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Geometric Aspects and Some Uses of Deformed Models of Thermostatistics

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Abstract: We consider diverse deformed Bose gas models (DBGMs) focusing on distributions and correlations of any order, and also on deformed thermodynamics. For so-called μ-deformed Bose gas model (μ-DBGM), main thermodynamic aspects are treated: total number of particles, deformed partition function, etc. Using a geometric approach, we confirm the existence of critical behavior—Bose-like condensation; we find the critical temperature $T_c^{(\mu)}$ depending on μ so that $T_c^{(\mu)} > T_c^{(\mathrm{Bose})}$ for $\mu > 0$. This fact and other advantages of μ-DBGM relative to the usual Bose gas, e.g., stronger effective inter-particle attraction (controlled by the parameter μ), allow us to consider the condensate in μ-DBGM as a candidate for modeling dark matter. As another, quite successful application we discuss the usage of the two-parameter $(\tilde{\mu}, q)$-deformed BGM for effective description of the peculiar (non-Bose like) behavior of two-pion correlations observed in the STAR experiment at RHIC (Brookhaven). Herein, we point out the transparent role of the two deformation parameters $\tilde{\mu}$ and q as being responsible for compositeness and (effective account of) interactions of pions, respectively.

Keywords: deformed Bose gas models; thermodynamic geometry; critical temperature; condensate; dark matter; higher-order distributions; correlation function intercept; two-pion correlations; RHIC/STAR experiment

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1. Introduction

Deformed analogs of the Bose gas model (BGM) find, either through their thermodynamics or by means of distributions and correlations, miscellaneous applications in many branches of modern quantum theory—from condensed matter physics to quantum field theory, black hole physics and cosmology. In this paper, we focus on theoretical, geometric and applied aspects of DBGMs. Though the deformed models are under study from early nineties, see, e.g., [1–7], their investigation still continues, and a number of diverse DBGMs' applications in several directions of physics were found. These include: description of the properties of phonon spectrum in 4 He [8]; physics of excitons [9]; usage of nonstandard statistics and DBGMs in the physics of black holes and dark matter [10–14]; attempts to explain the observed non-Bose features of pions shown by $\pi\,\pi$–correlations in the RHIC/STAR experiments (see, e.g., [15]). The latter line of application was initiated in [16–18], and the approach was further developed in [19–23], with main attention on explicit two- and n-particle distributions and respective correlations (intercepts).

Another aspect is the appearance of some new (generalized) or not often used special functions, which are basic for both deformed thermodynamics in DBGMs including geometric approaches, and also for treating quantum statistical quantities (distributions, correlations). This concerns the μ-DBGM, $(\tilde{\mu}, q)$-DBGM and other models (see below).

Usually, deformed models or DBGMs are based on respective deformed oscillator (DO) models. Best known among these are the Arik–Coon (AC) and Biedenharn–Macfarlane (BM)

q-oscillators [24–26]. The 2-parameter or p, q-deformed family of deformed oscillators [27] contains the AC type and BM type q-DOs as special cases. A plenty of other one-parameter DOs can be obtained [28] from the p, q-deformed family. In a variety of DOs, an important class constitute the polynomially deformed ones [29] with polynomial—in the excitation number operator N—function of deformation (nonlinearity function). Among the so-called quasi-Fibonacci oscillators [30], we also find the μ-deformed oscillator [31].

This paper is motivated by a number of already existing important results on various DBGMs, along with their successful applications for description of diverse complex phenomena from the realm of nonlinear quantum physics. It should be stressed that the involved, in each particular DBGM, one or two (or more) deformation parameter(s) may be used as the driving or control parameter(s) that obviously makes the deformed models much more rich, flexible and efficient in comparison with their non-deformed prototypes. Moreover, physical sense of those parameters is diversified and depends on particular DBGMs and concrete applications.

Our main purpose is to attract more attention to some most recent advances in this promising direction of modern quantum physics. More specifically, our main goals include (i) presenting exact analytical expressions for 1st, 2nd & higher order distributions and also nth order ($n \geq 2$) correlation intercepts in a number of DBGMs; (ii) study of thermodynamic geometry of μ-BGM in order to confirm Bose-like condensation and discuss the arguments stemming from certain advantages of the μ-deformed model over the Bose-condensate based approaches, which justify the use of μ-BGM for modeling main properties of dark matter (DM) surrounding dwarf galaxies; (iii) application of certain two-parameter $\tilde{\mu}, q$-DBGM, implying effective account of both interaction and compositeness of particles, for description of non-Bose like properties of 2-pion correlations in relativistic heavy ion collisions. We have to stress that the experimental data on $\pi^{\pm}\pi^{\pm}$-correlations [15] clearly show such key *regularities* as (A) monotonic growth of the two-particle correlation intercept $\lambda^{(2)}$ vs. transverse mass m_T, along with (B) concavity upwards, and (C) saturation by a constant < 1, contrary to $\lambda^{(2)} = 1$ valid for true bosons. We will demonstrate that the two-parameter $\tilde{\mu}, q$-DBGM successfully accounts for these non-Bose features of $\pi\,\pi$-correlations.

