Scalar-Tensor Black Holes Embedded in an Expanding Universe

Daria Tretyakova [1],* [ID] **and Boris Latosh** [2] [ID]

[1]   Department Theoretical Physics, Ural Federal University, Ekaterinburg 620075, Russia
[2]   Department of Physics & Astronomy, University of Sussex, Brighton BN1 9RH, UK; latosh.boris@gmail.com
*   Correspondence: daria.tretiakova@urfu.ru

Abstract: In this review, we focus our attention on scalar-tensor gravity models and their empirical verification in terms of black hole and wormhole physics. We focus on black holes, embedded in an expanding universe, describing both cosmological and astrophysical scales. We show that in scalar-tensor gravity it is quite common that the local geometry is isolated from the cosmological expansion, so that it does not backreact on the black hole metric. We try to extract common features of scalar-tensor black holes in an expanding universe and point out the issues that are not fully investigated.

---

1. Introduction

General Relativity (GR) is nowadays the most successful theory of gravity providing the best fit for all the observed gravitational phenomena. For a century, GR was only tested in the weak field regime [1–3] until the recent detection of gravitational waves [4–6] that provided a completely new way to study strong-field gravity [7–10] via optical and gravitational channels [11,12]. Furthermore, constantly increasing resolution of satellite and terrestrial observational facilities gives one hope to resolve the SgrA* in the near future [13,14].

However, on this successful background, we encounter several hints on the GR incompleteness [15,16] which are:

- dark matter,
- dark energy,
- inflation.

These issues can be approached in terms of the classical GR action or within the extended gravity framework [16,17], and we will now describe both ways in brief.

Dark matter is generally treated either as new gravitational effects lying beyond GR reach or as evidence of beyond standard model particle physics. Empirical evidence favours the former case [18,19], but both approaches seem to provide an adequate description of the dark matter [20–24].

Dark energy, on the other hand, appears to be a pure gravitational phenomenon. ΛCDM model provides the best fit for contemporary observational data [2,3]. The main problem here is that one cannot evaluate the cosmological constant from the first principles. Moreover, its value appears to be in a contradiction with the standard model: this is the cosmological constant problem [25,26].

Inflation is one of the key motivations to consider extended gravity, as it appears to be impossible to construct a self-consistent cosmological model without the inflationary phase [27,28]. Inflation does not occur in GR, so we are forced to consider extended gravity that offers two basic ways to

account for inflation. First, one is to consider quantum corrections to a matter Lagrangian in the realm of curved spacetime. Such corrections provide model renormalizability and result in pure gravitational terms. The approach is known as $f(R)$ gravity, and it was first applied in the paper [29] demonstrating the appearance of an inflationary phase. Another way to inflation is to introduce a new scalar degree of freedom, which is called inflation. During the early cosmological evolution, inflation is strongly coupled to gravity and its energy density drives the inflation [30]. This framework is called scalar-tensor (ST) gravity.

While dark matter and dark energy interpretations are ambiguous, inflation provides a direct way to extended gravity: $f(R)$ gravity or scalar-tensor one. By means of conformal transformations, one can translate an $f(R)$ gravity model into an ST model [31–34]. We discuss this matter in Section 2.1 in detail; here, we would like to briefly mention that one can treat conformal transformations as a way to map complicated form of $f(R)$ gravity effects with the scalar field. Therefore, ST models may be treated as a framework to account for beyond GR gravitational phenomena by means of new gravitational terms or new scalar degrees of freedom. In our paper, we therefore consider $f(R)$ gravity mostly to be another shape of the scalar-tensor theory and concentrate on the scalar-tensor formulation. We also prefer to focus on scalar-tensor models rather than on $f(R)$-gravity because the former mathematical framework has a power to treat the auxiliary degree of freedom separately, thereby providing a precise tool to track its influence on the gravitational phenomena. Moreover, recently discovered scalar-tensor models of the Horndeski class [35] may not be mapped on $f(R)$-gravity [36] (only on $f(R, R_{\mu\nu})$-gravity) because of the derivative coupling. Scalar-tensor gravity was also one of the first attempts to modify general relativity [37]. Therefore, scalar-tensor models represent the cornerstone of extended gravity and we focus on them in this review.

The research filed related to scalar-tensor gravity is vast and cannot be covered in one paper. Therefore, we focus our attention on ST models' empirical verification in terms of black hole and wormhole physics. The goal of this paper is to highlight that the most promising framework to test this models is given by a comprehensive approach, when one considers a black hole in the context of the cosmological background, i.e., embedded in an expanding universe. Speaking of such a black hole, we mean a self-consistent solution of the spherically symmetric field equations in vacuum in a model, describing the expanding Universe. Such a black hole will usually have the structure of the Schwarzschild–de Sitter solution describing both cosmological and astrophysical scales. However, as we shall see, in scalar-tensor gravity, it is quite common that the local geometry is isolated from the cosmological expansion, so that it does not backreact on the black hole metric. We try to extract common features of scalar-tensor black holes in an expanding universe and point out the issues that are not fully investigated. Fulfilling the requirements above, we are restricted to a small subset of scalar-tensor black holes, since most of the known solutions are vacuum or have no explicit analytical form due to the field equations complexity. For a more general picture of black hole solutions in extended gravity, we recommend the following reviews [33,34,38].

This paper is structured as follows. In Section 2, we discuss general features of ST models and their physical content. We proceed with Section 3 devoted to the standard Schwarzschild–de Sitter solution. In Section 4, we discuss no-hair theorems for ST gravity. We clarify that the absence of scalar hair constrains the black hole (but not wormhole) phenomenology and excludes any beyond Schwarzschild–anti-de Sitter black hole. Sections 5–8 are devoted to specific scalar-tensor frameworks, e.g., Brans–Dicke and Horndeski theories. We conclude in Section 9.

2. Scalar-Tensor Gravity

ST models occupy a huge branch of extended gravity and play a central role in gravity research. These models originate from Brans–Dicke theory [37] initially created to incorporate Mach's principle into the GR:

$$S_{\text{BD}} = \frac{1}{16\pi} \int d^4x \, \sqrt{-g} \left[\phi \mathcal{R} - \frac{\omega}{\phi} \partial_\mu \phi \partial^\mu \phi \right],$$

$$(1)$$

where ϕ is the Brans–Dicke scalar field, which manifests itself thorough the variation of the gravitational constant, R is the scalar curvature, and ω is a constant. The following generalization of Brans–Dicke action (1) is often considered as a general form of an ST model Lagrangian [39–41]:

$$S_{\text{ST}} = \frac{1}{16\pi} \int d^4x \ \sqrt{-g} \left[\phi R - \frac{\omega(\phi)}{\phi} \partial_\mu \phi \partial^\mu \phi - 2\Lambda(\phi) \right]. \tag{2}$$

Here, $\omega(\phi)$ is the coupling parameter and $\Lambda(\phi)$ is a ϕ-dependent generalization of the cosmological constant, i.e., the scalar field potential. Multiple ST gravity theories were developed in the past years and new ones emerge constantly (see Section 3.1 of [42] and Section 2.2 of [15] for detailed reviews of ST gravity). Through the years, they were all subject to theoretical and observational tests and Brans–Dicke theory among them was tested with the greatest care. The sole result is the bound on the model parameter $|\omega| > 50,000$ imposed via the parametrized post-Newtonian (PPN) expansion of the theory [17,43] (even $|\omega| > 10^{10}$ according to [44], and the paper [45] proposes $|\omega| > 10^{40}$).

The scalar field and related values enter field equations and affect the space-time geometry. In what follows, we briefly point out the main test beds for scalar–tensor gravity.

At the scale of stellar systems, the scalar field changes both internal structure and external gravitational field of a star. An example is the spontaneous scalarization process [46,47] occurring in massive neutron stars in the strong field regime. The scalar field receives an effective position-dependent mass due to its nonlinear coupling to matter thereby affecting the star internal structure. Spontaneous scalarization was demonstrated both in analytical [46,47] and numerical studies [48–50]. This allows one to confront ST theories with observations by modelling the stellar interior.

At the level of black holes the scalar field can violate no-hair argument and produce scalar hairs. We discuss this issue in Section 4; here, we would like to mention that a scalar field is an essential component for a no-hair theorem violation, as it can bifurcate the theorem requirements in a healthy manner by means of the nonlinear coupling with gravity. The wormhole is also a quite common result of such bifurcation. The scalar field plays a role of an exotic matter supporting a wormhole throat, thereby opening a new class of gravitational phenomena. Moreover, new scalar degree of freedom affects the black hole thermodynamics. One cannot simply adopt Bekenstein–Hawking formula $S = 1/4A$, as scalar field may bring an additional contribution on black hole entropy. We discuss this issue in more detail in Sections 5 and 5.2.

The scalar field also manifests itself at the scales of galaxies and galaxy clusters. In an expanding universe, the size of a gravitationally coupled object is defined by the balance of the gravitational attraction and the force expanding the universe. The maximal size of a gravitationally coupled object is known as a turnaround radius (see also zero-velocity surface) [51,52]. The scalar field affects the value of the turnaround radius, as it contributes to the total amount of energy density in the area [53–55]. The value of the turnaround radius can be established using the galaxy dust density profile, which experiences the leap at the border of a gravitationally coupled area. Contemporary methods do not yet allow one to measure this leap rigorously, but some constraints on model parameters can nevertheless be placed [53].

Finally, ST models trigger new phenomenology at the cosmological scale [45,56–58]. Contributing to the matter stress-energy tensor, the scalar field affects the cosmological dynamics and ST theories can thereby be tested using the entire cosmological dataset [59–61].

Some of ST models are equipped with the Vainshtein mechanism [62], which screens out the scalar field due to its nonlinear coupling to matter. In the high density area, the scalar field particles interact strongly with matter and do not propagate prohibiting scalar field influence. As a result of screening around localized objects, a transition region must appear where the solution interpolates between the cosmological and the local values of the parameters. This means that the local solution does not feel the cosmological scale properly and, at the same time, the long-range cosmological solution should not be read off directly from the large r hevahior of the metric, as it can be done in the GR [63]. However,

the Vainshtein mechanism requires a specific form of the scalar field interaction to take place and cannot wipe out the scalar field influence in the low matter density regime. Therefore, one may expect beyond GR effects even in models with the screening mechanism.

Most of the research papers attempting to test ST gravity models consider a single selected scale (and hence a single observational dataset). An extended gravitational theory must provide a uniform description for the phenomena taking place at all scales. We would like to focus our attention on the role of a comprehensive approach to both small (though astronomical) and large scale physics in ST gravity. To do so, we consider models, where such an approach can result in valuable constraints and compare bounds imposed by cosmological, astrophysical and weak field observations. A commonly emerging class of solutions describing an object in an expanding universe is the asymptotically anti-de Sitter solution (AdS), so we concentrate on such solutions hereafter.

