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Gauge Non-Invariant Higher-Spin Currents in AdS_4

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Abstract: Conserved currents of any spin $t > 0$ built from bosonic symmetric massless gauge fields of arbitrary integer spins $s_1 + s_2 > t$ in AdS_4 are found. Analogous to the case of $4d$ Minkowski space, currents considered in this paper are not gauge invariant, but generate gauge-invariant conserved charges.

Keywords: higher-spin theory; conserved currents; conserved charges

1. Introduction

Gauge-invariant conserved currents are well known and were deeply studied in the literature [1–9]. In the general case, a conserved current carries a set of three spins (t, s_1, s_2), where t is a spin of the current itself, and s_1 and s_2 are spins of the fields it is constructed from. For example, the so-called gravitational stress pseudo-tensor [10] ($s = t = 2$ conserved current) is not gauge invariant. The same fact is shown in [5] for the $t = 2$ current built from massless fields of spins $s > 2$. The spin-zero field has no gauge symmetry; thus, the currents with $s < 1$ are gauge invariant, while the spin-one current built from two massless spin-one fields is not.

The aim of this paper is to extend the Minkowski-space results of [11], presenting the full list of gauge non-invariant currents with integer spins in AdS_4 such that $t < s_1 + s_2$. Being gauge non-invariant, these currents give rise to the gauge-invariant conserved charges. Gauge non-invariant currents will be derived from the variation of the cubic action of [12,13], which is gauge invariant in the lowest order.

Conventions

In this paper, we consider AdS_4 space-time. Greek indices $\mu, \nu, \rho, \lambda, \sigma$ are the base and range from 0–3. Other Greek indices are spinorial and take values of one and two. The latter are raised and lowered by the $sp(2)$ antisymmetric forms: $\varepsilon_{\alpha\beta}, \varepsilon^{\alpha\beta}, \varepsilon_{\dot\alpha\dot\beta}, \varepsilon^{\dot\alpha\dot\beta}$

$$\varepsilon^{\alpha\beta}\varepsilon_{\alpha\gamma} = \delta^{\beta}_{\gamma}, \quad \varepsilon^{\dot\alpha\dot\beta}\varepsilon_{\dot\alpha\dot\gamma} = \delta^{\dot\beta}_{\dot\gamma}, \tag{1}$$

$$A_\alpha = A^\beta \varepsilon_{\beta\alpha}, \quad A^\alpha = A_\beta \varepsilon^{\alpha\beta}, \quad A_{\dot\alpha} = A^{\dot\beta}\varepsilon_{\dot\beta\dot\alpha}, \quad A^{\dot\alpha} = A_{\dot\beta}\varepsilon^{\dot\alpha\dot\beta}. \tag{2}$$

Complex conjugation $\bar{A}$ relates dotted and undotted spinors. Brackets ([...]) {...} imply complete (anti)symmetrization, i.e.,

$$A_{[\alpha}B_{\beta]} = \frac{1}{2}(A_\alpha B_\beta - A_\beta B_\alpha), \qquad A_{\{\alpha}B_{\beta\}} = \frac{1}{2}(A_\alpha B_\beta + A_\beta B_\alpha). \tag{3}$$

$A^{\alpha(m)}$ denotes a totally symmetric multispinor $A^{\{\alpha_1 \ldots \alpha_m\}}$.

The wedge symbol $\wedge$ is implicit.

2. Fields, Equations, Actions

In the four-dimensional case considered in this paper, it is convenient to use the frame-like formalism in two-component spinor notation. In these terms, a bosonic spin-s Fronsdal field [14] is represented by multispinor one-forms [15]:

$$s \geq 1: \quad \varphi_{\mu_1 \dots \mu_s} \to \{\omega^{\alpha(m)}{}_{,}{}^{\dot\beta(n)} \mid m+n = 2(s-1)\}, \quad \omega^{\alpha(m)}{}_{,}{}^{\dot\beta(n)} = dx^\mu \omega_\mu{}^{\alpha(m)}{}_{,}{}^{\dot\beta(n)},$$

which are symmetric in all dotted and all undotted spinor indices and obey the reality condition [15]:

$$\omega^\dagger_{\alpha(m),\dot\beta(n)} = \omega_{\beta(n),\dot\alpha(m)}. \tag{4}$$

The frame-like field is a particular connection at $n = m = s-1$ (s is integer):

$$h_\mu{}^{\alpha(s-1)}{}_{,}{}^{\dot\beta(s-1)} dx^\mu := \omega_\mu{}^{\alpha(s-1)}{}_{,}{}^{\dot\beta(s-1)} dx^\mu . \tag{5}$$

By imposing appropriate constraints, the connections $\omega^{\alpha(m),\dot\beta(n)}$ can be expressed via $t = \frac{1}{2}|m-n|$ derivatives of the frame-like field [15].

Background gravity is described by the vierbein one-form $\tilde h^{\alpha}{}_{,}{}^{\dot\beta}$ and one-form connections $\tilde\omega^{\dot\alpha\dot\beta}$, $\tilde\omega^{\alpha\beta}$. Lorentz covariant derivative $\tilde D$ acts as usual:

$$\tilde D A^{\alpha(m)}{}_{,}{}^{\dot\beta(n)} = dA^{\alpha(m)}{}_{,}{}^{\dot\beta(n)} + m\tilde\omega^{\alpha}{}_{\gamma} A^{\alpha(m-1)\gamma}{}_{,}{}^{\dot\beta(n)} + n\tilde\omega^{\dot\beta}{}_{\dot\delta} A^{\alpha(m)}{}_{,}{}^{\dot\beta(n-1)\dot\delta} \tag{6}$$

for any multispinor $A^{\alpha(m),\dot\beta(n)}$. The torsion and curvature two-forms are:

$$\tilde R^{\alpha}{}_{,}{}^{\dot\beta} = d\tilde h^{\alpha}{}_{,}{}^{\dot\beta} + \tilde\omega^{\alpha}{}_{\gamma}\tilde h^{\gamma}{}_{,}{}^{\dot\beta} + \tilde\omega^{\dot\beta}{}_{\dot\delta}\tilde h^{\alpha}{}_{,}{}^{\dot\delta}, \tag{7}$$

$$\tilde R^{\alpha\alpha} = d\tilde\omega^{\alpha\alpha} + \tilde\omega^{\alpha}{}_{\gamma}\tilde\omega^{\alpha\gamma} + \lambda^2 \tilde h^{\alpha}{}_{,\dot\delta}\tilde h^{\alpha}{}_{,}{}^{\dot\delta}, \tag{8}$$

$$\tilde R^{\dot\beta\dot\beta} = d\tilde\omega^{\dot\beta\dot\beta} + \tilde\omega^{\dot\beta}{}_{\dot\gamma}\tilde\omega^{\dot\beta\dot\gamma} + \lambda^2 \tilde h_{\gamma,}{}^{\dot\beta}\tilde h^{\gamma}{}_{,}{}^{\dot\beta}, \tag{9}$$

where the parameter λ is proportional to the inverse AdS radius $\lambda \sim r^{-1}$. AdS_4 space is described by the vierbein and connections obeying the equations:

$$\tilde R^{\alpha}{}_{,}{}^{\dot\beta} = 0, \qquad \tilde R^{\alpha\alpha} = 0, \qquad \tilde R^{\dot\beta\dot\beta} = 0. \tag{10}$$

Linearized higher-spin (HS) curvatures are:

$$\begin{aligned} R_1{}^{\alpha(m)}{}_{,}{}^{\dot\beta(n)} = {}& \tilde D\omega^{\alpha(m)}{}_{,}{}^{\dot\beta(n)} + n(\theta(m-n) + \lambda^2\theta(n-m-2))\tilde h_{\gamma,}{}^{\dot\beta}\omega^{\gamma\alpha(m)}{}_{,}{}^{\dot\beta(n-1)} \\ & + m(\theta(n-m) + \lambda^2\theta(m-n-2))\tilde h^{\alpha}{}_{,\dot\delta}\omega^{\alpha(m-1)}{}_{,}{}^{\dot\beta(n)\dot\delta}, \end{aligned} \tag{11}$$

where $\theta(x)$ is the step-function:

$$\theta(x) = \begin{cases} 1 & \text{at } x \geq 0; \\ 0 & \text{at } x < 0. \end{cases} \tag{12}$$

Curvatures (11) obey the Bianchi identities [15]:

$$\begin{aligned} \tilde D R_1{}^{\alpha(m)}{}_{,}{}^{\dot\beta(n)} = {}& -\lambda^{(|m-n|/2)+1}(m\lambda^{-|m-n-2|/2}\tilde h^{\alpha}{}_{,\dot\delta} R_1{}^{\alpha(m-1)}{}_{,}{}^{\dot\beta(n)\dot\delta} \\ & + n\lambda^{-|m-n+2|/2}\tilde h_{\gamma,}{}^{\dot\beta} R_1{}^{\alpha(m)\gamma}{}_{,}{}^{\dot\beta(n-1)}). \end{aligned} \tag{13}$$

It is convenient to introduce two-forms $H_{\alpha\beta}$ and $\bar{H}_{\dot\alpha\dot\beta}$:

$$\tilde{h}_{\alpha,\dot\beta}\tilde{h}_{\gamma,\dot\delta} = \frac{1}{2}\epsilon_{\alpha\gamma}\bar{H}_{\dot\beta\dot\delta} + \frac{1}{2}\epsilon_{\dot\beta\dot\delta}H_{\alpha\gamma}\,, \tag{14}$$

$$H_{\alpha\beta} := \tilde{h}_{\alpha,\dot\gamma}\tilde{h}_{\beta,}{}^{\dot\gamma}\,, \quad \bar{H}_{\dot\alpha\dot\beta} := \tilde{h}_{\gamma,\dot\alpha}\tilde{h}^{\gamma}{}_{,\dot\beta}. \tag{15}$$

Free field equations for massless fields of spins $s \geq 2$ in Minkowski space can be written in the form [15]:

$$R_1{}^{\alpha(m)}{}_{,}{}^{\dot\beta(n)} = 0 \qquad \text{for} \qquad n > 0,\ m > 0,\ n+m = 2(s-1); \tag{16}$$

$$R_1{}^{\alpha(m)} = C^{\alpha(m)\gamma\delta}\ H_{\gamma\delta} \qquad \text{for} \qquad m = 2(s-1); \tag{17}$$

$$R_1{}^{\dot\beta(n)} = \bar{C}^{\dot\beta(n)\dot\gamma\dot\delta}\ \bar{H}_{\dot\gamma\dot\delta} \qquad \text{for} \qquad n = 2(s-1). \tag{18}$$

Equations (16)–(18) are equivalent to the equations of motion, which follow from the Fronsdal action [14] supplemented with certain algebraic constraints, which express connections $\omega_{\alpha(m),\dot\beta(n)}$ via $\frac{1}{2}|m-n|$ derivatives of the dynamical frame-like HS field. The multispinor zero-forms $C^{\alpha(2s)}$ and $\bar{C}^{\dot\beta(2s)}$, which remain non-zero on-shell, are spin-s analogues of the Weyl tensor in gravity.

