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On the Causal and Topological Structure of the 2-Dimensional Minkowski Space

Kyriakos Papadopoulos [1,*] **, Nazli Kurt** [2] **and Basil K. Papadopoulos** [3]

[1] Department of Mathematics, Kuwait University, PO Box 5969, Safat 13060, Kuwait
[2] Faculty of Science, Open University, PO Box 297, Milton Keynes MK7 6BJ, UK; nazkrt.96@gmail.com
[3] Department of Civil Engineering, Democritus University of Thrace, 67 100 Xanthi, Greece; papadob@civil.duth.gr
[*] Correspondence: kyriakos@sci.kuniv.edu.kw

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Abstract: A list of all possible causal relations in the two-dimensional Minkowski space M is exhausted, based on the duality between timelike and spacelike in this particular case, and thirty topologies are introduced, all of them encapsulating the causal structure of M. Generalisations of these results are discussed, as well as their significance in a discussion on spacetime singularities.

Keywords: Minkowski space; causal relations; interval topology; Alexandrov topology; intersection topology

1. Preliminaries

Throughout the text, unless otherwise stated, we consider the two-dimensional Minkowski spacetime M, which is the two-dimensional real Euclidean space equipped with the characteristic quadratic form Q, where for $x = (x_0, x_1) \in M$, $Q(x) = x_0^2 - x_1^2$.

We denote the *light cone* through an event x by $C^L(x)$, and define it to be the set $C^L(x) = \{y : Q(y - x) = 0\}$. Similarly, we define the *time cone* as $C^T(x) = \{y : y = x \text{ or } Q(y - x) > 0\}$ and the *space cone* as $C^S(x) = \{y : y = x \text{ or } Q(y - x) < 0\}$. We call *causal cone* the set $C^T(x) \cup C^L(x)$ and we observe that the event x partitions its time/light/causal cone into future and past time/light/causal cones, respectively, while it divides the space cone into $-$ and $+$, respectively.

In [1] (Paragraph 1.4), we intuitively (i.e., in a topological sense, invariantly from a change in the geometry) partitioned the light-cone so that, apart from future and past, we also achieved a spacelike separation of $+$ and $-$. This space-like separation is more obvious in the two-dimensional Minkowski spacetime M. Let $x \in M$ be an event. Then, we consider the future and past time-cones, $C^T_+(x)$ and $C^T_-(x)$, respectively, as north and south in a compass, while the space-cones $C^S_+(x)$ and $C^S_-(x)$, respectively, as east and west.

We denote the Euclidean topology on $\mathbb{R}^2$ by E; this topology has a base of open sets which are open balls $B_\epsilon(x)$, of radius ϵ and centre x. Arbitrary unions of such open balls give the open sets in $\mathbb{R}^2$ under E.

Zeeman [2] (as a result of his previous work [3]) questioned the use of Topology E in four-dimensional Minkowski space, as its "natural" topology, listing a number of issues, including that the Euclidean topology is locally homogeneous (while M is not) and the group of all homeomorphisms of (four-dimensional) Euclidean space is of no physical significance. Zeeman proposed a topology, his "Fine" topology, under which the group of all homeomorphisms is generated by the (inhomogeneous) Lorentz group and dilatations. In addition, the light, time and space cones through a point can be deduced from this topology. Göbel [4] generalised Zeeman's results for curved spacetime manifolds, and obtained that, under a general relativistic frame, the Fine topology

gives the significant result that a homeomorphism is an isometry. Hawking, King and McCarthy [5] introduced the "Path" topology, which determines the causal, differential and conformal structure of a space-time, but it was proven by Low [6] that the Limit Curve Theorem under the Path topology fails to hold, and thus the formation of basic singularity theorems. Given that the questions that were raised by Zeeman [2] are of a tremendous significance for problems related to the topological, geometrical and analytical structure of a spacetime, the topologisation problem for spacetimes is still open and significant.