2. Deformed Oscillators (DOs) and the Structure Function of Deformation

For each DO, we retain two relations valid for the usual harmonic oscillator: $[N, a^\dagger] = a^\dagger$, $[N, a] = -a$. The remaining basic commutation relation for AC type resp. BM type DOs are deformed as [24–26]:

$$[a, a^\dagger]_q \equiv aa^\dagger - qa^\dagger a = 1 \qquad \text{respectively,} \qquad [a, a^\dagger]_q = q^N.$$

These relations can be rewritten using the "structure function of deformation" (DSF) defined as $a^\dagger a \equiv \phi(N)$ (see [32]). Then, deformed commutation relations can be presented in the form

$$[a, a^\dagger] = \phi^{\mathrm{AC}}(N+1) - \phi^{\mathrm{AC}}(N) \qquad \text{respectively,} \qquad [a, a^\dagger] = \phi^{\mathrm{BM}}(N+1) - \phi^{\mathrm{BM}}(N),$$

where the DSFs are given as $\phi^{\mathrm{AC}}(N) \equiv \dfrac{1-q^N}{1-q}$, respectively, $\phi^{\mathrm{BM}}(N) \equiv \dfrac{q^N - q^{-N}}{q - q^{-1}}$.

In the ϕ-analog of Fock space, $a|0\rangle = 0$, $|n\rangle = \dfrac{(a^\dagger)^n}{\sqrt{\phi(N)!}}|0\rangle$, $\phi(N)|n\rangle = \phi(n)|n\rangle$, and $a|n\rangle = \sqrt{\phi(N)}\,|n-1\rangle$, $a^\dagger|n\rangle = \sqrt{\phi(N+1)}\,|n+1\rangle$, with $\phi(N)! = \phi(N)\phi(N-1)\cdots\phi(1)$, $\phi(0)! = 0$. The DO models often possess unusual properties, e.g., various energy level degeneracies, nontrivial recurrent relations for energy spectra, etc. Nonstandard features of DOs motivate their application in diverse fields of quantum physics.

3. Two and n-Particle Correlations in Some Deformed Bose Gas Models

Ideal q, p-Bose gas: intercepts of correlation functions [16,19].

Deformed distributions and correlation functions are calculated using standard thermal averages, with the Hamiltonian linear in number operator of *deformed* bosons:

$$\langle A\rangle = \frac{\mathrm{Sp}(A\cdot e^{-\beta H})}{\mathrm{Sp}(e^{-\beta H})}, \quad H=\sum_i \omega_i N_i^{(qp)}, \quad \omega_i=\sqrt{m^2+\mathbf{k}_i^2} \tag{1}$$

(the ω_i is chosen for subsequent goal). From this formula, turning to creation/annihilation operators of q,p-deformed bosons, the one-particle distribution of q,p-Bose gas model can be obtained [7,16]:

$$n_i(q,p) = \langle A_i^\dagger A_i\rangle = \frac{e^{\beta\omega_i}-1}{(e^{\beta\omega_i}-p)(e^{\beta\omega_i}-q)}. \tag{2}$$

The two-particle correlation function of bosons is defined (see e.g., [33]) as

$$C^{(2)}(\mathbf{k}_1,\mathbf{k}_2) = \gamma\frac{P_2(\mathbf{k}_1,\mathbf{k}_2)}{P_1(\mathbf{k}_1)P_1(\mathbf{k}_2)} \tag{3}$$

in terms of one- and two-particle distributions $P_1(\mathbf{k})$ and $P_2(\mathbf{k}_1,\mathbf{k}_2)$. The multiplier $\gamma=\frac{\langle N\rangle^2}{\langle N(N-1)\rangle}$ is usually close to unity (from now on we put $\gamma=1$). Correlation function $C^{(2)}(\mathbf{k}_1,\mathbf{k}_2)$ rewritten as

$$C^{(2)}(\mathbf{q},\mathbf{K}) \overset{\mathbf{k}_1=\mathbf{k}_2}{\longrightarrow} C^{(2)}(\mathbf{q}=0,\mathbf{K}) = 1+\lambda^{(2)}(m,\mathbf{K}), \qquad \text{with} \qquad \mathbf{q}=\mathbf{k}_1-\mathbf{k}_2,\ \mathbf{K}=(\mathbf{k}_1+\mathbf{k}_2)/2, \tag{4}$$

involves $\lambda^{(2)}$—the *intercept* of two-particle correlation function (characterisic value at fixed $\mathbf{K}$). The (two- and) n-particle correlation function intercepts are calculated by the formulas [33]

$$\lambda_{\mathbf{k}}^{(n)} = \frac{\langle a_{\mathbf{k}}^{\dagger n} a_{\mathbf{k}}^n\rangle}{\langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}}\rangle^n} - 1, \quad n\geq 2 \qquad (\text{for Bose gas}, \lambda_{\mathbf{k}}^{(2)}\equiv 1 \ \text{ and } \ \lambda_{\mathbf{k}}^{(n)} = n!-1). \tag{5}$$

The intercepts of two-particle correlation functions and their asymptotics for p,q-Bose gas model, its particular Arik–Coon (AC), Biedenharn–Macfarlane (BM) and the $p=q$ or Tamm-Dankoff (TD) cases, are given in Table 1. Note the asymptotics for AC type q-DBGM is exactly q (giving physical sense for q).

