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Review

Holographic Entanglement in Group Field Theory

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Received: 5 August 2019; Accepted: 4 October 2019; Published: 9 October 2019

Abstract: This work is meant as a review summary of a series of recent results concerning the derivation of a *holographic* entanglement entropy formula for generic open spin network states in the group field theory (GFT) approach to quantum gravity. The statistical group-field computation of the Rényi entropy for a bipartite network state for a simple interacting GFT is reviewed, within a recently proposed dictionary between group field theories and random tensor networks, and with an emphasis on the problem of a consistent characterisation of the entanglement entropy in the GFT second quantisation formalism.

Keywords: holographic entanglement; quantum gravity; group field theory; random tensor networks; quantum many-body physics; quantum geometry

1. Introduction

Two potentially revolutionary ideas have inspired much work in contemporary theoretical physics. Both ideas herald from the use of the general information theoretic approach to the problem of quantum gravity. The first idea is that—*the world is holographic*—with the physics of several semi-classical systems (by which we mean systems in which matter is treated quantum-mechanically while spacetime and geometry are treated classically) entirely captured on spacetime regions of one dimension lower.

The evidence for holography is already suggestive when considering classical gravitational systems like black holes, or more general causal horizons, and the semi-classical physics of quantum fields in their vicinity. This evidence is strengthened by numerous results in the context of the AdS/CFT correspondence [1,2], where, in particular, holography as found in semi-classical gravitational systems is put in correspondence with general properties of non-gravitational, purely quantum mechanic *dual* many-body systems. Indeed, holographic features deeply characterise condensed matter physics—hence the suggestion that there may be a *purely quantum mechanic origin of holography* that may in fact underlie classical, gravitational, geometric physics as studied in the general relativistic context.

The second idea is that *geometry itself originates from entanglement*. The recent quantum information-theoretic paradigm for gravity has provided a new vision of the cosmos wherein the universe, together with its topology, its geometry and its macroscopic dynamics, arise from the entanglement between the fundamental constituents of some exotic underlying quantum system.

In this direction, in particular, along with the increasing impact of condensed matter physics in string theory and gauge/gravity duality, people have started exploring the use of tensor network (TN) algorithms [3–12] from condensed matter theory in quantum gravity [13–20], providing interesting insights on holographic duality and its generalisation in terms of geometry/entanglement correspondence [21,22]. Approaches like the AdS/MERA [10,23], where the geometry of the auxiliary tensor network decomposition of the quantum many-body vacuum state is interpreted as a representation of the dual spatial geometry, are providing an intriguing constructive framework for investigating holography beyond AdS/CFT [13–16,24].

However, if everything is quantum at its root, and this is true not only for ordinary systems living in spacetime but for *space, time and geometry themselves*, then this implies that the very holographic behaviour of the universe is the result of purely quantum properties of the microscopic constituents of spacetime. In addition, this line of research, therefore, indicates a strong need for a quantum foundation of holography.

In fact, background independence naturally leads to a quantum description of the universe in terms of fundamental quantum many-body physics of discrete and purely algebraic microscopic constituents [25–34], from which spacetime emerges only at an effective, approximate level, out of a texture of quantum correlations [35–43]. This means that the two suggestions that holography has a purely quantum origin and that geometry itself comes from entanglement are extremely natural when seen from the perspective of quantum gravity formalisms, in which spacetime and geometry are ultimately emergent notions. However, more than that, differently from the semiclassical framework, in the non-perturbative scheme, holography as detected in gravitational systems, as well as any macroscopic feature of our geometric universe, not only would result from purely quantum properties of the microscopic constituents of spacetime, but they can— *only*—be understood in this light.

This perspective is manifest in the *Group Field Theory* (GFT) formalism, a promising convergence of the insights and results from matrix models [44,45], loop quantum gravity and simplicial approaches into a background independent quantum field theory setup. The GFT approach to quantum gravity [30, 46–49] provides a very general quantum many-body formulation of the spacetime micro-structure, for instance of the spin networks and discrete quantum geometry states of Loop Quantum Gravity [25–27, 29], with a Fock space description where the quantum GFT fields create and annihilate elementary building blocks of space, interpreted as $(d-1)$-simplices in d spacetime dimensions, organised in nontrivial combinatorial tensor network structures.

As a higher order generalisation of matrix models, the GFT formalism at the same time provides a field-theoretic and inherently covariant framework for generalising the *tensor networks* approach to the holographic aspects of quantum many-body systems in condensed matter and in the AdS/CFT context. This makes GFT a very effective framework to investigate how space-time geometry, together with its holographic behaviour and macroscopic dynamics, arise from entanglement between the fundamental constituents.

In this paper, we review a series of recent results [18–20] concerning the definition of entanglement entropy in the GFT framework and the characterisation of its holographic behaviour. In particular, we focus on the definition of the notion of entanglement entropy in the second quantised formalism of GFT setting and on the set of choices which eventually lead to a holographic behaviour for the entanglement entropy.

The manuscript is organised as follows: Section 2 shortly reviews the framework of group field theory while focussing the attention of the reader on those aspects of the GFT fields that play a major role in the forthcoming derivation. Section 3 introduces the second quantisation formalism for group field theory, defines the notion of multi-particle state observable for quantum geometry in the GFT Fock space and specifies a class of GFT coherent state basis, necessary for the definition of the entanglement entropy expectation value. In Section 4, the statistical derivation of the Rényi entropy for a bipartite GFT open spin network state is reviewed, with an emphasis on the Fock space setting. The formal mapping of the expectation values of the Rényi entropy to BF theory amplitudes is described and the resulting divergence degree and scaling of the entropy analysed. Section 5 briefly comments on the results on the holographic scaling for the interacting GFT case. A brief discussion and an appendix on the notion of coherent states over-completeness close the manuscript.

2. Group Field Theory

Group field theories (GFT) are quantum field theories defined on d copies of a compact Lie group G with combinatorially non-local interactions. The dynamics of the GFT field

$$\phi : G^{\times d} \to \mathbb{C}$$

are specified by a probability measure

$$d\mu_C(\phi, \overline{\phi}) \exp\left(-S_{\text{int}}[\phi, \overline{\phi}]\right), \tag{1}$$

comprised of a *Gaussian measure* $d\mu_C$, associated with a positive covariance kernel operator C defining the propagator of the theory [45],[1] and a *perturbation* around it, given by an interaction term $S_{\text{int}}[\phi, \overline{\phi}]$, generically parametrized as

$$S_{\text{int}}[\phi, \overline{\phi}] = \sum_{\mathcal{I}} \sum_{\substack{p+q=\mathcal{I} \\ p,q\geq 0}} \int \prod_{\mathfrak{p}=1}^{p} d\mathbf{g}'_{\mathfrak{p}}\, \bar{\phi}(\mathbf{g}'_{\mathfrak{p}}) \prod_{\mathfrak{q}=1}^{q} d\mathbf{g}_{\mathfrak{q}}\, \phi(\mathbf{g}_{\mathfrak{q}})\, \lambda_{\mathcal{I}} V_{\mathcal{I}}(\mathbf{g}'_1, \cdots, \mathbf{g}'_p; \mathbf{g}_1, \cdots, \mathbf{g}_q),$$

where $\mathcal{I}$ denotes a term in the set of elementary interactions, $V_{\mathcal{I}}$ is the specific monomial in the fields associated with the interaction, and $\lambda_{\mathcal{I}}$ is the respective coupling constant. Together with the field valence d and symmetry, the specific choice of the covariance and interaction kernels identifies the GFT model.

We are particularly concerned with three peculiar aspects of the GFT formalism, which will combine at the hearth of the following derivation. The first is that the dynamical d-valent GFT field ϕ combinatorially behaves as an infinite dimensional rank-d tensor, with indices labelled by elements of the compact Lie group G [18]. This is apparent in the combinatorially non-local structure of the interaction kernels $V_{\mathcal{I}}$, as functions on $G^{d\times|\mathcal{I}|}$ for finite sets of interactions. The kernels $V_{\mathcal{I}}$ do not impose coincidence of points in the group space $G^{\times d}$, but the whole set of the $d \times |\mathcal{I}|$ field arguments is partitioned into pairs, convoluted "strandwise" by the kernel,

$$V_{\mathcal{I}}(\{\mathbf{g}\}_{\mathcal{I}}) = V(\{\mathbf{g}_{\mathfrak{p}}\}_{\mathbf{i}}\{\mathbf{g}_{\mathfrak{q}}^{-1}\}_{\mathbf{j}}). \tag{2}$$

Indeed, one can see GFTs as higher-rank, infinite dimensional generalizations of random matrix models [45]. For instance, if we take G as $\mathbb{Z}_N$, then group fields identically reduce to rank-d tensors,[2] where integrability with respect to the discrete Dirac measure μ is satisfied for all fields considered.

The second aspect is that specific GFT partition functions, where the group G is identified with the local gauge subgroup of gravity and the kernels properly chosen, define generating functions

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[1] We use a vector notation for the configuration space variables and its Haar measure (We will also use the short-hand notation:

$$\overline{\varphi}_1 \cdot \varphi_2 = \int d\mathbf{g} \overline{\varphi}_1(\mathbf{g}) \varphi_2(\mathbf{g}),$$

for any two square-integrable functions φ_1 and φ_2 on G^d.)

$$\mathbf{g} = (g_1, \dots, g_d) \in G^d, \qquad d\mathbf{g} = dg_1 \cdots dg_d.$$

[2] A rank-d tensor T with index cardinality N is a complex field on d copies of the cyclic group $\mathbb{Z}_N$:

$$T : \mathbb{Z}_N^{\times d} \to \mathbb{C},$$

which defines a state in $\mathcal{H}_{d,N}$ the space of

tensors with fixed rank d and index cardinality N. Neglecting the structure of the cyclic group, $\mathcal{H}_{d,N}$ is reduced to $\mathbb{C}^{N^d}$.

for the covariant quantization of Loop Quantum Gravity (LQG) in terms of spin foam models (for instance, [28]). In particular, LQG spin network states, describing three-dimensional discrete quantum geometries at the boundary of the spin foam transition amplitudes, can be expressed as expectation values of specific GFT operators. In a formalism of second quantisation for the GFT, spin network boundary states then become elements of the GFT Fock space and spin network vertices, intended as atoms of space that can be put in direct correspondence with fundamental GFT quanta that are created or annihilated by the field operators of GFT.

When G is set to correspond to the local gauge group of GR, e.g., the Lorentz group $SO(1, d-1)$ or its universal covering, or $SU(2)$ in dimension $d = 3$, the gauge symmetry leads to an invariance of the GFT action under the (right) diagonal action of G. GFT fields are constrained to satisfy a *gauge invariance condition*, defined as a global symmetry of the GFT field under simultaneous translation of its group variables:

$$\forall\, h \in G, \qquad \phi(g_1 h, \ldots, g_d h) = \phi(g_1, \ldots, g_d). \tag{3}$$

The gauge invariance condition, or *closure constraint*, is the main dynamical ingredient of GFT models for quantum BF theory in arbitrary dimension. In $d = 3$, $SU(2)$ BF theory can be interpreted as a theory of Euclidean gravity, and therefore $SU(2)$ GFT with closure constraint provides a natural arena in which to formulate 3D Euclidean quantum gravity models. A typical example is the Boulatov model [50] (which generates Ponzano–Regge spin foam amplitudes [51]), which will constitute an important ingredient of the following derivation.

The third aspect is that, for such (simplicial) GFT models, endowed with a geometric interpretation of the dynamical fields, the very field-theoretic nature of the GFT formalism provides a powerful tool to describe quantum geometry states as a peculiar *quantum many-body* systems in the formalism of *second quantisation*. For large systems in quantum mechanics, we know that the concept of a particle fades away and is replaced by the notion of an excitation of a given mode of the field representing the particle. Similarly, in the GFT description, we expect the solid graph description of spin networks quantum geometry to fade into a dynamical net of excitations of the GFT field over a vacuum. Given the tensorial behaviour of the GFT field, such a quantum many-body description turns quantum geometry states into collective, purely combinatorial and algebraic analogues of quantum *tensor networks states*.