HS gauge transformation is:

$$\delta\omega^{\alpha(m)}{}_{,}{}^{\dot\beta(n)} = \tilde{D}\epsilon^{\alpha(m)}{}_{,}{}^{\dot\beta(n)} + n(\theta(m-n) + \lambda^2\theta(n-m-2))\tilde{h}_{\gamma,}{}^{\dot\beta}\epsilon^{\gamma\alpha(m)}{}_{,}{}^{\dot\beta(n-1)} + m(\theta(n-m) + \lambda^2\theta(m-n-2))\tilde{h}^{\alpha}{}_{,\dot\delta}\epsilon^{\alpha(m-1)}{}_{,}{}^{\dot\beta(n)\dot\delta}, \tag{19}$$

where a gauge parameter $\epsilon^{\alpha(m)}{}_{,}{}^{\dot\beta(n)}(x)$ is an arbitrary function of x. Note that the limit $\lambda \to 0$ gives the proper description of HS fields in 4d Minkowski space.

As explained in [11], to obtain currents with odd and even spins, the connections $\omega^{i;\alpha(m),\dot\beta(n)}$ and curvatures $R^{i;\alpha(m),\dot\beta(n)}$ should be endowed with a color index $i = 1\ldots N$, which labels independent dynamical fields. To contract color indices, we introduce the real tensor c_{ijk}, which can be either symmetric or antisymmetric. Color indices are raised and lowered by the Euclidean metric g_{ij}. It is convenient to set $g_{ij} = \delta_{ij}$.

Free fields are described by the quadratic action [15]:

$$S_2 = \frac{1}{2}\int \sum_{m,n\geq 0} \frac{1}{m!n!}\ \varepsilon(m-n)\ \lambda^{-|m-n|}\ R_1{}^{i;\alpha(m)}{}_{,}{}^{\dot\beta(n)}R_{1\ i;\alpha(m),\dot\beta(n)}, \tag{20}$$

where $\varepsilon(x) = \theta(x) - \theta(-x)$ and $m+n = 2(s-1)$, s being a spin of the field.

Following [12,13], to obtain a cubic deformation of the quadratic action, the linear curvature R_1 in the action (20) has to be replaced by $R = R_1 + R_2$ where:

$$R_2{}^{i;\alpha(m)}{}_{,}{}^{\dot\beta(n)} = \sum_{p,q,k,l,u,v\geq 0} \lambda^{1+d_0-d_1-d_2}\ \frac{m!n!}{p!q!k!l!u!v!}\ c^{i}{}_{jk}\ \delta_{p+q,m}\delta_{u+v,n} \times \omega^{j;\alpha(p)}{}_{\gamma(k),\dot\delta(l)}{}^{\dot\beta(u)}\ \omega^{k;\alpha(q)\gamma(k)}{}_{,}{}^{\dot\delta(l)\dot\beta(v)}, \tag{21}$$

$$d_0 = \frac{|m-n|}{2}, \quad d_1 = \frac{|p+k-l-u|}{2}, \quad d_2 = \frac{|q+k-l-v|}{2}.$$

The nonlinear action is:

$$S = \frac{1}{2}\int \sum_{m,n\geq 0} \frac{1}{m!n!}\ \varepsilon(m-n)\ \lambda^{-|m-n|}\ R^{i;\alpha(m)}{}_{,}{}^{\dot\beta(n)}R_{i;\alpha(m),\dot\beta(n)}. \tag{22}$$

3. Problem

It is convenient to describe currents as Hodge-dual differential forms. The on-shell closure condition for the latter is traded for the current conservation condition. In this paper, we consider spin-t currents in AdS_4 built from two connections of integer spins $s_1, s_2 > 0$ such that $t \leq s_1 + s_2 - 1$. Such currents contain the minimal possible number of derivatives of the dynamical fields. The analogous problem in $4d$ Minkowski space has been solved in [11] for the case of $s_1 = s_2$. The form of the currents will be derived from the nonlinear action (22).

An arbitrary variation of the action (22) can be represented in the form:

$$\delta S = \int \sum_{t,s_1,s_2} \sum_{m,n} \delta(m+n-2(t-1)) J_{t,s_1,s_2}{}^{i;\alpha(m)}{}_{,}{}^{\dot\beta(n)}\ \delta\omega_{i;\alpha(m),\dot\beta(n)}\,. \tag{23}$$

The current $J_{t,s_1,s_2}{}^{i;\alpha(m)}{}_{,}{}^{\dot\beta(n)}$ carries the color index i. Actually, there are N copies of a current, one for each value of i, and we can set $i = 1$ without loss of generality. In what follows, this index $i = 1$ will be omitted in all current forms. Furthermore, it is convenient to set $c_{jk} := c_{1jk}$ with c_{jk} being either symmetric or antisymmetric, i.e.,

$$c_{jk} = \eta c_{kj}\,, \qquad \eta^2 = 1\,. \tag{24}$$

To define a nontrivial HS charge as an integral over a $3d$ space, one should find such a current three-form $J_{t,s_1,s_2}(x)$ built from dynamical HS fields that is closed by virtue of HS field Equations (16)–(18), but not exact. The closed current three-form is:

$$J_{t,s_1,s_2}(x) = \sum_{m,n} \frac{\lambda^{-|m-n|}}{m!n!} \delta(m+n-2(t-1)) \xi_{\alpha(m),\dot\beta(n)}(x) J_{t,s_1,s_2}{}^{\alpha(m)}{}_{,}{}^{\dot\beta(n)}(x), \tag{25}$$

where the factor of $\frac{\lambda^{-|m-n|}}{m!n!}$ is introduced for convenience and $\xi_{\alpha(m),\dot\beta(n)}$ are global symmetry parameters, which can be identified with those gauge symmetry parameters that leave the background gauge fields invariant. In accordance with (19), these parameters obey:

$$D\xi^{\alpha(m)}{}_{,}{}^{\dot\beta(n)} := \tilde D\xi^{\alpha(m)}{}_{,}{}^{\dot\beta(n)} + n(\theta(m-n) + \lambda^2\theta(n-m-2))\tilde h_{\gamma,}{}^{\dot\beta}\xi^{\gamma\alpha(m)}{}_{,}{}^{\dot\beta(n-1)} \\ + m(\theta(n-m) + \lambda^2\theta(m-n-2))\tilde h^{\alpha}{}_{,\dot\delta}\xi^{\alpha(m-1)}{}_{,}{}^{\dot\beta(n)\dot\delta} = 0. \tag{26}$$

One can see that:

$$dJ_{t,s_1,s_2} = \sum_{m,n} \frac{\lambda^{-|m-n|}}{m!n!} \left(D\xi_{\alpha(m),\dot\beta(n)} J_{t,s_1,s_2}{}^{\alpha(m)}{}_{,}{}^{\dot\beta(n)} + \xi_{\alpha(m),\dot\beta(n)} D J_{t,s_1,s_2}{}^{\alpha(m)}{}_{,}{}^{\dot\beta(n)} \right). \tag{27}$$

Hence, for parameters obeying (26), the conservation condition amounts to equations:

$$D J_{t,s_1,s_2}{}^{\alpha(m)}{}_{,}{}^{\dot\beta(n)} \simeq 0, \qquad m+n = 2(t-1). \tag{28}$$

For the currents defined via (23), the conservation condition (28) holds as a consequence of the gauge invariance of the action proven in [12].

Conserved currents generate conserved charges. By the Noether theorem, the latter are generators of global symmetries. Hence, one should expect as many conserved charges as global symmetry parameters. For a spin t, there are as many global symmetry parameters as the gauge parameters $\epsilon_{\alpha(m),\dot\beta(n)}$ with $m+n = 2(t-1)$.