Table 1. Two-particle correlation intercept and asymptotics: exponentially deformed models.

Case	**Intercept** $\boldsymbol{\lambda_{\mathbf{k}}^{(2)} = \frac{\langle a_{\mathbf{k}}^{\dagger 2} a_{\mathbf{k}}^2\rangle}{\langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}}\rangle^2} - 1}$	$\boldsymbol{\lambda_{\mathbf{k}}^{(2)}}$ **Asympt. at** $\boldsymbol{\beta\omega\to\infty}$
AC	$\lambda_{\mathrm{AC}}^{(2)} = q\frac{e^{\beta\omega}-1}{e^{\beta\omega}-q^2}$	$\lambda_{\mathrm{AC},\infty}^{(2)}=q$
BM	$\lambda_{\mathrm{BM}}^{(2)} = \frac{2\cos\theta(e^{2\beta\omega}-2\cos\theta\, e^{\beta\omega}+1)^2}{(e^{\beta\omega}-1)^2(e^{2\beta\omega}-2\cos(2\theta)e^{\beta\omega}+1)}-1,\quad q=e^{i\theta}$	$\lambda_{\mathrm{BM},\infty}^{(2)}=2\cos\theta-1$
q,p	$\lambda_{q,p}^{(2)} = \frac{(p+q)(e^{\beta\omega}-p)^2(e^{\beta\omega}-q)^2}{(e^{\beta\omega}-1)(e^{\beta\omega}-q^2)(e^{\beta\omega}-pq)(e^{\beta\omega}-p^2)}-1$	$\lambda_{q,p,\infty}^{(2)}=(p+q)-1$
TD	$\lambda_{\mathrm{TD}}^{(2)} = \frac{2q(e^{\beta\omega}-q)^4}{(e^{\beta\omega}-1)(e^{\beta\omega}-q^2)^3}-1$	$\lambda_{\mathrm{TD},\infty}^{(2)}=2q-1$

General n-th order correlation function intercept (5) for (p,q)-Bose gas model derived in [19] is

$$\lambda_{q,p}^{(n)} = [\![n]\!]_{qp}!\frac{(e^{\beta\omega}-p)^n(e^{\beta\omega}-q)^n}{(e^{\beta\omega}-1)^{n-1}\prod_{k=0}^n(e^{\beta\omega}-q^{n-k}p^k)}-1, \quad [\![n]\!]_{qp}\equiv\frac{q^n-p^n}{q-p}, \quad [\![n]\!]_{qp}!\equiv\prod_{j=1}^n [\![j]\!]_{qp}. \tag{6}$$

Asymptotics ($\beta\omega \to \infty$) of the intercept is given solely by the deformation parameters q and p:

$$\lambda^{(n)}_{q,p,\infty} = -1 + [\![n]\!]_{qp}! = -1 + \prod_{k=1}^{n-1} \Big(\sum_{r=0}^{k} q^r p^{k-r} \Big). \tag{7}$$

If $p \to 1$ and $q \to 1$, we recover familiar Bose case, with asymptotics given by usual factorial.

4. Distributions and Correlations in μ-Bose Gas Model: Exact Results

The model first proposed in [34] employs the DSF $\varphi_\mu(N) \equiv [N]_\mu = \frac{N}{1+\mu N}$ of μ-oscillator [31]. The deformed one-particle momentum distribution was calculated in [35]:

$$\langle a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \rangle = \langle \varphi_\mu(N_{\mathbf{k}}) \rangle = \mu^{-1} - \mu^{-2}(1-e^{-\beta\omega})\Phi(e^{-\beta\omega}, 1, \mu^{-1}), \quad \mu > 0, \tag{8}$$

along with r-particle ones $\langle (a^\dagger_{\mathbf{k}})^r (a_{\mathbf{k}})^r \rangle$ at same momenta; $\Phi(z, s, \alpha) \equiv \sum_{n=0}^{\infty} z^n/(n+\alpha)^s$ is the Lerch transcendent [36].

