τ has the dimensionality of length, then the number of self-intersections of the trajectory is given by the formula $\frac{1}{2\pi}\int_0^L d\tau\, \varepsilon_{\mu\nu}\ddot{x}_\mu\dot{x}_\nu$, where L is the length of the trajectory and the dot stands for $\frac{d}{d\tau}$. This number increases by 1 every time the trajectory winds counterclockwise, and decreases by 1 every time the trajectory winds clockwise. Furthermore, one can show that the number of self-intersections enters the world-line representation of the fermionic propagator with the coefficient equal to π (cf. Refs. [48,109]). Consequently, contributions to the world-line integral representing the fermionic propagator, which are produced by some two trajectories whose lengths are nearly the same but the numbers of self-intersections differ from each other by 1, cancel each other. For this reason, fermionic trajectories in 2D are much smoother than bosonic trajectories. Quantitatively, their Hausdorff dimension is equal to 1, i.e., the length of a fermionic trajectory grows linearly with the distance between its end-points, whereas the Hausdorff dimension of bosonic trajectories is equal to 2, i.e., the length of a bosonic trajectory grows as a square of the distance between the end-points. Note that, for a bosonic trajectory to lower its Hausdorff dimension to 1, the world-line action of the corresponding random walk should contain an additional term proportional to the absolute value of the curvature of the trajectory. In the presence of such a term, bending of a trajectory costs additional energy, whose amount is precisely such as to make the bosonic trajectories as smooth as the fermionic ones [109].

Coming closer to the 4D gauge theories with the Θ-term, let us consider next the 3D Maxwell theory extended by the Chern-Simons term [110] $i\Theta\varepsilon_{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda$. In this theory, the Wilson-loop average has the form

$$\langle W(C)\rangle = \int \mathcal{D}A_\mu\, \mathrm{e}^{-\int_x\left[\frac{1}{4g^2}F_{\mu\nu}^2 + i\Theta\varepsilon_{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda - iA_\mu j_\mu\right]}. \tag{66}$$

here, the electric coupling g has the dimensionality of $(\mathrm{mass})^{1/2}$, while the parameter Θ is dimensionless, and we use the notation $\int_x$ for $\int d^3x$. Furthermore, $j_\mu(\vec{\mathrm{x}}) \equiv j_\mu(\vec{\mathrm{x}}, C) = \oint_C dx_\mu(\tau)\,\delta(\vec{\mathrm{x}} - \vec{\mathrm{x}}(\tau))$ is a conserved current, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field-strength tensor. Performing the A_μ-integration in Equation (66), one arrives at the following expression (for details, see Appendix C):

$$\langle W(C)\rangle = \exp\left\{\frac{1}{2}\int_{x,y}\left[-g^2 j_\mu(\vec{\mathrm{x}})D_m(\vec{\mathrm{x}}-\vec{\mathrm{y}})j_\mu(\vec{\mathrm{y}}) + \frac{i}{2\Theta}\varepsilon_{\mu\nu\lambda}j_\mu(\vec{\mathrm{x}})j_\lambda(\vec{\mathrm{y}})\partial_\nu^x\left[D_0(\vec{\mathrm{x}}-\vec{\mathrm{y}}) - D_m(\vec{\mathrm{x}}-\vec{\mathrm{y}})\right]\right]\right\}. \tag{67}$$

Equation (67) yields a self-linkage of the contour C, as well as a short-range self-interaction of this contour through the Yukawa propagator $D_m(\vec{\mathrm{x}} - \vec{\mathrm{y}})$. By virtue of the expression for the Gauss' linking number of two contours, C and C', which has the form

$$\hat{L}(C, C') = \varepsilon_{\mu\nu\lambda}\int_{x,y} j_\mu(\vec{\mathrm{x}}, C)\, j_\nu(\vec{\mathrm{y}}, C')\, \partial_\lambda^x D_0(\vec{\mathrm{x}} - \vec{\mathrm{y}}),$$

we can represent the self-linkage term in Equation (67) as

$$\frac{i}{4\Theta}\,\varepsilon_{\mu\nu\lambda}\int_{x,y} j_\mu(\vec{x})j_\lambda(\vec{y})\partial_\nu^x D_0(\vec{x}-\vec{y}) = -\frac{i}{4\Theta}\hat{L}(C,C).$$

Thus, if the contour C of the Wilson loop is knotted k times, one gets a non-trivial phase factor of the Wilson-loop average provided $\Theta \neq \frac{k}{8\pi n}$, where n is some other integer [27]. Indeed, if this condition is not fulfilled, we have $-\frac{i}{4\Theta}\hat{L}(C,C) = -2\pi n i$, and the resulting phase factor becomes trivial, namely $\mathrm{e}^{-2\pi n i} = 1$. In general, it is a remarkable feature of the Chern-Simons term in the 3D Maxwell theory that it yields for the Wilson-loop average a phase factor which contains the number of self-linkings of the contour.

Let us now proceed to the 4D $[\mathrm{U}(1)]^{N-1}$-invariant compact QED. As has been discussed at the beginning of Section 1.4, the 4D compact QED provides confinement of external electrically charged particles only if the values of the electric coupling g are larger than a certain critical value. The reason for this fact is that only in such a strong-coupling regime can monopole loops become long enough as to create in the system the degree of disorder sufficient for confinement. Because of the Dirac quantization condition, the magnetic coupling g_m is small in this regime, so that the Coulomb interaction between monopole loops is weak. Thus, one has an ensemble of long monopole loops, which nevertheless interact with each other only weakly. Furthermore, unlike the 3D case, the dual photon in 4D is no longer a Lorentz scalar, but a Lorentz vector $\vec{\chi}_\mu$. Similar to Equation (34), one can consider a collective current corresponding to $n > 0$ monopole and/or antimonopole loops, which has the form

$$\vec{\mathrm{j}}_\mu^{(n)}(x) = g_m \sum_{k=1}^{n} \vec{\mathrm{q}}_{i_k} \oint dz_\mu^k(\tau)\delta(x - x^k(\tau)).$$

In this expression, we have parameterized k-th monopole loop by the vector $x_\mu^k(\tau) = y_\mu^k + z_\mu^k(\tau)$, with $y_\mu^k = \int_0^1 d\tau\, x_\mu^k(\tau)$ describing the position of the loop, and $z_\mu^k(\tau)$ describing its shape. In the presence of the Θ-term, the action describing the n-monopole configuration and the free photons $\vec{\mathrm{A}}_\mu$, reads

$$S[\vec{\mathrm{j}}_\mu^{(n)}, \vec{\mathrm{A}}_\mu] = \frac{1}{2}\int d^4x d^4y\, \vec{\mathrm{j}}_\mu^{(n)}(x) D_0(x-y)\vec{\mathrm{j}}_\mu^{(n)}(y) +$$

$$\frac{1}{4}\int d^4x\, \vec{\mathrm{F}}_{\mu\nu}^2 - \frac{i\Theta g^2}{32\pi^2}\int d^4x \left(\vec{\mathrm{F}}_{\mu\nu} + \vec{\mathrm{F}}_{\mu\nu}^{(n)}\right)\left(\tilde{\vec{\mathrm{F}}}_{\mu\nu} + \tilde{\vec{\mathrm{F}}}_{\mu\nu}^{(n)}\right).$$

Here the field-strength tensor $\vec{\mathrm{F}}_{\mu\nu}^{(n)}$, which describes the n-monopole configuration, violates the Bianchi identities in such a way as to yield the current $\vec{\mathrm{j}}_\mu^{(n)}$, namely $\partial_\mu \tilde{\vec{\mathrm{F}}}_{\mu\nu}^{(n)} = \vec{\mathrm{j}}_\nu^{(n)}$. Owing to this relation, the Θ-term can be rewritten up to a full derivative as

$$-\frac{i\Theta g^2}{32\pi^2}\int d^4x \left(\vec{\mathrm{F}}_{\mu\nu} + \vec{\mathrm{F}}_{\mu\nu}^{(n)}\right)\left(\tilde{\vec{\mathrm{F}}}_{\mu\nu} + \tilde{\vec{\mathrm{F}}}_{\mu\nu}^{(n)}\right) = \frac{i\Theta g^2}{8\pi^2}\int d^4x\, \vec{\mathrm{A}}_\mu \vec{\mathrm{j}}_\mu^{(n)}.$$

The partition function of the system can be obtained through the summation over the grand canonical ensemble of monopoles and antimonopoles, along with the integration over the free photons. The result can be represented in the form [111]

$$\mathcal{Z} = \int \mathcal{D}\vec{\mathrm{A}}_\mu \mathcal{D}\vec{\mathrm{j}}_\mu\, \mathrm{e}^{-S[\vec{\mathrm{j}}_\mu, \vec{\mathrm{A}}_\mu]} \int \mathcal{D}\vec{\chi}_\mu\, \mathrm{e}^{\int d^4x \left[2\zeta \sum_{i=1}^{(N^2-N)/2} \cos\left(\frac{\vec{\mathrm{q}}_i |\vec{\chi}_\mu|}{\Lambda}\right) + i\vec{\chi}_\mu \vec{\mathrm{j}}_\mu\right]}. \quad (68)$$

[27] Clearly, the same result holds also for the correlation function of two Wilson loops, whose contours are linked with each other k times.

Here $|\vec{\chi}_\mu|$ denotes the absolute value only with respect to the space-time (but not color) indices, i.e. $|\vec{\chi}_\mu| \equiv \left(\sqrt{\vec{\chi}_\mu^1\vec{\chi}_\mu^1}, \cdots, \sqrt{\vec{\chi}_\mu^{N-1}\vec{\chi}_\mu^{N-1}}\right)$. Furthermore, the dynamical monopole currents $\vec{\mathrm{j}}_\mu$'s represent a 4D generalization of the 3D dynamical monopole densities $\vec{\rho}$'s, which were introduced in Equation (39). Next, Λ is an ultra-violet cut-off, which unavoidably appears in the course of the summation over the grand canonical ensemble of monopole loops in 4D, and ζ is the monopole fugacity of dimensionality $(\text{mass})^4$. Clearly, in the absence of the Θ-term, the integration over $\vec{\mathrm{j}}_\mu$'s in Equation (68) recovers the standard kinetic term of the dual-photon field $\vec{\chi}_\mu$. In the presence of the Θ-term, due to the emerging coupling of $\vec{\mathrm{j}}_\mu$ to $\vec{\mathrm{A}}_\mu$, this is no longer the case. In general, instead of integrating over $\vec{\mathrm{j}}_\mu$'s, one can perform in Equation (68) a saddle-point integration over $\vec{\chi}_\mu$, which yields for the currents $\vec{\mathrm{j}}_\mu$'s a potential of the type (57).

We consider further the Wilson-loop average corresponding to a test particle which transforms under the fundamental representation of the group SU(N). Introducing, instead of $\vec{\mathrm{j}}_\mu$, an antisymmetric-tensor field $\vec{\mathrm{h}}_{\mu\nu}$ according to the relation $\partial_\mu \tilde{\vec{\mathrm{h}}}_{\mu\nu} = \vec{\mathrm{j}}_\nu$, we have

$$W_\alpha = \frac{1}{\mathcal{Z}} \int \mathcal{D}\vec{\mathrm{h}}_{\mu\nu}\, \mathrm{e}^{-S[\vec{\mathrm{h}}_{\mu\nu}] + \frac{ig}{2}\vec{\mu}_\alpha \int_S d\sigma_{\mu\nu}\vec{\mathrm{h}}_{\mu\nu}}. \tag{69}$$

In this expression, the action of the antisymmetric-tensor field has the form

$$S[\vec{\mathrm{h}}_{\mu\nu}] = \int d^4x \left(\frac{1}{4}\vec{\mathrm{h}}_{\mu\nu}^2 - \frac{i\Theta g^2}{32\pi^2}\vec{\mathrm{h}}_{\mu\nu}\tilde{\vec{\mathrm{h}}}_{\mu\nu}\right) + V[\vec{\mathrm{h}}_{\mu\nu}], \tag{70}$$

where the potential $V[\vec{\mathrm{h}}_{\mu\nu}]$ is given by Equation (57) with [28] $H_\alpha = \frac{g\Lambda}{(N-1)\zeta}\vec{\mu}_\alpha \left|\partial_\mu \tilde{\vec{\mathrm{h}}}_{\mu\nu}\right|$ and $\int d^3x$ replaced by $\int d^4x$. The mass acquired by the dual-photon field is equal to the mass of the antisymmetric-tensor field following from Equation (70), and reads

$$M = \frac{\eta}{4\pi}\sqrt{g_m^2 + \left(\frac{\Theta g}{2\pi}\right)^2},$$

where $\eta \equiv \frac{\sqrt{\zeta N}}{\Lambda}$. As one can see from this expression, the dual-photon field acquires in addition to the magnetic charge g_m also the electric charge $\frac{\Theta g}{2\pi}$, i.e., due to the Θ-term, it becomes a dyon.

