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Generalization of Nambu–Hamilton Equation and Extension of Nambu–Poisson Bracket to Superspace

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Abstract: We propose a generalization of the Nambu–Hamilton equation in superspace $\mathbb{R}^{3|2}$ with three real and two Grassmann coordinates. We construct the even degree vector field in the superspace $\mathbb{R}^{3|2}$ by means of the right-hand sides of the proposed generalization of the Nambu–Hamilton equation and show that this vector field is divergenceless in superspace. Then we show that our generalization of the Nambu–Hamilton equation in superspace leads to a family of ternary brackets of even degree functions defined with the help of a Berezinian. This family of ternary brackets is parametrized by the infinite dimensional group of invertible second order matrices, whose entries are differentiable functions on the space $\mathbb{R}^3$. We study the structure of the ternary bracket in a more general case of a superspace $\mathbb{R}^{n|2}$ with n real and two Grassmann coordinates and show that for any invertible second order functional matrix it splits into the sum of two ternary brackets, where one is the usual Nambu–Poisson bracket, extended in a natural way to even degree functions in a superspace $\mathbb{R}^{n|2}$, and the second is a new ternary bracket, which we call the Ψ-bracket, where Ψ can be identified with an invertible second order functional matrix. We prove that the ternary Ψ-bracket as well as the whole ternary bracket (the sum of the Ψ-bracket with the usual Nambu–Poisson bracket) is totally skew-symmetric, and satisfies the Leibniz rule and the Filippov–Jacobi identity (Fundamental Identity).

Keywords: Nambu–Hamilton equation; Nambu–Poisson bracket; superspace; Filippov–Jacobi identity; Nambu–Hamilton mechanics

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1. Introduction

In Reference [1], Nambu proposed a generalization of the Hamilton equation, which contained a pair of Hamiltonians. Now this generalization is called the Nambu–Hamilton equation. Nambu showed that his generalization of the Hamilton equation led in a natural way to a ternary bracket of three functions, defined by means of third order determinant, whose entries are the derivatives of functions with respect to coordinates of $\mathbb{R}^3$. Now this ternary bracket (or, more generally, n-ary bracket in the case of n-dimensional space) is called a ternary Nambu–Poisson bracket. An important fact of Nambu's approach was that in his generalization of Hamiltonian mechanics, Liouville's theorem (which is an important part of Hamiltonian dynamics) was also valid. We remind you of Liouville's theorem [2], which asserts that the flow induced by a Hamiltonian vector field (one-parameter group of diffeomorphisms induced by a vector field) preserves a volume. Liouville's theorem can be proved by means of a more general statement: if a vector field is divergenceless then it preserves a volume. It is easy to check that the vector field constructed by means of the right-hand sides of the generalization of the Hamilton equation proposed by Nambu is divergenceless. An excellent introduction to this field of research is given in Reference [3].

Independently of Nambu, Filippov proposed a notion of n-Lie algebra, which is a generalization of the notion of Lie algebra based on the n-ary Lie bracket [4]. The basic part of the definition of n-Lie algebra is generalized Jacobi identity, which is now referred to as either Fundamental Identity or Filippov–Jacobi identity. Later it turned out that a generalization of Hamiltonian mechanics proposed by Nambu and a generalization of Lie algebra proposed by Filippov are closely related. Particularly, it was shown that the ternary Nambu bracket (or, more generally, n-ary bracket) satisfies the Filippov–Jacobi identity. The question of quantization of Nambu–Poisson bracket has been considered in a number of papers, but so far this is the outstanding problem. In Reference [5], the authors propose the realization of quantum Nambu–Poisson bracket by means of nth order matrices, where the triple commutator is defined with the help of the usual commutator and the trace of a matrix. This approach is extended to the super Nambu–Poisson bracket by means of supermatrices, where the $\mathbb{Z}_2$-graded triple commutator is defined with the help of the supertrace of a supermatrix [6,7].

A ternary generalization of the Poisson bracket proposed by Nambu is defined with the help of the Jacobian of a mapping

$$(x, y, z) \to (f(x, y, z), g(x, y, z), h(x, y, z))$$

as follows

$$\{F, H, G\} = \frac{\partial(F, H, G)}{\partial(x, y, z)} = \mathrm{Det}\begin{pmatrix} \partial_x F & \partial_y F & \partial_z F \\ \partial_x H & \partial_y H & \partial_z H \\ \partial_x G & \partial_y G & \partial_z G \end{pmatrix}, \tag{1}$$

where x, y, z are the coordinates in $\mathbb{R}^3$ and F, H, G are differentiable functions. Evidently this ternary bracket is totally skew-symmetric. It can also be verified that it satisfies the Leibniz rule

$$\{G\,H, F^1, F^2\} = G\,\{H, F^1, F^2\} + H\,\{G, F^1, F^2\},$$

and the identity

$$\{G, H, \{F^1, F^2, F^3\}\} = \{\{G, H, F^1\}, F^2, F^3\} + \{F^1, \{G, H, F^2\}, F^3\} + \{F^1, F^2, \{G, H, F^3\}\}.$$

This identity is called either Fundamental Identity or Filippov–Jacobi identity and its n-ary version is the basic component of a concept of n-Lie algebra proposed by Filippov in Reference [4].

In this paper, we continue to study an analog of the ternary Nambu–Poisson bracket in superspace, which was proposed in Reference [8]. We propose a generalization of the Nambu–Hamilton equation in superspace $\mathbb{R}^{3|2}$ with three real and two Grassmann coordinates $\theta, \bar{\theta}$. The right-hand sides of this generalization of the Nambu–Hamilton equation are constructed by means of a Berezinian of two even degree functions H, G, which play the role of a pair of Hamiltonians, and two odd degree functions ϕ, ψ, which we consider as parameters of generalization of the Nambu–Hamilton equation in superspace. In analogy with the Nambu approach, we show that the even degree vector field, constructed by means of the right-hand sides of generalization of the Nambu–Hamilton equation in superspace, is divergenceless. We then show that the proposed generalization of the Nambu–Hamilton equation in superspace leads in a natural way to a family of ternary brackets defined with the help of a Berezinian. This family is parametrized by the group of invertible second order matrices, whose entries are differentiable functions on three dimensional space $\mathbb{R}^3$. We show that for any invertible second order functional matrix this ternary bracket splits into the sum of the usual ternary Nambu–Poisson bracket (extended in natural way to even degree functions in superspace) and the new ternary bracket, which we call the ternary Ψ-bracket, where Ψ can be identified either with a pair of odd degree functions or with the invertible second order functional matrix. We prove that the ternary Ψ-bracket as well as the whole sum (the usual Nambu–Poisson bracket plus ternary Ψ-bracket) is totally skew-symmetric, and satisfies the Leibniz rule and the Filippov–Jacobi identity. This gives us grounds to consider our family of ternary brackets defined by means of a Berezinian as an extension of the Nambu–Poisson bracket to superspace.