These three ingredients together make the GFT second quantised formalism an specially convenient setting to *quantitatively* investigate the relation between geometry and entanglement in quantum gravity, taking advantage of the most recent techniques and tools of quantum statistical mechanics, information theory and condensed matter theory. In particular, quantum tensor network algorithms provide a constructive tool to investigate the roots of the holographic behaviour of gravity at the quantum level.

3. The GFT Fock Space

In a second quantisation scheme [30], multi-particle states of the quantum GFT field $\phi(\mathbf{g})$ can be organised in a Fock space $\mathbb{F}$ generated by a Fock vacuum $|0\rangle$ and field operators

$$\widehat{\varphi}(\mathbf{g}) \equiv \widehat{\varphi}(g_1, \cdots, g_d), \qquad \widehat{\varphi}^\dagger(\mathbf{g}) \equiv \widehat{\varphi}^\dagger(g_1, \cdots, g_d), \tag{4}$$

assumed to be invariant under the diagonal action of the group $\widehat{\varphi}(\mathbf{g}h) \equiv \widehat{\varphi}(g_1 h, \cdots, g_V h) = \widehat{\varphi}(\mathbf{g})$, for $h \in G$, consistently with (3), and to obey canonical commutation relations (bosonic statistics)

$$\begin{aligned} [\widehat{\varphi}(\mathbf{g}), \widehat{\varphi}^\dagger(\mathbf{g}')] &= \int \mathrm{d}h \prod_1^d \delta\left(g_i h ({g_i}'^{-1}\right) = \mathbb{1}_G(g_i, {g_i}'), \\ [\widehat{\varphi}(\mathbf{g}), \widehat{\varphi}(\mathbf{g}')] &= [\widehat{\varphi}^\dagger(\mathbf{g}), \widehat{\varphi}^\dagger(\mathbf{g}')] = 0. \end{aligned} \tag{5}$$

In these terms, the Fock vacuum is the state with no quantum geometrical or matter degrees of freedom, satisfying $\hat{\varphi}(\mathbf{g})|0\rangle = 0$ for all arguments. A generic single-particle state $|\phi\rangle$ with wavefunction ϕ, consisting of a d-valent node with links labelled by group elements $\mathbf{g} \equiv (g_1, ..., g_d)$, is written as

$$|\phi\rangle = \int_{G^{\times d}} \mathrm{d}\mathbf{g}\, \phi(\mathbf{g})\, |\mathbf{g}\rangle, \tag{6}$$

where $\mathrm{d}\mathbf{g} = \prod_{i=1}^{d} \mathrm{d}g_i$ is the Haar measure, ϕ is an element of the single-particle Hilbert space $\mathbb{H} = L^2(G^{\times d})$, and $|\mathbf{g}\rangle = |g_1\rangle \times ... \times |g_d\rangle$ a basis (of Dirac distributions) in $\mathbb{H}$. For $d = 4$, this is the space of states of a quantum tetrahedron [52]. Notice then that one can think of (6) as the analogue of a quantum tensor state where each group element g corresponds to an index i variable in a continuous (∞-dim) index space $\mathbb{H}_i = L^2(G)$.[3]

The complete Fock space is given by the direct sum of n-particle sectors $\mathbb{H}^{\otimes n}$, restricted to states that are invariant under graph automorphisms of vertex relabelling in the spin network picture, in order to consider multi-particle states that only depend on the intrinsic combinatorial structure of their interaction pattern (a discrete counterpart of continuum diffeomorphisms). Therefore, one has

$$\mathbb{F} \equiv \bigoplus_{n\geq 0}^{\infty} \mathrm{sym}[\mathbb{H}^{\otimes n}]. \tag{9}$$

The symmetry condition is consistent with the assumed *bosonic statistics* and it implies *indistinguishability* for the quanta of the quantum many-body system.

Generic GFT observables $\widehat{\mathcal{O}}[\hat{\varphi}, \hat{\varphi}^\dagger]$ in the Fock space are defined in terms of a series of many-body operators expressed as a function of the field operators (or of the basic creation/annihilation operators). For instance, a (n)-body operator $\mathcal{O}_n$ acting on $\mathfrak{p}$ vertices and resulting in $\mathfrak{q}$ particles is written as [30]

$$\widehat{\mathcal{O}}_n[\hat{\varphi}, \hat{\varphi}^\dagger] = \sum_{n=2}^{\infty} \sum_{\substack{p+q=n \\ p,q\geq 0}} \int \prod_{\mathfrak{p}=1}^{p} \mathrm{d}\mathbf{g}'_{\mathfrak{p}}\; \hat{\varphi}^\dagger(\mathbf{g}'_{\mathfrak{p}}) \prod_{\mathfrak{q}=1}^{q} \mathrm{d}\mathbf{g}_{\mathfrak{q}}\; \hat{\varphi}(\mathbf{g}_{\mathfrak{q}})\; \mathcal{O}_n(\mathbf{g}'_1, \cdots, \mathbf{g}'_p; \mathbf{g}_1, \cdots, \mathbf{g}_q), \tag{10}$$

where $\mathcal{O}_n(\mathbf{g}'_1, \cdots, \mathbf{g}'_p; \mathbf{g}_1, \cdots, \mathbf{g}_q)$ denote the matrix elements of a corresponding first-quantized operator.

3.1. Multi-Particle State Observables

Analogous with the case of the single particle state in (6), we can think of a quantum many-body system as a collective state generated by the action of a *multi-particle group-field operator* on the Fock vacuum. We can define a **product** n-particle state, comprising n disconnected nodes, by the multiple action of the creation field operator in the group representation of the Fock space, e.g.,

$$|\mathbf{g}_1, \mathbf{g}_2, \cdots, \mathbf{g}_n\rangle = \frac{1}{\sqrt{n!}} \prod_{a=1}^{n} \hat{\varphi}^\dagger(\mathbf{g}_a)\, |0\rangle\,. \tag{11}$$

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[3] The analogy with a tensor state is apparent again for $G = Z_N$. Let $\mathcal{H}_{d,N}$ be the space of tensors with fixed rank d and index cardinality N. Neglecting the structure of the cyclic group, $\mathcal{H}_{d,N}$ is reduced to $\mathbb{C}^{N^d}$. The linear structure, the scalar product and the completeness of $\mathbb{C}^{N^d}$ establish $\mathcal{H}_{d,N}$ to be a Hilbert space. A basis of $\mathcal{H}_{d,N}$ is chosen by $|i_1, ..., i_d\rangle$, defined as:

$$\langle j_1, ... j_d | i_1, ..., i_d\rangle = \delta_{i_1,j_1} \cdot ... \cdot \delta_{i_d,j_d}. \tag{7}$$

With respect to this basis, we decompose a tensor T into its components $T_{i_1...i_d}$, which introduces an isomorphism to $\mathbb{C}^{N^d}$:

$$|T\rangle =: \sum_{i_1,...,i_d \in \mathbb{Z}_N} T_{i_1...i_d}\, |i_1, ..., i_d\rangle\,. \tag{8}$$

Because the $\widehat{\varphi}^\dagger(\mathbf{g}_a)$ commute with each other, the order of the particles does not affect the state and we have $|..., \mathbf{g}_a, ..., \mathbf{g}_b, ...\rangle = |..., \mathbf{g}_b, ..., \mathbf{g}_a, ...\rangle$. The n-particle state $|\mathbf{g}_1, ..., \mathbf{g}_n\rangle$ defines a multi-particle basis of the Fock space $\mathbb{F}$, with orthogonality relation given by

$$\langle \mathbf{g}'_1, ..., \mathbf{g}'_n | \mathbf{g}_1, ..., \mathbf{g}_n \rangle = \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} \prod_{a=1}^{n} \int \mathrm{d}h_a \; \delta^4(\mathbf{g}'_a h_a \mathbf{g}_{\pi(a)}^{-1}). \tag{12}$$

In addition, because of the required symmetry of the fields $\widehat{\varphi}(\mathbf{g}h) = \widehat{\varphi}(\mathbf{g})$, $h \in G$, the n-particle state is right invariant

$$|\cdots, \mathbf{g}_a h_a, \cdots\rangle = |\cdots, \mathbf{g}_a, \cdots\rangle \,. \tag{13}$$

Finally, the resolution of identity in the Fock space can be written in terms of the n-particle state as

$$\mathbb{1}_{\mathbb{F}} = |0\rangle \langle 0| + \sum_{n=1}^{\infty} \int \prod_{a=1}^{n} \mathrm{d}\mathbf{g}_a \, |\mathbf{g}_1, \cdots, \mathbf{g}_n\rangle \langle \mathbf{g}_1, \cdots, \mathbf{g}_n| \,. \tag{14}$$

One can check immediately that $\mathbb{1}_{\mathbb{F}} |\mathbf{g}_1, \cdots, \mathbf{g}_n\rangle = |\mathbf{g}_1, \cdots, \mathbf{g}_n\rangle$.

A generic GFT **multi-particle state**, analogous to (6), will then be based on a specific configuration of n fields, characterized by a multi-particle wavefunction Ψ_n, generated by a multi-particle operator in the GFT Fock space

$$\widehat{\Psi}_n[\widehat{\varphi}^\dagger] \;=\; \sum_{\substack{p+q=n \\ p,q \geq 0}} \int \prod_{\mathfrak{p}=1}^{p} \mathrm{d}\mathbf{g}'_{\mathfrak{p}} \; \widehat{\varphi}^\dagger(\mathbf{g}'_{\mathfrak{p}}) \prod_{\mathfrak{q}=1}^{q} \mathrm{d}\mathbf{g}_{\mathfrak{q}} \; \widehat{\varphi}(\mathbf{g}_{\mathfrak{q}}) \; \Psi_n(\mathbf{g}'_1, \cdots, \mathbf{g}'_p; \mathbf{g}_1, \cdots, \mathbf{g}_q) \tag{15}$$

via the repeated action of the GFT field operators on the Fock space.

We are interested in the structure of quantum correlations of the multi-particle state $\Psi_n(\mathbf{g}'_1, \cdots, \mathbf{g}'_p; \mathbf{g}_1, \cdots, \mathbf{g}_q)$. As it is the case for any highly entangled quantum many-body system, disentangling the information on the quantum correlations of Ψ_n is highly nontrivial. Therefore, we focus on a special class of multi-particle operators $\widehat{\Psi}_\Gamma$, where the wave-function is explicitly constructed via a pairwise contractions scheme of single node states, in correspondence with a given **network architecture** Γ. We write

$$\widehat{\Psi}_\Gamma[\widehat{\varphi}, \widehat{\varphi}^\dagger](\mathbf{g}_\partial) = \sum_{\substack{p+q=n \\ p,q \geq 0}} \int \prod_{\mathfrak{p}=1}^{p} \mathrm{d}\mathbf{g}'_{\mathfrak{p}} \; \widehat{\varphi}^\dagger(\mathbf{g}'_{\mathfrak{p}}) \prod_{\mathfrak{q}=1}^{q} \mathrm{d}\mathbf{g}_{\mathfrak{q}} \; \widehat{\varphi}(\mathbf{g}_{\mathfrak{q}}) \prod_{\ell \in \Gamma}^{L} \mathrm{d}h_\ell \; \prod_{\ell \in \Gamma} L_\ell \left(\mathbf{g}'_{s(\ell)} h_\ell \mathbf{g}_{t(\ell)}^{-1} \right) \tag{16}$$

with $\widehat{\varphi}^\dagger$ operators generating the nodes connected by link kernels L_ℓ to form an open network with $\mathbf{g}_\partial$ dangling indices, via an overall integration over $\mathbf{g}_{\ell \in \Gamma}$. The expression of the link convolution kernel connecting the nodes pairwise is left generic at this stage, with the only requirement to preserve the overall gauge invariance of the network state.