In what follows, we will use notations:

$$D^{top}\omega^{\alpha(m)}{}_{,}{}^{\dot\beta(n)} := n\theta(m-n)\tilde h_{\gamma,}{}^{\dot\beta}\ \omega^{\gamma\alpha(m)}{}_{,}{}^{\dot\beta(n-1)} + m\theta(n-m)\tilde h^{\alpha}{}_{,\dot\delta}\ \omega^{\alpha(m-1)}{}_{,}{}^{\dot\beta(n)\dot\delta}, \tag{29}$$

$$D^{sub}\omega^{\alpha(m)}{}_{,}{}^{\dot\beta(n)} := n\theta(n-m-2)\tilde h_{\gamma,}{}^{\dot\beta}\,\omega^{\gamma\alpha(m)}{}_{,}{}^{\dot\beta(n-1)} + m\theta(m-n-2)\tilde h^{\alpha}{}_{,\dot\delta}\,\omega^{\alpha(m-1)}{}_{,}{}^{\dot\beta(n)\dot\delta}, \tag{30}$$

$$D^{cur}\omega^{\alpha(m)}{}_{,}{}^{\dot\beta(n)} := R_1{}^{\alpha(m)}{}_{,}{}^{\dot\beta(n)}. \tag{31}$$

As a consequence of (11):

$$D^{cur} = \tilde D + D^{top} + \lambda^2 D^{sub}. \tag{32}$$

Since the λ-dependent term vanishes in the Minkowski case, it is convenient to introduce the "flat" part of the covariant derivative:

$$D^{fl} := \tilde D + D^{top}\,. \tag{33}$$

It is also convenient to denote:

$$D^{h} := D^{top} + \lambda^2 D^{sub}\,. \tag{34}$$

Free field equations (16) imply that:

$$D^{cur}\omega^{\alpha(m)}{}_{,}{}^{\dot\beta(n)} \simeq \delta_{n,0}C^{\alpha(m)\gamma\delta}\,H_{\gamma\delta} + \delta_{m,0}\bar C^{\dot\beta(n)\dot\gamma\dot\delta}\,\bar H_{\dot\gamma\dot\delta}\,, \tag{35}$$

where $\simeq$ implies on-shell equality.

If the three-form J_{t,s_1,s_2} verifies (28) on-shell, the charge:

$$Q_\xi = \int\limits_{M^3} J_{t,s_1,s_2} \tag{36}$$

is conserved by virtue of (26). As a result, there are as many conserved charges Q_ξ as independent global symmetry parameters ξ. Nontrivial charges are represented by the current cohomology, i.e., closed currents $J_{t,s_1,s_2}(x)$ modulo exact ones $J_{t,s_1,s_2} \simeq d\Psi_{t,s_1,s_2}$. Since the currents should be closed on-shell, i.e., by virtue of the free field Equations (16)–(18), analysis is greatly simplified by the fact that all linearized HS curvatures $R_1{}^{\alpha(m)}{}_{,}{}^{\dot\beta(n)}$ with $m>0,\ n>0$ are zero on-shell.

Conservation of currents does not imply that they are invariant under the gauge transformations (19). However, as shown below, the gauge variation of J_{t,s_1,s_2} is exact:

$$\delta J_{t,s_1,s_2}(x) \simeq dH_{t,s_1,s_2}(x) \tag{37}$$

so that the charge Q_ξ turns out to be gauge invariant.

Thus, the problem is:

- to find current three-forms (25) from the variation of action,
- to check that these forms obey the conservation condition (28),
- to check that in the flat limit $\lambda \to 0$, these forms give currents of [11],
- to check that the HS charges are gauge invariant.

4. Variation of the Action

Variation of the nonlinear curvature $R^{i;\alpha(m)}{}_{,}{}^{\dot\beta(n)}$ is:

$$\delta R^{i;\alpha(m)}{}_{,}{}^{\dot\beta(n)} = \delta R_1{}^{i;\alpha(m)}{}_{,}{}^{\dot\beta(n)} + \delta R_2{}^{i;\alpha(m)}{}_{,}{}^{\dot\beta(n)}, \tag{38}$$

where:

$$\begin{aligned}\delta R_1{}^{i;\alpha(m)}{}_{,}{}^{\dot\beta(n)} &= \tilde D\delta\omega^{i;\alpha(m)}{}_{,}{}^{\dot\beta(n)}\\ &\quad + n(\theta(m-n)+\lambda^2\theta(n-m-2))\,\tilde h_{\gamma,}{}^{\dot\beta}\,\delta\omega^{i;\gamma\alpha(m)}{}_{,}{}^{\dot\beta(n-1)}\\ &\quad + m(\theta(n-m)+\lambda^2\theta(m-n-2))\,\tilde h^{\alpha}{}_{,\dot\delta}\,\delta\omega^{i;\alpha(m-1)}{}_{,}{}^{\dot\beta(n)\dot\delta}\end{aligned} \tag{39}$$

and:

$$\delta R_2{}^{i;\alpha(m)}{}_{,}{}^{\dot\beta(n)} = \sum_{p,q,k,l,u,v} \lambda^{1+d_0-d_1-d_2} \frac{m!n!}{p!q!k!l!u!v!}(1+(-1)^{k+l}\eta)\, c^i{}_{jk}\, \delta_{p+q,m}\delta_{u+v,n} \times \delta\omega^{j;\alpha(p)}{}_{\gamma(k),\dot\delta(l)}{}^{\dot\beta(u)}\, \omega^{k;\alpha(q)\gamma(k)}{}_{,}{}^{\dot\delta(l)\dot\beta(v)} \quad (40)$$

with η defined in (24).

Variation of the action (22) is:

$$\delta S = \int \sum_{m,n} \varepsilon(m-n) \frac{\lambda^{-|m-n|}}{m!n!}(R_1{}^{i;\alpha(m)}{}_{,}{}^{\dot\beta(n)}\delta R_{1\ i;\alpha(m),\dot\beta(n)} + R_2{}^{i;\alpha(m)}{}_{,}{}^{\dot\beta(n)}\delta R_{1\ i;\alpha(m),\dot\beta(n)} + R_1{}^{i;\alpha(m)}{}_{,}{}^{\dot\beta(n)}\delta R_{2\ i;\alpha(m),\dot\beta(n)} + R_2{}^{i;\alpha(m)}{}_{,}{}^{\dot\beta(n)}\delta R_{2\ i;\alpha(m),\dot\beta(n)}). \quad (41)$$

The first term is the variation of the action S_2 (20), which vanishes on equations of motion (16)–(18). The last term is cubic in connections $\omega^{i;\alpha(m)}{}_{,}{}^{\dot\beta(n)}$, hence not contributing to bilinear currents. The second and third terms give rise to the currents. Using (11), (17), (18), (32), (39) and (40) and integrating by parts, we obtain:

$$\begin{aligned} \delta S \simeq \int \sum_{m,n} \varepsilon(m-n)\, \frac{\lambda^{-|m-n|}}{m!n!}\, [& -\tilde D R_2{}^{i;\alpha(m),\dot\beta(n)}\, \delta\omega_{i;\alpha(m),\dot\beta(n)} \\ & + n(\theta(m-n)+\lambda^2\theta(n-m-2))\, R_2{}^{i;\alpha(m),\dot\theta\dot\beta(n-1)}\, \tilde h^{\gamma}{}_{,\dot\theta}\, \delta\omega_{i;\gamma\alpha(m),\dot\beta(n-1)} \\ & + m(\theta(n-m)+\lambda^2\theta(m-n-2))\, R_2{}^{i;\alpha(m-1)\gamma,\dot\beta(n)}\, \tilde h_{\gamma,}{}^{\dot\delta}\, \delta\omega_{i;\alpha(m-1),\dot\beta(n)\dot\delta}] \\ & + \int \sum_{r>0} \frac{\lambda^{-r}}{r!}\, (C^{i;\alpha(r)\gamma\delta}\, H_{\gamma\delta}\, \delta R_{2\ i;\alpha(r)} - \bar C^{i;\dot\beta(r)\dot\gamma\dot\delta}\, \bar H_{\dot\gamma\dot\delta}\, \delta R_{2\ i;\dot\beta(r)}). \end{aligned} \quad (42)$$

Omitting the color index $i = 1$, this leads to the currents at $t > 1$ via:

$$J_{t,s_1,s_2} = \sum_{m,n} \delta(m+n-2(t-1))\xi_{\alpha(m),\dot\beta(n)} \frac{\delta S}{\delta\omega_{\alpha(m),\dot\beta(n)}}\,. \quad (43)$$

5. Examples

5.1. Spin-Two Current

To illustrate the structure of the current three-form and analyze the flat limit $\lambda \to 0$, consider a current with $t = 2$, $s_1 = s_2 = s > 1$:

$$J_{2,s} = \frac{\lambda^{-2}}{2}\xi_{\alpha\alpha}J_{2,s}{}^{\alpha\alpha} + \xi_{\alpha,\dot\beta}J_{2,s}{}^{\alpha}{}_{,}{}^{\dot\beta} + \frac{\lambda^{-2}}{2}\xi_{\dot\beta\dot\beta}J_{2,s}{}^{\dot\beta\dot\beta}\,, \quad (44)$$

where $J_{2,s} := J_{2,s,s}$. Using (21), (31), (35) and (40), we obtain:

$$\begin{aligned} J_{2,s}{}^{\alpha\alpha} = \sum_{m,n} \frac{4\lambda^{2-|m-n|}}{(m-1)!n!}c_{ij}[& n(\theta(m-n)+\lambda^2\theta(n-m-2))\omega^{i;\alpha\gamma(m-1)\varphi}{}_{,}{}^{\dot\delta(n-1)}\omega^{j;\alpha}{}_{\gamma(m-1),\dot\delta(n-1)\dot\theta}\tilde h_{\varphi,}{}^{\dot\theta} \\ & + (m-1)(\theta(n-m)+\lambda^2\theta(m-n-2))\omega^{i;\alpha\gamma(m-2)}{}_{,}{}^{\dot\delta(n)\dot\theta}\omega^{j;\alpha}{}_{\varphi\gamma(m-2),\dot\delta(n)}\tilde h^{\varphi}{}_{,\dot\theta} \\ & + (\theta(n-m)+\lambda^2\theta(m-n-2))\omega^{i;\gamma(m-1)}{}_{,}{}^{\dot\delta(n)\dot\theta}\omega^{j;\alpha}{}_{\gamma(m-1),\dot\delta(n)}\tilde h^{\alpha}{}_{,\dot\theta}]\,, \end{aligned} \quad (45)$$