2. Generalization of Nambu–Hamilton Equation in Superspace

Let us consider the superspace $\mathbb{R}^{3|2}$ with the real coordinates x, y, z and the Grassmann coordinates $\theta, \bar{\theta}$. In what follows, we will denote the collection of the even degree coordinates by r, the collection of the odd degree coordinates by ξ, and consider only smooth functions. The algebra of smooth functions on three dimensional space $\mathbb{R}^3$ will be denoted by $\mathfrak{C}$.

Let us consider a smooth curve $\alpha : I \to \mathbb{R}^{3|2}$, where

$$\alpha(t) = (x(t), y(t), z(t), \theta(t), \bar{\theta}(t)), \quad I \subset \mathbb{R},$$

and

$$\theta(t) = f_{11}(t)\,\theta + f_{12}(t)\,\bar{\theta}, \;\; \bar{\theta}(t) = f_{21}(t)\,\theta + f_{22}(t)\,\bar{\theta}. \tag{2}$$

Let us denote

$$f(t) = \begin{pmatrix} f_{11}(t) & f_{12}(t) \\ f_{21}(t) & f_{22}(t) \end{pmatrix}, \qquad |f(t)| = \mathrm{Det}\, f(t). \tag{3}$$

Then $\theta(t)\bar{\theta}(t) = |f(t)|\,\theta\bar{\theta}$. Let H, G be two even degree functions and ϕ, ψ two odd degree functions on the superspace $\mathbb{R}^{3|2}$. Functions ϕ, ψ can be written in terms of coordinates of superspace as follows

$$\phi(r,\xi) = \phi_1(r)\,\theta + \phi_2(r)\,\bar{\theta}, \quad \psi(r,\xi) = \psi_1(r)\,\theta + \psi_2(r)\,\bar{\theta}.$$

Let us denote by $\phi'_\theta, \phi'_{\bar{\theta}}, \psi'_\theta, \psi'_{\bar{\theta}}$ the partial derivatives of functions ϕ, ψ with respect to Grassmann coordinates $\theta, \bar{\theta}$. Then the determinant of the second order matrix

$$\Psi(r) = \frac{\partial(\phi,\psi)}{\partial(\theta,\bar{\theta})} = \begin{pmatrix} \phi'_\theta & \phi'_{\bar{\theta}} \\ \psi'_\theta & \psi'_{\bar{\theta}} \end{pmatrix} = \begin{pmatrix} \phi_1(r) & \phi_2(r) \\ \psi_1(r) & \psi_2(r) \end{pmatrix}, \;\; \phi_1(r), \phi_2(r), \psi_1(r), \psi_2(r) \in \mathfrak{C}, \tag{4}$$

will be denoted by Δ. We would like to emphasize that the symbol $\frac{\partial(\phi,\psi)}{\partial(\theta,\bar{\theta})}$ will denote the matrix of partial derivatives of corresponding functions, while the determinant of this matrix will be denoted by vertical lines. Hence,

$$\Delta = \Big|\frac{\partial(\phi,\psi)}{\partial(\theta,\bar{\theta})}\Big| = \begin{vmatrix} \phi'_\theta & \phi'_{\bar{\theta}} \\ \psi'_\theta & \psi'_{\bar{\theta}} \end{vmatrix} = \phi'_\theta\,\psi'_{\bar{\theta}} - \phi'_{\bar{\theta}}\,\psi'_\theta,$$

and we will assume that the functional matrix (4) is regular at any point r of $\mathbb{R}^3$, i.e., $\Delta \neq 0$. It is useful to denote the algebra of second order matrices, whose entries are smooth functions on the three dimensional space $\mathbb{R}^3$, by $\mathrm{Mat}_2(\mathfrak{C})$. Then the infinite dimensional group of regular matrices will be denoted by $\mathfrak{G}_2(\mathfrak{C})$, i.e.,

$$\mathfrak{G}_2(\mathfrak{C}) = \{\Psi(r) \in \mathrm{Mat}_2(\mathfrak{C}) : |\Psi(r)| \neq 0 \text{ at any point } r \in \mathbb{R}^3\}. \tag{5}$$

Thus $\Psi(r) \in \mathfrak{G}_2(\mathfrak{C})$.

Now let us consider the system of equations

$$\frac{dx}{dt} = \mathrm{Ber}\,\frac{(H,G,\phi,\psi)}{(y,z,\theta,\bar{\theta})}, \;\; \frac{dy}{dt} = \mathrm{Ber}\,\frac{(H,G,\phi,\psi)}{(z,x,\theta,\bar{\theta})}, \;\; \frac{dz}{dt} = \mathrm{Ber}\,\frac{(H,G,\phi,\psi)}{(x,y,\theta,\bar{\theta})}, \tag{6}$$

$$\frac{d\theta}{dt} = \frac{1}{\Delta^2}\Big(\Big|\frac{\partial(H,G)}{\partial(x,y)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(z,\bar{\theta})}\Big| + \Big|\frac{\partial(H,G)}{\partial(y,z)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(x,\bar{\theta})}\Big| + \Big|\frac{\partial(H,G)}{\partial(z,x)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(y,\bar{\theta})}\Big|\Big), \tag{7}$$

$$\frac{d\bar{\theta}}{dt} = \frac{1}{\Delta^2}\Big(\Big|\frac{\partial(H,G)}{\partial(x,y)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(z,\theta)}\Big| + \Big|\frac{\partial(H,G)}{\partial(y,z)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(x,\theta)}\Big| + \Big|\frac{\partial(H,G)}{\partial(z,x)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(y,\theta)}\Big|\Big), \tag{8}$$