We shall see such a class of multi-particle operators as *tensor networks* operators, where we reduce the entanglement structure of the multi-particle state to local correlations induced by the generic link kernels L, propagated non-locally via nodes.

Notice that the most generic tensor network state of the theory would involve superpositions of both network architectures (combinatorial structures) and number of particles corresponding to the same number of boundary degrees of freedom,

$$\widehat{\Psi}[\widehat{\varphi}, \widehat{\varphi}^\dagger](\mathbf{g}_\partial) = \sum_{\{\Gamma\}} \sum_{n=2}^{\infty} \sum_{\substack{p+q=n \\ p,q \geq 0}} \int \prod_{\mathfrak{p}=1}^{p} \mathrm{d}\mathbf{g}'_{\mathfrak{p}} \; \widehat{\varphi}^\dagger(\mathbf{g}'_{\mathfrak{p}}) \prod_{\mathfrak{q}=1}^{q} \mathrm{d}\mathbf{g}_{\mathfrak{q}} \; \widehat{\varphi}(\mathbf{g}_{\mathfrak{q}}) \; \mathcal{L}^{\Gamma}_{p,q}(\mathbf{g}'_1, \cdots, \mathbf{g}'_p; \mathbf{g}_1, \cdots, \mathbf{g}_q), \tag{17}$$

where, for simplicity, we indicate by $\mathcal{L}^{\Gamma}_{p,q}$ the set of pairwise link convolutions (gluing) functions associated with a given graph Γ, and some suitable symmetry quotient factor removing equivalent graph configurations is assumed.

3.2. GFT Coherent State Basis

In the context of quantum gravity, we specify the GFT formalism to the case $G = SU(2)$, the relevant local gauge subgroup of gravity, and we understand the group elements g as a generalisation of the embedded parallel transports (holonomies) of the gravitational G-connection of loop quantum gravity. The symmetric GFT d-valent fields as d-simplices (convex polyhedra, e.g., for $d = 4$, these are tetrahedra), with d number of faces labelled by dual Lie algebra-valued flux variables, become single "quanta" of twisted geometry states expressed in terms of quantum spin network basis [29].

In this setting, on the one hand, we are interested in working with states in the Fock space that can be eventually put in relation with extended macroscopic 3D geometries. To this aim, the natural choice consists of looking for a coherent state basis in $\mathbb{F}$, defined by exponential operators providing desirable coherence properties, having *macroscopic* occupation numbers for given modes controlled by the wave-function [53–56]. More concretely, this choice will allow us to compute quantum averages of many-body systems in thermal equilibrium using functional integrals over group field configurations [18–20].

The simplest class of such states is given by the single-particle (condensate) coherent states

$$\begin{aligned} |\varphi\rangle &\equiv \frac{1}{\mathcal{N}_\phi} \exp\left[\int \mathrm{d}\mathbf{g}\, \phi(\mathbf{g})\widehat{\varphi}^\dagger(\mathbf{g})\right]|0\rangle \equiv \frac{1}{\mathcal{N}_\varphi}\sum_{n=0}^{\infty}\frac{1}{n!}\prod_a^n\left[\int \mathrm{d}\mathbf{g}_a\, \phi(\mathbf{g}_a)\widehat{\varphi}^\dagger(\mathbf{g}_a)\right]|0\rangle \\ &\equiv \frac{1}{\mathcal{N}_\phi}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\int[\mathrm{d}\mathbf{g}]^n\, \phi(\mathbf{g}_1)\times\ldots\times\phi(\mathbf{g}_n)|\mathbf{g}_1,\cdots,\mathbf{g}_n\rangle. \end{aligned} \tag{18}$$

For the last equality, we use the definition of the n-particle state (11). $\phi(\mathbf{g})$ is the field on $\mathbb{H}$ that has the same gauge symmetry as $\widehat{\varphi}^\dagger(\mathbf{g})$, namely $\phi(\mathbf{g}h) = \phi(\mathbf{g})$, and $\mathcal{N}_\phi$ is the normalization [4]

$$\mathcal{N}_\phi^2 = \exp\left[\int \mathrm{d}\mathbf{g}\, \overline{\phi(\mathbf{g})}\phi(\mathbf{g})\right]. \tag{22}$$

One can show that $|\varphi\rangle$ is the eigenstate of the field operator $\widehat{\varphi}(\mathbf{g})$ such that

$$\widehat{\varphi}(\mathbf{g})\,|\varphi\rangle = \phi(\mathbf{g})\,|\varphi\rangle\,. \tag{23}$$

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4 Define an operator $\widehat{A}$ as

$$\widehat{A} = \int \mathrm{d}\mathbf{g}\, \phi(\mathbf{g})\widehat{\varphi}^\dagger(\mathbf{g}). \tag{19}$$

The commutator between $\widehat{A}$ and $\widehat{A}^\dagger$ is

$$[\widehat{A}^\dagger, \widehat{A}] = \int \mathrm{d}\mathbf{g}\, \overline{\phi(\mathbf{g})}\phi(\mathbf{g}). \tag{20}$$

Then, $\langle\varphi|\varphi\rangle$ can be given as

$$\begin{aligned} 1 &= \langle\varphi|\varphi\rangle = \mathcal{N}_\varphi^{-2}\,\langle 0|\, \mathrm{e}^{\widehat{A}^\dagger}\mathrm{e}^{\widehat{A}}\,|0\rangle \\ &= \mathcal{N}_\phi^{-2}\,\langle 0|\, \mathrm{e}^{\widehat{A}}\mathrm{e}^{\widehat{A}^\dagger}\mathrm{e}^{[\widehat{A}^\dagger,\widehat{A}]}\,|0\rangle \\ &= \mathcal{N}_\phi^{-2}\exp\left[\int \mathrm{d}\mathbf{g}\, \overline{\phi(\mathbf{g})}\phi(\mathbf{g})\right]. \end{aligned} \tag{21}$$

In the third equality, we use the Baker–Campbell–Hausdorff formula.

Indeed, we have

$$\widehat{\varphi}(\mathbf{g})\frac{1}{n!}\prod_{a}^{n}\left[\int \mathrm{d}\mathbf{g}_a\, \phi(\mathbf{g}_a)\widehat{\varphi}^\dagger(\mathbf{g}_a)\right] \tag{24}$$

$$= \frac{1}{n!}\sum_{k=1}^{n}\prod_{a\neq k}^{n}\left[\int \mathrm{d}\mathbf{g}_a\, \phi(\mathbf{g}_a)\widehat{\varphi}^\dagger(\mathbf{g}_a)\right]\int \mathrm{d}h\, \mathrm{d}\mathbf{g}_k\, \phi(\mathbf{g}_k)\delta(\mathbf{g}h\mathbf{g}_k^{-1}) \tag{25}$$

$$= \frac{1}{(n-1)!}\prod_{a}^{n-1}\left[\int \mathrm{d}\mathbf{g}_a\, \phi(\mathbf{g}_a)\widehat{\varphi}^\dagger(\mathbf{g}_a)\right]\int \mathrm{d}h\, \phi(\mathbf{g}h) \tag{26}$$

$$= \phi(\mathbf{g})\frac{1}{(n-1)!}\prod_{a}^{n-1}\left[\int \mathrm{d}\mathbf{g}_a\, \phi(\mathbf{g}_a)\widehat{\varphi}^\dagger(\mathbf{g}_a)\right]. \tag{27}$$

In the first equality, we use the commutator (5) between $\widehat{\varphi}$ and $\widehat{\varphi}^\dagger$. In the last equality, we use the fact that $\phi(\mathbf{g})$ is right invariant. Thus, when $\widehat{\varphi}(\mathbf{g})$ acts on $|\varphi\rangle$, it gives (23). In particular, coherent states $|\varphi\rangle$ provide an over-complete basis of the Fock space $\mathbb{F}$ (see Appendix A for details).

Via Equation (23), one immediately obtains the tensor fields $\phi(\mathbf{g})$ and $\overline{\phi(\mathbf{g})}$ in terms of expectation values of $\widehat{\varphi}(\mathbf{g})$ and $\widehat{\varphi}^\dagger(\mathbf{g})$ with respect to $|\varphi\rangle$

$$\langle\varphi|\,\widehat{\varphi}(\mathbf{g})\,|\varphi\rangle = \phi(\mathbf{g}), \quad \langle\varphi|\,\widehat{\varphi}^\dagger(\mathbf{g})\,|\varphi\rangle = \overline{\phi(\mathbf{g})}. \tag{28}$$

Accordingly, we can express the multi-particle state as the expectation value of a group-field network operator (16) in the 2nd quantised basis of eigenstates of the GFT quantum field operator.

For the network operator $\widehat{\Psi}_\Gamma[\widehat{\varphi},\widehat{\varphi}^\dagger](\mathbf{g}_\partial)$ defined in Equation (16), we get

$$\begin{aligned}\langle\varphi|\,\widehat{\Psi}_\Gamma[\widehat{\varphi},\widehat{\varphi}^\dagger]\,|\varphi\rangle &= \sum_{\substack{p+q=n\\ p,q\geq 0}}\int\prod_{\mathrm{p}=1}^{p}\mathrm{d}\mathbf{g}'_{\mathrm{p}}\,\overline{\phi(\mathbf{g}'_{\mathrm{p}})}\,\prod_{\mathrm{q}=1}^{q}\mathrm{d}\mathbf{g}_{\mathrm{q}}\,\phi(\mathbf{g}_{\mathrm{q}})\prod_{\ell\in\Gamma}^{L}\mathrm{d}h_\ell\,\prod_{\ell\in\Gamma}L_\ell\left(\mathbf{g}_{s(\ell)}h_\ell\mathbf{g}_{t(\ell)}^{-1}\right)\\ &= \Psi_\Gamma[\overline{\phi},\phi](\mathbf{g}_\partial).\end{aligned} \tag{29}$$

This is a group field tensor network state based on graph Γ with n nodes and L links. In particular, for $L_\ell\left(h_{s(\ell)}g_\ell h_{t(\ell)}^{-1}\right) = \delta\left(h_{s(\ell)}g_\ell h_{t(\ell)}^{-1}\right)$, we can think of the expectation values of the associated operators as peculiar projected entangled-pairs tensor network states (PEPS)

$$|\Psi_\Gamma\rangle \equiv \int \mathrm{d}\mathbf{g}_\partial\, \Psi_\Gamma(\mathbf{g}_\partial)\,|\mathbf{g}_\partial\rangle \equiv \bigotimes_{\ell\in\Gamma}\langle L_\ell|\bigotimes_{v\in\Gamma}|\phi_v\rangle \tag{30}$$

obtained by the contraction of maximally entangled link states

$$|L_\ell\rangle = \int \mathrm{d}g_{s(\ell)}\mathrm{d}g_{t(\ell)}\,\delta\left(g_{s(\ell)}h_\ell g_{t(\ell)}^{-1}\right)|g_{s(\ell)}\rangle\otimes\left|g_{t(\ell)}\right\rangle \tag{31}$$

via tensor states $|\phi\rangle$ on some generically open graph architecture Γ.

The basis $|\mathbf{g}_\partial\rangle$ labels the uncontracted dangling indices comprising the boundary of the auxiliary tensor network representation. Differently from standard PEPS, the GFT networks are further characterised by the inherently *random* character of the tensors $|\phi\rangle$, induced by their field-theoretic statistical descritpion. In this light, in particular, we can see states like (29) as a generalisation of the random tensor network states (RTNs) recently introduced in [15], where the statistical characterisation of the standard RTN gets mapped in the momenta of the GFT partition function.[5]

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5 When Γ is closed, one further recognises $\Psi_\Gamma[\overline{\phi},\phi](\mathbf{g}_{\ell\in\Gamma})$ to be equivalent to the cylindrical functions describing the quantum geometry of a closed spacial hypersurface in the kinematic Hilbert space of LQG [57].