In this article, we examine all possible (ten in number) causal relations that can appear in the two-dimensional Minkowski spacetime and the thirty topologies they induce. All these topologies incorporate the causal structure of spacetime, and we believe that a generalisation to curved four-dimensional spacetimes will equip modern problems of general relativity and cosmology with extra tools that can be used in attempts, for example, to describe the structure of the universe in the neighbourhood of the spacetime singularities that are predicted by the singularity theorems of general relativity (ambient cosmology) or contribute to the description of the transition from the quantum non-local theory to a classical local theory.

2. Causal Relations in the Two-Dimensional Minkowski Space

We consider the two-dimensional Minkowski spacetime M, equipped with the following relations:

1. $\ll$: The *chronological* partial order, defined as $x \ll y$, if $y \in C_+^T(x)$. We note that $\ll$ is irreflexive.
2. $\rightarrow$: The relation *horismos*, defined as $x \rightarrow y$, if $y \in C_+^L(x)$. Horismos is a reflexive relation.
3. $<$: The *chorological* ("choros" is the Greek for "space", similar to "chronos" is the Greek for "time") partial order, defined as $x < y$, if $y \in C_+^S(x)$. We note that $<$ is irreflexive.
4. $\rightarrow^{irr}$: We define the *irreflexive horismos* in a similar way as we defined $\rightarrow$, this time without permitting x to be at horismos with itself.
5. $\ll^{=}$: We define the *reflexive chronology* as we defined $\ll$, but this time we permit x to chronologically precede itself.
6. $\prec$: The *causal* order is a reflexive partial order defined as $x \prec y$ if $y \in C_+^T(x) \cup C_+^L(x)$.
7. $\ll^{\rightarrow^{irr}}$: We define the *irreflexive causal order* as we defined $\prec$, this time excluding the case that $x \prec x$.
8. $\leq$: We define the *reflexive chorology* as we defined $<$, but this time we permit x to chorologically precede itself.
9. $\ll^c$: The *complement of chronological* order is a reflexive partial order defined as $x \ll^c y$ if $y \in C_+^S(x) \cup C_+^L(x)$.
10. $<^{\rightarrow^{irr}}$: We define the *irreflexive complement of chronological order* as $\ll^c$ excluding the case that $x \ll^c x$.

Definition 1. *Let $f : M \rightarrow M$ be a one-to-one (and not necessarily continuous or linear) map. We say*

1. *f is a causal automorphism, if both f and f^{-1} preserve $\ll$, i.e., $x \ll y$ iff $f(x) \ll f(y)$; and*
2. *f is an acausal automorphism, if both f and f^{-1} preserve $<$, i.e., $x < y$ iff $f(x) < f(y)$.*

The causal automorphisms form the *causality group* and the acausal automorphisms form the *acausality group*.

The proofs of Lemmas 1 and 2 can be found in [3].

Lemma 1. *Let $f : M \rightarrow M$ be a one-to-one map. Then, f, f^{-1} preserve $\ll$ iff f, f^{-1} preserve $\rightarrow$.*

Lemma 1 does not hold for $<$, for the obvious reason that $x \rightarrow y$, iff either x does not chronologically precede y or $y \ll z$ implies $x \ll z$. Consequently, Lemma 1 does not hold for Relations 8, 9 and 10, while it holds for Relations 2, 6 and 7.

Lemma 2. *A causal automorphism maps:*

1.   *light rays to light rays;*
2.   *parallel light rays to parallel light rays;*
3.   *each light ray linearly; and*
4.   *parallel equal intervals on light rays to parallel equal intervals.*

Lemma 2 does not hold for an acausal automorphism, for similar reasons that $<$ fails to satisfy Lemma 1.

The *orthochronous* Lorentz group consists of all linear maps of M which leave Q invariant, preserve time orientation (south-to-north) but possible reverse space orientation. In the two-dimensional Minkowski space M, the *orthochorous* Lorentz group consists of all linear maps of M which leave Q invariant, preserve space-orientation (west-to-east) but possibly reverse time orientation.