where the right-hand sides of Equation (6) are the Berezinians of the corresponding supermatrices. For instance,

$$\mathrm{Ber}\,\frac{(H,G,\phi,\psi)}{(y,z,\theta,\bar\theta)} = \mathrm{Sdet}\frac{\partial(H,G,\phi,\psi)}{\partial(y,z,\theta,\bar\theta)} = \mathrm{Sdet}\begin{pmatrix} H'_y & H'_z & | & H'_\theta & H'_{\bar\theta} \\ G'_y & G'_z & | & G'_\theta & G'_{\bar\theta} \\ - & - & - & - & - \\ \phi'_y & \phi'_z & | & \phi'_\theta & \phi'_{\bar\theta} \\ \psi'_y & \psi'_z & | & \psi'_\theta & \psi'_{\bar\theta} \end{pmatrix}, \tag{9}$$

where Sdet stands for the superdeterminant of the supermatrix and the dashed lines show the structure of the supermatrix, i.e., they split the matrix into even degree and odd degree blocks. The elements of the upper-right block of this supermatrix are the right derivatives of functions H, G with respect to Grassmann variables $\theta, \bar\theta$, i.e.,

$$H'_\theta = H\,\frac{\overleftarrow{\partial}}{\partial\theta},\quad H'_{\bar\theta} = H\,\frac{\overleftarrow{\partial}}{\partial\bar\theta},\quad G'_\theta = G\,\frac{\overleftarrow{\partial}}{\partial\theta},\quad G'_{\bar\theta} = G\,\frac{\overleftarrow{\partial}}{\partial\bar\theta}.$$

Thus, according to the definition of superdeterminant [9] we have

$$\mathrm{Ber}\,\frac{(H,G,\phi,\psi)}{(y,z,\theta,\bar\theta)} = \Delta^{-1}\left|\frac{\partial(H,G)}{\partial(y,z)} - \frac{\partial(H,G)}{\partial(\theta,\bar\theta)}\left(\frac{\partial(\phi,\psi)}{\partial(\theta,\bar\theta)}\right)^{-1}\frac{\partial(\phi,\psi)}{\partial(y,z)}\right| \tag{10}$$

The expression at the right-hand side of Equation (7) contains the determinants of matrices

$$\frac{\partial(\phi,\psi)}{\partial(z,\bar\theta)} = \begin{pmatrix} \phi'_z & \phi'_{\bar\theta} \\ \psi'_z & \psi'_{\bar\theta} \end{pmatrix},\quad \frac{\partial(\phi,\psi)}{\partial(y,\bar\theta)} = \begin{pmatrix} \phi'_y & \phi'_{\bar\theta} \\ \psi'_y & \psi'_{\bar\theta} \end{pmatrix},\quad \frac{\partial(\phi,\psi)}{\partial(x,\bar\theta)} = \begin{pmatrix} \phi'_x & \phi'_{\bar\theta} \\ \psi'_x & \psi'_{\bar\theta} \end{pmatrix}. \tag{11}$$

These matrices have no structure of supermatrices because their first columns consist of the odd degree elements while the second columns consist of the even degree elements. But determinants of these matrices are correctly defined because the elements of the main diagonal, as well as the elements of the secondary diagonal, commute. It is worth mentioning that the values of these determinants are the odd degree functions and this is consistent with the left-hand side of Equation (7), which is also the odd degree function. This also holds for the right-hand side of the Equation (8).

Let us suppose that two even degree functions H, G do not depend on Grassmann coordinates $\theta, \bar\theta$ and two odd degree functions $\phi(r,\xi), \psi(r,\xi)$ do not depend on real coordinates x, y, z of the superspace $\mathbb{R}^{3|2}$, i.e., we have

$$\begin{aligned} \phi(r,\xi) &= \lambda_{11}\theta + \lambda_{12}\bar\theta, \\ \psi(r,\xi) &= \lambda_{21}\theta + \lambda_{22}\bar\theta, \end{aligned}$$

where λ_{ij} are real numbers. Then the matrix (4) has the form

$$\Psi = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix},$$

i.e., it does not depend on a point $r \in \mathbb{R}^3$ and its determinant Δ is the non-zero real number. In this case, the matrices (11) have zero columns and their determinants vanish. Hence, the right-hand sides of Equations (7) and (8) become zeroes and we get $\theta'_t = \bar\theta'_t = 0$. Hence, if $\alpha(t)$ is a solution of the system of Equations (6)–(8) in the case when odd degree functions $\phi(r,\xi), \psi(r,\xi)$ are constant functions in coordinates x, y, z, then Grassmann coordinates of solution $\alpha(t)$ do not depend on t and this solution can be considered as a parametrized curve $r(t)$ in the three dimensional space $\mathbb{R}^3$. Moreover, in this case the upper-right block of the supermatrix (9) is a zero matrix and it follows immediately from the definition of superdeterminant (10) that the right-hand side of the first equation in (6) turns into an ordinary determinant of the matrix $\frac{\partial(H,G)}{\partial(y,z)}$ with irrelevant numerical factor Δ^{-1}. Similar results hold in the case of the right-hand sides of the second and third equations in Equation (6). Thus, Equation (6) takes on the form

$$\frac{dx}{dt} = \Delta^{-1}\Big|\frac{\partial(H,G)}{\partial(y,z)}\Big|, \quad \frac{dy}{dt} = \Delta^{-1}\Big|\frac{\partial(H,G)}{\partial(z,x)}\Big|, \quad \frac{dz}{dt} = \Delta^{-1}\Big|\frac{\partial(H,G)}{\partial(x,y)}\Big|, \tag{12}$$

and we see that in this particular case the system of Equations (6)–(8) reduces to the Nambu–Hamilton equations in three dimensional space [1]. This gives us grounds to call the system of Equations (6)–(8) the generalization of the Nambu–Hamilton equation in the superspace $\mathbb{R}^{3|2}$.