4. Bipartite Entanglement of a GFT Network State

Given a group field tensor network state $|\Psi_\Gamma\rangle \in \mathbb{H}_\partial = \bigotimes_{\ell\in\partial} L^2_\ell[G]$, a bipartition of the boundary degrees of freedom corresponds to a factorisation of the boundary Hilbert space $\mathbb{H}_\partial$ into two subspaces $\mathbb{H}_A$ and $\mathbb{H}_B$, such that

$$\mathbb{H}_\partial = \mathbb{H}_A \otimes \mathbb{H}_B. \tag{32}$$

The entanglement of the boundary state $|\Psi_\Gamma\rangle$ across the bipartition in $\mathbb{H}_A$ and $\mathbb{H}_B$ is measured by the von Neumann entropy

$$S(A) = -\mathrm{Tr}\bar{\rho}_A \ln \bar{\rho}_A, \tag{33}$$

where

$$\bar{\rho}_A \equiv \frac{\rho_A}{\mathrm{Tr}\rho}, \quad \rho_A \equiv \mathrm{Tr}_B\rho = \mathrm{Tr}_B\, |\Psi_\Gamma\rangle \langle\Psi_\Gamma| \tag{34}$$

defines the (normalised) marginal on $\mathbb{H}_A$ of the GFT multi-particle density matrix,

$$\rho = |\Psi_\Gamma\rangle \langle\Psi_\Gamma| = \mathrm{Tr}\left[\bigotimes_\ell |L_\ell\rangle \langle L_\ell| \bigotimes_v |\phi_v\rangle \langle\phi_v|\right] \equiv \mathrm{Tr} \bigotimes_\ell \rho_\ell \bigotimes_v \rho_v. \tag{35}$$

A representation of ρ for the case of a simple 2-vertices graph is given in Figure 1.

It is computationally convenient to derive the von Neumann entropy as the limit of the Rényi entropy $S_N(A)$, via standard replica trick. The Rényi entropy is defined as

$$S_N(A) = \frac{1}{1-N} \ln \mathrm{Tr}\bar{\rho}_A^N = \frac{1}{1-N} \ln \frac{\mathrm{Tr}\rho_A^N}{(\mathrm{Tr}\rho)^N} \equiv \frac{1}{1-N} \ln \frac{\mathrm{Tr}(\rho^{\otimes N}\mathbb{P}_A)}{\mathrm{Tr}(\rho^{\otimes N})}, \tag{36}$$

where $\mathbb{P}_A = \mathbb{P}(\pi_A^0; n, d)$ is the 1-cycle permutation operator in S_N acting on the reduced Hilbert space $\mathbb{H}_A$,

$$\mathbb{P}(\pi_A^0; N, d) = \prod_{s=1}^{N} \delta_{\mu_A^{([s+1]_D)} \mu_A^{(s)}}, \tag{37}$$

and d is the dimension of the Hilbert space in the same region A (see Figure 1). Explicitly, one has

$$\mathbb{P}_A\, |a_1, b_1\rangle\, |a_2, b_2\rangle \cdots |a_N, b_N\rangle = |a_2, b_1\rangle\, |a_3, b_2\rangle \cdots |a_1, b_N\rangle \tag{38}$$

with $\bigotimes_i |a_i\rangle \in \mathbb{H}_A$ and $\bigotimes_i |b_i\rangle \in \mathbb{H}_B$.

The Rényi entropy $S_N(A)$ coincides with the von Neumann entropy $S(A)$ as N goes to 1, which is

$$S(A) = \lim_{N\to 1} S_N(A). \tag{39}$$

We are interested in carrying on this measurement directly in the Fock space of GFT.

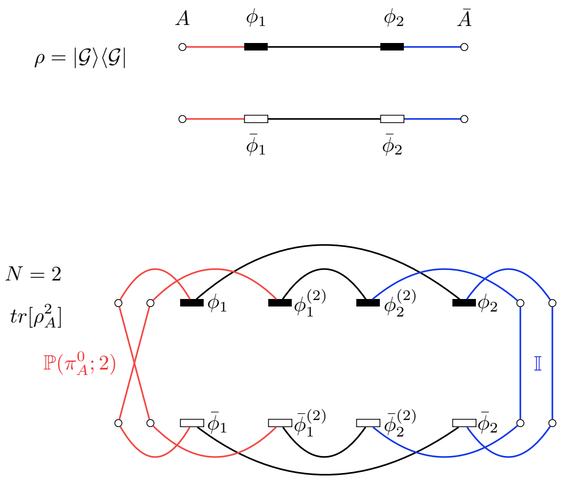


Figure 1. Graphical representation of the density matrix state for a simple bipartite GFT network state $|\mathcal{G}\rangle$, comprising two bivalent internal nodes, and trace of the $N = 2$ replica with the action of the cyclic permutation (swap) operator. Notice that the label $\bar{A}$, generally indicating the complementary marginal, corresponds to the B-labelling in the main text.

4.1. Expected Rènyi Entropy in the Fock Space

Due to the random character of the nodes $\{\phi_v\}$, induced by their dynamical GFT description, the measure of the entanglement will be necessarily given in expectation value.

Now, by construction, the Rényi entropy $S_N(A)$ is a functional of the GFT fields ϕ and $\overline{\phi}$, hence it can be promoted to an observable $S_N(A)[\widehat{\varphi}, \widehat{\varphi}^\dagger]$ in the GFT Fock space. We shall then derive the expectation value of the Rényi entropy in the Fock space using single-particle coherent state basis, by inserting the resolution of identity $\mathbb{1}_{\mathbb{F}}$ (and set $K = 0$, see Appendix A), with a normal ordering $:\cdots:$ such that all $\widehat{\varphi}^\dagger$ is to the left of $\widehat{\varphi}$:[6]

$$\mathbb{E}\left[S_N(A)[\widehat{\varphi}, \widehat{\varphi}^\dagger]\right] \equiv \frac{C}{Z_0}\mathrm{Tr}\left(:S_N(A)[\widehat{\varphi}, \widehat{\varphi}^\dagger]\, \mathrm{e}^{-\widehat{S}[\widehat{\varphi}, \widehat{\varphi}^\dagger]} : \mathbb{1}_{\mathbb{F}}\right), \tag{41}$$

where the very GFT action $\widehat{S}[\widehat{\varphi}, \widehat{\varphi}^\dagger]$ is constructed as a quantum many-body operator on the Fock space. By (36), the explicit form of the expectation value reads

$$\mathbb{E}[S_N(A)[\widehat{\varphi}, \widehat{\varphi}^\dagger]] = \frac{1}{1-N}\mathbb{E}\left[\ln \frac{\mathrm{Tr}(\rho\,[\widehat{\varphi}, \widehat{\varphi}^\dagger]^{\otimes N}\mathbb{P}_A)}{\mathrm{Tr}(\rho\,[\widehat{\varphi}, \widehat{\varphi}^\dagger])^N}\right] \quad . \tag{42}$$

If we choose to work in perturbative GFT regime, such that $S[\phi, \overline{\phi}] = S_0 + \lambda S_{\mathrm{int}}[\phi, \overline{\phi}]$, with $\lambda \ll 1$, then we deal with a polynomially perturbed generalised [45] Gaussian distribution for the random field, hence we can take advantage of the central limit theorem to get a good approximation of (42) in terms of a Taylor expansion around the mean values

$$\mathbb{E}[S_N(A)[\widehat{\varphi}, \widehat{\varphi}^\dagger]] \approx \frac{1}{1-N}\ln \frac{\mathbb{E}[\mathrm{Tr}(\rho\,[\widehat{\varphi}, \widehat{\varphi}^\dagger]^{\otimes N}\mathbb{P}_A)]}{\mathbb{E}[\mathrm{Tr}(\rho\,[\widehat{\varphi}, \widehat{\varphi}^\dagger]^{\otimes N})]} \quad , \tag{43}$$

---

[6] The GFT vacuum amplitude in the same basis reads

$$\begin{aligned} Z_0 &= C\mathrm{Tr}\left(:\mathrm{e}^{-\widehat{S}[\widehat{\varphi}, \widehat{\varphi}^\dagger]}:\right) = C\mathrm{Tr}\left(:\mathrm{e}^{-\widehat{S}[\widehat{\varphi}, \widehat{\varphi}^\dagger]}: \mathbb{1}_{\mathbb{F}}\right) \\ &= \int \mathcal{D}\phi\mathcal{D}\overline{\phi}\; \delta_{\mathbb{C}}[\phi - \phi_{\mathrm{GF}}]\; \mathrm{e}^{-S[\phi, \overline{\phi}]}. \end{aligned} \tag{40}$$

plus corrections due to fluctuations around the mean values that are suppressed whenever the dimensionality of our quantum system gets extremely large. For standard random tensor network states [15], such a *typical* regime is realised in the *large bond* limit, when the dimension of the random tensor index (bond) space $\mathcal{H}_\ell$ is extremely large, $\dim(\mathcal{H}_\ell) \gg 1$.

In the GFT setting, one has to deal with infinite dimensional bond spaces, $\mathcal{H}_\ell = L^2(G)$, which automatically set the derivation in the typicality regime of [15]. Nevertheless, link spaces are regularised via a cut-off on the group representation space, such that

$$\langle g|g'\rangle = D(\Lambda)\delta_{g,g'} \tag{44}$$

with $\delta_{g,g'}$ equal to $\mathbb{1}$ if $g = g'$ and 0, otherwise. Therefore, with $D(\Lambda)$, the dimension of the bonds (links) of the network, one can consistently assume typicality to hold in the $D(\Lambda) \gg 1$ regime.

Given (43), the expression of the N-th Rényi entropy is mapped into a ratio of averaged partition functions,

$$\frac{Z_N}{Z_0^N} \equiv \frac{\mathbb{E}\left[\mathrm{Tr}(\rho\,[\widehat{\varphi},\widehat{\varphi}^\dagger]^{\otimes N}\mathbb{P}_A)\right]}{\mathbb{E}\left[\mathrm{Tr}(\rho\,[\widehat{\varphi},\widehat{\varphi}^\dagger]^{\otimes N})\right]}, \tag{45}$$

where we removed the bars over Z_N, Z_0^N to simplify the notation.