In the weak-field limit of $|H_\alpha| \ll 1$, where the absolute value is now defined with respect to the color index, Equation (69) takes the form

$$W_\alpha \simeq \frac{1}{\mathcal{Z}} \int \mathcal{D}\vec{\mathrm{h}}_{\mu\nu} \exp\left[-\int d^4x \left(\frac{1}{12\eta^2}\vec{\mathrm{H}}_{\mu\nu\lambda}^2 + \frac{1}{4g^2}\vec{\mathrm{h}}_{\mu\nu}^2 - \frac{i\Theta}{32\pi^2}\vec{\mathrm{h}}_{\mu\nu}\tilde{\vec{\mathrm{h}}}_{\mu\nu}\right) + \frac{i}{2}\vec{\mu}_\alpha \int_S d\sigma_{\mu\nu}\vec{\mathrm{h}}_{\mu\nu}\right],$$

which generalizes Equation (59) to the 4D case with the Θ-term. The $\vec{\mathrm{h}}_{\mu\nu}$-integration in this expression is similar to the one in the absence of the Θ-term. Referring the reader for the details of this integration to Ref. [111], we present here the resulting formula for the Wilson-loop average. It reads

$$\begin{gathered} W_\alpha \simeq \exp\Bigg\{-\frac{\vec{\mu}_\alpha^2}{2}\Bigg[\frac{\eta^2}{2}\int_S d\sigma_{\mu\nu}(x)\int_S d\sigma_{\mu\nu}(y)\; + \\ \frac{i\Theta g^2}{8\pi^2}\int_S d\sigma_{\mu\nu}(x)\int_S d\tilde{\sigma}_{\mu\nu}(y) + g^2\oint_C dx_\mu \oint_C dy_\mu\Bigg] D_M(x-y)\Bigg\}, \end{gathered} \tag{71}$$

where $D_M(x) = MK_1(M|x|)/(4\pi^2|x|)$ is the 4D Yukawa propagator with K_1 standing for the Macdonald function. Clearly, Equation (71) represents a generalization of Equation (60) to the 4D case with

[28] In the following formula, the absolute value is again defined only with respect to the space-time indices, not the color ones.

the Θ-term. Furthermore, the interaction of two world-sheet elements corresponding to the term $\propto i\Theta \int_S d\sigma_{\mu\nu}(x) \int_S d\tilde{\sigma}_{\mu\nu}(y) D_M(x-y)$ in Equation (71), allows for a derivative expansion in a way similar to the one described in Section 1.4. Since $d\sigma_{\mu\nu}(x)d\tilde{\sigma}_{\mu\nu}(x) = 0$, the first term of this expansion, which could be analogous to the Nambu-Goto action, vanishes. The second term of the expansion has the same order in the derivatives as the rigidity term. It can be written as θn, where n is given by Equation (25) and represents the number of self-intersections of the string world sheet, while θ can be calculated through the first even-order integral moment of the Yukawa propagator, similarly to the coupling $1/\alpha$ from Equation (65). The so-obtained θ, much as the coupling α, comes out dimensionless and therefore independent of the cut-off Λ. It has the form

$$\theta = \frac{\Theta(g\vec{\mu}_\alpha)^2}{4\left[g_m^2 + \left(\frac{\Theta g}{2\pi}\right)^2\right]}.$$

Solving the equation $\theta = \pi$ with respect to Θ, we obtain the following critical values of the latter:

$$\Theta_\pm = \frac{\pi}{2}\left[\vec{\mu}_\alpha^2 \pm \sqrt{\vec{\mu}_\alpha^2 - \left(\frac{16\pi}{g^2}\right)^2}\right]. \tag{72}$$

Note that these values of Θ are expressed entirely in terms of g and N. Recalling that $\vec{\mu}_\alpha^2 = \frac{N-1}{2N}$, we find the lower bound $g_{\min} = 4\sqrt{\frac{2\pi N}{N-1}}$, starting from which the values $\Theta_\pm$ become accessible. The existence of such a lower bound for the electric coupling parallels the above-discussed strong-coupling regime, which is a necessary requirement for confinement in the 4D compact QED. In the particular case of an extreme strong-coupling limit imposed by the inequality $g \gg g_{\min}$, only the critical value $\Theta_+ \simeq \pi\vec{\mu}_\alpha^2$ remains relevant, while Θ_- vanishes, becoming thereby an unphysical solution. In the general case, once Θ is equal either to Θ_+ or Θ_- given by Equation (72), the statistical weight of an n-times self-intersecting world sheet in the functional sum (8) acquires an additional factor of $(-1)^n$. Thus, the possibility of obtaining the string θ-term from the confining gauge theory with a non-vanishing vacuum angle Θ, demonstrates that the presence of the vacuum angle can serve as a possible mechanism for the solution of the problem of crumpling of string world sheets.

As we have discussed, the 4D compact QED, as well as its $[\mathrm{U}(1)]^{N-1}$-invariant generalization, possesses confinement only in the strong-coupling regime. A natural question which therefore can be posed, is whether an Abelian gauge theory possessing confinement for *all* values of the coupling, can be constructed in 4D. By analogy with the 3D compact QED, one can argue [48] that such a theory should allow for the existence of the plasma of magnetically-charged objects, which are point-like but nevertheless possess a finite action. In the continuum limit, the grand canonical ensemble of such objects is described by a 4D sine-Gordon theory of a scalar "dual-photon" field φ. According to the duality relation, $\varepsilon_{\mu\nu\lambda\rho}\partial_\rho\varphi = \partial_\mu h_{\nu\lambda}$, the field dual to a scalar in 4D is an antisymmetric tensor. Therefore, such a theory can be viewed as an analogue of compact QED, where the role of the photon field A_μ is played by an antisymmetric-tensor field $h_{\mu\nu}$. Much as in compact QED, where the full field-strength tensor is given by the sum of the free-photon and the monopole field strengths, in the theory at issue the full strength tensor of the antisymmetric-tensor field is a sum of the strength tensor $H_{\mu\nu\lambda} = \partial_\mu h_{\nu\lambda} + \partial_\lambda h_{\mu\nu} + \partial_\nu h_{\lambda\mu}$ and the strength tensor which violates a 4D analog of the Bianchi identity for point-like magnetically charged objects. Specifically, for n such objects, the latter strength tensor obeys the relation $\varepsilon_{\mu\nu\lambda\rho}\partial_\mu H^{(n)}_{\nu\lambda\rho} \propto g_m \sum\limits_{a=1}^{n} q_a \delta(\vec{x} - \vec{z}_a)$, where q_a's are the charges of objects constituting the configuration, in the units of the magnetic coupling g_m. Furthermore, unlike the vector field, which couples to a world line, the antisymmetric-tensor field couples to a world sheet. For this reason, in the theory of an antisymmetric-tensor field, a counterpart of the Wilson loop, $\exp\left(ig \oint_C dx_\mu A_\mu\right)$, has the form $\exp\left(ig \oint_S d\sigma_{\mu\nu} h_{\mu\nu}\right)$, where S is some closed surface. Therefore, owing to the Gauss' theorem, the contribution of n magnetically charged objects to such a "Wilson loop" is equal to

$\exp\left(\frac{ig}{2}\int_\Sigma d\Sigma_{\mu\nu\lambda}H^{(n)}_{\mu\nu\lambda}\right)$, where Σ is some hypersurface bounded by S. Upon the summation over the grand canonical ensemble of magnetically charged objects, one gets for the corresponding "Wilson-loop average" an analog of the area law for the Wilson-loop average in compact QED, which can be called a volume law [112]. Clearly, this law means an exponential fall-off of the "Wilson-loop average" with the volume of the minimal hypersurface $\Sigma_{\rm min}$ bounded by a given closed surface S. Since the physical meaning of S can be the world sheet of a closed string, the volume law quantifies confinement of closed strings which carry electric fluxes. Thus, the 4D theory possessing confinement for all values of the coupling, describes actually confinement of closed strings rather than of electrically charged particles, and it can be referred to as a theory of confining membranes (for details, see [112]). In the general case of a higher-dimensional Euclidean space, the duality relation leads to a certain connection between the dimensionalities of magnetically charged objects, whose condensation can be described in terms of a grand canonical ensemble, and of the electrically charged objects confined owing to this condensation [113].

In particular, an important observation is that confinement of point particles most naturally occurs in 3D and 4D. Indeed, for any space-time dimensionality, the confinement criterion for a point particle is provided by the Wilson area law. As we have seen above, this law can be achieved through the coupling of an antisymmetric-tensor field to the world sheet of the confining string, $\int_S d\sigma_{\mu\nu}h_{\mu\nu}$. Therefore, since the string world sheet is two-dimensional, confinement of point particles is described in terms of an antisymmetric-tensor field in the space-time of any dimensionality. The Bianchi identities violated by the antisymmetric-tensor field in 3D and 4D, read $\varepsilon_{\mu\nu\lambda}\partial_\mu h_{\nu\lambda} \propto \rho$ and $\varepsilon_{\mu\nu\lambda\rho}\partial_\nu h_{\lambda\rho} \propto j_\mu$, where ρ and j_μ are the dynamical monopole density and the dynamical monopole current, respectively. Accordingly, in the space-time of some dimensionality $D > 4$, the Bianchi identities violated by the antisymmetric-tensor field, correspond to magnetically charged objects whose dynamical density is given by a totally antisymmetric tensor with $D-3$ indices. Clearly, such higher-dimensional magnetically charged objects can hardly be called monopoles, since the latter are normally assumed to be particle-like. This is the reason why the confinement scenario based on the grand canonical ensemble of magnetic monopoles is unlikely to hold for $D > 4$. Naturally, since the experimental and the lattice data provide evidence for confinement of quarks and gluons in the physically relevant case of $D = 4$, the above-presented argumentation suggests a reason for the four-dimensionality of the real world.

1.6. String Representation of the 't Hooft-Loop Average in the $[\mathrm{U}(1)]^{N-1}$-Invariant Dual Abelian Higgs Model Extended by the Θ-term

In the previous Section, we have considered the grand canonical ensemble of monopole loops, which form a quantum plasma. It turns out that, performing the summation over this grand canonical ensemble by imposing the property of a short-range repulsion of monopole loops, one obtains an effective description of the monopole condensate in terms of a magnetically charged Higgs field [114,115]. Accordingly, the resulting mean field theory is a dual Abelian Higgs model, in which the dual Higgs field minimally interacts with the dual gauge field. The derivation of such a dual Abelian Higgs model from the Yang-Mills theory starts with fixing the so-called maximal Abelian gauge [29]. In the 4D Yang-Mills theory, this gauge fixing leads to the same $\mathrm{SU}(N)\to[\mathrm{U}(1)]^{N-1}$ symmetry-breaking pattern as in the above-considered case of the 3D SU(N) Georgi-Glashow model. Upon this symmetry breaking, the N^2-N off-diagonal gluons of the Yang-Mills theory become massive, and therefore infra-red irrelevant, similarly to the W-bosons of the 3D Georgi-Glashow model. The resulting theory, emerging prior to the summation over the grand canonical ensemble of monopole loops, is therefore a $[\mathrm{U}(1)]^{N-1}$-invariant compact QED. For simplicity, let us start with considering the case of $N = 2$.

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29 Cf. Ref. [13].

As we have seen in the previous Section, monopoles in compact QED can be accounted for by adding to the Maxwell strength tensor $F_{\mu\nu}[A] = \partial_\mu A_\nu - \partial_\nu A_\mu$ the monopole one, $\mathcal{F}_{\mu\nu}$, which violates the Bianchi identities to yield the monopole current j_μ as $\partial_\nu \tilde{\mathcal{F}}_{\nu\mu} = j_\mu$. Accordingly, for the current j_μ corresponding to a certain contour C, along which a single monopole evolves in the Euclidean space, one can define the so-called 't Hooft-loop average

$$\langle H(C) \rangle = \int \mathcal{D}A_\mu \, \mathrm{e}^{-\frac{1}{4}\int d^4x\, (F_{\mu\nu}[A] + \mathcal{F}_{\mu\nu})^2}. \tag{73}$$

One can further apply to this expression a duality transformation. The purpose of this transformation is to represent $\langle H(C) \rangle$ in the form of a functional integral over the dual gauge field B_μ, which couples directly to the monopole current j_μ. To perform the transformation, one first represents the exponential in Equation (73) in terms of the functional integral over an auxiliary antisymmetric-tensor field $\lambda_{\mu\nu}$ as

$$\mathrm{e}^{-\frac{1}{4}\int d^4x\, (F_{\mu\nu}[A] + \mathcal{F}_{\mu\nu})^2} = \int \mathcal{D}\lambda_{\mu\nu} \, \mathrm{e}^{-\int d^4x \left[\frac{1}{4}\lambda_{\mu\nu}^2 + \frac{i}{2}\lambda_{\mu\nu}(\tilde{F}_{\mu\nu}[A] + \tilde{\mathcal{F}}_{\mu\nu})\right]}.$$

Performing in the term $\propto \int d^4x\, \lambda_{\mu\nu}\tilde{F}_{\mu\nu}[A]$ integration by parts, we can further carry out the functional integration over the A_μ-field as over a Lagrange multiplier. This yields the equation $\partial_\mu \tilde{\lambda}_{\mu\nu} = 0$, whose solution has the form $\lambda_{\mu\nu} = F_{\mu\nu}[B]$. Accordingly, for the term describing the interaction of the $\lambda_{\mu\nu}$-field with monopoles, we have $-\frac{i}{2}\int d^4x\, \lambda_{\mu\nu}\tilde{\mathcal{F}}_{\mu\nu} = i\int d^4x\, B_\mu j_\mu$. Altogether, in terms of the dual gauge field B_μ, the 't Hooft-loop average reads

$$\langle H(C) \rangle = \int \mathcal{D}B_\mu \, \mathrm{e}^{-\frac{1}{4}\int d^4x\, F_{\mu\nu}^2[B] + i\int d^4x\, B_\mu j_\mu}. \tag{74}$$

Thus, the duality transformation casts the 't Hooft-loop average to the form of a Wilson-loop average defined in terms of the dual gauge field, i.e., the 't Hooft- and the Wilson-loop averages are dual to each other.