In order to write the generalization of the Nambu–Hamilton equation in a more compact form, we introduce the functions $\mathfrak{K}, \mathfrak{L}, \mathfrak{M}, \mathfrak{N}, \mathfrak{S}$, where $\mathfrak{K}, \mathfrak{L}, \mathfrak{M}$ are the functions at the right-hand sides of the equations in (6) (from left to the right, respectively), and $\mathfrak{N}, \mathfrak{S}$ are the right-hand sides of the Equations (7) and (8), respectively. Thus,

$$\begin{aligned}
&\mathfrak{K} = \mathrm{Ber}\,\frac{(H,G,\phi,\psi)}{(y,z,\theta,\bar\theta)}, \qquad \mathfrak{L} = \mathrm{Ber}\,\frac{(H,G,\phi,\psi)}{(z,x,\theta,\bar\theta)}, \qquad \mathfrak{M} = \mathrm{Ber}\,\frac{(H,G,\phi,\psi)}{(x,y,\theta,\bar\theta)},\\
&\mathfrak{N} = \frac{1}{\Delta^2}\Big(\Big|\frac{\partial(H,G)}{\partial(x,y)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(z,\bar\theta)}\Big| + \Big|\frac{\partial(H,G)}{\partial(y,z)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(x,\bar\theta)}\Big| + \Big|\frac{\partial(H,G)}{\partial(z,x)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(y,\bar\theta)}\Big|\\
&\mathfrak{S} = \frac{1}{\Delta^2}\Big(\Big|\frac{\partial(H,G)}{\partial(x,y)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(z,\theta)}\Big| + \Big|\frac{\partial(H,G)}{\partial(y,z)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(x,\theta)}\Big| + \Big|\frac{\partial(H,G)}{\partial(z,x)}\Big|\Big|\frac{\partial(\phi,\psi)}{\partial(y,\theta)}\Big|.
\end{aligned}$$

The right-hand sides of the generalization of Nambu–Hamilton equation induce the even degree vector field on the superspace $\mathbb{R}^{3|2}$

$$\mathcal{X} = \mathfrak{K}\,\frac{\partial}{\partial x} + \mathfrak{L}\,\frac{\partial}{\partial y} + \mathfrak{M}\,\frac{\partial}{\partial z} + \frac{\overleftarrow{\partial}}{\partial\theta}\,\mathfrak{N} + \frac{\overleftarrow{\partial}}{\partial\bar\theta}\,\mathfrak{S}. \tag{13}$$

It is worth again mentioning that the vector field induced by the right-hand sides of the Nambu–Hamilton equation is divergenceless [1] and this motivated Nambu to develop his approach, because the divergenceless of the corresponding vector field is a sufficient and necessary condition for Liouville's theorem, which states that the volume of the flow generated by the Hamiltonian vector field is constant in time. In analogy with the Nambu–Hamilton equation, it can be shown by straightforward computations that the vector field of the generalization of the Nambu–Hamilton Equation (13) is also divergenceless in the superspace $\mathbb{R}^{3|2}$. Thus we have

$$\frac{\partial\mathfrak{K}}{\partial x} + \frac{\partial\mathfrak{L}}{\partial y} + \frac{\partial\mathfrak{M}}{\partial z} + \mathfrak{N}\,\frac{\overleftarrow{\partial}}{\partial\theta} + \mathfrak{S}\,\frac{\overleftarrow{\partial}}{\partial\bar\theta} = 0.$$

3. Extension of Nambu–Poisson Ternary Bracket to Superspace

The Nambu–Hamilton equations in three dimensional space $\mathbb{R}^3$ induce the ternary Nambu–Poisson bracket of smooth functions. This bracket is defined by means of the determinant of the matrix of partial derivatives of functions with respect to the coordinates of $\mathbb{R}^3$. The Nambu–Poisson bracket is totally skew-symmetric, and satisfies the Leibniz rule and the Filippov–Jacobi identity (Fundamental Identity) [3]. The aim of this section is to show that the generalization of Nambu–Hamilton Equations (6)–(8) introduced in the previous section leads to a ternary bracket of even degree functions, this ternary bracket depends on a pair of odd degree functions and can be defined by means of superdeterminant.

Let $F(r,\xi)$ be an even degree function, i.e., $F(r,\xi) = F_0(r) + F_1(r)\,\theta\bar\theta$. This function restricted to a curve $\alpha(t) = (x(t), y(t), z(t), \theta(t), \bar\theta(t))$, where

$$\theta(t) = f_{11}(t)\,\theta + f_{12}(t)\,\bar\theta, \;\; \bar\theta(t) = f_{21}(t)\,\theta + f_{22}(t)\,\bar\theta,$$

can be written in the form $F(t) = F_0(r(t)) + F_1(r(t))\,|f(t)|\,\theta\bar{\theta}$, where $|f(t)|$ is the determinant of matrix (3). The derivative of this function can be written in the form

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} + F\frac{\overleftarrow{\partial}}{\partial\theta}\frac{d\theta}{dt} + F\frac{\overleftarrow{\partial}}{\partial\bar{\theta}}\frac{d\bar{\theta}}{dt}. \tag{14}$$

Indeed we have

$$\begin{aligned}\frac{dF}{dt} &= \frac{dF_0}{dt} + \frac{dF_1}{dt}\,|f(t)|\,\theta\bar{\theta} + F_1\,\frac{d}{dt}\Big(|f(t)|\Big)\,\theta\bar{\theta} \\ &= \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} - (F_1\,\bar{\theta}(t))\,\frac{d\theta}{dt} + (F_1\,\theta(t))\,\frac{d\bar{\theta}}{dt} \\ &= \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} + F\frac{\overleftarrow{\partial}}{\partial\theta}\frac{d\theta}{dt} + F\frac{\overleftarrow{\partial}}{\partial\bar{\theta}}\frac{d\bar{\theta}}{dt}.\end{aligned} \tag{15}$$

Next we assert that if $\alpha(t)$ is a solution of generalization of Nambu–Hamilton Equations (6)–(8) in superspace then the derivative of any even degree function F can be expressed by means of a Berezinian as follows

$$\frac{dF}{dt} = \mathrm{Ber}\,\frac{(F,H,G,\phi,\psi)}{(x,y,z,\theta,\bar{\theta})} = \mathrm{Sdet}\begin{pmatrix} F'_x & F'_y & F'_z & | & F'_\theta & F'_{\bar{\theta}} \\ H'_x & H'_y & H'_z & | & H'_\theta & H'_{\bar{\theta}} \\ G'_x & G'_y & G'_z & | & G'_\theta & G'_{\bar{\theta}} \\ - & - & - & - & - & - \\ \phi'_x & \phi'_y & \phi'_z & | & \phi'_\theta & \phi'_{\bar{\theta}} \\ \psi'_x & \psi'_y & \psi'_z & | & \psi'_\theta & \psi'_{\bar{\theta}} \end{pmatrix}. \tag{16}$$