$$J_{2,s}{}^{\alpha}{}_{,}{}^{\dot\beta} = \sum_{m,n} 2\lambda^{2-|m-n|}\Big[\frac{1}{(m-1)!n!}c_{ij}\omega^{i;\alpha\gamma(m-1)}{}_{,}{}^{\dot\delta(n)}\omega^{j;\varphi}{}_{\gamma(m-1),\dot\delta(n)}\tilde h_{\varphi,}{}^{\dot\beta} - \frac{1}{m!(n-1)!}c_{ij}\omega^{i;\gamma(m)}{}_{,}{}^{\dot\delta(n-1)\dot\theta}\omega^{j;}{}_{\gamma(m),\dot\delta(n-1)}{}^{\dot\beta}\tilde h^{\alpha,}{}_{\dot\theta}\Big] + \frac{2\lambda^{4-2s}}{(2s-3)!}\big[c_{ij}C^{i;\alpha\gamma(2s-3)\varphi\rho}\,H_{\varphi\rho}\,\omega_{\gamma(2s-3)}{}^{\dot\beta} - c_{ij}\bar C^{i;\dot\delta(2s-3)\dot\beta\dot\psi\dot\theta}\,\bar H_{\dot\psi\dot\theta}\,\omega^{\alpha}{}_{\dot\delta(2s-3)}\big], \quad (46)$$

$$J_{2,s}{}^{\dot\beta\dot\beta} = \sum_{m,n}\frac{4\lambda^{2-|m-n|}}{m!(n-1)!}c_{ij}\big[m(\theta(n-m)+\lambda^2\theta(m-n-2))\omega^{i;\gamma(m-1)}{}_{,}{}^{\dot\delta(n-1)\dot\theta\dot\beta}\omega^{j;}{}_{\varphi\gamma(m-1),\dot\delta(n-1)}{}^{\dot\beta}\tilde h^{\varphi}{}_{,\dot\theta} + (n-1)(\theta(m-n)+\lambda^2\theta(n-m-2))\omega^{i;\varphi\gamma(m)}{}_{,}{}^{\dot\delta(n-2)\dot\beta}\omega^{j;}{}_{\gamma(m),\dot\delta(n-2)\dot\theta}{}^{\dot\beta}\tilde h_{\varphi,}{}^{\dot\theta} + (\theta(m-n)+\lambda^2\theta(n-m-2))\omega^{i;\varphi\gamma(m)}{}_{,}{}^{\dot\delta(n-1)}\omega^{j;}{}_{\gamma(m),\dot\delta(n-1)}{}^{\dot\beta}\tilde h_{\varphi,}{}^{\dot\beta}\big]. \quad (47)$$

Recall that $m+n = 2(s-1)$, $m, n \geq 0$.

The terms in (45), (46) and (47) that contain inverse powers of λ contain higher derivatives. To obtain a proper $\lambda \to 0$ limit, such terms should be compensated by an exact form $d\Psi_{2,s}$ with:

$$\Psi_{2,s} = \sum_{m=0}^{s-3}\frac{2\lambda^{4-2(s-m)}}{m!(2s-3-m!)}\big[\xi_{\alpha\alpha}c_{ij}\omega^{i;\alpha\gamma(m)}{}_{,}{}^{\dot\delta(2s-3-m)}\omega^{i;\alpha}{}_{\gamma(m),\dot\delta(2s-3-m)} + \xi_{\alpha,\dot\beta}\big(c_{ij}\omega^{i;\alpha\gamma(2s-3-m)}{}_{,}{}^{\dot\delta(m)}\omega^{j;}{}_{\gamma(2s-3-m),\dot\delta(m)}{}^{\dot\beta} - c_{ij}\omega^{i;\alpha\gamma(m)}{}_{,}{}^{\dot\delta(2s-3-m)}\omega^{j;}{}_{\gamma(m),\dot\delta(2s-3-m)}{}^{\dot\beta}\big) - \xi_{\dot\beta\dot\beta}c_{ij}\omega^{i;\alpha\gamma(2s-3-m)}{}_{,}{}^{\dot\delta(m)\dot\beta}\omega^{i;}{}_{\gamma(2s-3-m),\dot\delta(m)}{}^{\dot\beta}\big]. \quad (48)$$

At $s = 2$, it is not necessary to add this exact form since the current is regular in the flat limit.

The fact that complete antisymmetrization over any three two-component dotted or undotted indices gives zero yields the relation:

$$c_{ij}\omega^{i;\alpha\gamma(m-1)}{}_{,}{}^{\dot\delta(n)}\omega^{j;\varphi}{}_{\gamma(m-1),\dot\delta(n)}\,\tilde h_{\varphi,}{}^{\dot\beta} = -c_{ij}\omega^{i;\alpha\gamma(m-1)}{}_{,}{}^{\dot\delta(n-1)\dot\beta}\omega^{j;\varphi}{}_{\gamma(m-1),\dot\delta(n-1)}{}^{\dot\theta}\,\tilde h_{\varphi,\dot\theta} + c_{ij}\omega^{i;\alpha\gamma(m-1)}{}_{,}{}^{\dot\delta(n-1)\dot\theta}\omega^{j;\varphi}{}_{\gamma(m-1),\dot\delta(n-1)}{}^{\dot\beta}\,\tilde h_{\varphi,\dot\theta} \quad (49)$$

to be used in the sequel.

Straightforward calculation gives:

$$\hat J_{2,s} = J_{2,s} + d\Psi_{2,s} = \frac{\lambda^{-2}}{2}\xi_{\alpha\alpha}\hat J_{2,s}{}^{\alpha\alpha} + \xi_{\alpha,\dot\beta}\hat J_{2,s}{}^{\alpha}{}_{,}{}^{\dot\beta} + \frac{\lambda^{-2}}{2}\xi_{\dot\beta\dot\beta}\hat J_{2,s}{}^{\dot\beta\dot\beta}, \quad (50)$$

where

$$\hat J_{2,s}{}^{\alpha\alpha} = 2\lambda^2 c_{ij}\big[\omega^{i;\alpha\varphi\gamma(s-2)}{}_{,}{}^{\dot\delta(s-2)}\omega^{j;\alpha}{}_{\gamma(s-2),\dot\delta(s-2)\dot\theta}\tilde h_{\varphi,}{}^{\dot\theta} + \frac{s-2}{s-1}\omega^{i;\alpha\gamma(s-3)}{}_{,}{}^{\dot\delta(s-1)\dot\theta}\omega^{j;\alpha}{}_{\varphi\gamma(s-3),\dot\delta(s-1)}\tilde h^{\varphi}{}_{,\dot\theta} + \frac{1}{s-1}\omega^{i;\alpha\gamma(s-2)}{}_{,}{}^{\dot\delta(s-1)}\omega^{j;}{}_{\gamma(s-2),\dot\delta(s-1)}{}^{\dot\theta}\tilde h^{\alpha}{}_{,\dot\theta}\big], \quad (51)$$

$$\hat{J}_{2,s}{}^{\alpha}{}_{,}{}^{\dot\beta} = \frac{1}{s-1} c_{ij} [\omega^{i;\gamma(s-2)}{}_{,}{}^{\dot\delta(s-1)\dot\theta} \omega^{j;}{}_{\gamma(s-2),\dot\delta(s-1)}{}^{\dot\beta} \tilde{h}^{\alpha}{}_{,\dot\theta}$$
$$+ (s-2)\omega^{i;\alpha\gamma(s-3)}{}_{,}{}^{\dot\delta(s-1)\dot\theta} \omega^{j;}{}_{\varphi\gamma(s-3),\dot\delta(s-1)}{}^{\dot\beta} \tilde{h}^{\varphi}{}_{,\dot\theta} + (s-2)\omega^{i;\alpha\varphi\gamma(s-3)}{}_{,}{}^{\dot\delta(s-1)} \omega^{j;}{}_{\gamma(s-3),\dot\delta(s-1)}{}^{\dot\theta\dot\beta} \tilde{h}_{\varphi,\dot\theta}$$
$$+ (s-2)\omega^{i;\alpha\gamma(s-1)}{}_{,}{}^{\dot\delta(s-3)\dot\theta} \omega^{j;\varphi}{}_{\gamma(s-1),\dot\delta(s-3)}{}^{\dot\beta} \tilde{h}_{\varphi,\dot\theta} + (s-2)\omega^{i;\alpha\varphi\gamma(s-1)}{}_{,}{}^{\dot\delta(s-3)} \omega^{j;}{}_{\gamma(s-1),\dot\delta(s-3)\dot\theta}{}^{\dot\beta} \tilde{h}_{\varphi,}{}^{\dot\theta}$$
$$+ \omega^{i;\alpha\gamma(s-1)}{}_{,}{}^{\dot\delta(s-2)} \omega^{j;\varphi}{}_{\gamma(s-1),\dot\delta(s-2)} \tilde{h}_{\varphi,}{}^{\dot\beta}] \,, \quad (52)$$

$$\hat{J}_{2,s}{}^{\dot\beta\dot\beta} = 2\lambda^2 c_{ij} [\omega^{i;\varphi\gamma(s-2)}{}_{,}{}^{\dot\delta(s-2)\dot\beta} \omega^{j;}{}_{\gamma(s-2),\dot\delta(s-2)\dot\theta}{}^{\dot\beta} \tilde{h}_{\varphi,}{}^{\dot\theta}$$
$$+ \frac{s-2}{s-1} \omega^{i;\varphi\gamma(s-1)}{}_{,}{}^{\dot\delta(s-3)\dot\beta} \omega^{j;}{}_{\gamma(s-1),\dot\delta(s-3)\dot\theta}{}^{\dot\beta} \tilde{h}_{\varphi}{}^{\dot\theta}$$
$$+ \frac{1}{s-1} \omega^{i;\varphi\gamma(s-1)}{}_{,}{}^{\dot\delta(s-2)} \omega^{j;}{}_{\gamma(s-1),\dot\delta(s-2)}{}^{\dot\beta} \tilde{h}_{\varphi,}{}^{\dot\beta}] \,. \quad (53)$$

Note that $\hat{J}_{2,s}$ does not contain λ explicitly. One can check, that (51), (52) and (53) obey (28). As a result, the form $\hat{J}_{2,s}$ (50) is closed by virtue of (26).