2.1. Conformal Frame

Before proceeding to spherically symmetric solutions, we would like to make some comments on choosing the conformal frame.

One can use conformal transformations to map an $f(R)$ gravity model onto an ST one in the same way one can map one ST model onto another. Such mapping does not preserve a form of matter gravitational interaction. The frame in which matter only interacts with metric is called the Jordan frame, but in that frame the interaction between the scalar field and gravity may acquire a non-trivial form. The frame possessing the standard Hilbert–Einstein Lagrangian is called the Einstein frame, but in that frame the scalar field may interact with matter perturbing its propagation along the geodesics. Despite these frames being mathematically equivalent at the level of the classical action, the discussion on their physical equivalence is still going. Some authors claim that Jordan frame is not physically equivalent to the Einstein one because of the energy conditions' violation [64,65]. Others state that these frames are equivalent because one should compliment a conformal transformation with a unit transformation, thereby neglecting the apparent difference [66–68]. The Cauchy problem is well-stated both in Jordan and Einstein frames [69].

In the context of conformal equivalence, one also should consider conformal transformations of observable quantities, as a physical quantity available for measurement may not be invariant under the transformations. For instance, spacetime singularity may have a different form in Jordan and Einstein frames [70]. Another example is provided by [71], where it is proven that accelerated expansion of the universe in one model may be perceived as a decelerated expansion in the other. Therefore, one must choose a consistent method to relate observable quantities with the experimental data in the context of conformal frames' equivalence. Paper [64] presents the brief overview of the issue.

The situation is similar in the realm of quantum theory. In Ref. [72], the authors demonstrated that Jordan and Einstein frames are equivalent within the path integral approach, and Ref. [73] asserts that Jordan and Einstein frames of Brans–Dicke theory are equivalent at the quantum level. At the same time, the first loop contribution differs in some ST models for Jordan and Einstein frames [74].

In this review, we only consider classical implications of ST models, so we omit the issue of frame equivalence at the quantum level. At the classical level, we treat Jordan and Einstein frames as complimentary frameworks to map complex gravitational effects.

3. Schwarzschild–Anti-De Sitter Solution

We now recall general features of the black hole in the de Sitter universe. According to the Birkhoff theorem, spherically symmetric solutions of Einstein vacuum field equations with $\Lambda \neq 0$ reduce to Schwarzschild–de Sitter (SAdS) metric [75]:

$$ds^2 \;=\; U(r)\,dt^2 - \frac{dr^2}{U(r)} - r^2 d\Omega^2 \,, \tag{3}$$

$$U(r) \;=\; 1 - \frac{2m}{r} - \frac{\Lambda}{3}\,r^2 \,, \tag{4}$$

where $\Lambda > 0$ is the cosmological constant driving the large-scale expansion, and m is the black hole mass. Definitions of mass and angular momentum are complicated because of the asymptotic structure of de Sitter spacetime, the detailed discussion is given in [76]. Solution stability w.r.t. linear perturbations was proven in [77], and its thermodynamic properties were studied in [78]. The turnaround radius is given by the following expression:

$$r_{\mathrm{t}} = \sqrt[3]{\frac{3m}{\Lambda}} \,. \tag{5}$$

The equation above relates the black hole mass and the cosmological constant thereby showing connection between small and large scale physics and providing a uniform description for diverse physical phenomena. A similar feature occurs in thermodynamics: the black hole cannot have a temperature below the critical one

$$T_C = \frac{1}{2\pi}\sqrt{\Lambda} \tag{6}$$

because it is embedded into a thermal radiation background generated by the cosmological horizon. At the same time, the thermal radiation with the temperature below T_C cannot form a black hole. Despite the existence of a minimal temperature, the Bekenstein–Hawking formula for entropy holds. The laws of black hole thermodynamics in a de Sitter space resemble the standard features of a black hole in an asymptotically flat space [78].

4. No-Hair Argument

The no-hair argument plays a central role in black hole physics, constraining the number of non-trivial configurations available (see [79] for a recent review of no-hair theorems). A model admitting the no-hair theorem is indistinguishable from GR at the phenomenological level. Therefore, a violation of the no-hair argument is required to draw some conclusions on the extended gravity model by means of black hole observations. Here, we would like to discuss cornerstones of the argument.

The Hawking theorem [80] states that a rotating stationary black hole is axisymmetric and has a topologically spherical boundary. For an ST model, this means that the scalar field is constant and the resulting solution is indistinguishable from the standard GR one. In ST models with a Lagrangian (2), one can perform a conformal transformation to the Einstein frame. The resulting vacuum field equations are identical to the Einstein equations in the presence of the scalar field and this is where the Hawking theorem applies. Three main assumptions of the no-hair argument are crucial in our context.

First, one assumes that the spacetime is asymptotically flat or de Sitter. In Brans–Dicke theory, the cosmological expansion cannot be realised without the cosmological constant or the scalar field potential. Therefore, the Brans–Dicke model with negative cosmological constant is constrained by the no-hair argument and provides no new phenomenology at the black hole level. Note that, despite no-hair argument holds for an asymptotically de Sitter spacetime, it should be violated in the spacetime with anti-de Sitter asymptotic [81].

One should also exclude nontrivial interactions between the scalar field and gravity. Scalar field interacting with Gauss–Bonnet term, which we discuss in Section 6.2, provides an example of such

interaction [82,83]. Gauss–Bonnet contains a coupling between the Kretschmann scalar $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, vanishing only in the flat spacetime, and the scalar field. Gravity in the strong field regime induces the strong field regime on the scalar field and forms scalar hairs.

Finally, hair may develop due to the energy conditions violation. This can be achieved in the phantom as well as in the regular regime of the scalar field [57]. Therefore, more general theories, like (2), admit hairy asymptotically de Sitter black holes or wormholes, induced by the energy conditions' violation.

The no-hair argument was laid out for the most of the theories in question; however, there is usually a way to violate it without imposing any extraordinary requirements.

5. Brans–Dicke Theory

Spherically symmetric solutions within the Brans–Dicke theory (1) were obtained in [37,84] and divided into four classes connected by nontrivial transformations [85–88]. For a long time, they were thought to be black holes; however, depending on the value of ω, they describe either a wormhole (for the phantom scalar) or a naked singularity (for a normal non-trivial scalar field) [89,90]. Regular black hole geometry can only be achieved in the GR limit, when the scalar field ϕ is constant and $\omega = \infty$ (see [91] for a detailed study). Thus, if we are interested in the nontrivial geometry, yielding measurable effects and allowing us to test Brans–Dicke theory, we are forced to deal with wormholes.

Thermodynamic properties of the original Brans–Dicke black hole solutions [37,84] were studied in [92]. One cannot apply Bekenstein–Hawking entropy formula directly, as the scalar field can bring an additional entropy to the black hole (see [93] and references therein). The black hole entropy is related with the temperature as follows:

$$S = [2 - (Q + 3\chi)]\,\frac{r_0}{T_H}, \tag{7}$$

where T_H is the Hawking temperature, r_0 is the horizon radius, and Q and χ are solution parameters. A detailed study of the black hole thermodynamics [92] revealed that only solutions with $Q = 1$, $\chi = 0$ are physically relevant, as other configurations have zero entropy and infinite temperature.

On more general grounds, one can prove that the black hole entropy in Brans–Dicke theory is given by the following [94]:

$$S = \frac{1}{4}\int_{\Sigma} d^2x\,\sqrt{g^{(2)}}\,\phi, \tag{8}$$

where Σ denotes integration over black hole horizon, ϕ is the scalar field and $g^{(2)}$ is the metric on the horizon. In [94], it was also proven that the entropy remains constant under the conformal transformation. The fact that integral (8) should be evaluated at the horizon puts additional constraints on the solution. If the scalar field diverges at the horizon, one cannot attribute to it a finite entropy, so the solution should be considered as unphysical. Solutions with the vanishing scalar field at the horizon are known as cold black holes and they are subjected to a careful scrutiny [95–97].

Wormholes were initially designed as an object [98] embedded in a static universe and supported by exotic matter (We call exotic matter the one that violates the null energy condition (NEC)). Nonetheless, one can create a wormhole solution embedded in a dynamical universe in the presence of a phantom matter (Phantom matter violates the strong energy condition) [98–100]. The exterior gravitational field of a wormhole matches the (anti-)de Sitter spacetime far from the throat. A common feature of such wormholes is the time evolution of the throat, namely, a traversable wormhole may become un-traversable through the universe evolution [101]. Within GR, one can also construct a time-dependent wormhole embedded in an anisotropic traceless fluid, which does not require exotic matter until some critical time when the energy condition is violated inside the wormhole [102].

Before we proceed with describing wormholes in Brans–Dicke theory, we must give a few comments on the energy conditions violation normally needed to support a wormhole. Within Brans–Dicke theory, such a violation is usually achieved with $\omega < 0$. Negative values of ω are often ruled out because of the wrong sign before the kinetic term in the action leading to a ghost in the quantum theory (and to a phantom scalar field in cosmology [57]). Nevertheless, researchers do consider $\omega < 0$ for the following reasons. Modern observational data indicate that the phantom nature of dark energy is more likely [2]. Combining WMAP+eCMB+BAO+H0+SNe data yields the equation of state (EOS) parameter (The relation of energy and pressure density.) of the dark energy $w_{DE_0} = -1.17(+0.13 - 0.12)$ for the flat Universe at the significance level of 68%. This range of w_{DE_0} favours the phantom nature of the dark energy. A similar picture occurs for the non-flat universe model and for different dataset combinations, e.g., $w = -1.019 \; {}^{+0.075}_{-0.080}$ according to the PLANK data [1–3]. In the classical Brans–Dicke theory, a phantom regime can be achieved with $\omega < 0$. Finally, consideration of a ghost scalar field as an effective description for a theory with positive defined energy could eliminate quantum contradictions [103,104]. All of these arguments make the consideration of Brans–Dicke theory with $\omega < 0$ reasonable. Modern observational bounds only limit the absolute value of ω and modelling the gravitational collapse does not rule out negative ω values as well [105]. Thus, the range $\omega < 0$ is of great interest from the phenomenological point of view, since it can provide no-ghost but phantom cosmology in agreement with modern observations (see [57] for a no-ghost but phantom cosmological model). Therefore, we proceed with considering wormhole solutions on a cosmological background of the Brans–Dicke theory.

In the classical Brans–Dicke formalism, one can construct static as well as dynamic wormholes. Ref. [102] (see also references therein) introduced a time-dependent Brans–Dicke wormhole solution in an evolving cosmological background without the cosmological constant. Geometry is supported by a traceless matter, which in some cases can be non-exotic throughout space. The background spacetime conforms with the well-known radiation-dominated Friedmann model and does not describe the de Sitter expansion. For the solution to be non-trivial, one has to violate the weak energy condition by means of negative values of ω, and the resulting wormhole evolves along with the universe. This is a common feature for wormhole solutions embedded in a cosmological background. To describe the wormhole embedded in an expanding universe, the solution should possess de Sitter asymptotic, so we turn to such solutions now.