The obtained dual representation of the 't Hooft-loop average can further be used for the summation over the grand canonical ensemble of monopole loops. To perform such a summation, we specify the current j_μ to the form of a collective current of n monopole loops, namely $j_\mu^{(n)} = g_m \sum_{k=1}^{n} \oint dx_\mu(s_k)\delta(x - x(s_k))$, where g_m is the magnetic coupling. One further imposes the summation over the grand canonical ensemble of monopole loops in the form of the average of the phase factor in Equation (74) with the following path-integral measure [114,115]:

$$\langle \mathrm{e}^{i\int d^4x\, B_\mu j_\mu^{(n)}} \rangle_{j_\mu^{(n)}} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\prod_{k=1}^{n} \int_0^\infty \frac{ds_k}{s_k} \, \mathrm{e}^{2\lambda\eta^2 s_k} \oint \mathcal{D}x_\mu(s'_k)\right] \exp\left\{\sum_{k=1}^{n} \int_0^{s_k} ds'_k \times \right.$$
$$\left. \left[-\frac{1}{4}\dot{x}_\mu^2(s'_k) + ig_m \dot{x}_\mu(s'_k) B_\mu(x(s'_k))\right] - \lambda \sum_{k,l=1}^{n} \int_0^{s_k} ds'_k \int_0^{s_l} ds'_l \, \delta(x(s'_k) - x(s'_l))\right\},$$

where $\lambda > 0$. Comparing this expression with Equation (5), we observe that it additionally contains a term which leads to the short-range repulsion of monopole loops. While the effective action (5) corresponds to the partition function of a complex-valued field with the Lagrangian $\mathcal{L} = |D_\mu\varphi|^2$, the above average represents a generalization of that partition function to the case of a Langrangian which additionally contains the Higgs potential. This Lagrangian has the form $\mathcal{L} = |D_\mu\varphi|^2 + \lambda(|\varphi|^2 - \eta^2)^2$, where the magnetically charged Higgs field φ describes the monopole condensation, and the covariant derivative reads $D_\mu = \partial_\mu - ig_m B_\mu$. Thus, imposing the short-range repulsion property of monopole trajectories in the 4D compact QED, one arrives at a mean-field description of the grand canonical ensemble of these trajectories in terms of the dual Abelian Higgs model with the partition function

$$\mathcal{Z} = \int \mathcal{D}\varphi\mathcal{D}\varphi^*\mathcal{D}B_\mu\, \mathrm{e}^{-\int d^4x\left[\frac{1}{4}F_{\mu\nu}^2[B]+|D_\mu\varphi|^2+\lambda(|\varphi|^2-\eta^2)^2\right]}. \tag{75}$$

This model can therefore be viewed as an effective model of confinement, which stems from the 4D SU(2) Yang-Mills theory in the maximal Abelian gauge.

Within the dual description at issue, where the gauge field A_μ is replaced by the dual field B_μ, an electrically charged particle, which propagates along a closed contour C, is described through the 't Hooft-loop (and not the Wilson-loop) average. Furthermore, for the sake of generality, we will consider now the case of a $[\mathrm{U}(1)]^{N-1}$-invariant dual Abelian Higgs model, extended in addition by the Θ-term [116]. Therefore, the SU(N)-symmetry-breaking pattern in the case at issue is the same as in the above-considered SU(N)-invariant 3D Georgi-Glashow model. For this reason, in the present case too, the charges of quarks with respect to the maximal Abelian $[\mathrm{U}(1)]^{N-1}$ subgroup of the group SU(N) are distributed along N weight vectors $\vec{\mu}_\alpha$ of the group SU(N). Hence, the 't Hooft-loop average describing in the model at issue a quark of color α, is given by the formula

$$\begin{gathered}\langle H_\alpha(C)\rangle = \int\left(\prod_i |\Phi_i|\,\mathcal{D}\,|\Phi_i|\,\mathcal{D}\theta_i\right)\mathcal{D}\vec{\mathrm{B}}_\mu\,\delta\left(\sum_i\theta_i\right)\times\\ \exp\left\{-\int d^4x\left[\frac{1}{4}\left(\vec{\mathrm{F}}_{\mu\nu}+\vec{\mathrm{F}}_{\mu\nu}^{(\alpha)}\right)^2-\frac{i\Theta g_m^2}{16\pi^2}\left(\vec{\mathrm{F}}_{\mu\nu}+\vec{\mathrm{F}}_{\mu\nu}^{(\alpha)}\right)\left(\tilde{\vec{\mathrm{F}}}_{\mu\nu}+\tilde{\vec{\mathrm{F}}}_{\mu\nu}^{(\alpha)}\right)+\right.\right.\\ \left.\left.\sum_i\left[\left|\left(\partial_\mu-ig_m\vec{\mathrm{q}}_i\vec{\mathrm{B}}_\mu\right)\Phi_i\right|^2+\lambda\left(|\Phi_i|^2-\eta^2\right)^2\right]\right]\right\},\end{gathered} \tag{76}$$

where the index i in both the product and the sums runs from 1 to $(N^2-N)/2$. Clearly, the root vectors $\vec{\mathrm{q}}_i$'s appear in Equation (76) due to the fact that monopole charges are distributed along them. Furthermore, the dual Higgs fields, which describe monopole condensates, have been represented in the form $\Phi_i = |\Phi_i|\,\mathrm{e}^{i\theta_i}$. Since the SU($N$)-group is special, the phases θ_i's of the dual Higgs fields are subject to the constraint $\sum\limits_i \theta_i = 0$, which is imposed in Equation (76) by means of the corresponding δ-function [117]. Next, the field-strength tensor $\vec{\mathrm{F}}_{\mu\nu}^{(\alpha)}$ of the quark of color α violates the Bianchi identities, which are otherwise respected by the strength tensor $\vec{\mathrm{F}}_{\mu\nu} = \partial_\mu\vec{\mathrm{B}}_\nu - \partial_\nu\vec{\mathrm{B}}_\mu$ of the dual gauge field $\vec{\mathrm{B}}_\mu$. Such violated Bianchi identities have the form $\partial_\mu\tilde{\vec{\mathrm{F}}}_{\mu\nu}^{(\alpha)} = g\vec{\mu}_\alpha j_\nu$, where the electric coupling g is again related to the magnetic coupling g_m via the quantization condition $gg_m = 4\pi$, and [30] $j_\nu(x) = \oint_C dx_\nu(s)\delta(x-x(s))$, so that $g\vec{\mu}_\alpha j_\nu$ is the quark current. As can be readily checked with the use of the Stokes' theorem, these violated Bianchi identities are satisfied, e.g., by the strength tensor $\vec{\mathrm{F}}_{\mu\nu}^{(\alpha)} = -g\vec{\mu}_\alpha\tilde{\Sigma}_{\mu\nu}$, where the tensor $\Sigma_{\mu\nu}(x,S) = \int_S d\sigma_{\mu\nu}(y)\delta(x-y)$ is defined at an arbitrary surface S bounded by the contour C. Last but not least, we impose the normalization condition $\langle H_\alpha(0)\rangle = 1$.

We note further that the Θ-term in Equation (76) can be represented in the form

$$\frac{i\Theta g_m^2}{16\pi^2}\int d^4x\left(\vec{\mathrm{F}}_{\mu\nu}+\vec{\mathrm{F}}_{\mu\nu}^{(\alpha)}\right)\left(\tilde{\vec{\mathrm{F}}}_{\mu\nu}+\tilde{\vec{\mathrm{F}}}_{\mu\nu}^{(\alpha)}\right) = -\frac{i\Theta g_m}{\pi}\vec{\mu}_\alpha\int d^4x\,\vec{\mathrm{B}}_\mu j_\mu. \tag{77}$$

Thus, owing to the Θ-term, quarks acquire in addition to their electric charge also the magnetic charge $\Theta g_m/\pi$. Such particles, which possess both the electric and the magnetic charges, are called dyons. Consequently, the total charge of a dyon in our case reads

$$G \equiv \sqrt{\vec{\mu}_\alpha^2\left[g^2+(\Theta g_m/\pi)^2\right]}\;,$$

[30] For brevity, we omit the argument C of the current j_ν.

where again $\vec{\mu}_{\alpha}^{2} = \frac{N-1}{2N}$. As also follows from Equation (77), the acquired magnetic charge enables quarks to interact with the dual gauge field $\vec{\mathrm{B}}_{\mu}$ (cf. Ref. [118]).

Expanding the field Φ_i around the minimum of the Higgs potential, one obtains the masses of the dual Higgs field and of the dual vector boson, $m_H = 2\eta\sqrt{\lambda}$ and $m = g_m\eta\sqrt{N}$, respectively. In what follows, we will consider the 't Hooft-loop average (76) in the so-called London limit, which is characterized by the condition $\ln\kappa \gg 1$. Here $\kappa \equiv \frac{m_H}{m}$ is the Ginzburg-Landau parameter, which defines the type of dual superconductivity of the vacuum [8–10]. Thus, the London limit represents an extreme type-II dual superconductor. In this limit, not only the thickness of a dual Abrikosov-Nielsen-Olesen string, given by $1/m$, is much larger than the thickness of the string core, given by $1/m_H$, but even the logarithm of the ratio of these thicknesses is large [31]. It turns out (cf. Refs. [95,116]) that the London limit allows for a construction of an exact string representation of the 't Hooft-loop average (76). Furthermore, the consistency of the corresponding $[\mathrm{U}(1)]^{N-1}$-invariant dual Abelian Higgs model with the Yang-Mills theory in the large-N limit requires the coupling g to behave with the number of colors as

$$g = \sqrt{\bar{\lambda}/N}. \tag{78}$$

Here $\bar{\lambda}$ is the so-called 't Hooft coupling constant, which remains finite in the large-N limit [119]. The above definition of the London limit leads then to the following condition, which should be respected by the Higgs coupling λ in order for this limit to persist at large N: $\lambda \gg (2\pi e N)^2/\bar{\lambda}$. Following Refs. [95,116], we consider here the scaling behavior $\lambda = O(N^2)$, in which case κ stays N-independent in the large-N limit, i.e., the increase of N does not make the London limit deeper.

Integration over the radial parts $|\Phi_i|$ of the Higgs fields yields for Equation (76) in the London limit the following expression:

$$\begin{aligned}\langle H_\alpha(C)\rangle = \int \left(\prod_i \mathcal{D}\theta_i^{\mathrm{st}}\mathcal{D}\theta_i^{\mathrm{sm}}\right)\mathcal{D}\vec{\mathrm{B}}_\mu\,\mathcal{D}k\,\delta\left(\sum_i \theta_i^{\mathrm{st}}\right)\times \\ \exp\Bigg\{-\int d^4x\Bigg[\frac{1}{4}\left(\vec{\mathrm{F}}_{\mu\nu}+\vec{\mathrm{F}}_{\mu\nu}^{(\alpha)}\right)^2 - \frac{i\Theta g_m^2}{16\pi^2}\left(\vec{\mathrm{F}}_{\mu\nu}+\vec{\mathrm{F}}_{\mu\nu}^{(\alpha)}\right)\left(\tilde{\vec{\mathrm{F}}}_{\mu\nu}+\tilde{\vec{\mathrm{F}}}_{\mu\nu}^{(\alpha)}\right) - \\ ik\sum_i \theta_i^{\mathrm{sm}} + \eta^2\sum_i\left(\partial_\mu\theta_i - g_m\vec{\mathrm{q}}_i\vec{\mathrm{B}}_\mu\right)^2\Bigg]\Bigg\}.\end{aligned} \tag{79}$$

Here we have decomposed the phases of the dual Higgs fields into a multi-valued part θ_i^{st} and a single-valued part θ_i^{sm} as $\theta_i = \theta_i^{\mathrm{st}} + \theta_i^{\mathrm{sm}}$, where "st" and "sm" stand for "string" and "smooth", respectively. The fields θ_i^{st}'s describe closed dual strings, being related to the world sheets Σ_i's of those strings through the equation

$$\varepsilon_{\mu\nu\lambda\rho}\partial_\lambda\partial_\rho\theta_i^{\mathrm{st}}(x) = 2\pi\Sigma_{\mu\nu}^i(x) \equiv 2\pi\oint_{\Sigma_i} d\sigma_{\mu\nu}\left(x^{(i)}(\vec{\xi})\right)\delta\left(x - x^{(i)}(\vec{\xi})\right). \tag{80}$$