Since the AdS_4 current $\hat{J}_{2,s}$ (50) does not depend explicitly on λm it preserves its form in the flat limit $\lambda \to 0$. From (51)–(53), one can see that:

$$\hat{J}_{2,s} = J^M_{2,s} + D^{fl} \chi_{2,s}, \quad (54)$$

where $J^M_{2,s}$ at $\lambda = 0$ reproduces the spin-two current in Minkowski space and:

$$\chi_{2,s} = \frac{c_{ij}}{s-1} \Big(\xi_{\alpha\dot\beta} \big(\omega^{i;\alpha\gamma(s-2)}{}_{,}{}^{\dot\delta(s-1)} \omega^{j;}{}_{\gamma(s-2),\dot\delta(s-1)}{}^{\dot\beta} - \omega^{i;\alpha\gamma(s-1)}{}_{,}{}^{\dot\delta(s-2)} \omega^{j;}{}_{\gamma(s-1),\dot\delta(s-2)}{}^{\dot\beta} \big)$$
$$+ \lambda^2 \big(\xi_{\alpha\alpha} \omega^{i;\alpha\gamma(s-2)}{}_{,}{}^{\dot\delta(s-1)} \omega^{j;\alpha}{}_{\gamma(s-2),\dot\delta(s-1)} + \xi_{\dot\beta\dot\beta} \omega^{i;\gamma(s-1)}{}_{,}{}^{\dot\delta(s-2)\dot\beta} \omega^{j;}{}_{\gamma(s-1),\dot\delta(s-2)}{}^{\dot\beta} \big) \Big) \,.$$

This proves that the flat limit of the current (50) reproduces the results of [11].

We observe that the current is Hermitian. It is nonzero if c_{ij} is symmetric.

5.2. Spin-One Current

Since the action (22) does not contain a kinetic term for spin-one field ω^i carrying no spinor indices, following [12], it should be added separately in a standard way:

$$S_{EM} = \int R_i{}^* R^i \,, \quad (55)$$

where * is the Hodge star operator and, in agreement with (11) and (21),

$$R^i = d\omega^i + \sum_{k,l \geq 0} \frac{\lambda^{1-|m-n|}}{k!l!} c^i{}_{jk}\, \omega^{j;}{}_{\gamma(k),\dot\delta(l)}\, \omega^{k;\gamma(k)}{}_{,}{}^{\dot\delta(l)}. \quad (56)$$

The full action is $S_{full} = S + S_{EM}$ with S (22). The spin-one part of the variation $\delta S_{t=1}$ of this action is:

$$\delta S_{t=1} = \int \sum_{r>0} \frac{\lambda^{-r}}{r!} \left(C^{i;\alpha(r)\gamma\delta}\, H_{\gamma\delta}\, \delta R_{2\; i;\alpha(r)} - \bar{C}^{i;\dot\beta(r)\dot\gamma\dot\delta}\, \bar{H}_{\dot\gamma\dot\delta}\, \delta R_{2\; i;\dot\beta(r)} \right) + \delta S_{EM}, \quad (57)$$

where:

$$\delta S_{EM} = \int [R_{1i}{}^* \delta R_2{}^i + R_{2i}{}^* \delta R_1{}^i]. \quad (58)$$

In the spin-one case equations, (16)–(18) amount to:

$$R_1{}^i = C^{i;\gamma\delta} H_{\gamma\delta} + \bar{C}^{i;\dot\gamma\dot\delta} \bar{H}_{\dot\gamma\dot\delta}\,, \tag{59}$$

where $C^{i;\gamma\delta}$ and $\bar{C}^{i;\dot\gamma\dot\delta}$ parametrize self-dual and anti-self-dual components of the spin-one field tensor. Using properties of the Pauli matrices, one can also see that:

$$R_1{}^{i*} = \mathrm{i}(C^{i;\gamma\delta} H_{\gamma\delta} - \bar{C}^{i;\dot\gamma\dot\delta} \bar{H}_{\dot\gamma\dot\delta}). \tag{60}$$

The sum of spin-one ($t = 1$) currents of fields of arbitrary spins $s_1 = s_2 \geq 1$ can be expressed as (the color index i is omitted):

$$\xi \frac{\delta S_{t=1}}{\delta \omega} = J_{1,1} + \sum_{s>1} J_{1,s}, \tag{61}$$

where ξ is a global symmetry parameter zero-form (26) with no spinor indices and:

$$J_{1,1} = 2\xi\lambda \mathrm{i} c_{ij} (C^{i;\gamma\delta} H_{\gamma\delta} - \bar{C}^{i;\dot\gamma\dot\delta} \bar{H}_{\dot\gamma\dot\delta}) \omega^j - d(\xi R_2{}^{i*}), \tag{62}$$

$$J_{1,s} = 2\lambda^{3-2s} \xi c_{ij} (C^{i;\alpha(2s-2)\varphi\rho}\, H_{\varphi\rho}\, \omega^{j;}{}_{\alpha(2s-2)} - \bar{C}^{i;\dot\beta(2s-2)\dot\gamma\dot\delta}\, \bar{H}_{\dot\gamma\dot\delta}\, \omega^{j;}{}_{\dot\beta(2s-2)}\,). \tag{63}$$

In the case of $t = 1, s_1 = s_2 = s = 1$, one can transform $J_{1,1}$ into:

$$\hat{J}_{1,1} = \frac{1}{\lambda}(J_{1,1} + d(\xi R_2{}^{i*})) = 2\xi \mathrm{i} c_{ij} (C^{i;\gamma\delta} H_{\gamma\delta} - \bar{C}^{i;\dot\gamma\dot\delta} \bar{H}_{\dot\gamma\dot\delta}) \omega^j. \tag{64}$$

It is not hard to see that the C-dependent terms are not exact provided that c_{ij} is antisymmetric. The current (64) coincides with Minkowski current $J^M_{1,1}$ from [11] modulo an overall factor of two.

In the case of $t = 1, s_1 = s_2 = s > 1$, the current results from the C-dependent terms of (42) by virtue of (40):

$$J_{1,s} = 2\lambda^{3-2s} \xi c_{ij} (C^{i;\alpha(2s-2)\varphi\rho}\, H_{\varphi\rho}\, \omega^{j;}{}_{\alpha(2s-2)} - \bar{C}^{i;\dot\beta(2s-2)\dot\gamma\dot\delta}\, \bar{H}_{\dot\gamma\dot\delta}\, \omega^{j;}{}_{\dot\beta(2s-2)}\,). \tag{65}$$

Note, that there are no currents with $t = 1, s_1 \neq s_2$. Furthermore, note that Equation (64) is a particular case of (65) at $s = 1$. The current three-form $J_{1,s}$ (65) is nontrivial if c_{ij} is antisymmetric.

For $s > 1$, $J_{1,s}$ (65) can be rewritten in the bilinear form in connections by adding an exact form:

$$\begin{aligned}\hat{J}_{1,s} = -\frac{1}{\lambda(-2)^{s-1} s(s-1)!}(J_{1,s} + d\Psi_{1,s}) = \\ \xi c_{ij} [\omega^{i;\varphi\gamma(s-2)}{}_{,}{}^{\dot\delta(s-1)} \omega^{j;}{}_{\gamma(s-2),\dot\delta(s-1)\dot\theta} + \omega^{i;\varphi\gamma(s-1)}{}_{,}{}^{\dot\delta(s-2)} \omega^{j;}{}_{\gamma(s-1),\dot\delta(s-2)\dot\theta}] \tilde{h}_{\varphi,}{}^{\dot\theta}\,,\end{aligned} \tag{66}$$

with:

$$\begin{aligned}\Psi_{1,s} = 2\xi\lambda^{3-2s} \sum_{m=0}^{s-2} (-1)^{m+1} 2^m \lambda^{2m} \frac{(s-1)!}{(s-m-1)!} c_{ij} (\omega^{i;\alpha(2s-2-m)}{}_{,}{}^{\dot\beta(m)} \omega^{j;}{}_{\alpha(2s-2-m),\dot\beta(m)} \\ - \omega^{i;\alpha(m)}{}_{,}{}^{\dot\beta(2s-2-m)} \omega^{j;}{}_{\alpha(m),\dot\beta(2s-2-m)})\,.\end{aligned} \tag{67}$$

This three-form is λ-independent, on-shell-closed, Hermitian and reproduces the result of [11]. Hence, it is non-trivial.