5.1. Brans–Dicke Theory with the Non-Vanishing Cosmological Constant

Brans–Dicke theory successfully describes an expanding universe by means of the scalar field potential or the cosmological constant (BDΛ), i.e., $\omega = \text{const}$, $\Lambda(\phi) = \text{const} \times \phi$ [106–113]. However, to the best of our knowledge, no wormhole solutions within this framework were presented in the literature.

The value of a turnaround radius for BDΛ was calculated in [114]. Authors considered two different approaches: they studied spherically symmetric gravitational field of a point mass and cosmological perturbations. Calculations resulted in the following analytical expression for the turnaround radius:

$$r_{\text{t}} = \sqrt[3]{\frac{3m}{\Lambda}\frac{2\omega + 4}{2\omega + 1}} = r_{\text{t}}|_{\text{GR}} \times \left(1 + \frac{1}{2\omega} + O\left(\frac{1}{\omega^2}\right) \right) . \tag{9}$$

The standard GR expression for the turnaround radius is recovered in $\omega = \infty$ limit, receiving positive corrections suppressed by the ω^{-1} factor. According to [114], this correction is consistent with the contemporary observational data.

5.2. Brans–Dicke Theory with the Scalar Field Potential

5.2.1. Cosmology via the Scalar Field Potential

   A variety of cosmological phenomena in ST models with an arbitrary scalar field potential is much wider that in BDΛ. Moreover, such models are not excluded by the observations and serve as the cornerstone of inflation.

   The simplest example is given by quintessence models. One can treat the scalar field as a matter alongside the standard model particles. In that case, the EOS parameter w is given by the following:

$$w = \frac{\rho}{p} = \frac{\dot{\phi}^2 - 2V}{\dot{\phi}^2 + 2V}, \tag{10}$$

where $\dot{\phi}$ is a scalar field derivative w.r.t. time and V is the scalar field potential. In such a setting, w is constrained to lie between -1 and 1, which is allowed by the observational data. The quintessence scenario appears in models with a negatively defined scalar field potential, which satisfies the contemporary cosmological dataset.

5.2.2. Wormhole Solutions

   A wormhole solution for the Brans–Dicke setting with the $\Lambda\phi^2$ potential is given in [115]. It is obtained for $\phi = \phi(t)$ and the resulting wormhole is non-static, and its throat radius increases as the cosmic time increases. The wormhole connects two asymptotically flat regions and does not describe an expanding universe in the background. The situation is interesting, since the de Sitter expansion occurs in the corresponding cosmological model (and even a bounce for $\omega < -3/2$ [116], but does not show itself in the wormhole solution. However, Ref. [115] gives no explicit solution; therefore, the ability of the metric to describe the real world is unclear.

   Another non-trivial model with an exponential potential was considered in [57]

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g}\, e^{\alpha\phi} \left[R - \omega\partial_\mu\phi\partial^\mu\phi - V_0 \exp\left[\phi/\phi_0\right]\right], \tag{11}$$

where α, ω and ϕ_0 are constants. Performing the transformation to the Einstein frame, one can see that, even if ω is negative, in the case when

$$3\alpha^2/2 + \omega > 0, \tag{12}$$

the effective kinetic energy of ϕ remains positive and the ghost does not appear in spite of the phantom cosmology in the background [57]. This is a highly desirable feature, relaxing the main objection against the cosmological phantom scalar field, i.e., the ghost appearance. The authors also claim that quantum effects may prevent the finite-time future singularity associated with the phantom.

   Ref. [117] considers spherically symmetric solutions in the framework above. For the following combination of model parameters:

$$\phi_0 + \frac{\alpha}{2(\alpha^2 + \omega)} = 0. \tag{13}$$

The metric obtained by Agnese and La Camera [118] solves the field equations:

$$ds^2 = A^m(r)dt^2 - A^{-n}(r)dr^2 - A^{1-n}(r)\, r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{14}$$

$$A(r) = 1 - \frac{2\eta}{r}, \quad m = \sqrt{2/(1+\gamma)}, \quad n = \gamma\sqrt{2/(1+\gamma)}. \tag{15}$$

Parameter η eta relates to the solution mass M as:

$$\eta = M\sqrt{(1+\gamma)/2}. \tag{16}$$

The PPN parameter γ enters the metric and provides a direct way to relate it with observational data. Expressions for α, ω and ϕ are given in [117]. If we suppose that the scalar field potential acts as the cosmological constant, we can arrive at the conclusion that the accelerated expansion of the universe does not contribute to local dynamics, since the metric is exactly the same as in the classical Brans–Dicke setting. This effect is analogous to [115] and indifferent to the value of the cosmological constant. This property agrees with the fact that the cosmological expansion does not manifest itself in the solar system. Astrophysical properties were studied by Alexeyev et al. in [119] and were found to be in agreement with modern observations. Therefore, the metric above represents a valid candidate for a real-world object.

Following the scope of our paper, we complement the above with the weak field limit study. Evaluating the PPN parameter γ in terms of model parameters and applying the corresponding observational bound, we obtain:

$$\alpha^2/|\omega| < 10^{-5}, \tag{17}$$

where ω should be negative. Despite the model still providing a valuable description of compact objects, the constraint above comes in a contradiction with the no-ghost condition, thereby nullifying the main advantage of the model [117]. Both cosmological and astronomical solutions assert versatile bounds on model parameters, which resulted jointly in a more rigorous test of the model.

To sum up, the non-trivial phenomenology on the Brans–Dicke cosmological background requires unnatural scalar field configurations; however, the question regarding BDΛ is still open.

6. Horndeski Theories

The most general scalar-tensor action resulting in second order field equations (i.e., avoiding a ghost related to higher order terms) was proposed by Horndeski [120]. The Brans–Dicke theory, considered before, appears to be a narrow subclass of Horndeski models. In its modern reformulation, Horndeski theory is written as a generalized galileon Lagrangian:

$$
\begin{aligned}
L &= L_2 + L_3 + L_4 + L_5, \\
L_2 &= G_2, \\
L_3 &= -G_3 \Box\phi, \\
L_4 &= G_4 R + G_{4X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \\
L_5 &= G_5 G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G_{5X}\left[(\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right],
\end{aligned}
\tag{18}
$$

where G_2, G_3, G_4, G_5 are arbitrary functions of ϕ and $X = -\partial^\mu\phi\partial_\mu\phi/2$, the canonical kinetic term, R is the Ricci scalar, $G_{\mu\nu}$ is the Einstein tensor, and

$$(\nabla_\mu\nabla_\nu\phi)^2 = \nabla_\mu\nabla_\nu\phi\nabla^\nu\nabla^\mu\phi, \tag{19}$$

$$(\nabla_\mu\nabla_\nu\phi)^3 = \nabla_\mu\nabla_\nu\phi\nabla^\nu\nabla^\rho\phi\nabla_\rho\nabla^\mu\phi. \tag{20}$$

The scalar field has the property of admitting a special symmetry in flat (nondynamical) spacetime for $G_2 \sim G_3 \sim X$ and $G_4 \sim G_5 \sim X^2$, which resembles the Galilean symmetry, therefore the name galileon [121]. Galileon symmetry is broken for a curved background, and, for a general choice of G_i, the corresponding scalar is then refereed to as the "generalized" galileon [122]. It is important to note

that generalized galileon theory and Horndeski theory do not start from the same principle, but turn out to be identical [123].

We will consider the following ansatz:

$$ds^2 = h(r)dt^2 - \frac{dr^2}{f(r)} - r^2 d\Omega^2. \tag{21}$$

The class of Horndeski models is vast and cannot be explored for the spherically symmetric solutions in its general form. Such solutions were found for particular subclasses of Horndeski and we review them now.

6.1. Shift-Symmetric Subclass

When equipped with a remnant of the Galileon symmetry in the flat space-time [122,124], namely a shift symmetry for the scalar field $\phi \to \phi + c$ (with c arbitrary real constant) in arbitrary curved spacetime, the action for Horndeski/Galileons scalar-tensor model reads

$$S = \int d^4 x \sqrt{-g} (\mathcal{L}_2 + \mathcal{L}_4), \tag{22}$$

where

$$\mathcal{L}_2 = G_2(X), \tag{23}$$
$$\mathcal{L}_4 = G_4(X)R + G_{4X}[(\Box\phi)^2 - (\nabla_\mu \nabla_\nu \phi)^2]. \tag{24}$$

Here, R is the scalar curvature, $X = -(\nabla_\mu \phi)^2/2$ is the canonical kinetic term of the scalar field, G_2 and G_4 are arbitrary functions of X, and $G_{4X} \equiv dG_4/dX$. This theory is invariant under the shift $\phi \to \phi + c$ (with c being an arbitrary real constant) and the reflection $\phi \to -\phi$.

For the choice of the arbitrary functions,

$$G_2(X) = -2\Lambda + 2\eta X, \quad G_4(X) = \zeta + \beta X, \quad G_3(X) = G_5(X) = 0, \tag{25}$$

the action can be expressed in the form [125]

$$S = \int dx^4 \sqrt{-g} \left(\zeta R - \eta (\partial\phi)^2 + \beta G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 2\Lambda \right), \tag{26}$$

where $G^{\mu\nu}$ is the Einstein tensor, ϕ is the scalar field, $\zeta > 0$, η and β are model parameters. Though the Horndeski model is not exhausted by this action, we will restrict our consideration to it, since most of the static spherically symmetric solutions we obtained within this framework.

The model (26) is known to admit a rich spectrum of cosmological solutions (see [126] and references therein) describing the late-time acceleration and an inflationary phase. Moreover, for $\eta \neq 0$, it admits solutions for which the Λ-term is totally screened, while the metric is not flat but rather de Sitter with the Hubble rate proportional to η/β. It offers an exciting opportunity to describe the late time cosmic acceleration while screening the vacuum Λ-term and hence circumventing the cosmological constant problem. Henceforth, the model is very interesting on the cosmological level.