Exact formulas for r-th order intercepts can also be found [35]:

$$\lambda^{(r)}_\mu(\mathbf{K}) = \frac{1+\mu^{-1}(1-e^{-\beta\omega}) \sum_{l=0}^{r-1} A^{(r)}_l(\mu)\Phi(e^{-\beta\omega}, 1, \mu^{-1}-l)}{\big(1-\mu^{-1}(1-e^{-\beta\omega})\Phi(e^{-\beta\omega}, 1, \mu^{-1})\big)^r} - 1, \quad r = 2, 3, \ldots, \quad \mu > 0, \tag{9}$$

coefficients $A^{(r)}_l(\mu)$ being polynomials in μ^{-1} of order $r-1$. In the $|\mathbf{K}| \to \infty$ asymptotics

$$\lambda^{(r)}_\mu(\mathbf{K}, T, m) \underset{|\mathbf{K}|\to\infty}{\longrightarrow} \lambda^{(r)}_{\mu,\infty} = (1+\mu)^r [r]_\mu! - 1, \quad [r]_\mu! \equiv [r]_\mu [r-1]_\mu \ldots [1]_\mu. \tag{10}$$

5. Thermodynamics of μ-Bose Gas Model

The μ-BGM thermodynamics was first studied in [37]. In this and other deformed analogs of BGM, quantum statistical interaction modifies [38,39]. The deformation may also be viewed as absorbing [40] certain inter-bosonic interaction, present in the initially non-deformed system.

1. *Total number of particles*. To compute this basic quantity, we isolate the ground state, change summation to integration, and then integrate over 3-momenta in spherical coordinates:

$$N^{(\mu)} = \frac{4\pi V}{(2\pi\hbar^2)^3} \sum_{n=1}^{\infty} \frac{[n]_\mu z^n}{n} \int_0^\infty p^2 e^{-\frac{\beta p^2}{2m}} dp + \sum_{n=1}^{\infty} \frac{[n]_\mu z^n}{n} \tag{11}$$

(recall that $[n]_\mu = \frac{n}{1+\mu n}$). Integration leads to the μ-deformed total number of particles [37]

$$N^{(\mu)} = \frac{V}{\lambda^3} g^{(\mu)}_{3/2}(z) + g^{(\mu)}_0(z), \qquad g^{(\mu)}_0(z) = N^{(\mu)}_0, \tag{12}$$

where $\lambda = \sqrt{\dfrac{2\pi\hbar^2}{mkT}}$ is thermal wavelength, and the appearing μ-polylogarithm is defined as

$$g^{(\mu)}_l(z) = \sum_{n=1}^{\infty} \frac{[n]_\mu}{n^{l+1}} z^n. \tag{13}$$

If $\mu \to 0$, we recover usual $g_l(z)$ function (polylogarithm). Assuming that $\mu \geq 0$, the convergence properties are retained; like, for the usual g-function, there should be $|z| < 1$. It is convenient to rewrite the expression for total number of particles as

$$\frac{1}{v} = \frac{1}{\lambda^3} g^{(\mu)}_{3/2} + \frac{N^{(\mu)}_0}{V}, \qquad v \equiv \frac{V}{N^{(\mu)}}. \tag{14}$$

2. *Deformed grand partition function*. In the μ-BGM, all the thermodynamical functions including partition function are μ-dependent, though obeying the relations between thermodynamical functions similar to those of Bose gas thermodynamics. The μ-partition function plays a basic role as it allows to derive other thermodynamical functions and relations.

Deformed partition function $\ln Z^{(\mu)}$ is found [37] by inverting the relation $N^{(\mu)} = z\frac{d}{dz}\ln Z^{(\mu)}$:

$$\ln Z^{(\mu)} = \left(z\frac{d}{dz}\right)^{-1} N^{(\mu)} = \frac{V}{\lambda^3}\sum_{n=1}^{\infty}\frac{[n]_\mu}{n^{5/2}}(n)^{-1}z^n + \sum_{n=1}^{\infty}\frac{[n]_\mu}{n}(n)^{-1}z^n = \frac{V}{\lambda^3}g^{(\mu)}_{5/2} + g^{(\mu)}_1 . \tag{15}$$

To apply $\left(z\frac{d}{dz}\right)^{-1}$, we used the rule: $f\left(z\frac{d}{dz}\right)z^k = f(k)z^k$ (which is easy to verify).

3. *Critical temperature of condensation*. Starting with the equality $\frac{N_0^{(\mu)}}{V} = \frac{\lambda^3}{v} - g^{(\mu)}_{3/2}(z)$ (cf. (14)) and equating its r.h.s. to zero, one can find the critical temperature $T_c^{(\mu)}$ of μ-BGM [37]

$$T_c^{(\mu)} = \frac{2\pi\hbar^2/mk}{\left(v g^{(\mu)}_{3/2}(1)\right)^{2/3}}, \qquad \text{with} \qquad \frac{T_c^{(\mu)}}{T_c} = \left(\frac{2.61}{g^{(\mu)}_{3/2}(1)}\right)^{2/3} \tag{16}$$

being the ratio of $T_c^{(\mu)}$ to the usual Bose critical temperature T_c. It is easy to see that $g^{(\mu)}_{3/2}(1) < g^{(0)}_{3/2}(1)$ at $\mu > 0$. Hence, the ratio in (16), like in p,q-Bose gas model [38], shows the property: $T_c^{(\mu)}$ is higher for greater strength of deformation (here given by μ). If $\mu \to 0$ (no-deformation limit), we have $T_c^{(\mu)}/T_c = 1$, i.e., the μ-critical temperature reduces to the usual T_c.