6. General Spins

The AdS_4 conserved currents J_{t,s_1,s_2} with $1 < t \leq s_1 + s_2 - 1$ (for definiteness, we set $s_1 \geq s_2$) result from the variation of action (42):

$$J_{t,s_1,s_2} = \sum_{m,n} \varepsilon(m-n)\,\frac{\lambda^{-|m-n|}}{m!n!}\,[-\xi_{\alpha(m),\dot\beta(n)}\; D^h\; R_2{}^{\alpha(m),\dot\beta(n)}|_{s_1,s_2}$$
$$- n(\theta(m-n) + \lambda^2\theta(n-m-2))\;\xi_{\alpha(m+1),\dot\beta(n-1)}\; R_2{}^{\alpha(m),\dot\theta\dot\beta(n-1)}|_{s_1,s_2}\;\tilde{h}^{\alpha}{}_{,\dot\theta}$$
$$+ m(\theta(n-m) + \lambda^2\theta(m-n-2))\;\xi_{\alpha(m-1),\dot\beta(n+1)}\; R_2{}^{\alpha(m-1)\gamma,\dot\beta(n)}|_{s_1,s_2}\;\tilde{h}_{\gamma,}{}^{\dot\beta}]$$
$$+ \sum_{p,q,k,v} \frac{2\lambda^{1-|q+k-p-v|}}{p!q!k!v!}\delta_{2p+q+v,2(t-1)}\delta_{p+k,2(s_1-1)}\delta_{p+q+k+v,2(s_2-1)}c_{ij}\xi_{\alpha(p+q),\dot\beta(p+v)}$$
$$\times\,[C^{i;\alpha(p)\gamma(k)\varphi\rho}\; H_{\varphi\rho}\;\omega^{j;\alpha(q)}{}_{\gamma(k),}{}^{\dot\beta(p+v)} - \bar{C}^{i;\dot\beta(p)\dot\delta(k)\dot\varphi\dot\rho}\;\bar{H}_{\dot\varphi\dot\rho}\;\omega^{j;\alpha(p+q)}{}_{,\dot\delta(k)}{}^{\dot\beta(v)}], \tag{68}$$

where $R_2{}^{\alpha(m),\dot\beta(n)}|_{s_1,s_2}$ is the restriction of (21) to terms containing connections with spins s_1 and s_2. These currents contain $s_1 + s_2 - 2$ derivatives of the frame-like fields.

To check the non-exactness of the three-form (68), it suffices to add an exact form:

$$d\Psi_{t,s_1,s_2} = d(\sum_{m,n}\xi_{\alpha(m)\dot\beta(n)}\Psi_{t,s_1,s_2}{}^{\alpha(m),\dot\beta(n)})\,, \quad n+m = 2(t-1),$$

where:

$$\Psi_{t,s_1,s_2}{}^{\alpha(m),\dot\beta(n)} =$$
$$\sum_{p,q,k,l,u,v} \frac{2\lambda^{1-\frac{|p+k-l-u|-|q+k-l-v|}{2}}}{p!q!k!l!u!v!}\delta_{p+q,m}\delta_{u+v,n}\delta_{p+k+l+u,2(s_1-1)}\delta_{q+k+l+v,2(s_2-1)}$$
$$\times\,\theta(l+u-p-k-1)c_{ij}[\theta(m-n)\omega^{i;\alpha(p)\gamma(k),\dot\delta(l)\dot\beta(u)}\;\omega^{j;\alpha(q)}{}_{\gamma(k),\dot\delta(l)}{}^{\dot\beta(v)}$$
$$- \theta(n-m)\omega^{i;\alpha(u)\gamma(l),\dot\delta(k)\dot\beta(p)}\;\omega^{j;\alpha(v)}{}_{\gamma(l),\dot\delta(k)}{}^{\dot\beta(q)}], \tag{69}$$

One can see that $\Psi_{t,s_1,s_2}{}^{\alpha(m),\dot\beta(n)}$ is adjusted to cancel the C-dependent terms.

The resulting current three-form:

$$\hat{J}_{t,s_1,s_2} := J_{t,s_1,s_2} + d\Psi_{t,s_1,s_2} \tag{70}$$

is

$$\hat{J}_{t,s_1,s_2}{}^{\alpha(m),\dot\beta(n)} =$$
$$\sum_{p,q,k,l,u,v} \frac{2\lambda^{1+2|m-n|+|p+k-l-u|-|q+k-l-v|}}{p!q!k!l!u!v!}\delta_{p+q,m}\delta_{u+v,n}\delta_{p+k+l+u,2(s_1-1)}\delta_{q+k+l+v,2(s_2-1)}c_{ij}$$
$$\times\,[\theta(p+k-l-u-1)D^h(\theta(m-n)\omega^{i;\alpha(p)\gamma(k),\dot\delta(l)\dot\beta(u)}\;\omega^{j;\alpha(q)}{}_{\gamma(k),\dot\delta(l)}{}^{\dot\beta(v)}$$
$$- \theta(n-m)\omega^{i;\alpha(u)\gamma(l),\dot\delta(k)\dot\beta(p)}\;\omega^{j;\alpha(v)}{}_{\gamma(l),\dot\delta(k)}{}^{\dot\beta(q)})$$
$$- n(\theta(m-n) + \lambda^2\theta(n-m-2))((u+1)\omega^{i;\alpha(p-1)\gamma(k),\dot\delta(l)\dot\theta\dot\beta(u)}\;\omega^{j;\alpha(v)}{}_{\gamma(k),\dot\delta(l)}{}^{\dot\beta(v)}\tilde{h}^{\alpha}{}_{,\dot\theta}$$
$$+ (v+1)\omega^{i;\alpha(p)\gamma(k),\dot\delta(l)\dot\beta(u)}\;\omega^{j;\alpha(q-1)}{}_{\gamma(k),\dot\delta(l)}{}^{\dot\theta\dot\beta(v)}\tilde{h}^{\alpha}{}_{,\dot\theta})$$
$$+ m(\theta(n-m) + \lambda^2\theta(m-n-2))\;((p+1)\omega^{i;\alpha(p)\varphi\gamma(k),\dot\delta(l)\dot\beta(u)}\;\omega^{j;\alpha(q)}{}_{\gamma(k),\dot\delta(l)}{}^{\dot\beta(v)}\tilde{h}_{\varphi,}{}^{\dot\beta}$$
$$+ (q+1)\omega^{i;\alpha(p)\gamma(l),\dot\delta(k)\dot\beta(u-1)}\;\omega^{j;\alpha(q)\varphi}{}_{\gamma(l),\dot\delta(k)}{}^{\dot\beta(v-1)}\tilde{h}_{\varphi,}{}^{\dot\beta})]\,. \tag{71}$$

This current contains $t - |s_1 - s_2|$ derivatives, which is the minimal possible number. The non-exactness of the current three-form $\hat{J}_{t,s_1,s_2}$ can be checked in the flat limit $\lambda \to 0$ just as in [11].

In the case of $s_1 = s_2 = s$:

$$\hat{J}_{t,s} = \sum_{n,m} \frac{\lambda^{-|m-n|}}{m!n!} \xi_{\alpha(m),\dot{\beta}(n)} \hat{J}_{t,s}{}^{\alpha(m)}{}_{,}{}^{\dot{\beta}(n)}, \quad n+m = 2(t-1),$$

where:

$$\begin{aligned}
\hat{J}_{t,s}{}^{\alpha(m)}{}_{,}{}^{\dot{\beta}(n)} = \\
&\lambda^{|m-n|} m!n!(\theta(n-m-4)\,\hat{g}(n)\,c_{ij}\omega^{i;\alpha(m)\varphi\gamma(s-2)}{}_{,}{}^{\dot{\delta}(s-t)\dot{\beta}(n-t+1)}\omega^{j;}{}_{\gamma(s-2),\dot{\delta}(s-t)\dot{\theta}}{}^{\dot{\beta}(t-1)}\tilde{h}_{\varphi,}{}^{\dot{\theta}} \\
&\quad + \delta_{n,t}\,c_{ij}[2(t-1)\omega^{i;\alpha(m)\varphi\gamma(s-2)}{}_{,}{}^{\dot{\delta}(s-t)\dot{\beta}}\omega^{j;}{}_{\gamma(s-2),\dot{\delta}(s-t)\dot{\theta}}{}^{\dot{\beta}(n-1)}\tilde{h}_{\varphi,}{}^{\dot{\theta}} \\
&\quad + \sum_{p=1}^{t-2}\hat{f}(p)\omega^{i;\alpha(m)\varphi\gamma(s-p-1)}{}_{,}{}^{\dot{\delta}(s-t+p)}\omega^{j;}{}_{\gamma(s-p-1),\dot{\delta}(s-t+p)}{}^{\dot{\beta}(n-1)}\tilde{h}_{\varphi,}{}^{\dot{\beta}}] \\
&\quad + \delta_{m,t-1}\delta_{n,t-1}c_{ij}[\omega^{i;\alpha(t-1)\varphi\gamma(s-2)}{}_{,}{}^{\dot{\delta}(s-t)}\omega^{j;}{}_{\gamma(s-2),\dot{\delta}(s-t)\dot{\theta}}{}^{\dot{\beta}(t-1)}\tilde{h}_{\varphi,}{}^{\dot{\theta}} \\
&\qquad + \omega^{i;\alpha(t-1)\varphi\gamma(s-t)}{}_{,}{}^{\dot{\delta}(s-2)}\omega^{j;}{}_{\gamma(s-t),\dot{\delta}(s-2)\dot{\theta}}{}^{\dot{\beta}(t-1)}\tilde{h}_{\varphi,}{}^{\dot{\theta}}] \\
&+ \theta(m-n-4)\,\hat{g}(m)\,c_{ij}\omega^{i;\alpha(t-1)\varphi\gamma(s-t)}{}_{,}{}^{\dot{\delta}(s-2)}\omega^{j;\alpha(m-t+1)}{}_{\gamma(s-t),\dot{\delta}(s-2)\dot{\theta}}{}^{\dot{\beta}(n)}\tilde{h}_{\varphi,}{}^{\dot{\theta}} \\
&\quad + \delta_{m,t}\,c_{ij}[2(t-1)\omega^{i;\alpha(m-1)\varphi\gamma(s-t)}{}_{,}{}^{\dot{\delta}(s-2)}\omega^{j;\alpha}{}_{\gamma(s-t),\dot{\delta}(s-2)\dot{\theta}}{}^{\dot{\beta}(n)}\tilde{h}_{\varphi,}{}^{\dot{\theta}} \\
&\qquad + \sum_{p=1}^{t-2}\hat{f}(p)\omega^{i;\alpha(m-1)\gamma(s-t+p)}{}_{,}{}^{\dot{\delta}(s-p-1)}\omega^{j;}{}_{\gamma(s-t+p),\dot{\delta}(s-p-1)}{}^{\dot{\theta}\dot{\beta}(n)}\tilde{h}^{\alpha}{}_{,\dot{\theta}}]),
\end{aligned} \tag{72}$$

and:

$$\hat{g}(m) = \frac{2(t-1)!}{(2t-m-2)!(m-t+1)!}, \quad m \geq t+1, \tag{73}$$

$$\hat{f}(1) = \frac{t-1}{s-t+1}, \quad \hat{f}(p) = (t-1)\frac{(s-t)!(s-p)!}{(s-3)!(s-t+p)!}, \quad p > 1. \tag{74}$$

The second and last terms in (72) contribute to the special cases of $n = t$ and $m = t$, respectively.