Field equations of (26) can be integrated completely in static spherically symmetric sector [127] once the scalar field linear time dependence $\phi = \psi(r) + qt$ is introduced. The scalar field linear time evolution seems like a natural feature on a cosmological background and can be considered as an approximation for a more general theory. Although the Vainshtein screening mechanism is generally at work in Horndeski gravity, in the case of a minimal coupling of the scalar field to matter, no screening radius can be posed, so the solution must posses de Sitter asymptotic in Horndeski theory as well. Black hole solutions of the form (21) were found in the series of papers [128–132] (see [133] for a comprehensive review). These solutions are governed by the master equations:

$$f(r) = \frac{(\beta + \eta r^2)h(r)}{\beta (rh(r))'}, \tag{27}$$

$$h(r) = -\frac{\mu}{r} + \frac{1}{r}\int \frac{k(r)}{(\beta + \eta r^2)}dr, \tag{28}$$

$$\phi(r) = qt + \psi(r), \tag{29}$$

where μ plays a role of a mass term and k should be derived by means of the following constraint equation:

$$q^2\beta\left(\beta + \eta r^2\right)^2 - \left(2\zeta\beta + (2\zeta\eta - \lambda)r^2\right)k + C_0 k^{\frac{3}{2}} = 0. \tag{30}$$

Here, C_0 is an integration constant. By introducing a mild linear dependence in the time coordinate for the scalar field, one evades the scalar field being singular for its derivative on the horizon [134] and makes field equations bifurcate the no–hair theorem at the same time. The shift symmetry is keeping field equations time-independent and consistent with the static ansatz. The screening of the cosmological expansion at the black hole level appears here as well, given by the stealth Schwarzschild solution

$$f(r) = h(r) = 1 - \frac{\mu}{r}, \tag{31}$$

$$\Lambda = \eta = 0. \tag{32}$$

For this solution, the spatial part of the scalar is frozen and it only varies in time. By using different parameter combinations, the master equations can be integrated to give various solutions, among which is the Schwarzschild–de Sitter one,

$$f(r) = h(r) = 1 - \frac{\mu}{r} + \frac{\eta}{3\beta}r^2, \tag{33}$$

$$q^2 = (\zeta\eta + \beta\Lambda)/(\beta\eta), C_0 = (\zeta\eta - \beta\Lambda)\sqrt{\beta}/\eta, \tag{34}$$

with the effective cosmological constant being $\eta/3\beta$.

Stability analysis, however, showed that black holes described by the action (26) can only be stable for $q = 0$ in the anti de Sitter regime, when the effective cosmological constant is negative (see [135] and references therein) except for solution (33). With $q = 0$, the stealth Schwarzschild solution above becomes trivial with the frozen scalar field and displays therefore no phenomenology beyond the GR.

Not much is known about black hole thermodynamics in Horndeski models. As Lagrangians (18) depend on the arbitrary functions, the thermodynamic analysis is complicated. A simple case $G_4 = \frac{1}{16\pi G}$, $G_5 = $ const was discussed in [136], although the authors simply adopted the Bekenstein–Hawking formula, without deriving it. In Ref. [137], on the contrary, the authors used Wald formalism to derive the relation between the black hole entropy and the horizon area. Just as in the Brans–Dicke case, the black hole entropy obtains an additional contribution from the scalar field. Although no direct constraints were placed in [137], it seems possible to constrain a range of Horndeski model parameters by studying the black hole viscosity, as it is argued in [138]. Black hole entropy S and viscosity η are connected as $\eta/S = 1/4\pi a$ (where a is defined by the model parameters, see [138] and discussion there). Therefore, any black hole solutions must satisfy the viscosity relation to be considered physically relevant.

Wormhole solutions in the theory (26) were presented in [139,140]. It is shown that wormholes exist only if $\eta = -1$ (phantom case) and $\beta > 0$. The wormhole throat then connects two anti-de Sitter

spacetimes and the solution therefore is of no observational interest, since the model does not admit the Vainshtein screening mechanism.

6.2. Scalar-Tensor Models with Gauss–Bonnet Term

The second family of solutions in the Horndeski framework admitting hairy black holes involves the Gauss–Bonnet term G

$$\mathcal{L}_{GB} = \frac{M_P^2}{2} \left[R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \alpha\phi\hat{G} \right],$$ (35)

$$\hat{G} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2.$$ (36)

The Gauss–Bonnet term is a topological invariant and thus only contributes to the equations of motion in four-dimensional spacetime when coupled to the scalar. This model is a part of the G-inflation framework [123,141]. The action naturally supports de Sitter solutions equipped with an exit mechanism to a linearly expanding phase [142,143].

Sotiriou and Zhou insist that any Horndeski theory where this Gauss–Bonnet term is not forbidden must have hairy black holes [82,83]. They present a black hole for the time-independent scalar to the above theory using numerical and perturbative methods. The resulting solution contains the finite radius singularity, which can be hidden behind the horizon if the black hole is massive enough. The authors also noted that corrections to observables in this model are expected to be small w.r.t. GR. However, the solution was only explored in the vicinity of the horizon and its asymptotic behavior is unclear.

6.3. Quartic Horndeski Square Root Lagrangian

Ref. [144] considers black holes for the static and time-dependent scalar for the following action:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left\{ \left[\zeta + \beta\sqrt{(\partial\phi)^2/2} \right] R - \frac{\eta}{2}(\partial\phi)^2 - \frac{\beta}{\sqrt{2(\partial\phi)^2}} \left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2 \right] \right\}.$$ (37)

The resulting static scalar field solution takes the following form:

$$f(r) = h(r) = 1 - \frac{\mu}{r} - \frac{\beta^2}{2\zeta\eta r^2}.$$ (38)

The solution is of the Reissner–Nordstrom form. The "electric charge" is not an integration constant—it only depends on the parameters of the theory. The correction would be the exact same for any black hole. According to the authors, for the positive η (which would correspond to an imaginary charge of the RN) metric, we will always have an event horizon in (38). With a negative η, the solution behaves badly beyond the event horizon, since ϕ' becomes imaginary there.

The cosmological constant can be added to the initial action. The solution is then modified in the same way as it is in GR, and acquires anti-de Sitter or de Sitter asymptotic:

$$f(r) = h(r) = 1 - \frac{\mu}{r} - \frac{\beta^2}{2\zeta\eta r^2} - \frac{\Lambda}{3\zeta}r^2.$$ (39)

If one drops the staticity assumption, the asymptotic flatness must be abandoned at the same time. The result would again be the Reissner–Nordstrom type solution:

$$h \underset{q\to 0}{=} 1 - \frac{\mu}{r} - \frac{\beta^2}{2\eta\zeta r^2} + \frac{\eta q^2 r^2}{24\zeta} + \mathcal{O}(q^4)$$ (40)

for $\phi = qt + \psi(r)$. However, the properties of the spatial part of the scalar remain and η should be positive, and the solution is AdS in this case. The question of whether this asymptotic renders the solution unphysical is open, since we must first check that the theory admits no Vainshtein screening and the black hole asymptotic matters.

6.4. Cubic Galileon

Ref. [145] studies black hole solutions in a subclass of Horndeski theory, which contains the cubic Galileon term:

$$S = \int d^D x \sqrt{-g} \left[\zeta \left(R - 2\Lambda \right) - \eta \left(\partial \phi \right)^2 + \gamma \Box \phi \left(\partial \phi \right)^2 \right]. \tag{41}$$

The solutions found via perturbative methods can be interpreted as black holes immersed in a flat, or self-accelerated universe depending on the model parameter values:

$$f(r) = 1 - \frac{\mu}{r} - \frac{\Lambda_{\text{eff}}}{3} r^2 + O(r^{-4}), \tag{42}$$

$$h(r) = 1 - \frac{\mu}{r} - \frac{\Lambda_{\text{eff}}}{3} r^2 + O(r^{-6}). \tag{43}$$

The cosmological pure (A)dS solution is presented as well:

$$f(r) = h(r) = 1 - \frac{\Lambda_{\text{eff}}}{3} r^2. \tag{44}$$

The effective cosmological constant of the black hole metric is given solely by the model parameters and agrees with one of the cosmological solution perfectly:

$$\Lambda_{\text{eff}} = \frac{\eta^2}{3\gamma^2} \left[\frac{\zeta \Lambda}{\eta} \pm \sqrt{ \left(\frac{\zeta \Lambda}{\eta} \right)^2 - \frac{2\eta \zeta}{3\gamma^2} } \right]^{-1}. \tag{45}$$

We can see from the above that the effective cosmological constant vanishes rendering the metric asymptotically flat for $\eta = \Lambda = 0$ and dS otherwise. The following questions remain: does such a configuration actually take place during matter collapse, and is it free of ghost, gradient and tachyon instability?

7. Theories with Auxiliary Fields

There are models that expand GR with a larger number of auxiliary fields. The spectrum of such theories is broad and includes TeVeS (Tensor-Vector-Scalar gravity), MOG (MOdified Gravity), Einstein–Aether theory, biscalar-vector-tensor gravity, and many others. In this section, we focus our attention on the simplest examples of such theories, namely TeVeS and MOG. Both of these theories were meant to account for the dark matter and to reproduce MOND-like mechanics in the relativistic case.

The original paper devoted to MOG [146] proposed a way to explain the rotation velocity curves of galaxies via new scalar and vector fields. The model is given by the following action:

$$
\begin{aligned}
S &= \int d^4 x \sqrt{-g} \left[\frac{1}{16\pi G} (R + 2\Lambda) - \omega \left(\frac{1}{4} B^{\mu\nu} B_{\mu\nu} + V(\phi) \right) + \frac{1}{G^3} \left(\frac{1}{2} g^{\mu\nu} \nabla_\mu G \nabla_\nu G - V(G) \right) \right. \\
&\quad \left. + \frac{1}{G} \left(\frac{1}{2} g^{\mu\nu} \nabla_\mu \omega \nabla_\nu \omega - V(\omega) \right) + \frac{1}{\mu^2 G} \left(\frac{1}{2} g^{\alpha\beta} \nabla_\alpha \mu \nabla_\beta \mu - V(\mu) \right) \right],
\end{aligned} \tag{46}
$$

where ϕ_μ is the vector field, $B_{\mu\nu} = \nabla_\mu \phi_\nu - \nabla_\nu \phi_\mu$ is the vector field tensor, and G, ω, and μ are three scalar fields. Auxiliary scalar and vector fields contribute attractive and repulsive Yukawa components

to the Newton potential and thereby explain the dark matter phenomenology. The model appears to pass the Solar System tests [146–150], provide adequate cosmology [151], and describe some galaxy phenomenology [152,153] (see also [154–157]).

Black hole physics was considered [158,159]. The model contains multiple parameters that allow it to operate as a generalization of a nonlinear electrodynamics in a curved spacetime. It thereby admits black hole solutions obtained within the nonlinear electrodynamics model [158,160,161]. However, the authors of [158] studied a solitary black hole with zero cosmological constant and flat asymptotic behavior, therefore results of [158,159] relate to the stationary universe. For the best of our knowledge, there are no papers describing the black hole in an expanding universe; therefore, further research of the model is in order. The no-hair argument and the generalization of the Birkhoff theorem were also never considered; therefore, the role of the solutions [158,159] is not yet completely understood. Thermodynamics of rotating and non-rotating black holes in MOG was considered in [159]. General layout of MOG black hole thermodynamic is affected by the fact that some black hole solutions may possess a mass-dependent charge. This issue is discussed in great detail in [159]. The entropy also obtains logarithmic corrections due to quantum fluctuations.