Let us rewrite $\rho\,[\widehat{\varphi},\widehat{\varphi}^\dagger]^{\otimes N}$ as a trace contraction of individual link and nodes density matrices, similarly to (35). We restrict for simplicity to tensor network observables (16) with n vertices and L links, and with $p = n(q=0)$

$$\widehat{\Psi}_\Gamma[\widehat{\varphi}^\dagger](\mathbf{g}_\partial) = \int \prod_{\mathfrak{p}=1}^{n} \mathrm{d}\mathbf{g}_\mathfrak{p} \prod_{\mathfrak{p}=1}^{n} \widehat{\varphi}^\dagger(\mathbf{g}_\mathfrak{p}) \prod_{\ell\in\Gamma}^{L} \mathrm{d}h_\ell \prod_{\ell\in\Gamma}^{L} L_\ell\left(\mathbf{g}_{s(\ell)} h_\ell \mathbf{g}_{t(\ell)}^{-1}\right) \quad , \tag{46}$$

where now the $\mathfrak{p}$ label coincides with the v-label of the nodes of the graph Γ. The Nth replica of the density matrix operator describing a graph observable reads

$$\rho^{\otimes N} = \left(\int \prod_{v=1}^{n} \mathrm{d}\mathbf{g}_v \mathrm{d}\mathbf{g}'_v \prod_{v=1}^{n} \widehat{\varphi}^\dagger(\mathbf{g}_v)\, \widehat{\varphi}(\mathbf{g}'_v) \prod_{\ell}^{L} \mathrm{d}h_\ell\, \mathrm{d}h'_\ell \prod_{\ell}^{L} L_\ell L'_\ell\right)^{\otimes N} \tag{47}$$

$$= \int \left(\prod_{v=1}^{n} \mathrm{d}\mathbf{g}_v \mathrm{d}\mathbf{g}'_v \prod_{v=1}^{n} \widehat{\varphi}^\dagger(\mathbf{g}_v)\, \widehat{\varphi}(\mathbf{g}'_v)\right)^{\otimes N} \left(\prod_{\ell}^{L} \mathrm{d}h_\ell\, \mathrm{d}h'_\ell \prod_{\ell}^{L} L_\ell L'_\ell\right)^{\otimes N} \tag{48}$$

$$= \mathrm{Tr}\left[(\bigotimes_v \rho_v)^{\otimes N} (\bigotimes_\ell \rho_\ell)^{\otimes N}\right]. \tag{49}$$

The linearity of the trace allows for moving the expectation operator inside the integral and letting it act on the N replicas of the products of fields. We then write (45) as

$$\frac{\mathrm{Tr}\left[\otimes_\ell \rho_\ell^N \mathbb{E}[(\otimes_v \rho[\widehat{\varphi}^\dagger(\mathbf{g}_v),\ \widehat{\varphi}(\mathbf{g}'_v)])_v^{\otimes N}]\,\mathbb{P}_A\right]}{\mathrm{Tr}\left[\otimes_\ell \rho_\ell^N \mathbb{E}[(\otimes_v \rho[\widehat{\varphi}^\dagger(\mathbf{g}_v),\ \widehat{\varphi}(\mathbf{g}'_v)])_v^{\otimes N}]\right]} \tag{50}$$

and we focus on the calculation of $\mathbb{E}[\otimes_v \rho_v^{\otimes N}]$.

The derivation in [15,18] thereby proceeds by making two strongly simplifying assumptions. The first consists of the restriction to the case of a Gaussian (free) group field theory for describing the single node field statistics. The second assumes the expected value ϕ of the field operator at each node to be individually independently distributed (i.i.d.). The latter assumption corresponds, from a physical viewpoint, to considering a non-interacting quantum many body system. The latter condition translates in particular into a *local averaging* condition, namely

$$\mathbb{E}\left[\left(\bigotimes_v \rho_v\right)^{\otimes N}\right] \to \bigotimes_v \mathbb{E}\left[\rho_v^{\otimes N}\right] \quad , \tag{51}$$

which allows for an explicit calculation of the expectation value in terms of a product of n $2N$-point function of the free group field theory

$$\mathbb{E}\left[\rho_v^{\otimes N}\right] \equiv \frac{C}{Z_0}\mathrm{Tr}\left(\boldsymbol{:}\rho[\widehat{\varphi}^\dagger(\mathbf{g}_v),\widehat{\varphi}(\mathbf{g}'_v)]^{\otimes N}\, \mathrm{e}^{-\widehat{S}_0[\widehat{\varphi},\widehat{\varphi}^\dagger]}\boldsymbol{:}\,\mathbb{1}_{\mathbb{F}}\right) \tag{52}$$

$$= \frac{1}{Z_0}\int \mathcal{D}\phi\mathcal{D}\overline{\phi}\;\delta_C[\phi-\phi_{\mathrm{GF}}]\;\langle\varphi|\boldsymbol{:}[\widehat{\varphi}^\dagger(\mathbf{g}_v),\widehat{\varphi}(\mathbf{g}'_v)]^{\otimes N}\,\mathrm{e}^{-\widehat{S}_0[\widehat{\varphi},\widehat{\varphi}^\dagger]}\boldsymbol{:}\,|\varphi\rangle$$

$$= \frac{1}{Z_0}\int \mathcal{D}\phi\mathcal{D}\overline{\phi}\;\delta_C[\phi-\phi_{\mathrm{GF}}]\;(\phi,\overline{\phi})^N\,\mathrm{e}^{-S_0[\phi,\overline{\phi}]} \tag{53}$$

$$= \mathbb{E}_0\left[(\phi_v\bar{\phi}_v)^N\right] \quad . \tag{54}$$

In the free case, in particular, one can evaluate the $2N$-point functions at each node directly via Wick's theorem

$$\begin{aligned}\mathbb{E}_0\left[\prod_a^N \phi_v(\mathbf{g}_a)\overline{\phi_v(\mathbf{g}'_a)}\right] &= \mathcal{C}\sum_{\pi_v\in\mathcal{S}_N}\int\prod_a^N \mathrm{d}h_a \prod_a^N \delta_v\left(h_a\mathbf{g}_a{\mathbf{g}'_{\pi(a)}}^\dagger\right)\\ &= \mathcal{C}\sum_{\pi_v\in\mathcal{S}_N}\int\prod_a^N \mathrm{d}h_a\,\mathbb{P}_{\mathbf{h}_v}(\pi),\end{aligned}$$

where the permutation operator acts strandwise (locally on the link spaces)

$$\mathbb{P}_{\mathbf{g}}(\pi) \equiv \prod_a^N \delta\left(h_a\mathbf{g}_a{\mathbf{g}'_{\pi(a)}}^\dagger\right) = \prod_{s=1}^4\prod_a^N \delta\left(h_a g_{sa}{g'_{s\pi(a)}}^\dagger\right) \equiv \prod_s^4 \mathbb{P}^s_{\mathbf{g}}(\pi),$$

with $\mathbf{g}'$ independent from $\mathbf{g}$, a labelling the replica order at each node, $\mathbf{h}$ denoting the set of h_a, $a = 1,\cdots,N$.

When $h_a = \mathbb{1}$ for all a from 1 to N,

$$\begin{aligned}\mathbb{P}_{\mathbb{1}}(\pi) &= \prod_a^N \delta\left(\mathbf{g}_a{\mathbf{g}'_{\pi(a)}}^\dagger\right)\\ &= \mathbb{P}(\pi;N,D^4) = \prod_s^4 \mathbb{P}^s(\pi;N,D^4),\end{aligned} \tag{55}$$

where $\mathbb{P}(\pi;N,D^4)$ and $\mathbb{P}^s(\pi;N,D^4)$ are the representations of $\pi\in\mathcal{S}_N$ on $\mathbb{H}^{\otimes 4}$ and $\mathbb{H}$, respectively.

The averaged partition functions, Z_N and Z_0^N become

$$\begin{aligned} Z_N &\approx \mathcal{C}^{V_\Gamma} \sum_{\pi_v \in \mathcal{S}_N} \int \prod_v \mathrm{d}\mathbf{h}_v \operatorname{Tr}\left[\bigotimes_\ell \rho_\ell^N \bigotimes_v \mathbb{P}_{\mathbf{h}_v}(\pi_v)\mathbb{P}(\pi_A^0; N, d)\right] \\ &\equiv \mathcal{C}^{V_\Gamma} \sum_{\pi_v \in \mathcal{S}_N} \int \prod_v \mathrm{d}\mathbf{h}_v \, \mathcal{N}_A(\mathbf{h}_v, \boldsymbol{\pi}_v), \end{aligned} \tag{56}$$

$$\begin{aligned} Z_0^N &= \mathcal{C}^{V_\Gamma} \sum_{\pi_v \in \mathcal{S}_N} \int \prod_v \mathrm{d}\mathbf{h}_v \operatorname{Tr}\left[\bigotimes_\ell \rho_\ell^N \bigotimes_v \mathbb{P}_{\mathbf{h}_v}(\pi_v)\right] \\ &\equiv \mathcal{C}^{V_\Gamma} \sum_{\pi_v \in \mathcal{S}_N} \int \prod_v \mathrm{d}\mathbf{h}_v \, \mathcal{N}_0(\mathbf{h}_v, \boldsymbol{\pi}_v), \end{aligned} \tag{57}$$

respectively corresponding to summations of Feynman graphs $\mathcal{N}_A(\mathbf{h}_v, \boldsymbol{\pi}_v)$ and $\mathcal{N}_0(\mathbf{h}_v, \boldsymbol{\pi}_v)$ labelled by permutation operators $\mathbb{P}_{\mathbf{h}_v}(\pi_v)$, at each node v, contracted with the ρ_ℓ^N densities at each link ℓ (see Figure 2). The difference between the reduced and full (density matrix) networks is encoded in the boundary condition, as Z_N is defined with $\mathbb{P}(\pi_A^0; N, d)$ on A of $\partial\Gamma$ and $\mathbb{P}(\mathbb{1}; N, d)$ on B of $\partial\Gamma$, while Z_0^N is defined with $\mathbb{P}(\mathbb{1}; N, d)$ for all boundary region $\partial\Gamma$.

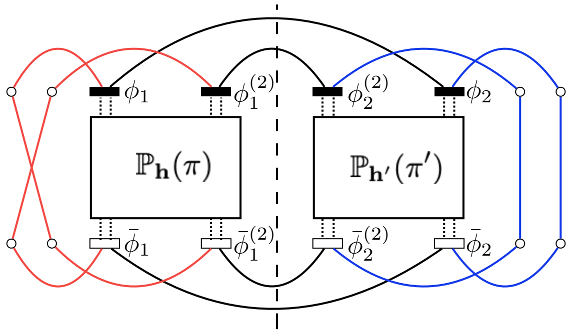


Figure 2. Free Group Field Theory propagators acting among two sets of N replicas for each node of the given graph.

It is easy to see at this stage what the impact of the i.i.d. assumption is in reducing the complexity of the propagator.[7] In this case, the permutation operator $\mathbb{P}$ acts only among the N replicas of the same node, independently taken across the graph. Clearly, $\otimes_v \mathbb{E}(\rho_v^N) \neq \mathbb{E}(\otimes_v^V \rho_v^N)$ as the permutation group $\mathcal{S}_{2N}$ is much smaller that $\mathcal{S}_{2NV}$, reducing to $(2N!)^V$ the number of permutation patterns with respect to the $(2NV)!$ allowed patterns of the indistinguishable case. Such a strong truncation is nevertheless consistent with a restriction to the tree-like, or "melonic", sector of the Feynmann diagrams of the $2NV$-point function, which we expect to provide the leading order contribution to the divergence degree.

From a qualitative point of view, a truly general result in GFT would require to consider a fully interacting GFT and to drop the i.i.d. assumption. Moreover, it would require to leave the *tensor*

[7] Note that, if we keep the indistinguishability condition, the calculation changes

$$\begin{aligned} \mathbb{E}_0\left[\bigotimes_v (|\phi_v\rangle\langle\phi_v|)^{\otimes N}\right] &= \mathcal{C} \sum_{\pi \in \mathcal{S}_{NV_\Gamma}} \mathbb{P}(\pi) \\ &= \mathcal{C} \sum_{\pi \in \mathcal{S}_{NV_\Gamma}} \prod_{a=1}^{N} \prod_{n=1}^{V_\Gamma} \int \mathrm{d}h_{na} \mathrm{d}\mathbf{g}_{na} \mathrm{d}\mathbf{g}'_{\pi(na)} \, \delta\left(h_{na}\mathbf{g}_{na}{\mathbf{g}'_{\pi(na)}}^{-1}\right) |\mathbf{g}_{na}\rangle\langle\mathbf{g}'_{\pi(na)}|, \end{aligned} \tag{58}$$

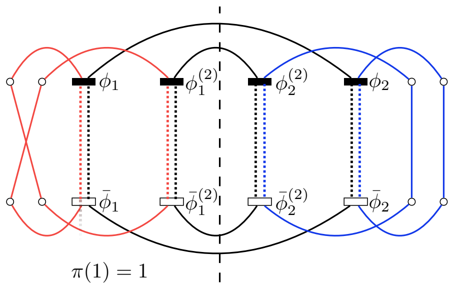
where $\mathcal{S}_{NV_\Gamma}$ is the permutation group of NV_Γ objects, which corresponds to the permutations of NV_Γ nodes; h_{na} comes from the required gauge symmetry of the propagator; $\mathcal{C}$ is a constant factor.

network representation of the GFT multi-particle operator, and derive the entropy for the generic operator given in (15).