This equation represents a local formulation of the Stokes' theorem for the vector field $\partial_\mu\theta_i$. In Equation (80), the vector $x^{(i)}(\vec{\xi}) \equiv x_\mu^{(i)}(\vec{\xi})$ parameterizes the world sheet Σ_i of a closed string, and $\vec{\xi}$ denotes the 2D coordinate. Owing to the one-to-one correspondence between θ_i^{st}'s and Σ_i's established by Equation (80), the integration over θ_i^{st}'s is implied in the sense of a certain prescription of the summation over string world sheets. A natural prescription of this kind corresponds to the dilute

[31] We recall that the radius of the usual Abrikosov vortex, i.e., the distance to the center of the vortex, at which the vortex magnetic field experiences an exponential fall-off, coincides with the penetration depth of a magnetic field into the superconductor. Rather, the thickness of the vortex core, also called the coherence length, is a distance to the center of the vortex, at which the Higgs field acquires its vacuum expectation value, while at the smaller distances the Higgs condensate is, to a certain extent, destroyed by the magnetic field of the vortex. In particular, the Higgs condensate vanishes altogether at the center of the vortex. Accordingly, type-II superconductors are those in which the thickness of a vortex is larger than the thickness of its core.

plasma of closed strings with winding numbers equal to ±1 (cf. Refs. [120,121]). Indeed, two parallel strings, with fluxes circulating in the same direction, experience an attractive interaction through the Higgs-boson exchanges, and a repulsive interaction through the vector-boson exchanges [8–10]. Since these interactions exponentially fall off at the distances equal, respectively, to $1/m_H$ and $1/m$, in the London limit at issue the interaction provided by the vector-boson exchanges is long-ranged compared to the interaction provided by the Higgs-boson exchanges. This leads to a strong repulsion of the likely oriented strings and to a decomposition of strings with winding numbers larger than the unit one into those with the unit winding number.

As for the single-valued parts of the phases, $\theta_i^{\rm sm}$'s, they describe fluctuations of the fields θ_i's around a string configuration described by the multi-valued fields $\theta_i^{\rm st}$'s. By virtue of Equation (80), one can readily see that the integration measure $\mathcal{D}\theta_i$ gets factorized into the product $\mathcal{D}\theta_i^{\rm st}\mathcal{D}\theta_i^{\rm sm}$. The functional δ-function $\delta\Big(\sum\limits_i \theta_i\Big)$ in Equation (76) also gets factorized into the product $\delta\Big(\sum\limits_i \theta_i^{\rm st}\Big)\delta\Big(\sum\limits_i \theta_i^{\rm sm}\Big)$. The first of these two δ-functions can further be written as $\delta\Big(\sum\limits_i \Sigma_{\mu\nu}^i\Big)$, where the Jacobian [122] emerging from the change of integration variables $\theta_i^{\rm st} \to x^{(i)}(\vec{\xi})$, can be included into the integration measure $\mathcal{D}x^{(i)}$. The other δ-function, namely $\delta\Big(\sum\limits_i \theta_i^{\rm sm}\Big)$, has been represented in Equation (79) through the integral over the Lagrange multiplier $k(x)$. Owing to the relation $\sum\limits_i \vec{\rm q}_i = 0$, one can nevertheless see [123] that the integration over $k(x)$ yields only an inessential constant factor, which can thus be accounted for by changing the normalization condition of the functional integration measure. By using the correspondence (80) between $\theta_i^{\rm st}$ and $\Sigma_{\mu\nu}^i$, we arrive then at the following intermediate result:

$$\int\left(\prod_i \mathcal{D}\theta_i^{\rm st}\mathcal{D}\theta_i^{\rm sm}\right)\mathcal{D}k\,\delta\left(\sum_i \theta_i^{\rm st}\right)\exp\left\{-\int d^4x\left[\eta^2\sum_i\left(\partial_\mu\theta_i - g_m\vec{\rm q}_i\vec{\rm B}_\mu\right)^2 - ik\sum_i \theta_i^{\rm sm}\right]\right\} =$$

$$\int\left(\prod_i \mathcal{D}x^{(i)}(\xi)\mathcal{D}h_{\mu\nu}^i\right)\delta\left(\sum_i \Sigma_{\mu\nu}^i\right)\exp\left\{-\int d^4x\left[\frac{1}{24\eta^2}\left(H_{\mu\nu\lambda}^i\right)^2 - i\pi h_{\mu\nu}^i\Sigma_{\mu\nu}^i + ig_m\vec{\rm q}_i\vec{\rm B}_\mu\partial_\nu\tilde{h}_{\mu\nu}^i\right]\right\}.$$

Here an antisymmetric-tensor field $h_{\mu\nu}^i$ appears as a field dual to $\partial_\mu\theta_i^{\rm sm}$, and $H_{\mu\nu\lambda}^i = \partial_\mu h_{\nu\lambda}^i + \partial_\lambda h_{\mu\nu}^i + \partial_\nu h_{\lambda\mu}^i$ is the strength tensor of this field. Referring the reader for the details of the subsequent integrations over $\vec{\rm B}_\mu$ and $h_{\mu\nu}^i$ to Refs. [116,123], we present here the final result. It has the form

$$\begin{aligned}\langle H_\alpha(C)\rangle = \exp&\left[-\tfrac{G^2}{2}\oint_C dx_\mu\oint_C dy_\mu D_m(x-y)\right]\int\Big(\prod_i \mathcal{D}x^{(i)}(\vec{\xi})\Big)\delta\Big(\sum_i \Sigma_{\mu\nu}^i\Big)\times\\ &\exp\Big\{-2(\pi\eta)^2\int_{x,y}\Sigma_{\mu\nu}^{i,\alpha}(x)D_m(x-y)\Sigma_{\mu\nu}^{i,\alpha}(y) - 2i\Theta s_i^\alpha\,\hat{L}\,(\Sigma_i,C) +\\ &2i\Theta\int_{x,y}\Big(2\vec{\mu}_\alpha^2\tilde{\Sigma}_{\mu\nu}(x) - s_i^\alpha\,\tilde{\Sigma}_{\mu\nu}^i(x)\Big)j_\mu(y)\partial_\nu^x D_m(x-y)\Big\},\end{aligned} \tag{81}$$

where $D_0(x) = \frac{1}{4\pi^2x^2}$ and $D_m(x) = \frac{m}{4\pi^2|x|}K_1(m|x|)$ are the Coulomb and the Yukawa propagator, respectively, and we have denoted for brevity $\int_x \equiv \int d^4x$. Furthermore,

$$\hat{L}(\Sigma_i,C) \equiv \int_{x,y}\tilde{\Sigma}_{\mu\nu}^i(x)j_\nu(y)\partial_\mu^x D_0(x-y)$$

in Equation (81) is the Gauss' linking number of the closed-string world sheet Σ_i and the contour C. Yet another notation used in Equation (81) is $\Sigma_{\mu\nu}^{i,\alpha} \equiv \Sigma_{\mu\nu}^i - Ns_i^\alpha\Sigma_{\mu\nu}$, where the coefficients s_i^α are defined through the relation $\vec{\mu}_\alpha = \sum\limits_i s_i^\alpha\vec{\rm q}_i$. As follows from this relation, for a given α, there exist $(N-1)$ non-vanishing coefficients s_i^α, which are equal to $\pm\frac{1}{N}$ (for details, see Ref. [116]). Therefore, given the value of the coefficient in front of $\hat{L}(\Sigma_i,C)$, one concludes that for $\Theta \neq N\pi \times$ integer, dyons experience a long-range topological interaction with closed dual strings, in accordance with the general arguments presented in Ref. [124,125]. Physicswise, this interaction represents the dual Aharonov-Bohm effect in 4D. That is, owing to the magnetic charge acquired by dyons through the Θ-term, they interact with electric fluxes carried by the dual Abrikosov-Nielsen-Olesen strings.

Furthermore, in Equation (81), the term quadratic in $\Sigma_{\mu\nu}^{i,\alpha}$ describes (self-)interactions of closed strings, as well as of the string that confines the dyon-antidyon pair, which are mediated by the dual-vector-boson exchanges. In particular, from the $\Sigma_{\mu\nu} \times \Sigma_{\mu\nu}$-interaction, we obtain through the general formulae (65) the following string tension and the inverse coupling constant of the rigidity term, which correspond to the confining-string world sheet Σ:

$$\sigma = 2\pi(N-1)\eta^2 \ln\kappa, \qquad \frac{1}{\alpha} = -\frac{\pi(N-1)}{4g_m^2 N}. \tag{82}$$

As we see, the string tension in the London limit receives an ultra-violet diverging contribution. For this reason, expression (82) for σ has been obtained within the logarithmic approximation of $\ln\kappa \gg 1$, which characterizes the London limit. One can prove that the so-obtained σ coincides with the energy density per the unit of length of an Abrikosov vortex. This energy density can be obtained by solving the Ginzburg-Landau equations, which describe the Higgs and the gauge fields of a vortex [8–10]. In the London limit, the corresponding solution can be found analytically. Another limit where the string tension can also be obtained analytically, is the so-called Bogomolny limit [86] of $\kappa = 1$, which thus borders between the type-II and the type-I superconductivity [32]. The result for the string tension in the Bogomolny limit follows from Equation (82) upon the replacement of $\ln\kappa$ by 1.

In the large-N limit, Equations (78) and (82) yield $\frac{1}{\alpha} = \mathcal{O}(1/N)$, making the anti-rigidity correction to the Nambu-Goto action additionally suppressed. As for the string tension, it is known to be N-independent in the large-N Yang-Mills theory, to the leading order of the strong-coupling expansion [22]. In our model, this condition can be fulfilled by imposing for η an N-dependence of the form $\eta \sim 1/\sqrt{N\ln\kappa}$. In particular, for the above-discussed large-N scaling of the Higgs coupling, $\lambda = \mathcal{O}(N^2)$, the N-dependence of η becomes simply $\eta = \mathcal{O}(N^{-1/2})$. Furthermore, since closed dual strings represent excitations of the vacuum, the mean sizes of their world sheets Σ_i's are much smaller than the mean size of the confining-string world sheet S. For this reason, closed strings and their interactions with the confining string can to the leading approximation be disregarded altogether. Within this approximation, one readily obtains Casimir scaling for the k-string tensions [95]. Corrections to this scaling, produced by closed dual strings, as well as by the deviation from the London limit, can also be obtained analytically (cf. Ref. [95]).

In conclusion of this Section, let us briefly discuss topological effects caused by the 3D counterpart of the Θ-term, i.e., the Chern-Simons term [110], in the 3D dual Abelian Higgs model. This model is nothing but the dual Landau-Ginzburg theory, whose partition function in the London limit has the form

$$\mathcal{Z} = \int \mathcal{D}\theta^{\mathrm{vor}} \mathcal{D}\theta^{\mathrm{sm}} \mathcal{D}B_\mu \, \mathrm{e}^{-\int d^3x \left[\frac{1}{4g_m^2} F_{\mu\nu}^2 + \eta^2(\partial_\mu\theta + mB_\mu)^2 + i\Theta\varepsilon_{\mu\nu\lambda}B_\mu\partial_\nu B_\lambda\right]},$$

where the dimensionalities of various parameters are

$$[\eta] = (\mathrm{mass})^{1/2}, \quad [m] = \mathrm{mass}, \quad [\Theta] = (\mathrm{mass})^2.$$

Here, θ^{vor} is the counterpart of θ_i^{st} in 3D, where "vor" stands for "vortex". Furthermore, the parameter m has been introduced for the purpose of providing to the magnetic coupling in 3D the correct dimensionality, $[g_m] = (\mathrm{mass})^{-1/2}$, while keeping the dual gauge field B_μ dimensionless.

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32 Note that the Ginzburg-Landau equations, while being in general second-order differential equations, get reduced in the Bogomolny limit to the first-order equations, which allow for a numerical solution [126].