One can check that the current (68) at $s_1 = s_2$ reproduces that of [11] up to a D^{fl}-exact form:

$$\chi_{t,s} = D^{fl}(\sum_{m,n}\xi_{\alpha(m)\dot{\beta}(n)}\chi_{t,s}{}^{\alpha(m),\dot{\beta}(n)}), \quad n+m = 2(t-1),$$

where:

$$\begin{aligned}
\chi_{t,s}{}^{\alpha(m)}{}_{,}{}^{\dot{\beta}(n)} = \\
&\theta(n-m-2)g(n)c_{ij}\sum_{p=1}^{m+1}\omega^{i;\alpha(m)\gamma(s-p),\dot{\delta}(s-t+p-1)\dot{\beta}(n-t+1)}\omega^{j;}{}_{\gamma(s-p),\dot{\delta}(s-t+p-1)}{}^{\dot{\beta}(t-1)} \\
&+ \theta(m-n-2)g(m)c_{ij}\sum_{p=1}^{n+1}\omega^{i;\alpha(t-1)\gamma(s-t+p-1),\dot{\delta}(s-p)}\omega^{j;\alpha(m-t+1)}{}_{\gamma(s-t+p-1),\dot{\delta}(s-p)}{}^{\dot{\beta}(n)} \\
&+ \delta_{m,t-1}\delta_{n,t-1}f c_{ij}\sum_{p=0}^{[\frac{t}{2}]}[\omega^{i;\alpha(t-p-1)\gamma(s-2),\dot{\delta}(s-t+1)\dot{\beta}(p)}\omega^{j;\alpha(p)}{}_{\gamma(s-2),\dot{\delta}(s-t+1)}{}^{\dot{\beta}(s-t-p)} \\
&\qquad + \omega^{i;\alpha(t-p-1)\gamma(s-1),\dot{\delta}(s-t)\dot{\beta}(p)}\omega^{j;\alpha(p)}{}_{\gamma(s-1),\dot{\delta}(s-t)}{}^{\dot{\beta}(s-t-p)}],
\end{aligned} \tag{75}$$

where:

$$f = \frac{1}{s-t+1}, \qquad g(m) = \frac{t-m}{s-p}. \tag{76}$$

The conserved currents are nontrivial if c_{ij} is antisymmetric for odd $t+s_1+s_2$ and symmetric for even.

Thus, the Hermitian current three-form $\hat{J}_{t,s_1,s_2}$ is on-shell closed, but not exact. It generates the corresponding real conserved charge $Q = \int \hat{J}_{t,s_1,s_2}$ that contains as many symmetry parameters as local HS gauge symmetries.

7. Gauge Transformations

Although the current three-form (68) is not invariant under the gauge transformations (19), its gauge variation is exact. Schematically, the proof consists of the following steps. By virtue of (32), the gauge variation of any term from (68) $\delta(\xi\omega_1\omega_2 h)$ can be written as:

$$\delta(\xi\omega_1\omega_2 h) = \xi\omega_1(\tilde{D}\varepsilon_2 + D^h\varepsilon_2)h + \xi(\tilde{D}\varepsilon_1 + D^h\varepsilon_1)\omega_2 h.$$

This gives:

$$\begin{aligned}\delta(\xi\omega_1\omega_2 h) = &-\tilde{D}(\xi\omega_1\varepsilon_2 h - \xi\varepsilon_1\omega_2 h) + \xi(D^{cur}\omega_1)\varepsilon_2 h + \xi\varepsilon_1(D^{cur}\omega_2)h \\ &+ (D^h\xi)\omega_1\varepsilon_2 h + \xi(D^h\omega_1)\varepsilon_2 h - \xi\omega_1(D^h\varepsilon_2)h + (D^h\xi)\varepsilon_1\omega_2 h + \xi\varepsilon_1(D^h\omega_2)h - \xi(D^h\varepsilon_1)\omega_2 h.\end{aligned}$$

All terms containing D^{cur} are canceled by $\delta(\xi C H\omega h)$. All terms with D^h cancel each other.

This gives:

$$\delta J_{t,s_1,s_2} \simeq dH_{t,s_1,s_2},$$

where:

$$\begin{aligned}H_{t,s_1,s_2} = &\\ &-\sum_{m,n}\varepsilon(m-n)\sum_{p,q,k,l,u,v}\delta_{p+q,m}\delta_{u+v,n}\delta_{p+k+l+u,2(s_1-1)}\delta_{q+k+l+v,2(s_2-1)}c_{ij}\xi_{\alpha(p+q),\dot\beta(u+v)} \\ &\qquad\times\frac{\lambda^{1-\frac{|m-n|}{2}-\frac{|p+k-l-u|}{2}-\frac{|q+k-l-v|}{2}}}{p!q!k!l!u!v!}[D^h\varepsilon^{i;\alpha(p)\gamma(k),\dot\delta(l)\dot\beta(u)}\,\omega^{j;\alpha(q)}{}_{\gamma(k),\dot\delta(l)}{}^{\dot\beta(v)} \\ &\qquad\qquad - n(\theta(m-n)+\lambda^2\theta(n-m-2))\,\xi_{\alpha(p+q+1),\dot\beta(u+v-1)} \\ &\times(\varepsilon^{i;\alpha(p)\gamma(k),\dot\delta(l)\dot\theta\dot\beta(u)}\,\omega^{j;\alpha(q)}{}_{\gamma(k),\dot\delta(l)}{}^{\dot\beta(v)}\,\tilde{h}^{\alpha}{}_{,\dot\theta} + \varepsilon^{i;\alpha(p)\gamma(k),\dot\delta(l)\dot\beta(u)}\,\omega^{j;\alpha(q)}{}_{\gamma(k),\dot\delta(l)}{}^{\dot\theta\dot\beta(v)}\,\tilde{h}^{\alpha}{}_{,\dot\theta}) \\ &\qquad\qquad - m(\theta(n-m)+\lambda^2\theta(m-n-2))\,\xi_{\alpha(p+q-1),\dot\beta(u+v+1)} \\ &\times(\varepsilon^{i;\alpha(p)\gamma(k),\dot\delta(l)\dot\beta(u)}\,\omega^{j;\alpha(q)}{}_{\gamma(k),\dot\delta(l)}{}^{\dot\beta(v)}\,\tilde{h}^{\alpha}{}_{,\dot\theta} + \varepsilon^{i;\alpha(p)\varphi\gamma(k),\dot\delta(l)\dot\beta(u)}\,\omega^{j;\alpha(q)\varphi}{}_{\gamma(k),\dot\delta(l)}{}^{\dot\beta(v)}\,\tilde{h}_{\varphi,}{}^{\dot\beta})] \\ &\quad + 2\sum_{p,q,k,v}\frac{\lambda^{2-s_1-\frac{|k-p|}{2}}}{p!q!k!v!}\delta_{2p+q+v,2(t-1)}\delta_{p+k,2(s_1-1)}\delta_{p+q+k+v,2(s_2-1)}c_{ij}\xi_{\alpha(p+q),\dot\beta(p+v)} \\ &\qquad\qquad\times[C^{i;\alpha(p)\gamma(k)\varphi\rho}\,H_{\varphi\rho}\,\varepsilon^{j;\alpha(q)}{}_{\gamma(k),}{}^{\dot\beta(p+v)} - \bar{C}^{i;\dot\beta(p)\dot\delta(k)\dot\varphi\dot\rho}\,\bar{H}_{\dot\varphi\dot\rho}\,\varepsilon^{j;\alpha(p+q)}{}_{,\dot\delta(k)}{}^{\dot\beta(v)}].\end{aligned} \tag{77}$$

Details of the derivation of this formula are given in the Appendix for the case of $t=2$, $s_1=s_2=s>1$.

Thus, $\delta J_{t,s_1,s_2}$ is exact on-shell. The same is true for the spin-one current (65). Hence, though the current J_t is not gauge invariant, the corresponding charge is:

$$\delta Q_\xi \simeq \int dH_{t,s_1,s_2} = 0.$$

8. Conclusion

In this paper, spin-t HS currents J_{t,s_1,s_2} in AdS_4, built from boson fields of arbitrary spins obeying $t \leq s_1 + s_2 - 1$ are found from the variation principle. Being represented as three-forms, J_{t,s_1,s_2} are closed, but not exact, hence leading to nontrivial HS charges. These charges are gauge invariant because $\delta J_{t,s_1,s_2}$ is shown to be exact.

In the 4d Minkowski case, in addition to natural parity-even currents, we found "mysterious" parity-odd currents [11]. In agreement with the conjecture of [11], we were not able to extend parity-odd currents to AdS_4. The $\lambda \to 0$ limit of $\hat{J}_{t,s}$ (72) reproduces the parity-even currents of [11].