6. Geometric Approach to μ-Bose Gas Model Thermodynamics

Thermodynamic geometry is useful for the study of system's thermodynamics: the curvature in $2d$ space with parameters β, $\gamma = -\mu\beta$ shows singularity at the phase transition point (see e.g., [41–43]. Thus, for the μ-BGM, we calculate its $2d$ scalar curvature. Metric components

$$G_{\beta\beta} = \frac{\partial^2 \ln Z}{\partial\beta^2} = -\left(\frac{\partial U}{\partial\beta}\right)_\gamma, \quad G_{\beta\gamma} = \frac{\partial^2 \ln Z}{\partial\beta\partial\gamma} = -\left(\frac{\partial N}{\partial\beta}\right)_\gamma, \quad G_{\gamma\gamma} = \frac{\partial^2 \ln Z}{\partial\gamma^2} = -\left(\frac{\partial N}{\partial\gamma}\right)_\beta \tag{17}$$

are calculated to yield the expressions [14]

$$G_{\beta\beta} = \frac{15}{4}\frac{V}{\lambda^3\beta^2}g^{(\mu)}_{\frac{5}{2}}(z), \qquad G_{\beta\gamma} = G_{\gamma\beta}\frac{3}{2}\frac{V}{\lambda^3\beta}g^{(\mu)}_{\frac{3}{2}}(z), \qquad G_{\gamma\gamma} = \frac{V}{\lambda^3}g^{(\mu)}_{\frac{1}{2}}(z) + g^{(\mu)}_{-1}(z), \tag{18}$$

$$det|G_{ij}| \equiv g = \frac{3}{4}\frac{V}{\lambda^3\beta^2}\left(5g^{(\mu)}_{\frac{5}{2}}(z)g^{(\mu)}_{-1}(z) + +\frac{V}{\lambda^3}\left(5g^{(\mu)}_{\frac{5}{2}}(z)\,g^{(\mu)}_{\frac{1}{2}}(z) - 3g^{(\mu)}_{\frac{3}{2}}(z)\,g^{(\mu)}_{\frac{3}{2}}(z)\right)\right). \tag{19}$$

Inverse metric is given as $G^{11} = G_{22}/g$, $G^{12} = -G_{12}/g$, $G^{22} = -G_{11}/g$. The Christoffel symbols are calculated as $\Gamma_{\alpha\beta\gamma} = \frac{1}{2}(\ln Z)_{,\alpha\beta\gamma}$. and the results involve $g^{(\mu)}_k(z) \equiv g^{(\mu)}_k$ from (13) with $k = \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}$ and -2. The Riemann tensor is found as $R_{\lambda\mu\nu\rho} \equiv g^{\kappa\tau}\left(\Gamma_{\kappa\lambda\rho}\Gamma_{\tau\mu\nu} - \Gamma_{\kappa\lambda\nu}\Gamma_{\tau\mu\rho}\right)$; scalar curvature (in $2d$) is $R = 2\,g^{-1}R_{1212}$. Calculation of R_{1212} and use of g^{-1} finally yield [14]

$$R = \frac{5}{2}\frac{\left[5g^\mu_{\frac{3}{2}}g^\mu_{\frac{3}{2}}g^\mu_{-1} - 7g^\mu_{\frac{5}{2}}g^\mu_{\frac{1}{2}}g^\mu_{-1} + 2g^\mu_{\frac{3}{2}}g^\mu_{\frac{5}{2}}g^\mu_{-2} + \frac{V}{\lambda^3}\left(2g^\mu_{\frac{3}{2}}g^\mu_{\frac{5}{2}}g^\mu_{-\frac{1}{2}} - 4g^\mu_{\frac{5}{2}}g^\mu_{\frac{1}{2}}g^\mu_{\frac{1}{2}} + 2g^\mu_{\frac{3}{2}}g^\mu_{\frac{3}{2}}g^\mu_{\frac{1}{2}}\right)\right]}{\left[5g^\mu_{\frac{5}{2}}g^\mu_{-1} + \frac{V}{\lambda^3}\left(5g^\mu_{\frac{5}{2}}g^\mu_{\frac{1}{2}} - 3g^\mu_{\frac{3}{2}}g^\mu_{\frac{3}{2}}\right)\right]}. \tag{20}$$

As seen, the whole thermodynamic geometry of μ-Bose gas involves μ-polylogarithms $g^{(\mu)}_l(z)$. Note that $g^{(\mu)}_l(z)$ is singular when $z \to 1$: for $l \leq 1$ in the case $\mu = 0$ and for $l \leq 0$ when $\mu \neq 0$.

The most interesting case of $V/\lambda^3 \gg 1$ implies neglect of the terms lacking the ratio V/λ^3. The plot of thermodynamic curvature $R(z)$ versus fugacity in isothermal process is shown in Figure 2 of Ref. [14]. As seen, the curvature $R(z)$ is singular at $z \to 1$. That is, the system undergoes phase transition of Bose-like condensation. Critical temperature $T_c^{(\mu)}$ of μ-BGM is given in Equation (16).