Accordingly, all dimensionful quantities can be naturally measured in the units of m. In the limit of $\frac{\Theta}{m^2} \gg g_m \eta$, one then finds for $\mathcal{Z}$ the following representation [127]:

$$\mathcal{Z} = \int \mathcal{D}\vec{x}(s)\, e^{-(2\pi)^2 \int_{x,y} [\eta^2 J_\mu^x J_\mu^y D_M^{xy} + i\frac{\Theta}{m^2}\varepsilon_{\mu\nu\lambda} J_\mu^x J_\lambda^y \partial_\nu^x (D_0^{xy} - D_M^{xy})]}. \tag{83}$$

In this expression, $D_0^{xy} = 1/(4\pi|\vec{x}-\vec{y}|)$ and $D_M^{xy} = e^{-M|\vec{x}-\vec{y}|}/(4\pi|\vec{x}-\vec{y}|)$ are the Coulomb and the Yukawa propagators, with the mass M given by the formula $M = \frac{(m\eta)^2}{\Theta}$, and we have used the notation $\int_x \equiv \int d^3x$. Furthermore, $J_\mu^x = \oint dx_\mu(s)\delta(\vec{x}-\vec{x}(s))$ in Equation (83) is the current of a dual Abrikosov vortex, which is related to $\theta^{\rm vor}$ through the equation analogous to Equation (80), namely as $(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\theta^{\rm vor} = 2\pi\varepsilon_{\mu\nu\lambda}J_\lambda$. We observe now that the exponential in Equation (83) contains the term $i(2\pi)^2\frac{\Theta}{m^2}\hat{L}$, where $\hat{L} = \varepsilon_{\mu\nu\lambda}\int_{x,y} J_\mu^x J_\nu^y \partial_\lambda^x D_0^{xy}$ is the Gauss' linking number of an Abrikosov vortex with itself. This term does contribute to Equation (83) provided it is not equal to $2\pi in$, where n is some integer. Thus, the topological effect produced by the Chern-Simons term is that a vortex with N self-intersections contributes to the partition function a non-trivial phase factor $e^{i(2\pi)^2\frac{\Theta}{m^2}N}$, unless $\frac{\Theta}{m^2}N \neq \frac{n}{2\pi}$.

2. Critical Properties of the Weakly Coupled 3D Confining Theories

2.1. Deconfinement Phase Transition in the 3D Compact QED

As has been discussed in the previous Section, there exists an analogy between the mechanisms which create disorder in the 2D XY model and in the 3D compact QED. Namely, disorder in the 2D XY model is produced (at high temperatures) by the condensation of vortices, while in the 3D compact QED it is produced (at low temperatures) by the condensation of magnetic monopoles. Since the finite-temperature behavior of the 2D XY model is known (for a review, see e.g., Ref. [128]), this analogy should allow one to explore the finite-temperature behavior of the 3D compact QED. Indeed, with the decrease of temperature in the 2D XY model, a phase transition associated with binding of monopoles and antimonopoles into molecules takes place [104–106]. Since the short-range fields of such molecules are unable to produce the sufficient degree of disorder, the spin-spin correlation functions in the molecular phase fall off with the distance only power-like, contrary to their exponential fall-off in the high-temperature phase. Therefore, the analogy between the two theories suggests that, with the *increase* of temperature in the 3D compact QED, monopoles and antimonopoles can bind into molecules. Accordingly, since the short-range magnetic fields of such molecules do not produce the sufficient degree of disorder in the system, the molecular phase for monopoles should correspond to the deconfinement phase for external electrically charged particles. In this way, as has been shown in Refs. [129–132], the deconfinement phase transition in the 3D compact QED can be studied analytically.

Furthermore, it turns out that the difference in the space dimensionalities between the 2D XY model and the 3D compact QED, which might have influenced the above considerations, is unimportant, since the 3D compact QED undergoes the so-called dimensional reduction to a 2D theory, and the temperatures at which this reduction occurs are exponentially smaller than the temperature of the deconfinement phase transition. In general, for every quantity in a certain finite-temperature (bosonic) field theory, the dimensional-reduction temperature can be defined in such a way that the total contribution to this quantity produced at higher temperatures by all Matsubara frequencies $\omega_k = 2\pi Tk$ with $k \neq 0$, becomes negligible in comparison with the contribution produced by ω_0. Still, although contributions of the non-zero modes at higher temperatures amount to at most few percent of the static-mode contribution, they are nevertheless always present. For this reason, and also because the choice of a temperature-dependent quantity is not unique, the dimensional reduction is not a phase transition with a definite critical temperature, which could be determined from the thermodynamic equations (such as the equality of pressures in the two phases). In the particular case of the 3D compact QED, an estimate for the temperature of the dimensional reduction can be obtained from the following consideration. In the finite-temperature version of the sine-Gordon theory (30), the dual-photon field χ becomes subject to the periodic boundary conditions in the temporal direction, with the period $\beta \equiv 1/T$. Therefore, the lines of the magnetic field originating from a monopole or an antimonopole, are confined inside the region of a temporal extension β. Accordingly, at spatial distances larger than β, these lines go almost parallel to the boundary of the aforementioned region, asymptotically approaching this boundary with the increase of the spatial distance. For this reason, for a monopole and an antimonopole separated from each other by a spatial distance $\gtrsim \beta$, the 3D Coulomb law gets changed to the 2D one. Since, on the other hand, the mean distance between monopoles and antimonopoles constituting the plasma, is $\rho_{\rm mean}^{-1/3} \propto \zeta^{-1/3}$, one concludes that the 3D plasma gets effectively reduced to the 2D one at the temperatures $T \gtrsim \zeta^{1/3}$. Thus, the temperature of the dimensional reduction in the 3D compact QED has the order of $O(\zeta^{1/3})$.

Upon the dimensional reduction in the theory (30), the field χ becomes dependent only on the spatial coordinates. Accordingly, the action of such a theory acquires an overall factor of β, and takes the form

$$\beta \int d^2x \left[\frac{1}{2}(\partial_\mu \chi)^2 - 2\zeta \cos(g_m \chi)\right] = \int d^2x \left[\frac{1}{2}(\partial_\mu \varphi)^2 - 2\xi \cos(g_m \sqrt{T}\varphi)\right]. \tag{84}$$

Here we have restored the conventional form of the kinetic term by introducing a rescaled field $\varphi = \sqrt{\beta}\chi$, and also defined in the dimensionally-reduced theory a rescaled monopole fugacity $\xi \equiv \beta\zeta$. As we see, the dimensionally-reduced theory describes the grand canonical ensemble of monopoles and antimonopoles, which interact with each other through the 2D Coulomb potential with the temperature-dependent coupling $g_m \sqrt{T}$. We emphasize that it is the temperature dependence of this coupling, which makes the Kosterlitz-Thouless phase transition in the finite-temperature 3D compact QED inverse with respect to the conventional Kosterlitz-Thouless phase transition in the 2D XY model, where the coupling is temperature independent. Namely, as was already discussed, monopoles in the 3D compact QED exist in the plasma phase at low temperatures and in the molecular phase at high temperatures. In order to obtain the critical temperature T_c, it suffices to consider the mean squared separation in the monopole-antimonopole molecule, which should diverge at $T < T_c$. To this end, we note that the action (84) corresponds to the monopole-antimonopole Coulomb interaction $V(r) = \frac{g_m^2 T}{2\pi} \ln(\mu r)$ in the dimensionally-reduced theory, where r is the 2D distance, and μ is an infra-red cut-off. Accordingly, the mean squared separation in the monopole-antimonopole molecule can be estimated as $\langle r^2\rangle \sim \int d^2r\, r^{2-\frac{g_m^2 T}{2\pi}}$. Therefore, this separation is infra-red finite provided $3 - \frac{g_m^2 T}{2\pi} < -1$, and T_c can be estimated as a temperature at which this relation turns into the equality. Furthermore, assuming that the 3D compact QED originates from the 3D Georgi-Glashow model, where the spontaneous SU(2)-symmetry breaking leads to the appearance of a compact U(1) group, one can use the corresponding quantization condition $gg_m = 4\pi$. This yields the following critical temperature [129]:

$$T_c = \frac{g^2}{2\pi}. \tag{85}$$

Thus, because of the exponential smallness of the parameter $\zeta^{1/3}$, the deconfinement phase transition in the 3D compact QED indeed occurs at the temperature which is exponentially larger than the temperature of the dimensional reduction. This observation fully justifies the use of the dimensionally-reduced theory for the study of the deconfinement phase transition in the 3D compact QED.

Furthermore, it looks interesting to explore how the above-discussed deconfinement phase transition in the 3D compact QED becomes affected by an extension of the underlying 3D Georgi-Glashow model by dynamical quarks. A particular interest to this problem is attracted by the observation that such a fermionic extension of the 3D Georgi-Glashow model is quite close to reality, being nothing but the 3D QCD with an extra Higgs field transforming under the adjoint representation. Hence, in this way one can address the question of how quarks in such a model affect their own deconfinement phase transition. One should however notice that, due to the presence of $W^{\pm}$-bosons in the 3D Georgi-Glashow model, the deconfinement phase transition in that model differs from the above-considered phase transition in the 3D compact QED. These bosons play an important role in the finite-temperature dynamics of the 3D Georgi-Glashow model, while being absent in the compact-QED action (84). The deconfinement phase transition in the 3D Georgi-Glashow model, as well as in its extension by the dynamical quarks, will be considered in the next Section. For what follows in this Section, we do not account for $W^{\pm}$-bosons, thereby addressing the issue of how a fermionic extension of the 3D Georgi-Glashow model affects the deconfinement phase transition in the resulting 3D compact QED.

An extension of the model (26) by the dynamical quarks transforming under the fundamental representation of the group SU(2), contains the following additional term:

$$\Delta S = -i\int d^3x\, \psi^\dagger \beta \left(\vec{\gamma}\vec{\mathrm{D}} + h\frac{\tau^a}{2}\Phi^a\right)\psi.$$

Here we simplify the notations by omitting the summation over the flavor indices, although considering the general case of an arbitrary number of flavors. Furthermore, in the above formula, $D_\mu\psi = \left(\partial_\mu - ig\frac{\tau^a}{2}A_\mu^a\right)\psi$, the Yukawa coupling h has the dimensionality of $[\text{mass}]^{1/2}$, and the Euclidean Dirac matrices are defined as $\vec{\gamma} = -i\beta\vec{\alpha}$, with $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\tau} \\ \vec{\tau} & 0 \end{pmatrix}$, where $\vec{\tau}$ are the Pauli matrices. The mechanism allowing quarks to affect the deconfinement phase transition in the 3D compact QED is based on an additional attractive force they provide to the constituents of a monopole-antimonopole molecule. This force is induced by the zero modes of the Dirac operator for a quark in the gauge field of such a molecule. The number of the zero modes is equal to the number of quark flavors. In particular, for the case of massless quarks, we will estimate below the number of flavors for which the mean size of a monopole-antimonopole molecule becomes as small as the inverse mass of a W-boson.

Another factor which determines the size of a monopole-antimonopole molecule, is the characteristic range of localization of quark zero modes. One can show that the stronger the zero modes are localized near the monopole center, the smaller becomes the mean size of a molecule. Let us consider the extreme case where the Yukawa coupling h vanishes, so that the initially massless quarks do not acquire any mass. In this case, the zero modes become maximally delocalized, and the strength of the quark-mediated interaction in a molecule becomes minimal. Owing to such a weakness of the quark-mediated interaction in the monopole-antimonopole molecules, the deconfinement phase transition in this case still occurs at a temperature of the order of g^2, provided the number of flavors is equal to one [130]. For any larger number of massless flavors, the deconfinement critical temperature becomes as exponentially small as the dimensional-reduction temperature, i.e., of the order of $\zeta^{1/3}$. This means that the monopole-antimonopole interaction mediated by $N_{\mathrm{f}} \geq 2$ massless quark flavors is sufficient to maintain deconfinement in the 3D compact QED throughout the entire dimensionally-reduced phase.

We proceed now with the derivation of these results. For a while, let us consider the zero-temperature case at $h \neq 0$. There, one can show that the Dirac equation in the field of the 't Hooft-Polyakov monopole gets split into two equations for the components of the SU(2)-doublet ψ. The masses of these two components stemming from those equations are equal to each other, and read $m = h\eta/2$. Furthermore, the Dirac operator in the gauge field of a monopole has a zero mode, whose asymptotic behavior at the 3D distances $r \gtrsim \frac{1}{m}$ is $O(\mathrm{e}^{-mr}/r)$ (cf. Ref. [133]). To find the effective action of a quark in the field of a monopole-antimonopole molecule, one can notice that, in the 3D case at issue, monopoles are actually instantons [84]. Owing to this fact, the sought effective action can be obtained in the same way as the quark effective action in the field of an instanton–anti-instanton molecule in QCD [134]. To apply the method of Ref. [134] to our case, we first fix the gauge $\Phi^a = \eta\delta^{a3}$, and define a free-quark propagator S_0 by the relation $S_0^{-1} = -i\left(\vec{\gamma}\vec{\partial} + m\tau^3\right)$. Next, we define the quark propagator S_M in the field $\vec{A}_M^a$ of a monopole M located at the origin. This can be done through the formula $S_M^{-1} = S_0^{-1} - g\vec{\gamma}\frac{\tau^a}{2}\vec{A}_M^a$. For an antimonopole M' located at a certain point $\vec{R}$, the gauge field reads $\vec{A}_{M'}^a(\vec{x}) = -\vec{A}_M^a(\vec{x}-\vec{R})$. Clearly, the quark propagator $S_{M'}$ in this field is defined by the above equation for S_M^{-1}, with the replacement $\vec{A}_M^a \to \vec{A}_{M'}^a$. The gauge field of a monopole-antimonopole molecule is given by a superposition of the fields of its constituents, i.e., $\vec{A}^a = \vec{A}_M^a + \vec{A}_{M'}^a$. Accordingly, the quark propagator S in this field can be defined through the above equation for S_M^{-1}, by replacing there $\vec{A}_M^a$ by $\vec{A}^a$.