3. Thirty Causal Topologies on the Two-Dimensional Minkowski Space

Consider an order relation R defined on a space X. Then, consider the sets $I^+(x) = \{y \in X : xRy\}$ and $I^-(x) = \{y \in X : yRx\}$, as well as the collections $\mathcal{S}^+ = \{X \setminus I^-(x) : x \in X\}$ and $\mathcal{S}^- = \{X \setminus I^+(x) : x \in X\}$. A basic-open set U in the *interval topology* T^{in} (see [7] and [8]) is defined as $U = A \cap B$, where $A \in \mathcal{S}^+$ and $B \in \mathcal{S}^-$; that is, $\mathcal{S}^+ \cup \mathcal{S}^-$ forms a subbase for T^{in}.

The four-dimensional Minkowski space in particular (and spacetimes in general) is not up-complete, and a topology $\mathcal{T}_{in}$ is weaker than the interval topology of [7], but for the particular case of two-dimensional Minkowski spacetime, $\mathcal{T}_{in}$ under the ten causal relations that we stated above is the actual interval topology defined in [7].

The *Alexandrov* topology (see [9]) is the topology that has basic open sets of the form $I^+(x) \cap I^-(y)$, where $I^+(x) = \{y \in M : x \ll y\}$ and $I^-(y) = \{x \in M : y \ll x\}$. In general, a spacetime manifold M is strongly causal iff the Alexandrov topology is Hausdorff iff the Alexandrov topology agrees with the manifold topology.

Finally, If T_1 and T_2 are two distinct topologies on a set X, then the *intersection topology* T^{int} (see [10,11]) with respect to T_1 and T_2, is the topology on X such that the set $\{U_1 \cap U_2 : U_1 \in T_1, U_2 \in T_2\}$ forms a base for (X, T).

Below, we list all possible order topologies that are generated by the ten causal relations above, either by defining the topology straight from the order (in a similar way the Alexandrov topology is induced by $\ll$-open diamonds) or as interval Topologies $\mathcal{T}_{in}$ or as intersection topologies (in the sense of Reed) between the natural Topology E of $\mathbb{R}^2$ and $\mathcal{T}_{in}$.

1.   The chronological order $\ll$ induces the Topology $T_\ll$, which has a subbase consisting of future time cones $C_+^T(x)$ or past time cones $C_-^T(y)$, where $x, y \in M$. The finite intersections of such subbasic-open sets give "open timelike diamonds", which are basic-open sets for the Alexandrov topology.
2. $\ll$ also induces the interval Topology $T_{in}^\ll$, with subbase consisting of sets $M \setminus C_+^T(x)$, which are complements of future time cones or sets $M \setminus C_-^T(x)$ which are complements of past time cones. This topology has basic-open sets of the form $C^S(x) \cup C^L(x)$ and it is easy to see that it is incomparable (neither finer, nor coarser, nor equal) to the natural Topology E, on M.
3.   The Topologies E and $T_{in}^\ll$, on M, give the intersection Topology $Z_{in}^\ll$, which has basic-open sets of the form $B_\epsilon(x) \cap [C^S(x) \cup C^L(x)]$ and is finer than the Topology E.
4.   The relation horismos $\rightarrow$ induces the Topology $T_\rightarrow$, which has a subbase consisting of future light cones $C_+^L(x) \cup \{x\}$ or past light cones $C_-^L(y) \cup \{y\}$, where $x, y \in M$. The finite intersections of such subbasic-open sets give the boundaries of "open diamonds" that we examined in topology 1.
5. $\rightarrow$ also induces the interval Topology $T_{in}^\rightarrow$, with subbase consisting of sets $M \setminus [C_+^L(x) \cup \{x\}]$, which are complements of future light cones union $\{x\}$ or sets $M \setminus [C_-^L(x) \cup \{x\}]$ which are

complements of past light cones union $\{x\}$. This topology has basic-open sets of the form $[C^S(x) \cup C^T(x)] \setminus \{x\}$ and it is incomparable to the natural topology of M.