This formula suggests that it is natural to introduce a new bracket, which can be considered as an analogue of the Nambu–Poisson ternary bracket [1,3] in the superspace $\mathbb{R}^{3|2}$. We consider even degree functions F, H, G in (16) as arguments and two odd degree functions ϕ, ψ as parameters of this new ternary bracket. Evidently, these two functions can be identified with the matrix $\Psi(r)$ (4). We denote this new ternary bracket by bold curly brackets and define it by

$$\boldsymbol{\{}F,H,G\boldsymbol{\}}_\Psi = \mathrm{Ber}\,\frac{(F,H,G,\phi,\psi)}{(x,y,z,\theta,\bar{\theta})}, \tag{17}$$

where F, H, G are even degree functions on the superspace $\mathbb{R}^{3|2}$ and Ψ shows dependence of ternary bracket on matrix $\Psi \in \mathfrak{G}_2(\mathfrak{C})$ associated to odd degree functions ϕ, ψ. Thus, we have associated to each element Ψ of the infinite dimensional group of invertible matrices $\mathfrak{G}_2(\mathfrak{C})$ the ternary bracket (17) of even degree functions on the superspace $\mathbb{R}^{3|2}$, that is

$$\Psi \in \mathfrak{G}_2(\mathfrak{C}) \;\mapsto\; \boldsymbol{\{}\;,\;,\;\boldsymbol{\}}_\Psi.$$

Now our aim is to prove Formula (16). In order to simplify the form of formulae we introduce the following notations

$$\begin{aligned}\epsilon^{H,G}_{x,y} &= \Big|\frac{\partial(H,G)}{\partial(x,y)}\Big|, \;\; \epsilon^{H,G}_{y,z} = \Big|\frac{\partial(H,G)}{\partial(y,z)}\Big|, \;\; \epsilon^{H,G}_{z,x} = \Big|\frac{\partial(H,G)}{\partial(z,x)}\Big| \\ \delta_{x,\theta} &= \Big|\frac{\partial(\phi,\psi)}{\partial(x,\theta)}\Big|, \;\; \delta_{y,\theta} = \Big|\frac{\partial(\phi,\psi)}{\partial(y,\theta)}\Big|, \;\; \delta_{z,\theta} = \Big|\frac{\partial(\phi,\psi)}{\partial(z,\theta)}\Big|, \\ \delta_{x,\bar{\theta}} &= \Big|\frac{\partial(\phi,\psi)}{\partial(x,\bar{\theta})}\Big|, \;\; \delta_{y,\bar{\theta}} = \Big|\frac{\partial(\phi,\psi)}{\partial(y,\bar{\theta})}\Big|, \;\; \delta_{z,\bar{\theta}} = \Big|\frac{\partial(\phi,\psi)}{\partial(z,\bar{\theta})}\Big|.\end{aligned}$$

Then the Berezinian of the supermatrix at the left-hand side of (16) can be written in the form of an ordinary determinant

$$\Delta^{-1}\begin{vmatrix} F'_x - F'_\theta\frac{\delta_{x,\bar\theta}}{\Delta} + F'_{\bar\theta}\frac{\delta_{x,\theta}}{\Delta} & F'_y - F'_\theta\frac{\delta_{y,\bar\theta}}{\Delta} + F'_{\bar\theta}\frac{\delta_{y,\theta}}{\Delta} & F'_z - F'_\theta\frac{\delta_{z,\bar\theta}}{\Delta} + F'_{\bar\theta}\frac{\delta_{z,\theta}}{\Delta} \\ H'_x - H'_\theta\frac{\delta_{x,\bar\theta}}{\Delta} + H'_{\bar\theta}\frac{\delta_{x,\theta}}{\Delta} & H'_y - H'_\theta\frac{\delta_{y,\bar\theta}}{\Delta} + H'_{\bar\theta}\frac{\delta_{y,\theta}}{\Delta} & H'_z - H'_\theta\frac{\delta_{z,\bar\theta}}{\Delta} + H'_{\bar\theta}\frac{\delta_{z,\theta}}{\Delta} \\ G'_x - G'_\theta\frac{\delta_{x,\bar\theta}}{\Delta} + G'_{\bar\theta}\frac{\delta_{x,\theta}}{\Delta} & G'_y - G'_\theta\frac{\delta_{y,\bar\theta}}{\Delta} + G'_{\bar\theta}\frac{\delta_{y,\theta}}{\Delta} & G'_z - G'_\theta\frac{\delta_{z,\bar\theta}}{\Delta} + G'_{\bar\theta}\frac{\delta_{z,\theta}}{\Delta} \end{vmatrix}, \tag{18}$$

where $H'_\theta, H'_{\bar\theta}, G'_\theta, G'_{\bar\theta}$ are right derivatives. If we expand this determinant along the first row we get

$$\begin{aligned} & F'_x \,\mathrm{Ber}\,\frac{(H,G,\phi,\psi)}{(y,z,\theta,\bar\theta)} + F'_y \,\mathrm{Ber}\,\frac{(H,G,\phi,\psi)}{(z,x,\theta,\bar\theta)} + F'_z \,\mathrm{Ber}\,\frac{(H,G,\phi,\psi)}{(x,y,\theta,\bar\theta)} \\ & + F'_\theta\,\frac{1}{\Delta^2}(\epsilon^{H,G}_{y,z}\,\delta_{x,\bar\theta} + \epsilon^{H,G}_{z,x}\,\delta_{y,\bar\theta} + \epsilon^{H,G}_{z,x}\,\delta_{y,\bar\theta}) + F'_{\bar\theta}\,\frac{1}{\Delta^2}(\epsilon^{H,G}_{y,z}\,\delta_{x,\theta} + \epsilon^{H,G}_{z,x}\,\delta_{y,\theta} + \epsilon^{H,G}_{z,x}\,\delta_{y,\theta}). \end{aligned} \tag{19}$$