Let us further consider the operator $-i\vec{\gamma}\vec{D}$ with the gauge field $\vec{A}^a$, and denote its eigenfunctions as $|\psi_n\rangle$, where $n = 0, 1, 2, \ldots$. That is, $-i\vec{\gamma}\vec{D}\,|\psi_n\rangle = \lambda_n\,|\psi_n\rangle$, where $\lambda_0 = 0$. This yields the following formal expression for the spectral representation of the propagator S:

$$S(\vec{x},\vec{y}) = \sum_{n=0}^{\infty} \frac{\big|\psi_n(\vec{x})\big\rangle\big\langle\psi_n(\vec{y})\big|}{\lambda_n - im\tau^3}\,.$$

The crucial property of zero modes is that they yield the dominant contribution to the quark propagator, which allows for the following approximation:

$$S(\vec{x},\vec{y}) \simeq \frac{\big|\psi_0(\vec{x})\big\rangle\big\langle\psi_0(\vec{y})\big|}{-im\tau^3} + S_0(\vec{x},\vec{y}). \tag{86}$$

Using further the definitions of the propagators introduced above, one has

$$S = \left(S_M^{-1} + S_{M'}^{-1} - S_0^{-1}\right)^{-1} = S_{M'}G^{-1}S_M.$$

In this expression, we have denoted

$$G = S_0 - (S_M - S_0)\,S_0^{-1}\,(S_{M'} - S_0) \simeq S_0 - \frac{\big|\psi_0^M\big\rangle\big\langle\psi_0^M\big|}{-im\tau^3}\,S_0^{-1}\,\frac{\big|\psi_0^{M'}\big\rangle\big\langle\psi_0^{M'}\big|}{-im\tau^3}\,.$$

Here $\big|\psi_0^M\big\rangle$, $\big|\psi_0^{M'}\big\rangle$ are the zero modes of the operator $-i\vec{\gamma}\vec{D}$ which is defined in the field of a monopole and an antimonopole, respectively, and Equation (86) has been used. Denoting further $c = \big\langle\psi_0^{M'}\big|S_0^{-1}\big|\psi_0^M\big\rangle$, we have $G = S_0 + \frac{c^*}{m^2}\big|\psi_0^M\big\rangle\big\langle\psi_0^{M'}\big|$, where the star means complex conjugation. Consequently, one has: $\det G = \left[1 + (|c|/m)^2\right] \cdot \det S_0$. Finally, in the general N_f-flavor case, the sought effective action $\Gamma = \ln\det S^{-1}$ has the form

$$\Gamma = \text{const} + N_f \ln\left(m^2 + |c|^2\right).$$

The constant in this formula cancels out from the normalized expression for the mean squared separation in the monopole-antimonopole molecule.

We set now h to zero, so that $m = 0$ too. By using the definition of the zero mode, one has in this case: $c = \big\langle\psi_0^{M'}\big|g\vec{\gamma}\frac{\tau^a}{2}\vec{A}_M^a\big|\psi_0^M\big\rangle$. By virtue of this expression, the dependence of $|c|$ on the monopole-antimonopole separation R can be readily estimated as $|c| \propto \int d^3r/\left(r^2\big|\vec{r}-\vec{R}\big|\right) \propto \ln(\mu R)$, where μ stands for the infra-red cutoff. We can now proceed to the finite-temperature dimensionally-reduced theory, and address the question of how the deconfinement critical temperature gets changed by the monopole-antimonopole interaction mediated by the quark zero modes. To this end, we need to know how this interaction, which is described by the above-obtained 3D matrix element $c(R)$, looks like in the dimensionally-reduced theory. After that, similarly to the case without quarks, the critical temperature can be determined as a temperature below which the mean squared separation in a molecule becomes infra-red divergent. To find a 2D counterpart of $c(R)$, we recall that, at finite temperature, the temporal coordinate gets compactified to a circle of circumference β. Windings around this circle are characterized by the winding modes n, which are dual to the Matsubara frequencies ω_k. Accordingly, in the high-temperature dimensionally-reduced theory, where only ω_0 is relevant, one should sum up over all the winding modes. For $c(R)$, this means that we should calculate the expression $\ln(\mu R) = \sum_{n=-\infty}^{+\infty} \ln\left[\mu\left(r^2+(\beta n)^2\right)^{1/2}\right]$, where n is the number of a winding mode, and a 2D vector $\vec{\mathrm{r}}$ is the spatial part of a (2+1)D vector $\vec{\mathrm{R}}$. One can show that this sum is equal to $\pi Tr + \ln\left[1-\exp(-2\pi Tr)\right] + \mathrm{const}$, where an infinite constant "const" is irrelevant, since it is r-independent. To obtain this expression for $\ln(\mu R)$, one can use the equality

$$x\sum_{n=-\infty}^{+\infty}\frac{1}{x^2+(2\pi n/b)^2} = \frac{b}{2}\coth\left(\frac{bx}{2}\right),$$

and notice that its left-hand side can be written as $\frac{1}{2}\frac{d}{dx}\sum_{n=-\infty}^{+\infty}\ln\left[x^2+\left(\frac{2\pi n}{b}\right)^2\right]$. Integrating then over x, one obtains

$$\sum_{n=-\infty}^{+\infty}\ln\left[x^2+\left(\frac{2\pi n}{b}\right)^2\right] = b\int dx\,\coth\left(\frac{bx}{2}\right) = bx + 2\ln\left(1-\mathrm{e}^{-bx}\right) + \mathrm{const}.$$

Setting here $\frac{2\pi}{b} = \mu\beta$ and $x = \mu r$, we prove the above formula:

$$\sum_{n=-\infty}^{+\infty}\ln\left[\mu\left(r^2+(\beta n)^2\right)^{1/2}\right] = \pi Tr + \ln\left[1-\exp(-2\pi Tr)\right] + \mathrm{const}.$$

Thus, for $r \gtrsim \beta$ of interest, the statistical weight of the interaction mediated in the monopole-antimonopole molecule by massless quark zero modes, reads $\mathrm{e}^{-\Gamma} \propto (Tr)^{-2N_{\mathrm{f}}}$. Accounting also for the Coulomb interaction between the constituents of the molecule, we obtain for the sought mean squared separation:

$$\langle r^2\rangle \sim \int d^2r\, r^{2-\frac{g_m^2 T}{2\pi}-2N_{\mathrm{f}}}. \tag{87}$$

Similarly to the above-considered case without quarks, the condition of the infra-red finiteness of this expression yields the deconfinement critical temperature in the presence of N_{f} massless quark flavors (cf. Equation (85)):

$$T_c = \frac{g^2}{4\pi}(2-N_{\mathrm{f}}).$$

We see that, for $N_{\mathrm{f}} = 1$, the deconfinement phase transition occurs at a twice smaller critical temperature than for $N_{\mathrm{f}} = 0$. For $N_{\mathrm{f}} \geq 2$, the mean squared separation in the molecule stays infra-red finite throughout the entire dimensionally-reduced phase. That is, for such a number of massless fundamentally charged quark flavors, quark zero modes keep the system in the deconfinement phase, characterized in particular by the deconfinement of these very quarks. Finally, one can show that, for $N_{\mathrm{f}} \gg \max\left\{1, 4\pi T/g^2\right\}$, the mean squared separation in the monopole-antimonopole molecule becomes as small as $1/M_W^2$, where M_W is the W-boson mass. Thus, for this number of flavors, the mean size of monopole-antimonopole molecules reaches its smallest possible value (cf. Ref. [130]).

In conclusion of this Section, let us mention that, in the 3D compact QED extended by the Chern-Simons term, the monopole plasma undergoes a Kosterlitz-Thouless phase transition to the molecular phase even at zero temperature [135]. The Lagrangian of such a theory has the form

$$\mathcal{L} = \frac{1}{4g^2}F_{\mu\nu}^2 + \frac{n}{8\pi}\varepsilon_{\mu\nu\lambda}F_{\mu\nu}A_\lambda. \tag{88}$$

The Chern-Simons term entering this Lagrangian is not invariant under the gauge transformations which do not vanish at the boundary of the 3D manifold. Consequently, for the parameter ϕ of gauge transformations, one obtains the following induced boundary action:

$$S = \frac{n}{8\pi} \int d^2x\, (\partial_i \phi)^2. \tag{89}$$

While in the non-compact case, where $\phi \in (-\infty, +\infty)$, this theory describes free massless bosons, in the compact case at issue the gauge parameter takes its values in the circle, i.e., $\phi \in S^1$. Such a 2D XY model (89) possesses vortices, which can be either in the plasma or in the molecular phase, with the phase transition between the two phases being the Kosterlitz-Thouless one. This phase transition takes place at the critical value $n_c = 8$ of the Chern-Simons coefficient n. Accordingly, in the theory (88) with the compact U(1) group, monopoles stay in the plasma phase only for $n < n_c$. However, even in that phase of the theory (88), there is no confinement of external electrically charged particles as long as $n > 0$ (cf. Ref. [135]). For $n > n_c$, the monopole vacuum of the theory (88) exists in the molecular phase, where a monopole and an antimonopole forming a molecule interact with each other through the linear potential.

2.2. Deconfinement Phase Transition in the 3D Georgi-Glashow Model

As has already been mentioned, $W^{\pm}$-bosons make the critical properties of the 3D Georgi-Glashow model different from those of the 3D compact QED. While, at zero temperature, these bosons are irrelevant for the monopole-based mechanism of confinement because of their large masses, at finite temperature they form a plasma, which significantly affects the deconfinement phase transition [136]. The density of this plasma can be readily calculated as

$$\rho_W = 6 \int \frac{d^2p}{(2\pi)^2} \frac{1}{\mathrm{e}^{\beta\sqrt{\vec{\mathrm{p}}^2 + M_W^2}} - 1} \simeq \frac{3M_W T}{\pi}\, \mathrm{e}^{-\beta M_W}.$$

Here, the factor of 6 describes the total number of spin states of W^+- and W^--bosons, and we have used the fact that the deconfinement temperature has the order of g^2, which is much smaller than $M_W = g\eta$ in the weak-coupling regime of $g \ll \eta$ at issue. To estimate the deconfinement temperature T_c, one can use the heuristic criterium that the thickness of the confining string at this temperature becomes equal to its length. While the thickness of the string in the dimensionally-reduced theory at issue is $\propto \xi^{-1/2}$, where ξ was introduced in Equation (84), the length of the string is of the order of the mean distance between $W^{\pm}$-bosons, which is $\rho_W^{-1/2}$. Therefore, with an exponential accuracy, the thickness and the length of the string become equal to each other at the temperature

$$T_c = \frac{g^2}{4\pi\varepsilon}, \tag{90}$$

where the function ε was defined in Equation (27).

We proceed now to the formal analysis of the deconfinement phase transition in the 3D Georgi-Glashow model. To this end, we first rewrite the Lagrangian corresponding to the partition function (30), in terms of the so-called vortex operator $V = \mathrm{e}^{-ig_m\chi/2}$ as

$$\mathcal{L}_{3\mathrm{D}} = \frac{1}{2}(\partial_\mu \chi)^2 - 2\zeta \cos(g_m\chi) = \frac{2}{g_m^2} \left|\partial_\mu V\right|^2 - \zeta \left[V^2 + (V^*)^2\right].$$

This representation makes manifest the magnetic Z_2-symmetry of the 3D compact QED [137,138]. At zero temperature, this symmetry is spontaneously broken, since $\left\langle V(\vec{\mathrm{x}})V^*(\vec{0})\right\rangle \overset{|\vec{\mathrm{x}}|\to\infty}{\longrightarrow} 1$. The breakdown of the magnetic symmetry is associated with the generation of the Debye mass of the dual photon, and, consequently, with confinement. Furthermore, since the 3D compact QED "inherits" the magnetic symmetry from the 3D Georgi-Glashow model, the deconfinement phase transition in the 3D Georgi-Glashow model should be associated with the restoration of this symmetry. Together with the dimensional reduction, this observation indicates that the deconfinement phase transition in the 3D Georgi-Glashow model should be of the same kind as in the 2D Ising model, where the latter model also possesses a Z_2 symmetry. The universality class of the 2D Ising model implies a second-order phase transition, with the value of 1 for the critical exponent ν characterizing the behavior of the correlation length at $T \to T_c$. As such, this universality class radically differs from the universality class of the Kosterlitz-Thouless phase transition in the 3D compact QED, which is an infinite-order phase transition with the correlation length possessing an essential singularity.