6. The Topologies E and $T_{in}^{\rightarrow}$, on M, give the intersection Topology $Z_{in}^{\rightarrow}$, which has basic-open sets of the form $B_\epsilon(x) \cap [(C^S(x) \cap C^T(x)) \setminus \{x\}]$ and is a finer topology than E.

7. The chorological order $<$ induces the Topology $T_<$, which has a subbase consisting of $+$-oriented (and deleted by definition, i.e., not including x) space cones $C_+^S(x)$ or $-$-oriented (deleted) space cones $C_-^S(y)$, where $x, y \in M$. The finite intersections of such subbasic-open sets give "open diamonds" that are spacelike.

8. $<$ induces the interval Topology $T_{in}^<$, with subbase consisting of sets $M \setminus C_+^S(x)$, which are complements of $+$-oriented space cones or sets $M \setminus C_-^S(x)$ which are complements of $-$-ve oriented space cones. This topology has basic-open sets of the form $C^T(x) \cup C^L(x)$ (causal cones) and it is easy to see that it is incomparable to the natural topology of M.

9. The Topologies E and $T_{in}^<$, on M, give the intersection Topology $Z_{in}^<$, which has basic-open sets of the form $B_\epsilon(x) \cap [C^T(x) \cup C^L(x)]$ and is a topology finer than E.

10. The irreflexive horismos $\rightarrow^{irr}$ induces the Topology $T_{\rightarrow^{irr}}$, which has a subbase consisting of deleted (that is, without $\{x\}$ future light cones $C_+^L(x) \setminus \{x\}$ or deleted past light cones $C_-^S(y) \setminus \{y\}$, where $x, y \in M$. The finite intersections of such subbasic-open sets give deleted boundaries of "open diamonds".

11. $\rightarrow^{irr}$ induces the interval Topology $T_{in}^{\rightarrow^{irr}}$, with subbase consisting of sets $M \setminus [C_+^L(x) \setminus \{x\}]$, which are complements of deleted future light cones or sets $M \setminus [C_-^L(x) \setminus \{x\}]$ which are complements of deleted past light cones. This topology has basic-open sets of the form $[C^T(x) \cup C^S(x)] \cup \{x\}$ and it is easy to see that it is incomparable to the natural topology of M.

12. The Topologies E and $T_{in}^{\rightarrow^{irr}}$, on M, give the intersection Topology $Z_{in}^{\rightarrow^{irr}}$, which has basic-open sets of the form $B_\epsilon(x) \cap [(C^T(x) \cup C^S(x)) \cup \{x\}]$ and is a topology finer than E.

13. The reflexive chronology $\ll^=$ induces the Topology $T_{\ll^=}$, which has a subbase consisting of future time cones $C_+^T(x) \cup \{x\}$ or past time cones $C_-^T(y) \cup \{y\}$, where $x, y \in M$. The finite intersections of such subbasic-open sets give "closed diamonds", in the sense of a closed interval containing its endpoints.

14. $\ll^=$ induces the interval Topology $T_{in}^{\ll^=}$, with subbase consisting of sets $M \setminus [C_+^T(x) \cup \{x\}]$, or sets $M \setminus [C_-^T(x) \cup \{x\}]$. This topology has basic-open sets of the form $[C^S(x) \cup C^L(x)] \setminus \{x\}$ and it is incomparable to the natural topology of M.

15. The Topologies E and $T_{in}^{\ll^=}$, on M, give the intersection Topology $Z_{in}^{\ll^=}$, which has basic-open sets of the form $B_\epsilon(x) \cap [(C^S(x) \cup C^L(x)) \setminus \{x\}]$ and it is a topology finer than E.

16. The irreflexive causal order $\ll^{\rightarrow^{irr}}$ induces the Topology $T_{\ll^{\rightarrow^{irr}}}$, which has a subbase consisting of (deleted) future causal cones $[C_+^T(x) \cup C_+^L(x)] \setminus \{x\}$ or (deleted) past causal cones $[C_-^T(y) \cup C_-^L(y)] \setminus \{y\}$, where $x, y \in M$. The finite intersections of such subbasic-open sets give "causal diamonds" which are open (causal diamonds, i.e., together with their light boundaries), but without the endpoints.