Now making use of the system of Equations (6)–(8) and Equation (16), we get Equation (16). Every column of determinant (18) is the linear combination of five columns

$$\mathfrak{R}_x = \begin{pmatrix} F'_x \\ H'_x \\ G'_x \end{pmatrix},\ \mathfrak{R}_y = \begin{pmatrix} F'_y \\ H'_y \\ G'_y \end{pmatrix},\ \mathfrak{R}_z = \begin{pmatrix} F'_z \\ H'_z \\ G'_z \end{pmatrix},\ \mathfrak{R}_\theta = \begin{pmatrix} F'_\theta \\ H'_\theta \\ G'_\theta \end{pmatrix},\ \mathfrak{R}_{\bar\theta} = \begin{pmatrix} F'_{\bar\theta} \\ H'_{\bar\theta} \\ G'_{\bar\theta} \end{pmatrix}, \tag{20}$$

with corresponding coefficients. Hence, we can write the determinant (18) in the form

$$\Delta^{-1}\,|\mathfrak{R}_x - \mathfrak{R}_\theta\frac{\delta_{x,\bar\theta}}{\Delta} + \mathfrak{R}_{\bar\theta}\frac{\delta_{x,\theta}}{\Delta} \quad \mathfrak{R}_y - \mathfrak{R}_\theta\frac{\delta_{y,\bar\theta}}{\Delta} + \mathfrak{R}_{\bar\theta}\frac{\delta_{y,\theta}}{\Delta} \quad \mathfrak{R}_z - \mathfrak{R}_\theta\frac{\delta_{z,\bar\theta}}{\Delta} + \mathfrak{R}_{\bar\theta}\frac{\delta_{z,\theta}}{\Delta}|. \tag{21}$$

Now using the properties of the ordinary determinant and taking all possible combinations of columns, we can write the determinant (21) as the sum of determinants, where every determinant is determined by a corresponding combination of columns (20). It follows from the property $\theta^2 = \bar\theta^2 = 0$ of Grassmann coordinates that the determinant of a combination of columns, which includes at least two columns $\mathfrak{R}_\theta, \mathfrak{R}_{\bar\theta}$, vanishes. Altogether, we have seven non-trivial combinations of columns (i.e., the determinant of this combination of columns does not vanish), which give the following expression for the ternary bracket (17)

$$\begin{aligned} \{F,H,G,\}_\Psi \;=\; & \frac{1}{\Delta}\,|\mathfrak{R}_x\ \mathfrak{R}_y\ \mathfrak{R}_z| - \frac{1}{\Delta^2}\Big(|\mathfrak{R}_x\ \mathfrak{R}_y\ \mathfrak{R}_\theta|\,\delta_{z,\bar\theta} + |\mathfrak{R}_x\ \mathfrak{R}_\theta\ \mathfrak{R}_z|\,\delta_{y,\bar\theta} + |\mathfrak{R}_\theta\ \mathfrak{R}_y\ \mathfrak{R}_z|\,\delta_{x,\bar\theta} \\ & - |\mathfrak{R}_x\ \mathfrak{R}_y\ \mathfrak{R}_{\bar\theta}|\,\delta_{z,\theta} - |\mathfrak{R}_x\ \mathfrak{R}_{\bar\theta}\ \mathfrak{R}_z|\,\delta_{y,\theta} - |\mathfrak{R}_{\bar\theta}\ \mathfrak{R}_y\ \mathfrak{R}_z|\,\delta_{x,\theta}\Big). \end{aligned} \tag{22}$$

The first term at the right-hand side of the above relation is the usual Nambu–Poisson ternary bracket of even degree functions F, H, G

$$\{F,H,G\} = |\mathfrak{R}_x\ \mathfrak{R}_y\ \mathfrak{R}_z| = \begin{vmatrix} F'_x & F'_y & F'_z \\ H'_x & H'_y & H'_z \\ G'_x & G'_y & G'_z \end{vmatrix}. \tag{23}$$

In addition to the usual Nambu–Poisson ternary bracket, the expression at the right-hand side of relation (22) also includes terms enclosed in parentheses. These terms depend on the derivatives of odd degree functions ϕ, ψ. This motivated us to introduce one more ternary bracket as follows

$$\begin{aligned} \{F,H,G\}_\Psi \;=\; & \left|\frac{\partial(F,H,G)}{\partial(x,y,\theta)}\right|\left|\frac{\partial(\phi,\psi)}{\partial(z,\bar\theta)}\right| + \left|\frac{\partial(F,H,G)}{\partial(x,\theta,z)}\right|\left|\frac{\partial(\phi,\psi)}{\partial(y,\bar\theta)}\right| + \left|\frac{\partial(F,H,G)}{\partial(\theta,y,z)}\right|\left|\frac{\partial(\phi,\psi)}{\partial(x,\bar\theta)}\right| \\ & - \left|\frac{\partial(F,H,G)}{\partial(x,y,\bar\theta)}\right|\left|\frac{\partial(\phi,\psi)}{\partial(z,\theta)}\right| - \left|\frac{\partial(F,H,G)}{\partial(x,\bar\theta,z)}\right|\left|\frac{\partial(\phi,\psi)}{\partial(y,\theta)}\right| - \left|\frac{\partial(F,H,G)}{\partial(\bar\theta,y,z)}\right|\left|\frac{\partial(\phi,\psi)}{\partial(x,\theta)}\right|. \end{aligned} \tag{24}$$

It should be noted that the order of cofactors in every product at the right-hand side of (24) is important, because cofactors are odd degree functions.

Now we can express the ternary bracket (17) as the sum of the usual Nambu–Poisson bracket (23) and the ternary bracket (24). Hence,

$$\mathbf{\{}F,H,G\mathbf{\}}_\Psi = \frac{1}{\Delta}\{F,H,G\} - \frac{1}{\Delta^2}\{F,H,G\}_\Psi. \tag{25}$$

Formula (25) gives grounds to consider the ternary bracket (17) introduced by means of superdeterminant as an extension of usual Nambu–Poisson bracket to the superspace $\mathbb{R}^{3|2}$. In the next section, we will prove that this extension preserves all the algebraic properties of the Nambu–Poisson bracket such as skew-symmetry, the Leibniz rule and the Filippov–Jacobi identity (Fundamental Identity). The ternary bracket (17) denoted by bold-face curly bracket and endowed with the subscript Ψ will be referred to as the ternary total Ψ-bracket and the ternary bracket (24) denoted by the usual curly bracket and endowed with the subscript Ψ will be referred to as ternary Ψ-bracket.