The situation is similar within TeVeS models (see [42,162] for reviews of TeVeS in the context of modified gravity). TeVeS model was incepted in [163] in order to construct a relativistic MOND in terms of the nonlinear field interaction (also known as the aquadratic Lagrangian theory). The theory describes a gravitational field with the physical metric $g_{\mu\nu}$, which is universally coupled to matter in order to satisfy the Einstein equivalence principle, the vector field A_μ, and the scalar field ϕ. One defines the Bekenstein metric $\widetilde{g}^{\mu\nu}$ by the following:

$$g_{\mu\nu} = e^{-2\phi}\widetilde{g}_{\mu\nu} - 2\sinh(2\phi)A_\mu A_\nu. \tag{47}$$

The metric couples to auxiliary fields and it should be used to operate with their indices. One also requires the vector field to be a time-like vector with respect to the Bekenstein metric. The model action is given by the following:

$$\begin{aligned} S &= \frac{1}{16\pi G}\int d^4x \sqrt{-\widetilde{g}}\widetilde{R} - \frac{1}{32\pi G}\int d^4x\sqrt{-\widetilde{g}}\left[K\,\widetilde{g}^{\mu\alpha}\widetilde{g}^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} - 2\lambda(\widetilde{g}^{\mu\nu}A_\mu A_\nu + 1)\right] \\ &\quad - \frac{1}{16\pi G}\int d^4\sqrt{-\widetilde{g}}\left[\mu(\widetilde{g}^{\mu\nu} - A^\mu A^\nu)\widetilde{\nabla}_\mu\phi\widetilde{\nabla}_\nu\phi + V(\mu)\right], \end{aligned} \tag{48}$$

where μ is a nondynamical dimensionless scalar field and λ is a Lagrange multiplier.

In order to obtain the late-time cosmological acceleration, one may introduce the cosmological constant as a linear term in the potential $V(\mu)$, but this only leads to the same cosmological problem, as the constant is going to be a free model parameter. At the same time, the model equipped with nonlinear field interaction that is able to produce the late-time acceleration with vanishing cosmological constant in a certain case [164]. Such phenomenology appears because the TeVeS action has a resemblance to the k-essence model [165,166], but we are not aware of any paper proving that the late-time acceleration phase takes place in the model with an arbitrary set of parameters. Therefore, a more detailed investigation of the model is required to establish its ability to explain dark energy phenomenology.

Despite the large number of papers devoted to black hole study [167–169], further research is required. The black hole solution presented in [167] violates causality and may not be considered as a physically-relevant. In Ref. [168], the authors constrained themselves only to an asymptotically flat spacetime, thereby excluding phenomenology of a black hole in an expanding universe. Only in Ref. [169] was a solution obtained without usage of any special approximations, and it appears to have a flat asymptotic. To the best of our knowledge, there are no papers giving the no-hair argument or Birkhoff theorem in TeVeS. It is also unclear if the mechanism generating the late-time acceleration in the cosmological regime [164] holds for local solutions such as [168,169]. Black hole thermodynamics were studied in [168], and black hole temperature and entropy appeared to be identical to the GR case.

There are several crucial issues of models with auxiliary fields that should be clarified. First, the no-hair argument should be extended on these theories. The argument defines if a model has any significant difference from the GR at the level of black hole physics. If the no-hair argument fails to constrain these models, one should seek asymptotically de Sitter black hole solutions to check whether the theory can provide a uniform description of small- and large-scale physics. Finally, one should clarify the role of an auxiliary vector field. Within the Einstein–Aether theory [170], the vector field leads to the violation of the Lorentz invariance, while TeVeS gravity nonlinear coupling of scalar and vector fields can lead to the late-time acceleration phase without the cosmological constant [164]. Therefore, one should clarify if TeVeS violates the Lorentz invariance and how the auxiliary vector field affects the scalar field phenomenology.

8. String Theory Framework

The string theory approach to the black hole physics differs extremely from the modified gravity framework. The discussion of black holes in the string theory lies far beyond the reach of this paper (see [171,172] for a detailed review); nonetheless, we present the basic features of the string theory black hole physics, since the scalar field is generally involved.

The string theory is a quantum theory describing the physical realm in terms of the fundamental superstrings interaction. It is widely accepted that the theory provides an adequate description of the gravitational phenomena at the Planck energy scale, thereby serving as a high energy gravity theory. In the low energy regime, the string theory should be replaced with some effective theory for the gravitational phenomena, such as black holes, in terms of the classical spacetime. Therefore, one may not be able to obtain a relevant description of a macroscopic black hole within the string theory framework; nonetheless, the so-called fuzzball conjecture [173,174] provides a way to do this. It is well established that the classical black hole has a non-zero entropy defined by the size of its event horizon. The fuzzball conjecture states that a classical black hole with entropy S should be treated as a manifestation of $\exp[S]$ black hole microstates [172,175]. This is the main reason why our method—study of black holes with Schwarzschild–de Sitter asymptotic—may hardly be applied in the context of the string theory. The fuzzball conjecture considers the black hole to be solitary object, which is not affected by the global cosmological expansion. Therefore, objects with de Sitter asymptotic lie beyond the scope of the fuzzball conjecture.

At the same time, the string theory should be replaced by an effective low energy theory for the sub-Planckian phenomena [176,177]. Such an effective theory must describe the gravitational phenomena in terms of the classical spacetime and fields, which is exactly what modified gravity is about (see [178–180] for examples). Therefore, one can study properties of black holes with de Sitter asymptotic obtained within the effective low-energy models generated by the string theory. On practical terms, such models may be treated on the same foot as modified gravity models and should be addressed with the discussion from the previous sections. As string theory itself is not yet complete, it is impossible to find a single modified gravity model providing a comprehensive low-energy description of the string theory. Therefore, one should investigate various string-inspired models.

9. Conclusions

In this paper, we reviewed the properties of black hole-like solutions in the scalar-tensor gravity. Scalar-tensor theories play an important role in the extended gravity layout. This key role is defined by the fact that such models provide a way to separate a new gravitational degree of freedom at the level of the model action. This is a powerful tool allowing one to trace the structure of a theory in a more detail. For instance, in the $f(R)$ gravity models, the new degree of freedom is encoded into a metric variable.

Although the theories in the scope of this paper are very different, black hole solutions display several general properties:

- the model, describing the de Sitter expansion of the universe, naturally contains the asymptotically de Sitter solution;
- the local phenomenology may be isolated from the cosmological expansion by means of the Vainshtein screening or the parameter fine tuning, which results in a flat asymptotic for the local solution.

Considering the cosmological and the black hole solutions at the same time yields a more comprehensive view of the model and may result in useful constraints. For example, in the Brans–Dicke theory, we know that we must account for the scalar field potential or the cosmological constant to obtain the cosmological expansion; therefore, we must consider black holes in terms of such models as well. In such a consideration, different constraints may contradict each other, thereby cancelling the model, as it occurred in [117,135].

However, such a versatile picture is rarely available. The unresolved questions about scalar-tensor theories in the scope of our paper are the following:

- There are no wormhole solutions for the BD+Λ framework.
- Scalar-tensor models with the Gauss–Bonnet term should be explored in more detail to clarify the concordance between the local and the cosmological solution.
- Gauss–Bonnet–Horndeski theory describes the late-time cosmology and inflation, but no analytic black hole solutions embedded in an expanding Universe are known. Such a solution could give a major insight into the model properties.
- Quartic Horndeski square root Lagrangian is explored for black holes; however, we are lacking cosmological research.
- Horndeski models require additional research of black hole thermodynamics.
- It is not clear if most of the scalar-tensor black hole configurations actually take place during the matter collapse.
- The no-hair argument should be considered for the theories with the auxiliary fields.

Unfortunately, the asymptotic bevahior of the solution is not always known, even if the cosmology in the model is known to be adequate, like in [83]. This prevents the conformity assessment between the cosmological and astrophysical scale.

Despite all the issues mentioned above, the scalar-tensor gravity remains one of the most fruitful extended gravity frameworks naturally emerging from modern theoretical and observational considerations.

Author Contributions: D.T. proposed the concept of the article. D.T. made leading contribution to chapters 5 and 6. B.N. made leading contribution to chapters 7 and 8. D.T. and B.L. made an equal contribution to chapters 1-4 and 9.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Will, C.M. The Confrontation between General Relativity and Experiment. *Living Rev. Relativ.* **2014**, *17*, 4.
2. Hinshaw, G.; Larson, D.; Komatsu, E.; Spergel, D.N.; Bennett, C.; Dunkley, J.; Nolta, M.R.; Halpern, M.; Hill, R.S.; Odegard, N.; et al. Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Parameter Results. *Astrophys. J. Suppl.* **2013**, *208*, 19.
3. Ade, P.A.R.; Aghanim, N.; Arnaud, M.; Ashdown, M.; Aumont, J.; Baccigalupi, C.; Banday, A.J.; Barreiro, R.B.; Bartlett, J.G.; Bartolo, N.; et al. Planck 2015 results. XIII. Cosmological parameters. *Astron. Astrophys. A* **2016**, *594*, A13.
4. Abbott, B.P.; Abbott, R.; Abbott, T.D.; Abernathy, M.R.; Acernese, F.; Ackley, K.; Adams, C.; Adams, T.; Addesso, P.; Adhikari, R.X.; et al. Observation of Gravitational Waves from a Binary Black Hole Merger. *Phys. Rev. Lett.* **2016**, *116*, 061102.

5. Abbott, B.P.; Abbott, R.; Abbott, T.D.; Abernathy, M.R.; Acernese, F.; Ackley, K.; Adams, C.; Adams, T.; Addesso, P.; Adhikari, R.X.; et al. GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence. *Phys. Rev. Lett.* **2016**, *116*, 241103.

6. Abbott, B.P.; Abbott, R.; Abbott, T.D.; Abernathy, M.R.; Acernese, F.; Ackley, K.; Adams, C.; Adams, T.; Addesso, P.; Adhikari, R.X.; et al. GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2. *Phys. Rev. Lett.* **2017**, *118*, 221101.

7. Abbott, B.P.; Abbott, R.; Abbott, T.D.; Abernathy, M.R.; Acernese, F.; Ackley, K.; Adams, C.; Adams, T.; Addesso, P.; Adhikari, R.X.; et al. Tests of general relativity with GW150914. *Phys. Rev. Lett.* **2016**, *116*, 221101.

8. Moffat, J.W. LIGO GW150914 and GW151226 gravitational wave detection and generalized gravitation theory (MOG). *Phys. Lett. B* **2016**, *763*, 427–433.

9. Cayuso, J.; Ortiz, N.; Lehner, L. Fixing extensions to General Relativity in the non-linear regime. *Phys. Rev. D* **2017**, *96*, 084043.