Currents constructed from fields of half-integer spins can be found analogously. How to operate with half-integer fields is shown in [15]. It is important to mention that for the currents built from fields of half-integer spin, the computations are essentially different, because Equation (11) for half-integer spins contains λ instead of λ^2 [15].

Let us stress that the derivation of the currents via the action applied in this paper leads to currents containing the non-minimal number of derivatives according to [16] with the higher-derivative terms corresponding to certain improvements. This is however anticipated since consistent cubic HS interactions are known [12] to contain higher-derivative terms allowing one to preserve HS gauge symmetries associated with gauge fields of different spins.

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Appendix Example of the Gauge Variation of $J^{t,s}$

Consider the case of $t = 2$, $s_1 = s_2 = s > 1$ (44):

$$J_{2,s} = \frac{\lambda^{-2}}{2}\xi_{\alpha\alpha} J_{2,s}{}^{\alpha\alpha} + \xi_{\alpha,\dot{\beta}} J_{2,s}{}^{\alpha}{}_{,}{}^{\dot{\beta}} + \frac{\lambda^{-2}}{2}\xi_{\dot{\beta}\dot{\beta}} J_{2,s}{}^{\dot{\beta}\dot{\beta}}.$$

The gauge variation of (44) under the gauge transformation (19) is:

$$\delta J_{2,s} = \frac{\lambda^{-2}}{2}\xi_{\alpha\alpha} \delta J_{2,s}{}^{\alpha\alpha} + \xi_{\alpha,\dot{\beta}} \delta J_{2,s}{}^{\alpha}{}_{,}{}^{\dot{\beta}} + \frac{\lambda^{-2}}{2}\xi_{\dot{\beta}\dot{\beta}} \delta J_{2,s}{}^{\dot{\beta}\dot{\beta}}, \tag{A1}$$

where:

$$\begin{aligned}\delta J_{2,s}{}^{\alpha\alpha} = -D^h \Big(\sum_{m,n} \frac{4\lambda^{2-|m-n|}}{(m-1)!n!} c_{ij} [\tilde{D}\varepsilon^{i;\alpha\gamma(m-1)}{}_{,}{}^{\dot{\delta}(n)} \omega^{j;\alpha}{}_{\gamma(m-1),\dot{\delta}(n)} \\ - D^{top}\varepsilon^{i;\alpha\gamma(m-1)}{}_{,}{}^{\dot{\delta}(n)} \omega^{j;\alpha}{}_{\gamma(m-1),\dot{\delta}(n)} - \lambda^2 D^{sub}\varepsilon^{i;\alpha\gamma(m-1)}{}_{,}{}^{\dot{\delta}(n)} \omega^{j;\alpha}{}_{\gamma(m-1),\dot{\delta}(n)}] \Big),\end{aligned} \tag{A2}$$

$$
\begin{aligned}
\delta J_{2,s}{}^{\alpha}{}_{,}{}^{\dot\beta} = \sum_{m,n} 2\lambda^{2-|m-n|}[& \frac{1}{(m-1)!n!} c_{ij} \tilde{D}\varepsilon^{i;\alpha\gamma(m-1)}{}_{,}{}^{\dot\delta(n)} \omega^{j;\varphi}{}_{\gamma(m-1),\dot\delta(n)} \tilde{h}_{\varphi,}{}^{\dot\beta} \\
& - \frac{1}{(m-1)!n!} c_{ij} D^{top} \varepsilon^{i;\alpha\gamma(m-1)}{}_{,}{}^{\dot\delta(n)} \omega^{j;\varphi}{}_{\gamma(m-1),\dot\delta(n)} \tilde{h}_{\varphi,}{}^{\dot\beta} \\
& - \frac{\lambda^2}{(m-1)!n!} c_{ij} D^{sub} \varepsilon^{i;\alpha\gamma(m-1)}{}_{,}{}^{\dot\delta(n)} \omega^{j;\varphi}{}_{\gamma(m-1),\dot\delta(n)} \tilde{h}_{\varphi,}{}^{\dot\beta} \\
& - \frac{1}{m!(n-1)!} c_{ij} \omega^{i;\gamma(m)}{}_{,}{}^{\dot\delta(n-1)\dot\theta} \tilde{D}\varepsilon^{j;}{}_{\gamma(m),\dot\delta(n-1)}{}^{\dot\beta} \tilde{h}^{\alpha,}{}_{\dot\theta} \\
& + \frac{1}{m!(n-1)!} c_{ij} \omega^{i;\gamma(m)}{}_{,}{}^{\dot\delta(n-1)\dot\theta} D^{top} \varepsilon^{j;}{}_{\gamma(m),\dot\delta(n-1)}{}^{\dot\beta} \tilde{h}^{\alpha,}{}_{\dot\theta}] \\
& + \frac{\lambda^2}{m!(n-1)!} c_{ij} \omega^{i;\gamma(m)}{}_{,}{}^{\dot\delta(n-1)\dot\theta} D^{sub} \varepsilon^{j;}{}_{\gamma(m),\dot\delta(n-1)}{}^{\dot\beta} \tilde{h}^{\alpha,}{}_{\dot\theta}] \\
+ \frac{2\lambda^{4-2s}}{(2s-3)!} c_{ij} [& C^{i;\alpha\gamma(2s-3)\varphi\rho}\, H_{\varphi\rho}\, \tilde{D}\varepsilon_{\gamma(2s-3)}{}^{\dot\beta} C^{i;\alpha\gamma(2s-3)\varphi\rho}\, H_{\varphi\rho}\, D^{top}\varepsilon_{\gamma(2s-3)}{}^{\dot\beta} \\
& - \bar{C}^{i;\dot\delta(2s-3)\dot\beta\dot\psi\dot\theta}\, \bar{H}_{\dot\psi\dot\theta}\, \tilde{D}\varepsilon^{\alpha}{}_{\dot\delta(2s-3)} + \bar{C}^{i;\dot\delta(2s-3)\dot\beta\dot\psi\dot\theta}\, \bar{H}_{\dot\psi\dot\theta}\, D^{top}\varepsilon^{\alpha}{}_{\dot\delta(2s-3)}] ,
\end{aligned} \tag{A3}
$$

$$
\begin{aligned}
\delta J_{2,s}{}^{\dot\beta\dot\beta} = -D^h \Big(\sum_{m,n} \frac{4\lambda^{2-|m-n|}}{m!(n-1)!} c_{ij} [& \tilde{D}\varepsilon^{i;\gamma(m)}{}_{,}{}^{\dot\delta(n-1)\dot\beta} \omega^{j;}{}_{\gamma(m),\dot\delta(n-1)}{}^{\dot\beta} \\
& - D^{top}\varepsilon^{i;\gamma(m)}{}_{,}{}^{\dot\delta(n-1)\dot\beta} \omega^{j;}{}_{\gamma(m),\dot\delta(n-1)}{}^{\dot\beta} - \lambda^2 D^{sub}\varepsilon^{i;\gamma(m)}{}_{,}{}^{\dot\delta(n-1)\dot\beta} \omega^{j;}{}_{\gamma(m),\dot\delta(n-1)}{}^{\dot\beta}] \Big) .
\end{aligned} \tag{A4}
$$

Rearranging terms in (A1), one can obtain:

$$
\delta J_{t,s} = dH + \chi,
$$

where $\chi \simeq 0$ (vanishes on-shell) and dH is exact with:

$$
\begin{aligned}
H = & -\xi_{\alpha\alpha} D^h \big(\sum_{m,n} \frac{4\lambda^{2-|m-n|}}{(m-1)!n!} c_{ij} \epsilon^{i;\alpha\gamma(m-1)}{}_{,}{}^{\dot\delta(n)} \omega^{j;\alpha}{}_{\gamma(m-1),\dot\delta(n)} \big) \\
& + \sum_{m,n} 2\xi_{\alpha,\dot\beta} \frac{\lambda^{2-|m-n|}}{(m-1)!n!} c_{ij} \big(\varepsilon^{i;\alpha\gamma(m-1)}{}_{,}{}^{\dot\delta(n)} \omega^{j;\varphi}{}_{\gamma(m-1),\dot\delta(n)} \tilde{h}_{\varphi,}{}^{\dot\beta} + \omega^{i;\alpha\gamma(m-1)}{}_{,}{}^{\dot\delta(n)} \varepsilon^{j;\varphi}{}_{\gamma(m-1),\dot\delta(n)} \tilde{h}_{\varphi,}{}^{\dot\beta} \big) \\
& - \sum_{m,n} 2\xi_{\alpha,\dot\beta} \frac{\lambda^{2-|m-n|}}{m!(n-1)!} c_{ij} \big(\varepsilon^{i;\gamma(m)}{}_{,}{}^{\dot\delta(n-1)\dot\theta} \omega^{j;}{}_{\gamma(m),\dot\delta(n-1)}{}^{\dot\beta} \tilde{h}^{\alpha,}{}_{\dot\theta} - \omega^{i;\gamma(m)}{}_{,}{}^{\dot\delta(n-1)\dot\theta} \varepsilon^{j;}{}_{\gamma(m),\dot\delta(n-1)}{}^{\dot\beta} \tilde{h}^{\alpha,}{}_{\dot\theta} \big) \\
& - \xi_{\dot\beta\dot\beta} D^h \big(\sum_{m,n} \frac{4\lambda^{2-|m-n|}}{m!(n-1)!} c_{ij} \varepsilon^{i;\gamma(m)}{}_{,}{}^{\dot\delta(n-1)\dot\beta} \omega^{j;}{}_{\gamma(m),\dot\delta(n-1)}{}^{\dot\beta} \big) .
\end{aligned} \tag{A5}
$$

Thus, the on-shell gauge variation of $J_{2,s}$ is exact.

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