One can account for $W^{\pm}$-bosons in the Lagrangian $\mathcal{L}_{2D} = \frac{1}{2}(\partial_\mu \varphi)^2 - 2\xi \cos(g_m \sqrt{T}\varphi)$ of the dimensionally-reduced 3D compact QED [cf. Equation (84)] by noticing that these bosons are nothing but vortices of the φ-field, described by the additional term $-2\mu \cos\tilde{\varphi}$ (cf. Ref. [136]). The field $\tilde{\varphi}$ dual to the field φ, is defined through the relation $i\partial_\mu \tilde{\varphi} = g\sqrt{\beta}\varepsilon_{\mu\nu}\partial_\nu \varphi$, and the fugacity μ of $W^{\pm}$-bosons is proportional to their density ρ_W, i.e., $\mu \propto M_W T \mathrm{e}^{-\beta M_W}$. Owing to this extra cosine term, the dual photon does not become massless even in the absence of the monopole plasma, as opposed to the phase of the 3D compact QED where monopoles and antimonopoles are bound into molecules. Instead, with the increase of temperature, the dual-photon Debye mass M_D increases, so that the correlation length in the system decreases, as expected on general grounds. To illustrate this, one can suppress monopoles by setting $\xi = 0$, which yields the Lagrangian

$$\mathcal{L}_{2D} = \frac{1}{2}(\partial_\mu \phi)^2 - 2\mu \cos\left(g\sqrt{\beta}\phi\right), \tag{91}$$

where $\phi = \sqrt{T}\tilde{\varphi}/g$. As follows from this Lagrangian, $M_D^2 = 2\mu\beta g^2 \propto M_W g^2 \mathrm{e}^{-\beta M_W}$, which shows that the Debye mass M_D indeed increases with the increase of T. Furthermore, one can find the critical temperature of the Kosterlitz-Thouless phase transition associated with the Lagrangian (91). To do so, it suffices to calculate the mean squared separation in the W^+-W^- molecule, which reads $\langle r^2 \rangle \sim \int d^2 r\, r^{2-\frac{g^2}{2\pi T}}$. The condition of the infra-red finiteness of this expression yields the critical temperature $\frac{g^2}{8\pi}$. Thus, at this temperature, $W^{\pm}$-bosons undergo a deconfinement Kosterlitz-Thouless phase transition from the molecular to the plasma phase.

We have considered above two limiting cases, where either $W^{\pm}$-bosons or monopoles are suppressed. In the first case, which corresponds to the 3D compact QED, the phase transition occurs at the temperature $\frac{g^2}{2\pi}$, while in the second case it occurs at the temperature $\frac{g^2}{8\pi}$. In the full 3D Georgi-Glashow model, which contains both $W^{\pm}$-bosons and monopoles, the deconfinement phase transition takes place at the intermediate temperature (90). Furthermore, the overlap of the inverse and the direct Kosterlitz-Thouless phase transitions turns out to yield the expected second-order deconfinement phase transition of the universality class of the 2D Ising model.

An independent condition of the deconfinement phase transition is given by the equality of the scaling dimensions of the operators $:\cos\left(g_m\sqrt{T}\varphi\right):$ and $:\cos\left(g\sqrt{\beta}\phi\right):$. These scaling dimensions read $\Delta = \frac{g_m^2 T}{4\pi}$ and $\tilde{\Delta} = \frac{g^2\beta}{4\pi}$, respectively, so that the monopole cosine term is relevant at $T < \frac{g^2}{2\pi}$, while the cosine term describing $W^{\pm}$-bosons is relevant at $T > \frac{g^2}{8\pi}$. The two scaling dimensions become equal to each other once the condition

$$g_m^2 T = g^2 \beta$$

is fulfilled. This extremely simple and symmetric relation illustrates the full duality that holds between magnetically charged monopoles and electrically charged $W^{\pm}$-bosons at finite temperature. It yields the critical temperature $\frac{g^2}{4\pi}$, which is indeed equal to T_c up to the factor of $\frac{1}{\varepsilon}$. At the temperature $\frac{g^2}{4\pi}$, scaling dimensions of the two cosine operators are both equal to 1, so that both these operators are relevant at this temperature, as well as in the whole range of temperatures $\frac{g^2}{8\pi} < T < \frac{g^2}{2\pi}$.

A small discrepancy between the values of $T_c = \frac{g^2}{4\pi\varepsilon}$ and $T_c = \frac{g^2}{4\pi}$, obtained from the equality of the length of the confining string to its width and from the equality of the two scaling dimensions, respectively, has been resolved in Ref. [139], where an exact value of T_c has been found from the renormalization-group equations. The critical temperature corresponds to an infra-red unstable fixed point of these equations, where the fugacities ξ and μ become equal to each other. In addition, one should impose the condition that both ξ/Λ^2 and μ/Λ^2 (where Λ is the momentum scale) should not exceed 1 at the critical point, since the opposite would contradict the dilute-plasma approximation for both monopoles and $W^\pm$-bosons. The renormalization-group equations read [140–142]

$$\frac{d}{d\lambda}\frac{\xi(\Lambda)}{\Lambda^2} = (2-\Delta)\frac{\xi(\Lambda)}{\Lambda^2}, \quad \frac{d}{d\lambda}\frac{\mu(\Lambda)}{\Lambda^2} = (2-\tilde{\Delta})\frac{\mu(\Lambda)}{\Lambda^2},$$

where $\lambda = \ln(T/\Lambda)$. Their integration yields

$$\frac{\xi(\Lambda)}{\Lambda^2} = \frac{\zeta}{T^3}\left(\frac{T}{\Lambda}\right)^{2-\Delta}, \quad \frac{\mu(\Lambda)}{\Lambda^2} = \frac{\mu}{T^2}\left(\frac{T}{\Lambda}\right)^{2-\tilde{\Delta}}.$$

Equating further these expressions to 1, one obtains with an exponential accuracy the following relation:

$$-S_0 + (2-\Delta)\lambda = -\beta M_W + (2-\tilde{\Delta})\lambda = 0, \tag{92}$$

where $S_0 = \frac{4\pi M_W}{g^2}\varepsilon$ is the monopole action (27). Unlike the two mutually independent criteria for the determination of T_c considered above, Equation (92) contains the information on the fugacities *and* on the scaling dimensions Δ and $\tilde{\Delta}$. With the use of the aforementioned explicit expressions for Δ and $\tilde{\Delta}$, one readily obtains the corresponding critical temperature:

$$T_c = \frac{g^2}{4\pi}\frac{2+\varepsilon}{1+2\varepsilon}. \tag{93}$$

In the formal limit of $\varepsilon \to 0$, where the density of monopoles is exponentially larger than the density of $W^\pm$-bosons, Equation (93) reproduces the critical temperature (85) in the 3D compact QED. Accordingly, in the opposite formal limit of $\varepsilon \to \infty$, where the density of monopoles is exponentially smaller than the density of $W^\pm$-bosons, Equation (93) reproduces the other above-obtained critical temperature of $\frac{g^2}{8\pi}$.

In the same way, one can find the deconfinement critical temperature in the 3D Georgi-Glashow model extended by $N_{\rm f}$ flavors of massless quarks which transform under the fundamental representation of the group SU(2). By using Equation (87) along with the above definition of the scaling dimension Δ, we observe that the monopole-antimonopole interaction mediated by the quark zero modes yields for Δ the following modified expression: $\Delta = \frac{g_m^2 T}{4\pi} + N_{\rm f}$. Substituting it into Equation (92), we obtain for the critical temperature in the presence of $N_{\rm f}$ massless flavors:

$$T_c = \frac{g^2}{4\pi}\frac{2+\varepsilon-N_{\rm f}}{1+2\varepsilon}.$$

Recalling the upper bound of $\varepsilon < 1.8$ (cf. Ref. [87]), one finds that, for $N_{\rm f} \geq 4$, the 3D Georgi-Glashow model extended by $N_{\rm f}$ massless quark flavors stays in the deconfinement phase throughout the entire dimensionally reduced phase, i.e., from the exponentially small temperatures $T \sim \zeta^{1/3}$ onwards. The critical number of flavors $N_{\rm f} = 4$ is larger than $N_{\rm f} = 2$, which was found in the previous Section for the case of the 3D compact QED. It is remarkably close to the critical number $\simeq 5$ of massless quark flavors, at which instantons and anti-instantons, within the instanton-liquid model of the QCD vacuum, form molecules already at the vanishingly small temperatures [143,144]. This coincidence does not look purely accidental in view of the already mentioned fact that monopoles in the 3D case at issue are actually instantons [84]. Nevertheless, physical phenomena associated with the phase transitions in the 3D Georgi-Glashow model and in the 4D QCD, are different. Namely, the formation of instanton–anti-instanton molecules in QCD corresponds not to the deconfinement but to the restoration of chiral symmetry. That is, it is the *chiral* phase transition in QCD, which occurs already in the hadronic phase if the number of massless quark flavors gets larger than the critical one.

3. Summary

In this review, we have discussed various topological effects and critical properties of the gauge theories where confinement is based on the condensation of magnetic monopoles. Those theories included the 3D SU(N) Georgi-Glashow model, the 4D $[\mathrm{U}(1)]^{N-1}$-invariant compact QED, and the $[\mathrm{U}(1)]^{N-1}$-invariant dual Abelian Higgs model. We have started with illustrating the importance of Wilson loops for the description of confinement, and proceeded further to the general properties of the static quark-antiquark potential and the associated Nambu-Goto and the rigid-string models of the confining string. In this way, we have in particular considered various models of the deconfinement phase transition. At the next step, we have provided a detailed treatment of confinement in the 3D SU(N) Georgi-Glashow model, along with the analytic study of the so-called k-string tensions in that model, demonstrating that these tensions obey Casimir scaling. In the corresponding 3D $[\mathrm{U}(1)]^{N-1}$-invariant compact QED, we have obtained the string representation of the Wilson loop, and presented the Nambu-Goto string tension and the rigid-string coupling constant. We have further proceeded to the discussion of the topological effects, which appear in 3D due to the Chern-Simons term and in 4D due to the Θ-term. While the effects of the Chern-Simons term in the 3D Maxwell theory reveal themselves for Wilson loops defined at knotted contours, the Θ-term in the strongly-coupled 4D $[\mathrm{U}(1)]^{N-1}$-invariant compact QED induces the string θ-term, which describes self-intersections of the string world sheet. We have also presented general arguments which explain why, in the models allowing for quantum plasmas of magnetically charged objects, confinement of point particles most naturally occurs in 3D and 4D. Next, we have derived the string representation of the 't Hooft-loop average in the $[\mathrm{U}(1)]^{N-1}$-invariant dual Abelian Higgs model. There, in the presence of the Θ-term, external electrically charged particles acquire the magnetic charge. Owing to this induced magnetic charge, such particles experience long-range Aharonov-Bohm-type interactions with the dual Abrikosov-Nielsen-Olesen strings. Rather, the Chern-Simons term in the 3D dual Abelian Higgs model leads to the appearance in the partition function of a non-trivial phase factor, which is caused by self-intersecting dual Abrikosov vortices. Finally, we have presented the analysis of critical properties, first of the 3D compact QED and the fermionic extension thereof, and then of the full 3D Georgi-Glashow model. Remarkably, the known vacuum structure of these weakly-coupled confining theories allowed for an analytic study of their finite-temperature behavior and of the corresponding deconfinement phase transitions.

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Appendix A. Derivation of Equation (48)

The sum (47) can be written as

$$S = \frac{N-1}{2N}\Big(\sum_{i=1}^{p} n_i^2 + n\Big) - \frac{1}{2N}\sum_{i=1}^{p}\sum_{\substack{i'=1\\ i'\neq i}}^{p} n_i n_{i'} - \frac{n}{2N}\sum_{i=1}^{p} n_i - \frac{1}{2N}\sum_{j=1}^{n}\sum_{\substack{j'=1\\ j'\neq j}}^{n} 1.$$

The two nested sums in this expression can be further represented as

$$\sum_{i=1}^{p}\sum_{\substack{i'=1\\ i'\neq i}}^{p} n_i n_{i'} = \Big(\sum_{i=1}^{p} n_i\Big)^2 - \sum_{i=1}^{p} n_i^2 \quad \text{and} \quad \sum_{j=1}^{n}\sum_{\substack{j'=1\\ j'\neq j}}^{n} 1 = \Big(\sum_{j=1}^{n} 1\Big)^2 - \sum_{j=1}^{n} 1 = n^2 - n,$$

which yields

$$S = \frac{N-1}{2N}\Big(\sum_{i=1}^{p} n_i^2 + n\Big) + \frac{1}{2N}\sum_{i=1}^{p} n_i^2 - \frac{n^2-n}{2N} - \frac{1}{2N}\Big(\sum_{i=1}^{p} n_i\Big)^2 - \frac{n}{2N}\sum_{i=1}^{p} n_i.$$

Representing the last two terms of this expression as

$$-\frac{1}{2N}\Big(\sum_{i=1}^{p} n_i\Big)\cdot\Big(\sum_{i=1}^{p} n_i + n\Big) = -\frac{1}{2N}\,(k-n)\cdot k,$$

we have

$$S = \frac{1}{2}\sum_{i=1}^{p} n_i^2 + n\cdot\frac{N-1}{2N} - \frac{1}{2N}\cdot\Big(n^2 - n + k^2 - nk\Big).$$

Simplifying this expression, we arrive at Equation (48) of the main text.