7. Application for Modeling Dark Matter

With the use of thermodynamic geometry, we have confirmed the presence of Bose-like condensation. This key property shows that μ-BGM may be used to model the properties of DM. In models of dark matter, the idea of using Bose–Einstein condensate (BEC) is quite popular, see e.g., [44–46], and the review [47] with Refs. therein. These models embody plausible features of cold dark matter, in some respects showing advantages, but they may also have their own difficulties such as e.g., the problem of (gravitational) collapse [48], or overestimated total mass of DM halos for dwarf galaxies [46]. This motivates researchers to explore other models that would be free of the weak points.

Precise nature of dark matter particles is unknown, so some authors adopt systems with nonstandard statistics to model the unusual objects—dark matter [12–14] or black holes [10,11]. It is worth examining other candidate models of nonstandard thermostatistics able to (effectively) describe main properties of dark matter—and to choose the appropriate one(s). The μ-BGM shows important merits, namely those discussed in [14]. Let us mention the main ones:

(1) In the case of μ-BGM, the quantum–statistics interaction between μ-bosons is attractive and, moreover, the deformation enhances it relative to usual bosons. This stems e.g., from the difference of the second virial coefficients $V_2^{(\mu)}$ of μ-BGM [37] and the Bose one $V_2^{(\mathrm{Bose})} = 2^{-5/2}$ that reads: $V_2^{(\mu)} - V_2^{(\mathrm{Bose})} = 2^{-5/2}\mu^2/(1+2\mu) > 0$. We see the enhanced quantum attraction, i.e., μ-bosons are even "more bosonic" than just bosons. This enforces the status of *strongly coupled* system of (quasi) particles composing dark matter, a property whose role was emphasized in [46]. Moreover, we have at our disposal the deformation parameter μ as the control parameter.

(2) Though unlimited growth of the parameter μ (expressing the strength of quantum-statistical attraction) could cause a collapse of the system under study, it is possible to find certain bound μ_0 for the set of values of μ, which prevents [14] negative pressure and allows avoiding the danger of collapse.

(3) Using dimensionless factor $f = \left(\frac{2\pi akT}{Gm^2}\right)^{1/2} \gg 1$ (with the s-wave scattering length a, $a < \lambda$, and gravit. constant G), we have the relations for total mass of DM halo and its "radius":

$$M^{(\mu)} = \frac{\pi}{6} m g_{3/2}^{(\mu)}(1) f^3, \qquad R = \frac{1}{2}\lambda f = \pi\left(\frac{\hbar^2 a}{Gm^3}\right)^{1/2}. \tag{21}$$

Since $g_{3/2}^{(\mu)}(1) < g_{3/2}^{(0)}(1)$, see text below (16), we hope for better agreement of μ-BGM predictions with experimental data (see e.g., [46]), in contrast to the overestimated result for the total mass $M^{BEC} \equiv M^{(0)}$ in a pure Bose case.

These and some other merits of the μ-BGM (e.g., those concerning the behavior of entropy or critical temperature of condensation as functions of μ) make it possible [14] to propose and further develop the use of μ-BGM in the role of viable model of dark matter.

8. Composite Bosons (Quasi-Bosons) as Deformed Oscillators

Quasi-bosons composed from two more elementary particles (e.g., two fermions) have integer total spin like for elementary bosons. However, their statistical behavior differs from bosonic ones and depends on boundness (entanglement) of the constituents (see e.g., [49]). This resembles DOs [32] for which the deviation from usual bosonic oscillator sits in (the deformation parameters involved in) their deformation structure function.

As known [50], the algebra of composite boson ("quasiboson") operators, built from constituents' fermionic operators a, $a^\dagger$, b, $b^\dagger$, is modified with respect to bosonic algebra:

$$\begin{array}{ll} A^\dagger_\alpha = \sum_{\mu\nu}\Phi^{\mu\nu}_\alpha a^\dagger_\mu b^\dagger_\nu, \quad A_\alpha = \sum_{\mu\nu}\overline{\Phi^{\mu\nu}_\alpha} b_\nu a_\mu, & \Longrightarrow \\ [A_\alpha, A^\dagger_\beta] = \delta_{\alpha\beta} - \Delta_{\alpha\beta}, & \Delta_{\alpha\beta} \equiv \sum_{\mu\nu}\overline{\Phi^{\mu\nu}_\alpha}\Big(\sum_{\mu'}\Phi^{\mu'\nu}_\beta a^\dagger_{\mu'} a_\mu + \sum_{\nu'}\Phi^{\mu\nu'}_\beta b^\dagger_{\nu'} b_\nu\Big). \end{array} \tag{22}$$