10. Abbott, B.P.; Abbott, R.; Abbott, T.D.; Acernese, F.; Ackley, K.; Adams, C.; Adams, T.; Addesso, P.; Adhikari, R.X.; Adya, V.B.; et al. GW170814: A Three-Detector Observation of Gravitational Waves from a Binary Black Hole Coalescence. *Phys. Rev. Lett.* **2017**, *119*, 141101.

11. Abbott, B.P.; Abbott, R.; Abbott, T.D.; Acernese, F.; Ackley, K.; Adams, C.; Adams, T.; Addesso, P.; Adhikari, R.X.; Adya, V.B.; et al. GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral. *Phys. Rev. Lett.* **2017**, *119*, 161101.

12. Fermi GBM; LIGO Scientific Collaboration; Virgo Collaboration; INTEGRAL, IceCube Collaboration; AstroSat Cadmium Zinc Telluride Imager Team; IPN Collaboration; The Insight-Hxmt Collaboration; ANTARES Collaboration; The Swift Collaboration; AGILE Team; et al. Multi-messenger Observations of a Binary Neutron Star Merger. *Astrophys. J. Lett.* **2017**, *848*, L12.

13. Fish, V.; Akiyama, K.; Bouman, K.; Chael, A.; Johnson, M.; Doeleman, S.; Blackburn, L.; Wardle, J.; Freeman, W. Observing—and Imaging—Active Galactic Nuclei with the Event Horizon Telescope. *Galaxies* **2016**, *4*, 54.

14. Ortiz-León, G.N.; Johnson, M.D.; Doeleman, S.S.; Blackburn, L.; Fish, V.L.; Loinard, L.; Reid, M.J.; Castillo, E.; Chael, A.A.; Hernández-Gómez, A.; et al. The Intrinsic Shape of Sagittarius A* at 3.5-mm Wavelength. *Astrophys. J.* **2016**, *824*, 10.

15. Berti, E.; Barausse, E.; Cardoso, V.; Gualtieri, L.; Pani, P.; Sperhake, U.; Stein, L.C.; Wex, N.; Yagi, K.; Baker, T.; et al. Testing General Relativity with Present and Future Astrophysical Observations. *Class. Quantum Gravity* **2015**, *32*, 243001.

16. Capozziello, S.; De Laurentis, M. Extended Theories of Gravity. *Phys. Rep.* **2011**, *509*, 167–321.

17. Bertotti, B.; Iess, L.; Tortora, P. A test of general relativity using radio links with the Cassini spacecraft. *Nature* **2003**, *425*, 374–376.

18. Clowe, D.; Bradac, M.; Gonzalez, A.H.; Markevitch, M.; Randall, S.W.; Jones, C.; Zaritsky, D. A direct empirical proof of the existence of dark matter. *Astrophys. J. Lett.* **2006**, *648*, L109.

19. Boran, S.; Desai, S.; Kahya, E.; Woodard, R. GW170817 Falsifies Dark Matter Emulators. *arXiv* **2017**, arXiv:1710.06168.

20. Blumenthal, G.R.; Faber, S.M.; Primack, J.R.; Rees, M.J. Formation of Galaxies and Large Scale Structure with Cold Dark Matter. *Nature* **1984**, *311*, 517–525.

21. Davis, M.; Efstathiou, G.; Frenk, C.S.; White, S.D.M. The Evolution of Large Scale Structure in a Universe Dominated by Cold Dark Matter. *Astrophys. J.* **1985**, *292*, 371–394.

22. Capozziello, S.; Cardone, V.F.; Carloni, S.; Troisi, A. Can higher order curvature theories explain rotation curves of galaxies? *Phys. Lett. A* **2004**, *326*, 292–296.

23. Capozziello, S.; Cardone, V.F.; Troisi, A. Dark energy and dark matter as curvature effects. *J. Cosmol. Astropart. Phys.* **2006**, *2006*, 1.

24. Capozziello, S.; Cardone, V.F.; Troisi, A. Low surface brightness galaxies rotation curves in the low energy limit of R_n gravity: no need for dark matter? *Mon. Not. R. Astron. Soc.* **2007**, *375*, 1423–1440.

25. Zel'dovich, Y.B.; Krasinski, A.; Zeldovich, Y.B. The Cosmological constant and the theory of elementary particles. *Sov. Phys. Usp.* **1968**, *11*, 381–393.

26. Weinberg, S. The Cosmological Constant Problem. *Rev. Mod. Phys.* **1989**, *61*, 1–23.

27. Linde, A.D. *Inflationary Cosmology*; Springer: Berlin, Germany, 2008.

28. Senatore, L. Lectures on Inflation. In Proceedings of the Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings (TASI 2015), Boulder, CO, USA, 1–26 June 2015; pp. 447–543.

29. Starobinsky, A.A. A New Type of Isotropic Cosmological Models Without Singularity. *Phys. Lett. B* **1980**, *91*, 99–102.

30. Linde, A.D. Chaotic Inflation. *Phys. Lett. B* **1983**, *129*, 177–181.

31. De Felice, A.; Tsujikawa, S. f(R) theories. *Living Rev. Relativ.* **2010**, *13*, 3.

32. Nojiri, S.; Odintsov, S.D.; Oikonomou, V.K. Modified Gravity Theories on a Nutshell: Inflation, Bounce and Late-time Evolution. *Phys. Rep.* **2017**, *692*, 1–104.

33. Nojiri, S.; Odintsov, S.D. Unified cosmic history in modified gravity: From F(R) theory to Lorentz non-invariant models. *Phys. Rept.* **2011**, *505*, 59–144.

34. Nojiri, S.; Odintsov, S.D. Introduction to modified gravity and gravitational alternative for dark energy. *Int. J. Geom. Meth. Mod. Phys.* **2007**, *4*, 115.

35. Charmousis, C.; Copeland, E.J.; Padilla, A.; Saffin, P.M. General second order scalar-tensor theory, self tuning, and the Fab Four. *Phys. Rev. Lett.* **2012**, *108*, 051101.

36. Momeni, D.; Houndjo, M.J.S.; Güdekli, E.; Rodrigues, M.E.; Alvarenga, F.G.; Myrzakulov, R. Spherically symmetric solutions of light Galileon. *Int. J. Theor. Phys.* **2016**, *55*, 1211–1221.

37. Brans, C.; Dicke, R.H. Mach's principle and a relativistic theory of gravitation. *Phys. Rev.* **1961**, *124*, 925–935.

38. De la Cruz-Dombriz, A.; Saez-Gomez, D. Black holes, cosmological solutions, future singularities, and their thermodynamical properties in modified gravity theories. *Entropy* **2012**, *14*, 1717–1770.

39. Bergmann, P.G. Comments on the scalar tensor theory. *Int. J. Theor. Phys.* **1968**, *1*, 25–36.

40. Wagoner, R.V. Scalar tensor theory and gravitational waves. *Phys. Rev. D* **1970**, *1*, 3209–3216.

41. Nordtvedt, K., Jr. PostNewtonian metric for a general class of scalar tensor gravitational theories and observational consequences. *Astrophys. J.* **1970**, *161*, 1059–1067.

42. Clifton, T.; Ferreira, P.G.; Padilla, A.; Skordis, C. Modified Gravity and Cosmology. *Phys. Rep.* **2012**, *513*, 1–189.

43. Hohmann, M.; Jarv, L.; Kuusk, P.; Randla, E. Post-Newtonian parameters γ and β of scalar-tensor gravity with a general potential. *Phys. Rev. D* **2013**, *88*, 084054.

44. Deng, X.M.; Xie, Y. Solar System tests of a scalar-tensor gravity with a general potential: Insensitivity of light deflection and Cassini tracking. *Phys. Rev. D* **2016**, *93*, 044013.

45. Tretyakova, D.; Shatskiy, A.A.; Novikov, I.; Alexeyev, S. Non-singular Brans–Dicke cosmology with cosmological constant. *Phys. Rev. D* **2012**, *85*, 124059.

46. Damour, T.; Esposito-Farese, G. Nonperturbative strong field effects in tensor-scalar theories of gravitation. *Phys. Rev. Lett.* **1993**, *70*, 2220–2223.

47. Damour, T.; Esposito-Farese, G. Tensor-scalar gravity and binary pulsar experiments. *Phys. Rev. D* **1996**, *54*, 1474–1491.

48. Shibata, M.; Nakao, K.i.; Nakamura, T. Scalar type gravitational wave emission from gravitational collapse in Brans–Dicke theory: Detectability by a laser interferometer. *Phys. Rev. D* **1994**, *50*, 7304–7317.

49. Harada, T.; Chiba, T.; Nakao, K.i.; Nakamura, T. Scalar gravitational wave from Oppenheimer-Snyder collapse in scalar-tensor theories of gravity. *Phys. Rev. D* **1997**, *55*, 2024–2037.

50. Novak, J. Spherical neutron star collapse in tensor - scalar theory of gravity. *Phys. Rev. D* **1998**, *57*, 4789–4801.

51. Faraoni, V.; Lapierre-Léonard, M.; Prain, A. Turnaround radius in an accelerated universe with quasi-local mass. *J. Cosmol. Astropart. Phys.* **2015**, *2015*, 013.

52. Faraoni, V. Turnaround radius in modified gravity. *Phys. Dark Universe* **2016**, *11*, 11–15.

53. Pavlidou, V.; Tomaras, T.N. Where the world stands still: turnaround as a strong test of ΛCDM cosmology. *J. Cosmol. Astropart. Phys.* **2014**, *2014*, 020.

54. Pavlidou, V.; Tetradis, N.; Tomaras, T.N. Constraining Dark Energy through the Stability of Cosmic Structures. *J. Cosmol. Astropart. Phys.* **2014**, *1405*, 017.

55. Tanoglidis, D.; Pavlidou, V.; Tomaras, T. Testing LambdaCDM cosmology at turnaround: Where to look for violations of the bound? *J. Cosmol. Astropart. Phys.* **2015**, *2015*, 060.

56. Damour, T.; Nordtvedt, K. General relativity as a cosmological attractor of tensor scalar theories. *Phys. Rev. Lett.* **1993**, *70*, 2217–2219.