17. $\ll^{\rightarrow^{irr}}$ induces the interval Topology $T_{in}^{\ll^{\rightarrow^{irr}}}$, with subbase consisting of sets $M \setminus [C_+^T(x) \cup C_+^L(x) \setminus \{x\}]$, which are complements of deleted future causal cones or sets $M \setminus [C_-^T(x) \cup C_-^L(x) \setminus \{x\}]$ which are complements of deleted past causal cones. This topology has basic-open sets of the form $C^S(x)$, that is space cones, and it is easy to see that it is incomparable to the natural topology of M.

18. The Topologies E and $T_{in}^{\ll^{\rightarrow^{irr}}}$, on M, give the intersection Topology $Z_{in}^{\ll^{\rightarrow^{irr}}}$, which has basic-open sets of the form $B_\epsilon(x) \cap C^S(x)$ (bounded space cones) and it is finer than E.

19. The causal order $\prec$ induces the Topology $T_\prec$, which has a subbase consisting of future causal cones $C_+^T(x) \cup C_+^L(x)$ or past causal cones $C_-^T(y) \cup C_-^L(y)$, where $x, y \in M$. The finite intersections of such subbasic-open sets give "causal diamonds", containing the endpoints.

20. $\prec$ induces the interval Topology $T_{in}^{\prec}$, with subbase consisting of sets $M \setminus [C_+^T(x) \cup C_+^L(x)]$, which are complements of future causal cones or sets $M \setminus [C_-^T(x) \cup C_-^L(x)]$ which are complements of past causal cones. This topology has basic-open sets of the form $C^S(x) \setminus \{x\}$ and it is easy to see that it is incomparable to the natural topology of M.

21. The Topologies E and $T_{in}^{\prec}$, on M, give the intersection Topology $Z_{in}^{\prec}$, which has basic-open sets of the form $B_\epsilon(x) \cap [C^S(x) \setminus \{x\}]$ and it is finer than E.

22. The reflexive chorological order $\leq$ induces the Topology $T_\leq$, which has a subbase consisting of $+$-oriented space cones $C_+^S(x)$ or $-$-oriented space cones $C_-^S(y)$, where $x, y \in M$. The finite intersections of such subbasic-open sets give "closed diamonds", that is diamonds containing the endpoints, that are spacelike.

23. $\leq$ induces the interval Topology $T_{in}^{\leq}$, with subbase consisting of sets $M \setminus [C_+^S(x) \cup \{x\}]$, or sets $M \setminus [C_-^S(x) \cup \{x\}]$. This topology has basic-open sets of the form $[C^T(x) \cup C^L(x)] \setminus \{x\}$ and it is easy to see that it is incomparable to the natural topology of M.

24. The Topologies E and $T_{in}^{\leq}$, on M, give the intersection Topology $Z_{in}^{\leq}$, which has basic-open sets of the form $B_\epsilon(x) \cap [(C^T(x) \cup C^L(x)) \setminus \{x\}]$ and it is a finer topology than E.

25. The irreflexive complement of the chronological order, namely $<^{\to irr}$, induces the Topology $T_{<^{\to irr}}$, which has a subbase consisting of $+$-oriented (deleted) space cones with their light boundary $[C_+^S(x) \cup C_+^L(x)] \setminus \{x\}$ or $-$-oriented (deleted) space cones with their light boundary $[C_-^S(y) \cup C_-L(y)] \setminus \{y\}$, where $x, y \in M$. The finite intersections of such subbasic-open sets give deleted "open diamonds" that are spacelike.

26. $<^{\to irr}$ induces the interval Topology $T_{in}^{<^{\to irr}}$, with subbase consisting of sets $M \setminus [C_+^S(x) \cup C_+^L(x) \setminus \{x\}]$, or sets $M \setminus [C_-^S(x) \cup C_-^L(x) \setminus \{x\}]$. This topology has basic-open sets of the form $C^T(x)$, i.e., time cones, and it is easy to see that it is incomparable to the natural topology of M.