Deviation $\Delta_{\alpha\beta}$ is easily modeled [51] through a deformed oscillator. The basic commutation relations for composite bosons should map (on their states) onto relations of DOs:

$$[A_\alpha, A^\dagger_\beta] \equiv -\epsilon\Delta_{\alpha\beta} = 0 \quad \text{for} \quad \alpha \neq \beta, \qquad [N_\alpha, A^\dagger_\alpha] = A^\dagger_\alpha, \quad [N_\alpha, A_\alpha] = -A_\alpha,$$

$$[A_\alpha, A^\dagger_\alpha] \equiv 1 - \epsilon\Delta_{\alpha\alpha} = \varphi(N_\alpha + 1) - \varphi(N_\alpha) \tag{23}$$

(the structure function $\varphi(N)$ was deduced in [51]; $\epsilon = +1$ for the both fermionic constituents as above, and $\epsilon = -1$ for bosonic ones). Related with these, defining relations for the system of mode independent deformed oscillators (deformed bosons) taken to build φ-deformed Bose-gas read

$$[N_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k}\mathbf{k}'} a^\dagger_{\mathbf{k}}, \quad [N_{\mathbf{k}}, a_{\mathbf{k}'}] = -\delta_{\mathbf{k}\mathbf{k}'} a_{\mathbf{k}},$$

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k}\mathbf{k}'}\big(\varphi(N_{\mathbf{k}}+1) - \varphi(N_{\mathbf{k}})\big), \qquad [a_{\mathbf{k}}, a_{\mathbf{k}'}] = 0, \qquad a^\dagger_{\mathbf{k}} a_{\mathbf{k}} = \varphi(N_{\mathbf{k}}). \tag{24}$$

The relations on Φ_α and $\varphi(N)$ derived and solved in [51] lead to the quadratic (in N) DSF $\varphi_{\tilde{\mu}}(N) = (1 + \tilde{\mu}N) - \tilde{\mu}N^2$, $\tilde{\mu} = \frac{1}{m}$ with m being the size of matrix Φ_α. Thus, using DOs (deformed bosons), it is possible to account for compositeness of quasibosons.

One is also able to effectively describe interparticle interactions by means of deformation (see e.g., [40]). That is, the account for interparticle interactions by the use of deformation implies the correspondence: *non-ideal Bose gas* $\leftrightarrow$ *ideal deformed Bose gas* model.

Intra-Quasibosonic Entanglement and Deformed Oscillators

As it was shown in [49], the existence of mapping of composite boson operator algebra onto the algebra of (quadratically nonlinear) DO enables obtaining the measures of entanglement inside quasiboson in terms of deformation parameter $\tilde{\mu}$. Say the entanglement entropy of one-quasiboson state reads $S_{\text{entang}} = -\sum_k(\lambda^\alpha_k)^2\ln(\lambda^\alpha_k)^2 = \ln(1/\tilde{\mu})$. The other measure of entanglement—*purity* takes the most simple form: $P \equiv \sum_k(\lambda^\alpha_k)^4 = \tilde{\mu}$ (recall that $\tilde{\mu} = \frac{1}{m}$).

Remark 1. *The both non-ideality factors of bosons: compositeness and interaction can be treated jointly, by means of "hybrid" DSF. For the hybrid case, we will use the DSF* $\varphi_{\tilde{\mu},q}(N) = (1+\tilde{\mu})\,[N]_q - \tilde{\mu}\,[N]^2_q$ *with* $[N]_q = \frac{q^N-1}{q-1}$, *when treating the distribution/correlation aspects of DBGM (see the next section), or likewise* $\varphi_{\tilde{\mu},q}(X)$ *in which* $X \equiv (z\frac{d}{dz})$, *when realizing deformed thermodynamics.*

9. Two-Particle Correlation Function Intercept in $\tilde{\mu}, q$-DBGM

For the average $\langle a^\dagger_{\mathbf{k}} a_{\mathbf{k}}\rangle$ ($\mathbf{k}$th mode is fixed; its label $\mathbf{k}$ is dropped) calculation yields [23]:

$$\langle a^\dagger a\rangle = \frac{w + \varphi_{\tilde{\mu},q}(2) - [3]_q}{(w-q)(w-q^2)}, \qquad w = \exp(\beta\omega), \qquad \omega = \sqrt{m^2 + \mathbf{k}^2}, \tag{25}$$

where $\varphi_{\tilde{\mu},q}(2) > [2]_q\frac{1+|q|-|q-1|}{2}$. Consider 1-parameter limit cases of $\langle a^\dagger a\rangle$. The limit $q \to 1$ gives

$$\langle a^\dagger a\rangle \xrightarrow[q\to 1]{} \frac{w-1-2\tilde{\mu}}{(w-1)^2}, \quad \tilde{\mu} < 0. \tag{26}$$

The limit $\tilde{\mu} \to 0$ (at $q > 1$ yields the same result as in q-DBGM of the AC case (see Table 1 above), i.e., $\langle a^{\dagger} a \rangle \xrightarrow[\tilde{\mu}\to 0]{} \frac{1}{e^x - q}$, $\quad x > \ln q$ (note that, if $|q| < 1$, the restriction $x > \ln q$ is dropped).