57. Elizalde, E.; Nojiri, S.; Odintsov, S.D. Late-time cosmology in (phantom) scalar-tensor theory: Dark energy and the cosmic speed-up. *Phys. Rev. D* **2004**, *70*, 043539.
58. Ratra, B.; Peebles, P.J.E. Cosmological Consequences of a Rolling Homogeneous Scalar Field. *Phys. Rev. D* **1988**, *37*, 3406.
59. Demianski, M.; Piedipalumbo, E.; Rubano, C.; Scudellaro, P. Cosmological models in scalar tensor theories of gravity and observations: A class of general solutions. *Astron. Astrophys.* **2008**, *481*, 279–294.
60. Nagata, R. Constraints on Scalar-Tensor Cosmology From Wmap Data. *Int. J. Mod. Phys.* **2011**, *1*, 183–188.
61. Alonso, D.; Bellini, E.; Ferreira, P.G.; Zumalacárregui, M. Observational future of cosmological scalar-tensor theories. *Phys. Rev. D* **2017**, *95*, 063502.
62. Babichev, E.; Deffayet, C. An introduction to the Vainshtein mechanism. *Class. Quantum Gravity* **2013**, *30*, 184001.
63. Bellini, E.; Sawicki, I. Maximal freedom at minimum cost: linear large-scale structure in general modifications of gravity. *J. Cosmol. Astropart. Phys.* **2014**, *2014*, 050.
64. Magnano, G.; Sokolowski, L.M. On physical equivalence between nonlinear gravity theories and a general relativistic selfgravitating scalar field. *Phys. Rev. D* **1994**, *50*, 5039–5059.
65. Faraoni, V.; Gunzig, E. Einstein frame or Jordan frame? *Int. J. Theor. Phys.* **1999**, *38*, 217–225.
66. Postma, M.; Volponi, M. Equivalence of the Einstein and Jordan frames. *Phys. Rev. D* **2014**, *90*, 103516.
67. Catena, R.; Pietroni, M.; Scarabello, L. Einstein and Jordan reconciled: a frame-invariant approach to scalar-tensor cosmology. *Phys. Rev. D* **2007**, *76*, 084039.
68. Faraoni, V.; Nadeau, S. The (pseudo)issue of the conformal frame revisited. *Phys. Rev. D* **2007**, *75*, 023501.
69. Salgado, M. The Cauchy problem of scalar tensor theories of gravity. *Class. Quantum Gravity* **2006**, *23*, 4719–4742.
70. Bahamonde, S.; Odintsov, S.D.; Oikonomou, V.K.; Wright, M. Correspondence of $F(R)$ Gravity Singularities in Jordan and Einstein Frames. *Annals Phys.* **2016**, *373*, 96–114.
71. Bahamonde, S.; Odintsov, S.D.; Oikonomou, V.K.; Tretyakov, P.V. Deceleration versus acceleration universe in different frames of $F(R)$ gravity. *Phys. Lett. B* **2017**, *766*, 225–230.
72. Calmet, X.; Yang, T.C. Frame Transformations of Gravitational Theories. *Int. J. Mod. Phys. A* **2013**, *28*, 1350042.
73. Pandey, S.; Banerjee, N. Equivalence of Jordan and Einstein frames at the quantum level. *Eur. Phys. J. Plus* **2017**, *132*, 107.
74. Kamenshchik, A.Yu.; Steinwachs, C.F. Question of quantum equivalence between Jordan frame and Einstein frame. *Phys. Rev. D* **2015**, *91*, 084033.
75. Kottler, F. The physical basis of Einstein's theory of gravitation. *Ann. Phys.* **1918**, *56*, 401.
76. Ashtekar, A.; Magnon, A. Asymptotically anti-de Sitter space-times. *Class. Quantum Gravity* **1984**, *1*, L39–L44.
77. Kodama, H.; Ishibashi, A. A Master equation for gravitational perturbations of maximally symmetric black holes in higher dimensions. *Prog. Theor. Phys.* **2003**, *110*, 701–722.
78. Hawking, S.W.; Page, D.N. Thermodynamics of Black Holes in anti-De Sitter Space. *Commun. Math. Phys.* **1983**, *87*, 577.
79. Sotiriou, T.P. Black Holes and Scalar Fields. *Class. Quantum Gravity* **2015**, *32*, 214002.
80. Hawking, S.W. Black holes in general relativity. *Commun. Math. Phys.* **1972**, *25*, 152–166.
81. Torii, T.; Maeda, K.; Narita, M. Scalar hair on the black hole in asymptotically ant-de Sitter spacetime. *Phys. Rev. D* **2001**, *64*, 044007.
82. Sotiriou, T.P.; Zhou, S.Y. Black hole hair in generalized scalar-tensor gravity. *Phys. Rev. Lett.* **2014**, *112*, 251102.
83. Sotiriou, T.P.; Zhou, S.Y. Black hole hair in generalized scalar-tensor gravity: An explicit example. *Phys. Rev. D* **2014**, *90*, 124063.
84. Brans, C.H. Mach's Principle and a Relativistic Theory of Gravitation. II. *Phys. Rev.* **1962**, *125*, 2194–2201.
85. Bhadra, A.; Nandi, K.K. On the equivalence of the Buchdahl and the Janis-Newman-Winnicour solutions. *Int. J. Mod. Phys. A* **2001**, *16*, 4543–4545.
86. Bhadra, A.; Sarkar, K. On static spherically symmetric solutions of the vacuum Brans–Dicke theory. *Gen. Relativ. Gravitation* **2005**, *37*, 2189–2199.
87. Wyman, M. Static Spherically Symmetric Scalar Fields in General Relativity. *Phys. Rev. D* **1981**, *24*, 839–841.

88. Buchdahl, H.A. Reciprocal Static Metrics and Scalar Fields in the General Theory of Relativity. *Phys. Rev.* **1959**, *115*, 1325–1328.

89. Agnese, A.G.; La Camera, M. Wormholes in the Brans–Dicke theory of gravitation. *Phys. Rev. D* **1995**, *51*, 2011–2013.

90. Faraoni, V.; Hammad, F.; Belknap-Keet, S.D. Revisiting the Brans solutions of scalar-tensor gravity. *Phys. Rev. D* **2016**, *94*, 104019.

91. Bhattacharya, S.; Dialektopoulos, K.F.; Romano, A.E.; Tomaras, T.N. Brans–Dicke Theory with $\Lambda > 0$: Black Holes and Large Scale Structures. *Phys. Rev. Lett.* **2015**, *115*, 181104.

92. Kim, H. Thermodynamics of black holes in Brans–Dicke gravity. *Nuovo Cim. B* **1997**, *112*, 329–338.

93. Faraoni, V. Black hole entropy in scalar-tensor and $f(R)$ gravity: An Overview. *Entropy* **2010**, *12*, 1246.

94. Kang, G. On black hole area in Brans–Dicke theory. *Phys. Rev. D* **1996**, *54*, 7483–7489.

95. Zaslavskii, O.B. Thermodynamics of black holes with an infinite effective area of a horizon. *Class. Quantum Gravity* **2002**, *19*, 3783–3798.

96. Nandi, K.K.; Bhadra, A.; Alsing, P.M.; Nayak, T.B. Tidal forces in cold black hole space-times. *Int. J. Mod. Phys. D* **2001**, *10*, 529–538.

97. Bronnikov, K.A.; Clement, G.; Constantinidis, C.P.; Fabris, J.C. Cold scalar tensor black holes: Causal structure, geodesics, stability. *Gravity Cosmol.* **1998**, *4*, 128–138.

98. Morris, M.S.; Thorne, K.S. Wormholes in space-time and their use for interstellar travel: A tool for teaching general relativity. *Am. J. Phys.* **1988**, *56*, 395–412.

99. Heydarzade, Y.; Riazi, N.; Moradpour, H. Phantom Wormhole Solutions in a Generic Cosmological Constant Background. *Can. J. Phys.* **2015**, *93*, 1523–1531.

100. Lemos, J.P.S.; Lobo, F.S.N.; Quinet de Oliveira, S. Morris-Thorne wormholes with a cosmological constant. *Phys. Rev. D* **2003**, *68*, 064004.

101. Narahara, K.; Furihata, Y.; Sato, K. Traversable wormhole in the expanding universe. *Phys. Lett. B* **1994**, *336*, 319–323.

102. Riazi, N.; Esfahani, B.N. Time dependent wormholes in an expanding universe dominated by traceless matter. *Astrophys. Space Sci.* **2000**, *271*, 237–243.

103. Nojiri, S.; Odintsov, S.D. Quantum de Sitter cosmology and phantom matter. *Phys. Lett. B* **2003**, *562*, 147–152.

104. Carroll, S.M.; Hoffman, M.; Trodden, M. Can the dark energy equation-of-state parameter w be less than −1? *Phys. Rev. D* **2003**, *68*, 023509.

105. Scheel, M.; Shapiro, S.; Teukolsky, S. Collapse to black holes in Brans–Dicke theory. II. Comparison with general relativity. *Phys. Rev. D* **1995**, *51*, 4236–4249.

106. Uehara, K.; Kim, C.W. Brans-dicke Cosmology With the Cosmological Constant. *Phys. Rev. D* **1982**, *26*, 2575.

107. Riazi, N.; Ahmadi-Azar, E. A Class of Exact Cosmological Solutions of Brans–Dicke Theory with Cosmological Constant. *Astrophys. Space Sci.* **1995**, *226*, 1–15.

108. Pimentel, L.O. Exact cosmological solutions in the scalar-tensor theory with cosmological constant. *Astrophys. Space Sci.* **1985**, *112*, 175–183.

109. Ram, S.; Singh, C.P. Early cosmological models with bulk viscosity in Brans–Dicke theory. *Astrophys. Space Sci.* **1997**, *254*, 143–150.

110. Pandey, S.N. Brans–Dicke cosmology with non-vanishing cosmological constant and non-zero curvature. *Astrophys. Space Sci.* **2001**, *277*, 403–408.

111. Romero, C.; Barros, A. Brans–Dicke vacuum solutions and the cosmological constant: A Qualitative analysis. *Gen. Relativ. Gravitation* **1993**, *25*, 491–502.

112. Romero, C.; Barros, A. Brans–Dicke cosmology and the cosmological constant: the spectrum of vacuum solutions. *Astrophys. Space Sci.* **1992**, *192*, 263–274.

113. Cerveró, J.M.; Estévez, P.G. General solutions for a cosmological Robertsonwalker metric in the Brans–Dicke theory. *Gen. Relativ. Gravitation* **1983**, *15*, 351–356.

114. Bhattacharya, S.; Dialektopoulos, K.F.; Romano, A.E.; Skordis, C.; Tomaras, T.N. The maximum sizes of large scale structures in alternative theories of gravity. *J. Cosmol. Astropart. Phys.* **2017**, *1707*, 018.

115. Xiao, X.G.; Zhu, J.Y. Wormhole solution in vacuum Brans–Dicke theory with cosmological constant. *Chin. Phys. Lett.* **1996**, *13*, 405–408.

116. Hrycyna, O.; Szydłowski, M. Dynamical complexity of the Brans–Dicke cosmology. *J. Cosmol. Astropart. Phys.* **2013**, *1312*, 016.

117. Tretyakova, D.A.; Latosh, B.N.; Alexeyev, S.O. Wormholes and naked singularities in Brans–Dicke cosmology. *Class. Quantum Gravity* **2015**, *32*, 185002.

118. Agnese, A.G.; La Camera, M. Schwarzschild metrics and quasiuniverses. *Found. Phys. Lett.* **2001**, *14*, 581–587.

119. Alexeyev, S.O.; Rannu, K.A.; Gareeva, D.V. Possible observation sequences of Brans–Dicke wormholes. *J. Exp. Theor. Phys.* **2011**, *113*, 628–636.