27. The Topologies E and $T_{in}^{<^{\to irr}}$, on M, give the intersection Topology $Z_{in}^{<^{\to irr}}$, which has basic-open sets of the form $B_\epsilon(x) \cap C^T(x)$. This intersection topology is the special relativistic analogue of the Path topology, introduced in [5] and it is finer than E.

28. The complement of the chronological order, namely $\ll^c$, induces the Topology $T_{\prec^a}$, which has a subbase consisting of $+$-oriented space cones with their light boundary $C_+^S(x) \cup C_+^L(x)$ or $-$-oriented space cones with their light boundary $C_-^S(y) \cup C_-L(y)$, where $x, y \in M$. The finite intersections of such subbasic-open sets give "closed diamonds" that are spacelike.

29. $\ll^c$ induces the interval Topology $T_{in}^{\ll^c}$, with subbase consisting of sets $M \setminus [C_+^S(x) \cup C_+^L(x)]$, which are complements of $+$-oriented space cones with their light boundary or sets $M \setminus [C_-^S(x) \cup C_-^L(x)]$ which are complements of $-$-ve oriented space cones with their light boundary. This topology has basic-open sets of the form $C^T(x) \setminus \{x\}$, i.e., deleted time cones, and it is easy to see that it is incomparable to the natural topology of M.

30. The Topologies E and $T_{in}^{\prec^a}$, on M, give the intersection Topology $Z_{in}^{\prec^a}$, which has basic-open sets of the form $B_\epsilon(x) \cap (C^T(x) \setminus \{x\})$ and it is finer than E.

4. Discussion

4.1. Curved Spacetimes

A first question is: Can one generalise the thirty above mentioned topologies to curved spacetimes? The answer is positive. Indeed, from a topological perspective, and without any extra condition or restriction, one can consider the general relativistic analogue of each one of the mentioned Topologies 1–30, since as soon as there exists spacetime there are events and for each event there are time/light/causal cone assigned to it; the point-set topology is independent of the curvature and the tilt of the cones and, since the mentioned topologies are generated from the causal relations of the spacetime, one has to only choose an arbitrary Riemannian metric h, on the spacetime manifold M.

For example, the Path topology of Hawking–King–McCarthy (see [5]) will be the generalisation of Topology 3 of our list, as follows.

Consider the chronological order $\ll$, on a relativistic spacetime manifold M. Then, $\ll$ will induce the interval Topology $T_{in}^{\ll}$, with subbase consisting of sets $M \setminus C_+^T(x)$, which are complements of future time cones, or sets $M \setminus C_-^T(x)$, which are complements of past time cones. This topology, exactly as with our Topology 3 of the list, has basic-open sets of the form $C^S(x) \cup C^L(x)$. Now, consider the manifold Topology $\mathcal{M}$ and for a Riemannian metric h consider the base of $\mathcal{M}$-open sets of the form $B_\epsilon^h(x)$, the open balls centered at x and radius ϵ with respect to h. Then, a basic-open set for the Path topology will be of the form $T_{in}^{\ll} \cap B_\epsilon^h(x)$. Low (see [6]) has shown that the Limit Curve Theorem fails to hold for the Path topology, thus the formation of a basic contradiction present in the proofs of all singularity theorems fails as well (for a more extensive discussion see [12–14]).