4. Infinite Dimensional Family of Nambu–Poisson Superspaces

In this section, we consider a more general superspace $\mathbb{R}^{n|2}$, whose n real coordinates will be denoted by $x^1, x^2, \ldots, x^n$ and Grassmann coordinates as before by $\theta, \bar{\theta}$. The algebra of smooth functions will be denoted by $C^\infty(\mathbb{R}^{n|2})$, its subalgebra of even degree functions by $C_0^\infty(\mathbb{R}^{n|2})$ and the subspace of odd degree functions by $C_1^\infty(\mathbb{R}^{n|2})$. As in the previous section, we fix two odd degree functions $\phi, \psi \in C_1^\infty(\mathbb{R}^{n|2})$ and define the total Ψ-bracket of n even degree functions $F^1, F^2, \ldots, F^n$ by means of the superdeterminant

$$\mathbf{\{}F^1,F^2,\ldots,F^n\mathbf{\}}_\Psi = \mathrm{Sdet}\begin{pmatrix} F^1_{x^1} & F^1_{x^2} & \ldots & F^1_{x^n} & | & F^1_\theta & F^1_{\bar\theta} \\ F^2_{x^1} & F^2_{x^2} & \ldots & F^2_{x^n} & | & F^2_\theta & F^2_{\bar\theta} \\ \ldots & \ldots & \ldots & \ldots & | & \ldots & \ldots \\ F^n_{x^1} & F^n_{x^2} & \ldots & F^n_{x^n} & | & F^n_\theta & F^n_{\bar\theta} \\ -- & -- & -- & -- & -- & -- & -- \\ \phi_{x^1} & \phi_{x^2} & \ldots & \phi_{x^n} & | & \phi_\theta & \phi_{\bar\theta} \\ \psi_{x^1} & \psi_{x^2} & \ldots & \psi_{x^n} & | & \psi_\theta & \psi_{\bar\theta} \end{pmatrix}, \tag{26}$$

where we slightly simplified the notions for derivatives

$$F^i_{x^j} = \frac{\partial F^i}{\partial x^j},\ F^i_\theta = F^i\frac{\overleftarrow{\partial}}{\partial\theta},\ F^i_{\bar\theta} = F^i\frac{\overleftarrow{\partial}}{\partial\bar\theta},\ \phi_{x^j} = \frac{\partial\phi}{\partial x^j},\ \phi_\theta = \frac{\partial\phi}{\partial\theta},\ \phi_{\bar\theta} = \frac{\partial\phi}{\partial\bar\theta},\ \psi_{x^j} = \frac{\partial\psi}{\partial x^j},\ \psi_\theta = \frac{\partial\psi}{\partial\theta},\ \psi_{\bar\theta} = \frac{\partial\psi}{\partial\bar\theta}.$$

It can be shown that the n-ary total Ψ-bracket (26) splits into the sum of n-ary Nambu–Poisson bracket and the n-ary Ψ-bracket

$$\mathbf{\{}F^1,F^2,\ldots,F^n\mathbf{\}}_\Psi = \frac{1}{\Delta}\{F^1,F^2,\ldots,F^n\} - \frac{1}{\Delta^2}\{F^1,F^2,\ldots,F^n\}_\Psi, \tag{27}$$

where

$$\begin{aligned}\{F^1,F^2,\ldots,F^n\}_\Psi \ = \ \textstyle\sum_{i=1}^n \Big(& \left|\frac{\partial(F^1,F^2,\ldots,F^i,\ldots,F^n)}{\partial(x^1,x^2,\ldots,\underset{i}{\theta},\ldots,x^n)}\right| \left|\frac{\partial(\phi,\psi)}{\partial(x^i,\bar\theta)}\right| \\ & - \left|\frac{\partial(F^1,F^2,\ldots,F^i,\ldots,F^n)}{\partial(x^1,x^2,\ldots,\underset{i}{\bar\theta},\ldots,x^n)}\right| \left|\frac{\partial(\phi,\psi)}{\partial(x^i,\theta)}\right| \Big).\end{aligned} \tag{28}$$

Now we give the following general definition:

Definition 1. *A superspace $\mathbb{R}^{n|m}$ with n real coordinates and m Grassmann coordinates is said to be a Nambu–Poisson superspace if it is endowed with a multilinear mapping*

$$\{\ ,\ ,\dots,\ \}:\underbrace{C_0^\infty(\mathbb{R}^{n|m})\times C_0^\infty(\mathbb{R}^{n|m})\times\dots\times C_0^\infty(\mathbb{R}^{n|m})}_{n\ times}\to C_0^\infty(\mathbb{R}^{n|m}), \tag{29}$$

such that

1. *it is totally skew-symmetric*

$$\{F^1,F^2,\dots,F^n\}=(-1)^{|\sigma|}\,\{F^{\sigma(1)},F^{\sigma(2)},\dots,F^{\sigma(n)}\},$$

   where σ is a permutation of n integers and $|\sigma|$ is its parity,
2. *it satisfies the Leibniz rule*

$$\{H\,G,F^1,\dots,F^{n-1}\}=H\,\{G,F^1,\dots,F^{n-1}\}+G\,\{H,F^1,\dots,F^{n-1}\},$$

3. *it satisfies the Filippov–Jacobi identity*

$$\{F^1,\dots,F^{n-1},\{G^1,G^2,\ \dots\ ,G^n\}\}=\sum_{i=1}^{n}\{G^1,\dots,G^{i-1},\{F^1,\dots,F^{n-1},G^i\},G^{i+1},\dots,G^n\}.$$

Then a linear mapping (29) is referred to as n-ary Nambu–Poisson bracket in superspace $\mathbb{R}^{n|m}$. The Nambu–Poisson superspace, whose Nambu–Poisson structure is determined by n-ary Nambu–Poisson bracket (29), will be denoted by $(\mathbb{R}^{n|m},\{\ ,\ ,\dots,\ \})$.

Obviously the usual Nambu–Poisson bracket (extended to even degree functions in a natural way) defines the Nambu–Poisson structure in superspace $\mathbb{R}^{n|m}$. We will call this structure canonical Nambu–Poisson structure in superspace $\mathbb{R}^{n|m}$.