4.2. Mapping to BF Theory Partition Function

The partition functions Z_N and Z_0^N correspond to summations of two *auxiliary* networks $\mathcal{N}_A(\mathbf{h}_v, \boldsymbol{\pi}_v)$ and $\mathcal{N}_0(\mathbf{h}_v, \boldsymbol{\pi}_v)$. We shall proceed by giving the main ingredients of the derivation, while referring the reader to the original literature (see e.g., [18]) for a fully detailed description of the combinatorics of the calculation.

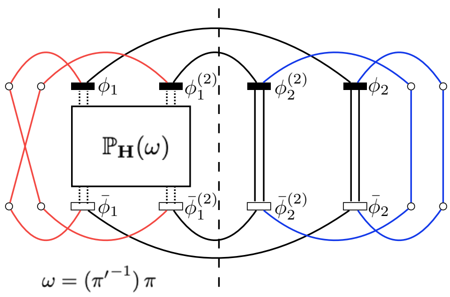
First of all, in opening the expressions for Z_N and Z_0^N, we shall notice that the action of $\mathbb{P}_{\mathbf{h}_v}(\pi_v)$ at each node is decoupled among the incident legs. Due to the strandwise action of the propagator, the value of the networks $\mathcal{N}_A(\mathbf{h}_v, \boldsymbol{\pi}_v)$ and $\mathcal{N}_0(\mathbf{h}_v, \boldsymbol{\pi}_v)$ can be written as factorised products over internal (e.g., trivial propagators acting in the picture below) and boundary links



$$\mathcal{N}_A(\mathbf{h}_v, \boldsymbol{\pi}_v) = \prod_{e\in\Gamma} \mathcal{L}_e(\pi_v, \pi_{v'}; \mathbf{h}_v, \mathbf{h}_{v'}) \prod_{e\in A} \mathcal{L}_e(\pi_v, \pi_A^0; \mathbf{h}_v) \prod_{e\in B} \mathcal{L}_e(\pi_v, \mathbb{1}; \mathbf{h}_v), \tag{59}$$

$$\mathcal{N}_0(\mathbf{h}_v, \boldsymbol{\pi}_v) = \prod_{e\in\Gamma} \mathcal{L}_e(\pi_v, \pi_{v'}; \mathbf{h}_v, \mathbf{h}_{v'}) \prod_{e\in\partial\Gamma} \mathcal{L}_e(\pi_v, \mathbb{1}; \mathbf{h}_v). \tag{60}$$

On the internal (dotted) links, $\mathcal{L}(\pi, \pi', \mathbf{h}, \mathbf{h}')$ can be written as a trace of a modified representation of a permutation group element $\varpi \equiv (\pi')^{-1}\pi$, such that



$$\mathcal{L}(\pi, \pi'; \mathbf{h}, \mathbf{h}') = \mathrm{Tr}\left[\mathbb{P}_{\mathbf{h}}(\pi)\rho_e^N \mathbb{P}_{\mathbf{h}'}(\pi)\right] = \mathrm{Tr}\left[\mathbb{P}_{\mathbf{h}}\left((\pi')^{-1}\pi\right)\right] \equiv \mathrm{Tr}\left[\mathbb{P}_{\mathbf{h}}\left(\varpi\right)\right], \tag{61}$$

where

$$\mathbf{h} = \left\{ H_a \mid H_a \equiv \left(h'_{\varpi(a)}\right)^{\dagger} h_a,\ \forall a = 1, \cdots, N \right\}.$$

In particular, any element $\varpi \in \mathcal{S}_N$ can be expressed as the product of disjoint cycles $\mathcal{C}_i$

$$\varpi \equiv \prod_i^{\chi(\varpi)} \mathcal{C}_i, \tag{62}$$

leading to a specific simple form for the individual link contributions

$$\begin{aligned}\mathcal{L}(\pi,\pi';\mathbf{g},\mathbf{g}') &= \mathrm{Tr}\left[\mathbb{P}_{\mathbf{g}}(\varpi)\right] = \prod_i \mathrm{Tr}\left[\mathbb{P}_{\mathbf{g}}(\mathcal{C}_i)\right] \\ &= \prod_i \int \prod_{k=1}^{r_i} \mathrm{d}g_{a^i_k}\, \delta\left(H_{a^i_k} g_{a^i_k} g^\dagger_{a^i_{[k]r_i+1}}\right) = \prod_i^{\chi(\varpi)} \delta\left(\overleftarrow{\prod_{k=1}^{r_i}} H_{a^i_k}\right),\end{aligned} \tag{63}$$

which is expressed as the product of the traces of the individual cycles $\mathcal{C}_i$. Indeed, one can eventually realise [18,20] that the integral of the pattern networks $\mathcal{N}(\mathbf{h}_v, \boldsymbol{\pi}_v)$ on the gauge holonomies are equivalent to the amplitudes of a three-dimensional topological BF field theory [50], with given boundary condition. Such amplitudes are discretized on a specific 2-complex comprised by the N replicas of the networks, with each different pattern $\mathbb{P}$ corresponding to a different 2-complex. The simple form of the various functions entering the calculation of the entropy directly follows from the specific approximations used in the calculation of expectation values, namely the choice to neglect the GFT interactions and to consider tensor-network like observables with simple delta functions associated with the links' kernels.

In the local averaging setting, the expectation values of the partition functions therefore reduce to contractions of single node $2N$-point functions of the GFT model, expanded in series of Feynman amplitudes and corresponding to a local BF gravity spin-foams (see Figure 3). In particular, if one specifies to a 3-valent GFT field theory, with simplicial four-field interaction kernels, each domain amplitude will correspond to a Boulatov (or 3-*d* BF theory) spinfoam amplitude, whose semi-classical limit coincides with Ponzano–Regge gravity [50].

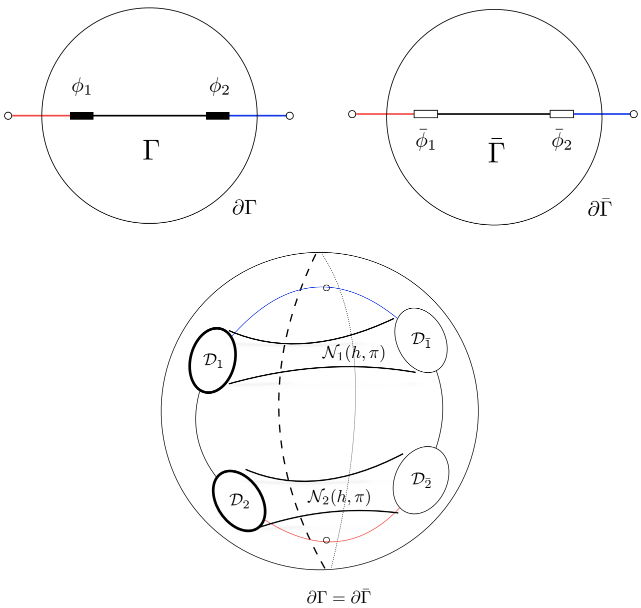


Figure 3. (Above) Simple open graph associated with a boundary tensor state (and its conjugate). (Down) Cartoon picture of the expectation value of the partition function defined by the trace of the boundary tensor state density matrix. The N replicas of the nodes of the graph define domain regions $\mathcal{D}$, connected among them via (replicas of) the graph links. The boundary links are contracted among the conjugate graphs via a global trace. The expectation value generates a set of independent spin-foam channels $\mathcal{N}$ among conjugate domains (GFT $2N$-point functions), due to the i.i.d. assumption considered. Indistinguishability would instead merge the different spin-foams into a single one, raising the degree of correlations of the boundary degrees of freedom. Nevertheless, the local approximation seems to convey the leading order contribution of the quantum entanglement of the bipartite system.

4.3. Entanglement Scaling and Divergence Degree

Given the expression for the Rényi entropy, $e^{(1-N)S_N} = Z_N/Z_0^N$, finding the scaling of the entanglement entropy amounts to identifying the most divergent terms of the partition functions Z_N and Z_0^N, which corresponds to the divergence degree of the BF theory amplitudes discretized on a lattice [58–62]. The recipe given in [18] consists of a combination of coarse graining and combinatorics.

Once given global boundary conditions for $\mathcal{N}_0$, $\pi = \mathbb{1}$ and $\boldsymbol{h} = \mathbb{1}$, one can coarse-grain the boundary of $\mathcal{N}_0$ into a single node with $\pi = \mathbb{1}$ and $\boldsymbol{h} = \mathbb{1}$. Accordingly, the boundary of $\mathcal{N}_A$ gets coarse-grained into two nodes, one of which corresponds to A with $\pi = \mathcal{C}_0$, $\boldsymbol{h} = \mathbb{1}$ and the other to B with $\pi = \mathbb{1}$ and $\boldsymbol{h} = \mathbb{1}$. The corresponding closed graphs are denoted as Γ_0 and Γ_{AB}. Once we take the expectation value on the N replicas at each node, then a given pattern $\mathbb{P}(\boldsymbol{\pi_v})$ divides Γ_0 and Γ_{AB} into regions (set of nodes), each one coloured with permutation group $\boldsymbol{\pi_m}$ and N integrals over $\boldsymbol{h_m}$ (see Figure 4 below).

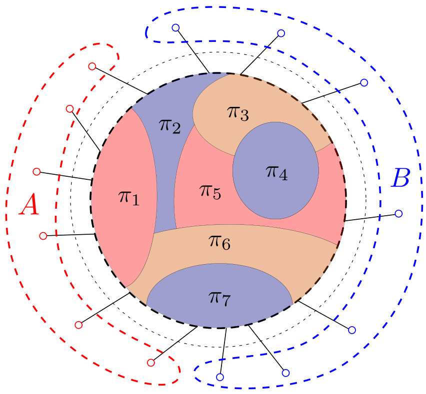


Figure 4. Example of a permutation pattern $\mathbb{P}(\boldsymbol{\pi_v})$ dividing Γ_{AB} into regions (set of nodes), each one colored with permutation group element $\boldsymbol{\pi}$ and N integrals over holonomies $\mathbf{h}$.

In full analogy with the S_N Ising model [15], links connecting different regions identify boundaries which can be interpreted as domain walls. We are interested in finding the scaling behaviour of the Rényi entropy in the large bond regime $D(\Lambda) \gg 1$, classically corresponding to the long-range ordered phase for such an Ising-like model, where the entropy of a boundary region is known to be directly related to the energy of the domain wall between domains of the order parameter [15].

Different local regions R, with uniform boundary conditions, can be further coarse-grained via gauge invariance to single block nodes. For a given region, the degree of divergence is counted by the number of links in the region, minus the number of trivialised links on a maximal spanning tree T_R

$$\#_R = (\#_{e\in R} - \#_{T_R})\, N. \tag{64}$$

One can then generally show that the pattern with no domain walls, Γ_0, corresponding to the ordered phase in which all nodes are assigned with the same permutation group, has the highest degree of divergence

$$\#_0 = (\#_{e\in\Gamma_0} - \#_{T_{\Gamma_0}})N, \tag{65}$$

where $\#_{e\in\Gamma_0}$ is the number of links in the Γ_0 graph.