For the two-particle distribution, the formula was also derived in [23], namely

$$\langle (a^{\dagger})^2 a^2 \rangle = \frac{\varphi_{\tilde{\mu},q}(2)}{(w-q)(w-q^2)} \cdot \left\{ 1 + \frac{(\varphi_{\tilde{\mu},q}(3) - [3]_q)\left(w - q^2\left(\frac{[4]_q}{\varphi_{\tilde{\mu},q}(2)} - 1\right)\right)}{(w-q^3)(w-q^4)} \right\}. \tag{27}$$

Joining this with one-particle distribution (25) yields the two-particle correlation intercept

$$\lambda^{(2)}_{\tilde{\mu},q} = -1 + \frac{\varphi_{\tilde{\mu},q}(2)(w-q)(w-q^2)}{(w-[3]_q+\varphi_{\tilde{\mu},q}(2))^2(w-q^3)(w-q^4)} \cdot \left\{ (w-q^3)(w-q^4) + (\varphi_{\tilde{\mu},q}(3) - [3]_q)\left(w + q^2 - \frac{q^2[4]_q}{\varphi_{\tilde{\mu},q}(2)} \right) \right\}. \tag{28}$$

The latter formula gives explicit dependence of the intercept $\lambda^{(2)}_{\tilde{\mu},q}$ on the space momentum. The asymptotics of $\lambda^{(2)}_{\tilde{\mu},q}$ results: $\lambda^{(2)}_{as} = \varphi_{\tilde{\mu},q}(2)! - 1 = q[1 - \tilde{\mu}(1+q)]$. It depends on $\tilde{\mu}$ and q only.

Confronting the Latter Result in $\tilde{\mu}, q$-DBGM with Experimental Data

The following Figure 1 (see also [23]) gives the plot for $\lambda^{(2)}_{\tilde{\mu},q}$ vs. transverse momentum confronted with data from STAR/RHIC from [15], and a remarkable agreement is seen. To the best of our knowledge, no other model is able to provide that.

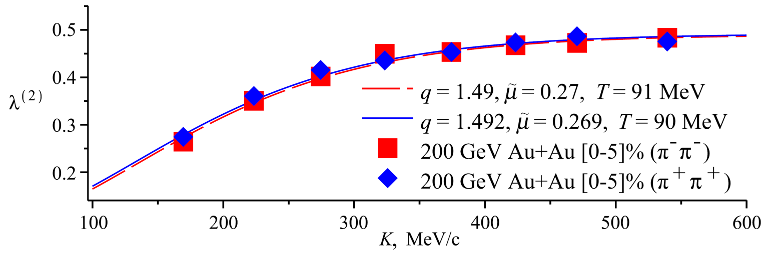

Figure 1. Intercept $\lambda^{(2)}_{\tilde{\mu},q}(K)$ vs. momentum K_T, for different values of q, $\tilde{\mu}$ and T chosen to fit experimental data. Experimental dots taken from [15] are shown by boxes.

10. Discussion

A number of results on the momentum distributions and correlations of nth order are presented. One may stress that the expressions given in Equations (6)–(10) and (25)–(28) are exact although the classes of deformation are rather different: exponential-type deformation versus rational-type ones. However, we encounter special functions (Lerch transcendent) for the rational case, in contrast to elementary functions in the exponential case.

Thermodynamics developed within μ-BGM, including the use of geometric tools, resulted in confirming the existence of Bose-like condensation. This basic fact along with a number of important features of μ-deformed model considered in Section 6 above, and showing its advantages with respect to the usual Bose case, make it possible to suggest the application of μ-BGM in order to model the properties of dark matter surrounding dwarf galaxies.

Such very nice agreement of the theory curve for $\lambda^{(2)}_{\tilde{\mu},q}$, given in Equation (28) with the set of experimental points implies well developed, significant deformation, witnessed by rather large values of $\tilde{\mu}$ and $q - 1$. Remembering physical sense of the two parameters, that means: in the experiment, both the interactions between π-mesons and their composite nature are clearly manifested and thus essential. In addition, the experiment "exhibits" intra-pionic entanglement and, as the value of $\tilde{\mu}$ (related to compositeness) is close to $\frac{1}{4}$, this implies two-qubit form of entanglement between quark and antiquark inside pions.

A remarkable property stemming from the asymptotical (at large mean momenta) behavior of the correlation intercepts within each DBGMs should be specially emphasized. This is the asymptotical independence of intercepts from the temperature and from the mass of particles (see respective formulas in Sections 3, 4 and 9). As result, we come to the experimentally verifiable statement: the *intercepts of pions and kaons at large mean momenta should merge* (note that, for the q-deformed BGM, this fact, and respective prediction, was pointed out in [17]).

An interesting question is whether it is possible to relate the parameters $\tilde{\mu}$ and q of our model with the couple of Lévy parameters of a different approach to pion femtoscopy based on Lévy distributions [52,53].

In summary, miscellaneous deformed analogs of BGM are both efficient and, due to their unusual and nice properties, remain very perspective tools from the viewpoint of diverse applications.

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