Lemma 1. *For any pair of odd degree functions $\Psi=(\phi,\psi)\in C_1^\infty(\mathbb{R}^{n|2})\times C_1^\infty(\mathbb{R}^{n|2})$ such that the functional matrix*

$$\begin{pmatrix}\phi_\theta & \phi_{\bar\theta}\\ \psi_\theta & \psi_{\bar\theta}\end{pmatrix} \tag{30}$$

is regular the n-ary Ψ-bracket (28) is the Nambu–Poisson bracket in the superspace $\mathbb{R}^{n|2}$, i.e., it is totally skew-symmetric, satisfies the Leibniz rule and the Filippov–Jacobi identity. Hence, there is the family of Nambu–Poisson superspaces $(\mathbb{R}^{n|2},\{\ ,\ ,\dots,\ \}_\Psi)$ parametrized by the infinite dimensional group of regular functional matrices (30).

Proof. Let $F^i=F_0^i+F_1^i\,\theta\,\bar\theta$. We shall use the symbol η for both θ and $\bar\theta$. Our proof is based on the properties of a determinant of the type

$$\Big|\frac{\partial(F^1,F^2,\dots,F^i,\dots,F^n)}{\partial(x^1,x^2,\dots,\underset{i}{\eta},\dots,x^n)}\Big|. \tag{31}$$

The entries of this determinant are even degree functions except for ith column, where we have odd degree functions F^i_η either $F^i_\theta=-F^i_1\,\bar\theta$ or $F^i_{\bar\theta}=F^i_1\,\theta$ (right partial derivatives with respect to Grassmann coordinates). However, F^i_η commute with other entries of determinant. Thus, the determinant (31) has the properties of a usual determinant. Hence, if we do a permutation of functions $F^1,F^2,\dots,F^n$ in the n-ary Ψ-bracket (28) then this is equivalent to the permutation of corresponding rows in every determinant (31). Hence, a sign of the whole expression at the right-hand side of (28) will change according to the parity of permutation. This implies the total skew-symmetry of n-ary Ψ-bracket.

In order to prove the Leibniz rule, we consider the bracket, where the first argument is the product of two even degree functions $H\,G$ (Definition 1, condition 2). Then, the first cofactor in every product of the expression at the right hand-side of (28) is the determinant of the type (31), where the entries of the first row are the derivatives of the product $H\,G$. For any integer $i = 1, 2, \ldots, n$ we have

$$(H\,G)_{x^j} = H_{x^j}\,G + H\,G_{x^j}\;(j \neq i), \quad (H\,G)_\eta = H\,G_\eta + H_\eta\,G.$$

Using the above formula and the properties of usual determinant we get

$$\left|\frac{\partial(H\,G, F^2, \ldots, F^i, \ldots, F^n)}{\partial(x^1, x^2, \ldots, \underset{i}{\eta}, \ldots, x^n)}\right. = H\left|\frac{\partial(G, F^2, \ldots, F^i, \ldots, F^n)}{\partial(x^1, x^2, \ldots, \underset{i}{\eta}, \ldots, x^n)}\right. + G\left|\frac{\partial(H, F^2, \ldots, F^i, \ldots, F^n)}{\partial(x^1, x^2, \ldots, \underset{i}{\eta}, \ldots, x^n)}\right.,$$

which immediately gives the Leibniz rule for n-ary Ψ-bracket (28).

Similarly, we can prove the Filippov–Jacobi identity. Indeed, the first cofactor in every product in the sum at the right-hand side of (28) satisfies the Filippov–Jacobi identity because it has the properties of an ordinary determinant. □

Theorem 1. *For any pair of odd degree functions $\Psi = (\phi, \psi) \in C_1^\infty(\mathbb{R}^{n|2}) \times C_1^\infty(\mathbb{R}^{n|2})$ such that the functional matrix*

$$\begin{pmatrix} \phi_\theta & \phi_{\bar\theta} \\ \psi_\theta & \psi_{\bar\theta} \end{pmatrix}$$

is regular the n-ary total Ψ-bracket (26) is the Nambu–Poisson bracket in the superspace $\mathbb{R}^{n|2}$, i.e., it is totally skew-symmetric, satisfies the Leibniz rule and the Filippov–Jacobi identity. Hence, there is the family of Nambu–Poisson superspaces $(\mathbb{R}^{n|2}, \{\ ,\ ,\ldots,\ \}_\Psi)$ parametrized by infinite dimensional group of regular functional matrices (30).

This theorem can be proved by means of Lemma 1.

5. Discussion

The algebra of functions on a superspace $\mathbb{R}^{n|m}$ is the superalgebra, i.e., it is the direct sum of the subalgebra of even degree functions and the subspace of odd degree functions. In Reference [10], the author constructs a graded Nambu–Poisson bracket (or super Nambu bracket) on the whole superalgebra of functions, that is, a bracket that takes into account the graded structure of the superalgebra of functions. This means that the bracket is graded skew-symmetric, satisfies the graded Leibniz rule and the graded Filippov–Jacobi identity. The number of arguments of this graded Nambu–Poisson bracket is $n + m$. In the case of the superspace $\mathbb{R}^{3|2}$, the graded n-ary bracket proposed in Reference [10] yields the graded quintuple Nambu–Poisson bracket. The generalization of Nambu–Poisson equation in the superspace $\mathbb{R}^{3|2}$, proposed in the present paper, leads to a ternary Nambu–Poisson bracket, which depends on two odd degree functions. Thus, formally, we employ five functions in our Nambu–Poisson bracket (defined with the help of a Berezinian), just as in the case of graded quintuple bracket in Reference [10], but our construction is different from the one proposed in Reference [10], because

1. we treat three even degree functions as arguments of ternary Nambu–Poisson bracket and two odd degree functions as parameters of this ternary bracket,

2. the dependence of our ternary Nambu–Poisson bracket on two odd degree functions (parameters) is skew-symmetric (this immediately follows from Formula (24)), while a graded quintuple bracket, proposed in Reference [10], is symmetric with respect two odd degree functions.

We think that the advantage of our ternary Nambu–Poisson bracket in superspace is that it naturally arises from the generalization of the Nambu–Poisson equation in superspace, proposed in the present paper.

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