120. Horndeski, G.W. Second-order scalar-tensor field equations in a four-dimensional space. *Int. J. Theor. Phys.* **1974**, *10*, 363–384.

121. Nicolis, A.; Rattazzi, R.; Trincherini, E. The Galileon as a local modification of gravity. *Phys. Rev. D* **2009**, *79*, 064036.

122. Deffayet, C.; Esposito-Farèse, G.; Vikman, A. Covariant Galileon. *Phys. Rev. D* **2009**, *79*, 084003.

123. Kobayashi, T.; Yamaguchi, M.; Yokoyama, J. Generalized G-inflation: Inflation with the most general second-order field equations. *Prog. Theor. Phys.* **2011**, *126*, 511–529.

124. Deffayet, C.; Deser, S.; Esposito-Farèse, G. Generalized Galileons: All scalar models whose curved background extensions maintain second-order field equations and stress-tensors. *Phys. Rev. D* **2009**, *80*, 064015.

125. Kobayashi, T.; Tanahashi, N. Exact black hole solutions in shift symmetric scalar–tensor theories. *Prog. Theor. Exp. Phys.* **2014**, *2014*, 073E02,

126. Starobinsky, A.A.; Sushkov, S.V.; Volkov, M.S. The screening Horndeski cosmologies. *J. Cosmol. Astropart. Phys.* **2016**, *1606*, 007.

127. Charmousis, C.; Iosifidis, D. Self tuning scalar tensor black holes. *J. Phys. Conf. Ser.* **2015**, *600*, 012003.

128. Rinaldi, M. Black holes with non-minimal derivative coupling. *Phys. Rev. D* **2012**, *86*, 084048.

129. Babichev, E.; Charmousis, C. Dressing a black hole with a time-dependent Galileon. *J. High Energy Phys.* **2014**, *2014*, 106.

130. Babichev, E.; Charmousis, C.; Hassaine, M. Charged Galileon black holes. *J. Cosmol. Astropart. Phys.* **2015**, *2015*, 031.

131. Anabalon, A.; Cisterna, A.; Oliva, J. Asymptotically locally AdS and flat black holes in Horndeski theory. *Phys. Rev. D* **2014**, *89*, 084050.

132. Minamitsuji, M. Solutions in the scalar-tensor theory with nonminimal derivative coupling. *Phys. Rev. D* **2014**, *89*, 064017.

133. Babichev, E.; Charmousis, C.; Lehébel, A. Black holes and stars in Horndeski theory. *Class. Quantum Gravity* **2016**, *33*, 154002.

134. Charmousis, C.; Kolyvaris, T.; Papantonopoulos, E.; Tsoukalas, M. Black Holes in Bi-scalar Extensions of Horndeski Theories. *J. High Energy Phys.* **2014**, *2014*, 85.

135. Tretyakova, D.A.; Takahashi, K. Stable black holes in shift-symmetric Horndeski theories. *Class. Quantum Gravity* **2017**, *34*, 175007.

136. Miao, Y.G.; Xu, Z.M. Thermodynamics of Horndeski black holes with non-minimal derivative coupling. *Eur. Phys. J. C* **2016**, *76*, 638.

137. Feng, X.H.; Liu, H.S.; Lü, H.; Pope, C.N. Thermodynamics of Charged Black Holes in Einstein-Horndeski-Maxwell Theory. *Phys. Rev. D* **2016**, *93*, 044030.

138. Feng, X.H.; Liu, H.S.; Lü, H.; Pope, C.N. Black Hole Entropy and Viscosity Bound in Horndeski Gravity. *J. High Energy Phys.* **2015**, *2015*, 176.

139. Sushkov, S.V. Horndeski Wormholes. *Fundam. Theor. Phys.* **2017**, *189*, 89–109.

140. Korolev, R.V.; Sushkov, S.V. Exact wormhole solutions with nonminimal kinetic coupling. *Phys. Rev. D* **2014**, *90*, 124025.

141. Kobayashi, T.; Yamaguchi, M.; Yokoyama, J. G-inflation: Inflation driven by the Galileon field. *Phys. Rev. Lett.* **2010**, *105*, 231302.

142. Kanti, P.; Gannouji, R.; Dadhich, N. Gauss–Bonnet Inflation. *Phys. Rev. D* **2015**, *92*, 041302.

143. Fomin, I.V.; Chervon, S.V. Exact Inflation in Einstein-Gauss–Bonnet Gravity. In Proceedings of the 5th Ulyanovsk International School Seminar: Problems of Theoretical and Observational Cosmology (UISS 2016), Ulyanovsk, Russia, 19–30 September 2016.

144. Babichev, E.; Charmousis, C.; Lehébel, A. Asymptotically flat black holes in Horndeski theory and beyond. *J. Cosmol. Astropart. Phys.* **2017**, *2017*, 27.

145. Babichev, E.; Charmousis, C.; Lehébel, A.; Moskalets, T. Black holes in a cubic Galileon universe. *J. Cosmol. Astropart. Phys.* **2016**, *2016*, 011.

146. Moffat, J.W. Scalar-tensor-vector gravity theory. *J. Cosmol. Astropart. Phys.* **2006**, *2006*, 4.

147. Moffat, J.W.; Rahvar, S. The MOG weak field approximation and observational test of galaxy rotation curves. *Mon. Not. R. Astron. Soc.* **2013**, *436*, 1439–1451.

148. Iorio, L. Putting Yukawa-like Modified Gravity (MOG) on the test in the Solar System. *Schol. Res. Exch.* **2008**, *2008*, 238385.

149. Iorio, L. Constraints to a Yukawa gravitational potential from laser data to LAGEOS satellites. *Phys. Lett. A* **2002**, *298*, 315–318.

150. Iorio, L. Constraints on the range lambda of Yukawa-like modifications to the Newtonian inverse-square law of gravitation from Solar System planetary motions. *J. High Energy Phys.* **2007**, *2007*, 041.

151. Moffat, J.W.; Toth, V.T. Cosmological observations in a modified theory of gravity (MOG). *Galaxies* **2013**, *1*, 65–82.

152. Moffat, J.W.; Rahvar, S.; Toth, V.T. Applying MOG to lensing: Einstein rings, Abell 520 and the Bullet Cluster. *arXiv* **2012**, arXiv:1204.2985.

153. Moffat, J.W.; Toth, V.T. Testing modified gravity with motion of satellites around galaxies. *arXiv* **2007**, arXiv:0708.1264.

154. Moffat, J.W.; Toth, V.T. The Bending of light and lensing in modified gravity. *Mon. Not. R. Astron. Soc.* **2009**, *397*, 1885–1992.

155. Moffat, J.W. Time delay predictions in a modified gravity theory. *Class. Quantum Gravity* **2006**, *23*, 6767–6772.

156. Moffat, J.W.; Toth, V.T. Fundamental parameter-free solutions in modified gravity. *Class. Quantum Gravity* **2009**, *26*, 085002.

157. Moffat, J.W. A Modified Gravity and its Consequences for the Solar System, Astrophysics and Cosmology. *Int. J. Mod. Phys. D* **2008**, *16*, 2075–2090.

158. Moffat, J.W. Black Holes in Modified Gravity (MOG). *Eur. Phys. J. C* **2015**, *75*, 175.

159. Mureika, J.R.; Moffat, J.W.; Faizal, M. Black hole thermodynamics in MOdified Gravity (MOG). *Phys. Lett B.* **2016**, *757*, 528–536.

160. Ayon-Beato, E.; Garcia, A. Regular black hole in general relativity coupled to nonlinear electrodynamics. *Phys. Rev. Lett.* **1998**, *80*, 5056–5059.

161. Ayon-Beato, E.; Garcia, A. Nonsingular charged black hole solution for nonlinear source. *Gen. Relativ. Gravitation* **1999**, *31*, 629–633.

162. Skordis, C. The Tensor-Vector-Scalar theory and its cosmology. *Class. Quantum Gravity* **2009**, *26*, 143001.

163. Bekenstein, J.D. Relativistic gravitation theory for the MOND paradigm. *Phys. Rev. D* **2004**, *70*, 083509.

164. Zhao, H. Constraining TeVeS Gravity as Effective Dark Matter and Dark Energy. *Int. J. Mod. Phys. D* **2008**, *16*, 2055–2063.

165. Armendariz-Picon, C.; Damour, T.; Mukhanov, V.F. *k*-inflation. *Phys. Lett. B* **1999**, *458*, 209–218.

166. Armendariz-Picon, C.; Mukhanov, V.F.; Steinhardt, P.J. A Dynamical solution to the problem of a small cosmological constant and late time cosmic acceleration. *Phys. Rev. Lett.* **2000**, *85*, 4438–4441.

167. Giannios, D. Spherically symmetric, static spacetimes in TeVeS. *Phys. Rev. D* **2005**, *71*, 103511.

168. Sagi, E.; Bekenstein, J.D. Black holes in the TeVeS theory of gravity and their thermodynamics. *Phys. Rev. D* **2008**, *77*, 024010.

169. Skordis, C.; Zlosnik, T. The Geometry Of Modified Newtonian Dynamics. *Phys. Rev. D* **2012**, *85*, 044044.

170. Blas, D.; Lim, E. Phenomenology of theories of gravity without Lorentz invariance: the preferred frame case. *Int. J. Mod. Phys. D* **2015**, *23*, 1443009.

171. Skenderis, K. *Black holes in string theory*; Springer: Binlin, Germany, 2000; pp. 325–364.

172. Mathur, S.D. The Fuzzball proposal for black holes: An Elementary review. *Fortsch. Phys.* **2005**, *53*, 793–827.

173. Susskind, L. *Some speculations about black hole entropy in string theory*; World Scientific: Singapore, Singapore, 1998; pp. 118–131.

174. Russo, J.G.; Susskind, L. Asymptotic level density in heterotic string theory and rotating black holes. *Nucl. Phys. B* **1995**, *437*, 611–626.

175. Skenderis, K.; Taylor, M. The fuzzball proposal for black holes. *Phys. Rep.* **2008**, *467*, 117–171.

176. Donoghue, J.F. General relativity as an effective field theory: The leading quantum corrections. *Phys. Rev. D* **1994**, *50*, 3874–3888.

177. Burgess, C.P. Quantum gravity in everyday life: General relativity as an effective field theory. *Living Rev. Relativ.* **2004**, *7*, 5–56.

178. Horowitz, G.T.; Maldacena, J.M.; Strominger, A. Nonextremal black hole microstates and U duality. *Phys. Lett. B* **1996**, *383*, 151–159.

179. Youm, D. Black holes and solitons in string theory. *Phys. Rep.* **1999**, *316*, 1–232.

180. Cvetic, M. Properties of black holes in toroidally compactified string theory. *Nucl. Phys. B Proc. Suppl.* **1997**, *56*, 1–10.