4.2. Singularities

Furthermore, we observe that the Limit Curve Theorem holds for Topologies $2, 3, 8, 9, 14, 15, 23, 24$ of our list, but not for Topologies $5, 6, 11, 12, 17, 18, 20, 21, 26, 27, 29, 30$. Following the argument of Low ([6], Paragraph V), we can easily see if U is a basic-open set of one of Topologies $2, 3, 8, 9, 14, 15, 23$ or 24, then this set does not contain the light cone of the event which defines it. Consider a sequence of null vectors p_n converging to p in the usual topology. Let γ_n be the null geodesic through the origin with tangent p_n and γ the null geodesic through the origin with tangent vector p. Clearly, γ is the unique limit curve of the sequence $\{\gamma_n\}$ in the usual topology, for all n. However, γ_n intersected with an open set (not containing the origin) of either of the basic-open sets defined in Topologies $2, 3, 8, 9, 14, 15, 23$ or 24 will give empty set, thus γ will be not a limit curve of the sequence γ_n under specified Topologies $2, 3, 8, 9, 14, 15, 23$, and 24 and so the Limit Curve Theorem will fail for each of these topologies. On the contrary, following the same argument, the Limit Curve Theorem will hold for Topologies $5, 6, 11, 12, 17, 18, 20, 21, 26, 27, 29$ and 30, since each of them has basic-open sets containing the light-cone for each event.

4.3. Ambient Cosmology

The significance of the above remarks is that, unlike the manifold topology (which merely characterises continuity properties according to Hawking et al.), there are thirty topologies (those listed in this article) that determine the causal and conformal structures of spacetime and are most appealing than the Fine topology of Zeeman (which does not admit a countable base of open sets). In addition, there are no other topologies that can be defined immediately from the causal relations in spacetime.

A question that is now raised is: Which topology is the most appropriate one, if one can set it in this way, or the most physical one? The remark that for eight of these topologies the Limit Curve Theorem fails to hold, could bring the discussion on the need for an Ambient Cosmology to a different level. For example, the very own construction of the ambient boundary-ambient space model (see [15]) was an attempt to consider a four-dimensional spacetime as the conformal infinity of a five-dimensional ambient space, to show that singularities are absent and the Cosmic Censorship becomes valid by construction. In the frame of topologies such as some ones that we mention in this paper, however, this is achieved without the need of working in extra dimensions (all due to the reason that LCT fails under them).

Finally, the topologies from our list could be linked to the study of sliced spaces (see [16]). In particular, a sliced spacetime V can be considered as a product of a smooth manifold M of dimension 3 times the real line $\mathbb{R}$, where V is equipped with with a four-dimensional Lorentzian metric that splits in a particular way with respect to a shift and a lapse function. Thus far, global hyperbolicity has been studied with respect to the Alexandrov topology and the natural Product topology of V, but not with respect to causal topologies. This would be particularly interesting in the case of Ambient Cosmology,

since one can consider *M* as a four-dimensional spacetime, the conformal boundary at infinity of an ambient space $V = M \times \mathbb{R}$.

4.4. Girders and Twistor Spaces

Lastly, Topologies 4 and 10 seem to fit well in spaces consisted of girders, hypergirders and links (see [17]). Although they depend on the structure of the light cone, the question that has to be addressed is how they could be used in a description of the transition from quantum non-local theory to a classical local theory. Certainly, there is not a definite answer to this question at present but we believe that methods of point-set topology will contribute significantly, as one can work using topological tools invariantly from the geometry of a spacetime. Topologies that are constructed through light-rays, e.g., Topologies 4 and 10 in our list, could also place a role in twistor theory (see [18]). More particularly, two points P, R in the Minkowski space that are incident with the same non-zero (null) twistor Z must be null-separated by each other which means that Z defines a light ray in the Minkowski space. This gives an inside of how one could transfer topological properties from the Minkowski space to the twistor space $\mathbb{T}$ or, better, to the projective twistor space $\mathbb{PT}$.

4.5. On Abstract Conformally Invariant Boundary Constructions in Relativity Theory

Some other possible applications of the topologies in the list in Section 3 might appear for abstract conformally invariant constructions in Relativity theory, namely boundaries of Low, Geroch, Kronheirmer and Penrose (see, for example, [19–24]). These topologies are based purely on the causal structure of the spacetime, but it is not clear which one would be the canonical choice of the topology. Low proposed a new such topology in [21], where the causal boundary has been developed further in relation to the classical conformal boundary. A further study could be based, for example, on the thorough study in [22], through links with deep constructions in pure mathematics in [23] and on a recent revision in [24].

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