For the bipartite graph Γ_{AB}, defined by the assignment of different boundary conditions for A and B, patterns with a single domain wall have higher divergence degree than the multi-domain walls configurations. The coarse-grained graph contains only two block nodes and one finds [18]

$$\#_{\Gamma_{AB}(\pi_m)} \leq \quad \#_{AB} = \#_0 + (1-N)\min(\#_{e\in\partial_{AB}}). \tag{66}$$

The leading order divergence terms of the amplitudes are given by

$$\begin{aligned} Z_0^N &= C^{V_\Gamma}[D(\Lambda)]^{\#_0}, \\ Z_N &= C^{V_\Gamma}[D(\Lambda)]^{\#_0+(1-N)\min(\#_{e\in\partial_{AB}})}\left[1+\mathcal{O}(D(\Lambda)^{-1})\right] \quad , \end{aligned}$$

and the Nth order Rényi entropy S_N eventually reads

$$e^{(1-N)S_N} = \frac{Z_N}{Z_0^N} = [D(\Lambda)]^{(1-N)\min(\#_{e\in\partial_{AB}})}\left[1+\mathcal{O}(D(\Lambda)^{-1})\right]. \tag{67}$$

When N goes to 1, the leading term of the entanglement entropy S_{EE} is given by

$$S_{\text{EE}} = \min(\#_{e\in\partial_{AB}})\ln D(\Lambda), \tag{68}$$

where $D(\Lambda)$ with $\Lambda \gg 1$ reads as a regularisation of each BF bubble divergence $\delta(\mathbb{1})$.

The result, within a different formal setting, reproduces the universal behaviour typical of random tensor network states [15]. However, differently from the case of a standard random tensor network, the considered GFT states carry an inherent geometric characterisation. The graph Γ is dual to a 2D simplicial complex. Each node is dual to a triangle and each link is dual to an edge of this complex, and the specific GFT model endows the simplicial complex with dynamical geometric data. In particular, the proportionality of the entropy to the cardinality of the minimal domain wall $\sigma_{\min} \equiv \min(\#_{e\in\partial_{AB}})$ has a clear geometric interpretation, which becomes apparent in passing from a group element to a spin representation description of the dynamical fields, in the sense of discrete geometry. In this case, indeed, we have

$$\text{Area}(\sigma_{\min}) = \sum_{e\in\sigma_{\min}} \ell_e(j_e) = \langle \ell_{j_e}\rangle |\sigma_{\min}|, \tag{69}$$

where the length of each edge ℓ_j, in any given eigenstate of the length operator [26], is a function of the irreducible representation j_e associated with it, and to the dual link. Therefore, the cardinality of the minimal global domain wall can be interpreted as the $1D$-area (length) of a dual discrete minimal one-dimensional path and we can write $|\sigma_{\min|} = \text{Area}(\sigma_{\min})/\langle \ell_{j_e}\rangle$. [20].

Equation (68) can then be understood as the discrete tensor network analogue of the Ryu–Takayanagi formula in the context of group field theory [21], if we consider the path integral averaging over the open network Γ as a simplified model of a bulk/boundary (spinfoam/network state) duality [18].

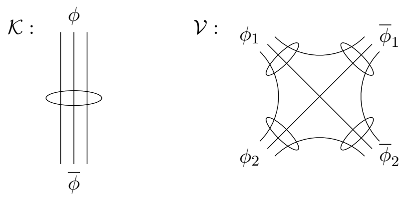
5. Holographic Scaling for Interacting with GFT

A complete geometric characterisation of the result requires working with a fully interacting GFT model. Indeed, it is the simplicial character of the interaction kernels that actually provides a connection between GFT and quantum gravity spin-foams. In this sense, both consistency and robustness of the result in (68) require exploring the possible modifications to the RT formula induced by group field interactions.

At the perturbative level, this amounts to calculating the first order correction of $\mathbb{E}[S_N(A)]$ in the GFT coupling constant λ, namely $\mathbb{E}[\text{Tr}(\rho^{\otimes N}\mathbb{P}_A)]$ and $\mathbb{E}[\text{Tr}(\rho^{\otimes N})]$, with

$$\begin{aligned} \mathbb{E}[\text{Tr}(\rho^{\otimes N}\mathbb{P}_A)] &= \mathbb{E}_0[\text{Tr}(\rho^{\otimes N}\mathbb{P}_A)] + \lambda\mathbb{E}_0\left[S_{\text{int}}[\phi,\overline{\phi}]\text{Tr}(\rho^{\otimes N}\mathbb{P}_A)\right], \\ \mathbb{E}[\text{Tr}(\rho^{\otimes N})] &= \mathbb{E}_0[\text{Tr}(\rho^{\otimes N})] + \lambda\mathbb{E}_0\left[S_{\text{int}}[\phi,\overline{\phi}]\text{Tr}(\rho^{\otimes N})\right]. \end{aligned} \tag{70}$$

This is a combinatorially highly non-trivial problem, as the interaction processes correspond to further stranded diagrams that contribute to the expectation value of $Z^{(N)}_{A/0}$ (see Figure 5). The behaviour of the amplitudes for rank-3 fields and Boulatov (simplicial) four fields interaction kernel $\mathcal{V}^{\text{sym}}(\mathbf{g}^{(1)}\mathbf{g}^{(2)}\underline{\mathbf{g}}^{(1)}\underline{\mathbf{g}}^{(2)})$



$$\int \prod_{\ell=1}^{4} dh_\ell\, \delta(h_1 g_1^{(1)}, h_3 \underline{g}_1^{(1)})\delta(h_1 g_2^{(1)}, h_4 \underline{g}_2^{(2)})\delta(h_1 g_3^{(1)}, h_2 g_3^{(2)})$$
$$\delta(h_2 g_1^{(2)}, h_4 \underline{g}_1^{(2)})\delta(h_2 g_2^{(2)}, h_3 \underline{g}_2^{(1)})\delta(h_3 \underline{g}_3^{(1)}, h_4 \underline{g}_3^{(2)})$$

is studied in [20]. Therein, the authors provide a series of theorems aimed at constraining the complexity of the combinatorial pattern. In particular, it was shown that patterns with a single interaction happening between two incoming and two outgoing fields of the same network node v leads at most to the same number of divergences as in a maximal case of the free theory, where the interaction at v is replaced by a free propagation [20].

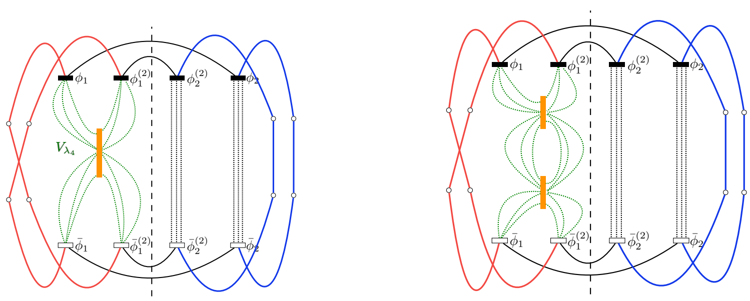


Figure 5. Examples of insertions of Boulatov 4-fields interaction vertices $\mathcal{V}$ in the amplitude among replicas of the individual node of the graph. Multiple of interaction vertices generate new faces in the amplitudes.

Interestingly, the situation changes if one drops the assumption of gauge invariance (closure) for the GFT field. In this case, the expected Nth Rényi entanglement entropy is estimated to be

$$\mathbb{E}_{\text{n-sym}}[S_N(\rho_A)] \approx [\ln D(\Lambda) - \lambda N] \min(\#_{e\in\partial_{AB}}). \tag{71}$$

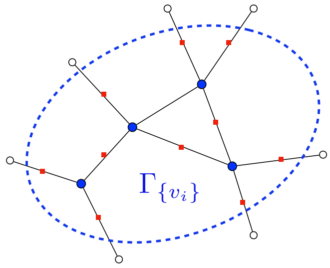
The linear order corrections do modify the asymptotical scaling of the Rènyi entanglement entropy with the area of a minimal surface. In particular, the results in [20] show that the proportionality factor is corrected by an additive term linear in the perturbed group field theory coupling constant λ. However, no additive leading order correction to the area scaling entropy formula emerges from the analysis. A more systematic analysis of such dynamical regularisation and its relation with the emergence of an effective gravitational coupling is an open interesting issue for future work.

6. Discussion and Conclusions

Group field theories realise a kinematical description of quantum space geometry in terms of discrete, pre-geometric degrees of freedom of combinatorial and algebraic nature, described in terms of spin-network states. In loop quantum gravity (LQG), such theories play the role of *auxiliary* field theories, whose partition functions, for appropriate choices of the kernels, provide generating functionals for the LQG spinfoams: a covariant path integral realisation of spacetime as a transition amplitude between boundary spin network states. More generally, GFTs provide a versatile field-theoretic tool to study the very emergence of space-time quantum geometry via path integral techniques and a quantum many-body approach associated with their second quantised formalism [57,63]. Moreover, as higher order generalisations of matrix models, group field theories can be put in direct formal relation with tensor network algorithms [10–12,17,18] in condensed matter theory.

In this review, we considered a setting *similar* to that of a gauge field theory on a lattice, which in the background independent quantum gravity context consists of nets of fundamental quanta of space that admit an interpretation as quantized fuzzy geometries.

$$|\Psi_\Gamma\rangle = \sum_{\{v_i\}\in\mathcal{V}_\partial}$$



$$\bigotimes_i^{|V_\partial|} |v_i\rangle$$

The entanglement structure of the wave-function for such collections of quanta is encoded into a ∞-rank tensor of coefficients, which we interpret here as an *open* GFT network state. In [18–20], such an open GFT spin network is interpreted as a *tensor network representation* of some quantum geometry wave-function, with physical indices corresponding to boundary degrees of freedom of the "auxiliary" GFT network state, realised by a spin network.

In this setting, the degree of entanglement of a generic quantum region of space can be measured *holographically*, in terms of the entanglement entropy of the bipartite auxiliary spin network. Such entanglement is directly related to the topology of the internal network, which, differently from many similar derivations in quantum gravity, is only partially fixed by the choice of the auxiliary network architecture. Indeed, due to the random character of the nodes, expectation values imply a dynamical characterisation of the internal graph topology, which end up being directly specified by the choice of the GFT model. Therefore, different GFT models a priori induce different architecture for the internal network, with higher order interactions corresponding to a higher degree of connectivity of the graph.

The GFT formalism, along the lines first proposed by [15], allows for an explicit computation of the entanglement entropy, which in [18–20] is realised within a set of simplifying assumptions:

1. indistinguishability and *i.i.d.*: the symmetry of the group field naturally selects a bosonic quantum many-body characterisation for the quanta comprising our network states. This translates into indistinguishability for the nodes comprising the network. Furthermore, along with the results in [15], in the statistical evaluation of the averaged Rényi entropy, 'tensor' fields at the nodes are individually independently distributed. This leads to a sensible simplification of the auxiliary bulk averages, reduced to a set of *local* amplitudes defined among conjugate domains of replicas of the fields at each node. Such i.i.d. assumption is a natural choice in [15], where random tensor networks are associated with maximally mixed states. In our setting, accordingly, the *i.i.d.* assumption presupposes actual *absence of dynamical interaction* among the quanta sitting at the nodes of the graph. Effectively, quanta couples through the network only via local adjacent link entanglement. Such a non-interacting quantum many-body assumption is very strong. Nevertheless, it is consistent with the simplifying choice of dealing with *product* coherent states in the adopted 2nd quantised formalism. On the other hand, the i.i.d. assumption embodies some a priori knowledge of the position of the individual nodes (their replicas and conjugates) on the graph, hence somehow violating indistinguishability.