Appendix B. Integration over the Field $\vec{\mathrm{h}}_{\mu\nu}$ in Equation (59)

The Gaussian functional integration in Equation (59) can be performed by finding the saddle-point expression for $\vec{\mathrm{h}}_{\mu\nu}$, and substituting further this expression back into the action

$$\mathcal{A} \equiv \int d^3x\left(\frac{1}{12M_D^2}\vec{\mathrm{H}}^2_{\mu\nu\lambda} + \frac{1}{4}\vec{\mathrm{h}}^2_{\mu\nu} - \frac{ig\vec{\mu}_\alpha}{2}\vec{\mathrm{h}}_{\mu\nu}\Sigma_{\mu\nu}\right), \tag{A1}$$

where $\Sigma_{\mu\nu}(\vec{\mathrm{x}}) \equiv \int_S d\sigma_{\mu\nu}(\vec{\mathrm{x}}(\vec{\xi}))\delta(\vec{\mathrm{x}} - \vec{\mathrm{x}}(\vec{\xi}))$. The saddle-point equation corresponding to the action (A1) reads [33]

$$-\frac{1}{M_D^2}\left(\partial_\mu^2\vec{\mathrm{h}}_{\alpha\beta} + \partial_\mu\partial_\beta\vec{\mathrm{h}}_{\mu\alpha} + \partial_\mu\partial_\alpha\vec{\mathrm{h}}_{\beta\mu}\right) + \vec{\mathrm{h}}_{\alpha\beta} = ig\vec{\mu}_\alpha\Sigma_{\alpha\beta}.$$

Defining the Fourier-transformed field $\vec{\mathrm{h}}_{\mu\nu}(\vec{\mathrm{p}}) = \int d^3x\, \mathrm{e}^{-i\vec{\mathrm{p}}\vec{\mathrm{x}}}\vec{\mathrm{h}}_{\mu\nu}(\vec{\mathrm{x}})$, one can write the saddle-point equation in the momentum representation as

$$\frac{1}{M_D^2}\left[\vec{\mathrm{p}}^2\,\vec{\mathrm{h}}_{\alpha\beta}(\vec{\mathrm{p}}) + p_\mu p_\beta\vec{\mathrm{h}}_{\mu\alpha}(\vec{\mathrm{p}}) + p_\mu p_\alpha\vec{\mathrm{h}}_{\beta\mu}(\vec{\mathrm{p}})\right] + \vec{\mathrm{h}}_{\alpha\beta}(\vec{\mathrm{p}}) = ig\vec{\mu}_\alpha\Sigma_{\alpha\beta}(\vec{\mathrm{p}}). \tag{A2}$$

This equation can be solved by further rewriting it with the use of the operators

$$\hat{1}_{\mu\nu,\lambda\rho} = \frac{1}{2}(\delta_{\mu\lambda}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\lambda}) \quad \text{and} \quad \hat{P}_{\mu\nu,\lambda\rho} = \frac{1}{2}(\mathcal{P}_{\mu\lambda}\mathcal{P}_{\nu\rho} - \mathcal{P}_{\mu\rho}\mathcal{P}_{\nu\lambda}),$$

---

[33] The color index α of the weight vector $\vec{\mu}_\alpha$, which takes the values $1, \ldots, N$, should not be confused with a Lorentz index. A confusion regarding various types of vectors should also be avoided, namely $\vec{\xi}$ is a 2-dimensional vector, $\vec{\mathrm{x}}$ and $\vec{\mathrm{x}}(\vec{\xi})$ are 3-dimensional vectors, and $\vec{\mu}_\alpha$ and $\vec{\mathrm{h}}_{\mu\nu}$ are $(N-1)$-dimensional vectors.

where $\mathcal{P}_{\mu\nu} = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{\vec{p}^2}$. The operators $\hat{1}_{\mu\nu,\lambda\rho}$ and $\hat{P}_{\mu\nu,\lambda\rho}$ are antisymmetric with respect to permutations inside the first and the second pair of indices, while remaining symmetric with respect to permutations of these pairs of indices themselves, e.g.,

$$\hat{1}_{\mu\nu,\lambda\rho} = -\hat{1}_{\nu\mu,\lambda\rho} = -\hat{1}_{\mu\nu,\rho\lambda} = \hat{1}_{\lambda\rho,\mu\nu}.$$

Furthermore, the operators $\hat{P}$ and $(\hat{1} - \hat{P})$ possess the properties of projection operators, namely

$$\hat{P}_{\mu\nu,\lambda\rho}\hat{P}_{\lambda\rho,\alpha\beta} = \hat{P}_{\mu\nu,\alpha\beta}, \;\; (\hat{1} - \hat{P})_{\mu\nu,\lambda\rho}(\hat{1} - \hat{P})_{\lambda\rho,\alpha\beta} = (\hat{1} - \hat{P})_{\mu\nu,\alpha\beta}, \;\; \hat{P}_{\mu\nu,\lambda\rho}(\hat{1} - \hat{P})_{\lambda\rho,\alpha\beta} = 0. \tag{A3}$$

The saddle-point equation (A2) can then be represented as

$$\left(\frac{\vec{p}^2}{M_D^2}\hat{P}_{\alpha\beta,\mu\nu} + \hat{1}_{\alpha\beta,\mu\nu}\right)\vec{h}_{\mu\nu}(\vec{p}) = ig\vec{\mu}_\alpha \Sigma_{\alpha\beta}(\vec{p}).$$

A solution to this equation can be sought in the form

$$\vec{h}_{\mu\nu}(\vec{p}) = ig\vec{\mu}_\alpha \left[A(\vec{p}) \cdot \hat{1}_{\mu\nu,\lambda\rho} + B(\vec{p}) \cdot (\hat{1} - \hat{P})_{\mu\nu,\lambda\rho}\right] \Sigma_{\lambda\rho}(\vec{p}).$$

With the use of the properties (A3) of the operators $\hat{P}$ and $(\hat{1} - \hat{P})$, this ansatz for $\vec{h}_{\mu\nu}(\vec{p})$ yields the following coefficient functions:

$$A(\vec{p}) = \frac{M_D^2}{\vec{p}^2 + M_D^2}, \quad B(\vec{p}) = \frac{\vec{p}^2}{\vec{p}^2 + M_D^2}.$$

Therefore, the saddle-point expression for the field $\vec{H}_{\mu\nu}(\vec{p})$ becomes

$$\vec{h}_{\mu\nu}(\vec{p}) = ig\vec{\mu}_\alpha \left[\hat{1}_{\mu\nu,\lambda\rho} - \frac{\vec{p}^2}{\vec{p}^2 + M_D^2}\hat{P}_{\mu\nu,\lambda\rho}\right] \Sigma_{\lambda\rho}(\vec{p}). \tag{A4}$$

The action (A1) can also be rewritten in the momentum representation, where it takes the form

$$\mathcal{A} = \int \frac{d^3p}{(2\pi)^3}\left[\frac{1}{4}\left(\frac{\vec{p}^2}{M_D^2} + 1\right)\vec{h}_{\mu\nu}(\vec{p})\vec{h}_{\mu\nu}(-\vec{p}) - \frac{\vec{p}^2}{4M_D^2}\left(\hat{1} - \hat{P}\right)_{\mu\nu,\lambda\rho}\vec{h}_{\mu\nu}(\vec{p})\vec{h}_{\lambda\rho}(-\vec{p}) - \right.$$
$$\left. \frac{ig\vec{\mu}_\alpha}{2}\vec{h}_{\mu\nu}(\vec{p})\Sigma_{\mu\nu}(-\vec{p})\right].$$

Substituting now into this formula the saddle-point expression (A4), we obtain

$$\mathcal{A} = \frac{(g\vec{\mu}_\alpha)^2}{4}\int \frac{d^3p}{(2\pi)^3}\Sigma_{\mu\nu}(\vec{p})\Sigma_{\lambda\rho}(-\vec{p}) \cdot \frac{1}{\vec{p}^2 + M_D^2}\left[M_D^2 \cdot \hat{1} + \vec{p}^2 \cdot (\hat{1} - \hat{P})\right]_{\mu\nu,\lambda\rho}.$$

Performing the inverse Fourier transform to the coordinate representation, one obtains the result of the $\vec{h}_{\mu\nu}$-integration in Equation (59) in the form of the following non-local action:

$$\mathcal{A} = \frac{(g\vec{\mu}_\alpha)^2}{2}\int d^3x \int d^3y\, D_{M_D}(\vec{x} - \vec{y})\left[\frac{M_D^2}{2}\Sigma_{\mu\nu}(\vec{x})\Sigma_{\mu\nu}(\vec{y}) + j_\mu(\vec{x})j_\mu(\vec{y})\right], \tag{A5}$$

where $j_\mu(\vec{x}) \equiv \oint_C dx_\mu(\tau)\,\delta(\vec{x} - \vec{x}(\tau))$. Note finally that, in the course of the derivation of Equation (A5), the local form of the Stokes' theorem, $\partial_\nu \Sigma_{\mu\nu} = j_\mu$, has been used.

Appendix C. Integration over the A_μ-Field in Equation (66)

The A_μ-integration in Equation (66) can be performed by imposing the Lorenz gauge-fixing condition $\partial_\mu A_\mu = 0$, which yields the saddle-point equation $-\partial^2 A_\mu + im\varepsilon_{\mu\nu\lambda}\partial_\nu A_\lambda = ig^2 j_\mu$, where $m = 2g^2\Theta$. Seeking a solution in the form $A_\mu = U_\mu + iV_\mu$, we get a system of equations

$$\partial^2 U_\mu + m\varepsilon_{\mu\nu\lambda}\partial_\nu V_\lambda = 0, \qquad -\partial^2 V_\mu + m\varepsilon_{\mu\nu\lambda}\partial_\nu U_\lambda = g^2 j_\mu. \tag{A6}$$

The first of these equations can be solved with respect to U_μ as

$$U_\mu(\vec{x}) = m\varepsilon_{\mu\nu\lambda}\int_y D_0(\vec{x}-\vec{y})\partial_\nu V_\lambda(\vec{y}). \tag{A7}$$

Differentiating now the second Equation (A6), and applying the maximum principle, one gets $\partial_\mu V_\mu = 0$. Using this relation, one further obtains from Equation (A7): $\varepsilon_{\mu\nu\lambda}\partial_\nu U_\lambda = mV_\mu$. The substitution of this formula into the second Equation (A6) yields for that equation a remarkably simple form: $(-\partial^2 + m^2)V_\mu = g^2 j_\mu$. Therefore, one obtains $V_\mu(\vec{x}) = g^2\int_y D_m(\vec{x}-\vec{y})j_\mu(\vec{y})$, while $U_\mu(\vec{x})$, given by Equation (A7), can be calculated through the relation

$$\int_y D_m(\vec{x}-\vec{y})D_0(\vec{y}-\vec{u}) =$$

$$\int_y\int_p \frac{e^{i\vec{p}(\vec{x}-\vec{y})}}{\vec{p}^2+m^2}\int_q \frac{e^{i\vec{q}(\vec{y}-\vec{u})}}{\vec{q}^2} = \int_p \frac{e^{i\vec{p}(\vec{x}-\vec{u})}}{\vec{p}^2(\vec{p}^2+m^2)} = \frac{1}{m^2}\left[D_0(\vec{x}-\vec{u}) - D_m(\vec{x}-\vec{u})\right],$$

where $\int_p \equiv \int \frac{d^3p}{(2\pi)^3}$, and the equality $\frac{1}{\vec{p}^2(\vec{p}^2+m^2)} = \frac{1}{m^2}\left(\frac{1}{\vec{p}^2} - \frac{1}{\vec{p}^2+m^2}\right)$ has been used at the last step. The resulting $U_\mu(\vec{x})$ reads $U_\mu(\vec{x}) = \frac{1}{2\Theta}\varepsilon_{\mu\nu\lambda}\int_y[D_0(\vec{x}-\vec{y}) - D_m(\vec{x}-\vec{y})]\partial_\nu j_\lambda(\vec{y})$, which yields for the Wilson-loop average the expression (67) from the main text (for a review, see [145]). Note that, in the limit of $\Theta \to 0$, Equation (67) recovers the expression for the Wilson-loop average in the 3D Maxwell theory without the Chern-Simons term, namely

$$\langle W(C)\rangle = \exp\left[-\frac{g^2}{2}\oint_C dx_\mu \oint_C dy_\mu\, D_0(\vec{x}-\vec{y})\right].$$

Indeed, in this limit, one has $D_0(\vec{x}-\vec{y}) - D_m(\vec{x}-\vec{y}) \to \frac{m}{4\pi}$, so that the expression

$$\int_{x,y} j_\mu(\vec{x})j_\lambda(\vec{y})\partial_\nu^x[D_0(\vec{x}-\vec{y}) - D_m(\vec{x}-\vec{y})] =$$

$$\int_{x,y} j_\mu(\vec{x})[D_0(\vec{x}-\vec{y}) - D_m(\vec{x}-\vec{y})]\partial_\nu j_\lambda(\vec{y}) \to \frac{m}{4\pi}\int_{x,y} j_\mu(\vec{x})\partial_\nu j_\lambda(\vec{y})$$

vanishes, since $\int_x j_\mu(\vec{x}) = 0$.

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