2. maximally entangled bonds: the class of GFT tensor networks considered is further radically characterised by link states given by maximally entangled pair states, associated with gluing kernels realised in terms of bivalent intertwiners with $\delta\left(g_{s(\ell)}h_\ell g_{t(\ell)}^{-1}\right)$ coefficients. In principle, the choice of the link state can be generalised to a more general kernel, while still being expressed in terms of the mutual information among the half link states [14].
3. propagator holonomies are set to $h_\ell = \mathbb{1}$ for all link $\ell \in \Gamma$: this assumption makes the state $|\Psi_\Gamma\rangle$ lying in the flat vacuum state recently considered in the context of Loop Quantum Gravity [64].
4. finite link space (bond) dimension D: while taking a large leg space dimension $D \gg 1$ limit, we always deal with a finite-dimensional restriction of the leg spaces $L^2[G,\mu]$, obtained through the introduction of a sharp cut-off Λ in the group representation, such that $\delta(g) = D(\Lambda)$, for $g \in G$. More radically, one could regularise the divergences via "box" normalization of $\delta(g) \in L^2[G,\mu]$ by using quantum groups. Interestingly, such a quantum deformation can be related to the cosmological constant Λ in the semi-classical regime of the spinfoam formalism (see e.g., [65,66]).
5. tensor network setting: we work with regular networks of 3-valent nodes. The GFT interaction kernels adopted correspond to the ones defining the Boulatov model for 3D gravity. The coupling λ of the interaction term is always assumed to be much smaller than 1, allowing for a perturbative expansion of the expectation value of the Rényi entropy in λ.

As a result, the set of assumptions above induces a structure of correlations that is essentially local in its *leading* contribution to the entropy, even when the interaction among fields at different nodes is considered [18,20]. This suggests that the emergence of the dominant holographic behaviour for the entropy is intertwined with the universality features of the large bond regime. In the presence of (weak) interactions, such area scaling behaviour is shown to remain a solid feature of the *quantum typical regime* [20]. [8]

Many aspects can be tuned to further test such a holographic behaviour in the specific aproach. At the level of the free theory, it would be interesting to look at changes in the area scaling due to minimal modification of the very free propagator, via more general heat-kernel techniques, possibly expressed in terms of a different choice of mutual information among the half link states [14]. For higher order perturbations, where the diagrams are dominated by the bulk structure, induced by the creation of new interaction vertices [20], we may expect significant changes in the area scaling behaviour, even if the local averaging scheme induced by the i.i.d. assumption is preserved. Outside the typicality regime, it is natural to expect that fluctuations in the annealed average (43) become more and more relevant and the area scaling behaviour is eventually lost. A detailed study in this sense requires strong computational efforts, necessary to explore the properties and combinatorics of the dynamically induced bulk auxiliary networks.

More generally, concerning the potential of the GFT formalism in the study of holographic entanglement in quantum geometry, much still remain to be understood. We showed how the formalism of *second quantization* for GFT, and the use of a coherent state basis, allow for a new approach to the study of entanglement in quantum geometry. Standard quantum mechanical correlations among links and nodes of the quantum many-body network state are replaced by correlations among *excitations* of modes of the tensor field representing the particle. This is likely the right formalism for understanding the diverse roles played by entanglement in quantum gravity, from the local entanglement among single quanta responsible for the connectivity of a quantum geometry configuration, to the local and non-local entanglement among modes of the GFT field, intended as elementary excitations induced by the *collective* behaviour of the underlying many-body quantum system. This would effectively allow for maximally disregarding the auxiliary spin network structures and work in full indistinguishability with bosonic GFT quanta of geometry in the Fock space.

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8 A possibly deep relation between such universal regime and the universality proper of tensor models in the large D regime (see e.g., [67]) has been pointed out and discussed in [18].

However, harnessing entanglement in the 2nd quantisation formalism is still a fundamental open problem already in condensed matter and quantum information theory [68–70]. In our derivation, the second quantisation scheme was in fact only partially realised. Indistinguishability would have actually prevented us from defining an AB-factorisation of the Hilbert space. In this regard, we proceeded by selecting a single graph Hilbert space within the full Fock space, hence coming back to a first quantisation formalism. Indistinguishability was further violated by the local averaging approach, in Section 4.1, induced by the i.i.d. assumption. A GFT derivation that makes full use of the mode aspect of second quantization is therefore substantially left for future work. Success in this sense would be evocative of (and consistent with) the general perspective that sees continuum spacetime and geometry as emergent from the collective, quantum many-body description of the fundamental GFT degrees of freedom, the same perspective motivating a large part of the literature and in particular the one concerned with GFT renormalization (both perturbative and non-perturbative) [71–82].

Most recent work in this direction focussed on entanglement among GFT modes induced by interaction, starting from a generalization of the Bogoliubov description of a weakly interacting Bose gas to the GFT framework [83]. Analogous insights on the inherently dynamic character of entanglement among GFT modes in second quantization appear in the series of recent results proposing a general procedure for constructing states that describe macroscopic, spatially homogeneous universes as Bose–Einstein condensates (see e.g., [53]). Strong results in this sense would provide a new quantitative tool to unravel the relation between entanglement, holography and emergence of spacetime geometry in quantum gravity.

For instance, an interesting direct correspondence between black holes and Bose–Einstein condensates (BECs) of gravitons at the point of maximal packing was recently proposed in [84], though within a standard *effective field theory framework*. In this case, the physics of maximally-packed gravitational systems is identified with the general behaviour of BECs at the critical point of the quantum phase transition, while collective nearly gapless excitations of the quantum condensate are shown to define the holographic degrees of freedom responsible for the known semiclassical holographic properties of black holes (BH) [84]. In this exemplary case, GFTs would provide a unique *non-perturbative* formalism to investigate the foundations of black hole entropy [85] and holography as a general quantum phenomenon of nature.

Funding: This research received no external funding.

Acknowledgments: It is a pleasure to acknowledge Daniele Oriti, Mingyi Zhang and Alex Goeßmann for their precious contribution in the exploration of such a fascinating topic. I am particularly grateful to Hermann Nicolai and to the Albert Einstein Institute in Potsdam, for the support received during the preparation of the manuscript.

Conflicts of Interest: The author declare no conflict of interest.

Appendix A. Coherent States Over-Completeness

Consider two coherent state $|\varphi\rangle$ and $|\varphi'\rangle$. One can prove that they are not orthogonal

$$\langle\varphi|\varphi'\rangle = \frac{1}{\mathcal{N}_\varphi \mathcal{N}_{\varphi'}} \exp\left[\int \mathrm{d}\mathbf{g}\, \overline{\varphi(\mathbf{g})}\varphi'(\mathbf{g})\right]. \tag{A1}$$

The state $|\varphi\rangle$ can be decomposed by the n-particle state basis $|\cdots, \mathbf{g}_a, \cdots\rangle$ as

$$|\varphi\rangle = |0\rangle\langle 0|\varphi\rangle + \sum_{n=1}^{\infty} \int \prod_{a=1}^{n} \mathrm{d}\mathbf{g}_a\, |\mathbf{g}_1, \cdots, \mathbf{g}_n\rangle \langle \mathbf{g}_1, \cdots, \mathbf{g}_n|\varphi\rangle, \tag{A2}$$

where $\langle \mathbf{g}_1, \cdots, \mathbf{g}_n | \varphi \rangle$ is given as

$$\begin{aligned} \langle \mathbf{g}_1, \cdots, \mathbf{g}_n | \varphi \rangle &= \frac{1}{\mathcal{N}_\varphi} \frac{1}{\sqrt{n!}} \prod_a^n \left[\int \mathrm{d}\mathbf{g}'_a \, \varphi(\mathbf{g}'_a) \right] \langle \mathbf{g}_1, \cdots, \mathbf{g}_n | \mathbf{g}'_1, \cdots, \mathbf{g}'_n \rangle \\ &= \frac{1}{\mathcal{N}_\varphi} \frac{1}{(n!)^{3/2}} \sum_{\pi \in \mathcal{S}_n} \prod_a^n \left[\int \mathrm{d}\mathbf{g}'_a \, \mathrm{d}h_a \, \varphi(\mathbf{g}'_a) \delta(\mathbf{g}_{\pi(a)} h_a (\mathbf{g}'_a)^\dagger) \right] \\ &= \frac{1}{\mathcal{N}_\varphi} \frac{1}{\sqrt{n!}} \prod_a^n \varphi(\mathbf{g}_a). \end{aligned} \tag{A3}$$

In order to obtain the resolution of identity in terms of $|\varphi\rangle$, let us first introduce the gauge fixed field φ_{GF}

$$\varphi_{\mathrm{GF}}(\mathbf{g}) \equiv \varphi(\mathbf{g} h_1^{-1}) = \varphi(\mathbb{1}, \, h_2 h_1^{-1}, \, h_3 h_1^{-1}, \, h_4 h_1^{-1}) \equiv \varphi([\mathbf{g}]). \tag{A4}$$

Then, we have the identities

$$\begin{aligned} &\int \mathcal{D}\varphi \mathcal{D}\overline{\varphi} \, \delta_{\mathbb{C}}[\varphi - \varphi_{\mathrm{GF}}] \, \mathrm{e}^{-K \int \mathrm{d}\mathbf{g} \, \overline{\varphi(\mathbf{g})} \varphi(\mathbf{g})} \langle \mathbf{g}_1, \cdots, \mathbf{g}_n | \varphi \rangle \langle \varphi | \mathbf{g}'_1, \cdots, \mathbf{g}'_n \rangle \\ =\ & \frac{1}{n!} \int \mathcal{D}\varphi \mathcal{D}\overline{\varphi} \, \delta_{\mathbb{C}}[\varphi - \varphi_{\mathrm{GF}}] \, \mathrm{e}^{(-K-1) \int \mathrm{d}\mathbf{g} \, \overline{\varphi(\mathbf{g})} \varphi(\mathbf{g})} \prod_a^n \varphi(\mathbf{g}_a) \overline{\varphi(\mathbf{g}'_a)} \\ =\ & \frac{1}{n!} \int \mathcal{D}\varphi_{\mathrm{GF}} \mathcal{D}\overline{\varphi_{\mathrm{GF}}} \, \mathrm{e}^{(-K-1) \int \mathrm{d}\mathbf{g} \, \overline{\varphi_{\mathrm{GF}}(\mathbf{g})} \varphi_{\mathrm{GF}}(\mathbf{g})} \prod_a^n \varphi_{\mathrm{GF}}(\mathbf{g}_a) \overline{\varphi_{\mathrm{GF}}(\mathbf{g}'_a)} \\ =\ & C \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} \prod_{a=1}^n \delta^3([\mathbf{g}'_a][\mathbf{g}^\dagger_{\pi(a)}]) \\ =\ & C \frac{1}{n!} \sum_{\pi \in \mathcal{S}_n} \prod_{a=1}^n \int \mathrm{d}h_a \, \delta^4(\mathbf{g}'_a h_a \mathbf{g}^\dagger_{\pi(a)}) \\ =\ & C \langle \mathbf{g}_1, \cdots, \mathbf{g}_n | \mathbf{g}'_1, \cdots, \mathbf{g}'_n \rangle, \end{aligned} \tag{A5}$$

where K is a parameter which we assume to be real, and C is a constant number

$$C \equiv \int \mathcal{D}\varphi \mathcal{D}\overline{\varphi} \, \delta_{\mathbb{C}}[\varphi - \varphi_{\mathrm{GF}}] \, \exp\left[(-K-1) \int \mathrm{d}\mathbf{g} \, \overline{\varphi(\mathbf{g})} \varphi(\mathbf{g}) \right]. \tag{A6}$$

For the third equality, we use the Wick's theorem, and, for the fourth equality, we reintroduce the gauge symmetry such that the arguments of φ are equal in weight.

The resolution of identity in the Fock space $\mathbb{F}$ is in terms of $|\varphi\rangle$

$$\mathbb{1}_{\mathbb{F}} = C^{-1} \int \mathcal{D}\varphi \mathcal{D}\overline{\varphi} \, \delta_{\mathbb{C}}[\varphi - \varphi_{\mathrm{GF}}] \, \mathrm{e}^{-K \int \mathrm{d}\mathbf{g} \, \overline{\varphi(\mathbf{g})} \varphi(\mathbf{g})} \, |\varphi\rangle \, \langle \varphi| \, . \tag{A7}$$

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