Article

Gravity-Induced Geometric Phases and Entanglement in Spinors and Neutrinos: Gravitational Zeeman Effect

Banibrata Mukhopadhyay *[ID] and **Soumya Kanti Ganguly**

Department of Physics, Indian Institute of Science, Bangalore 560012, India; soumya09ganguly@gmail.com
* Correspondence: bm@iisc.ac.in

check for
updates

Abstract: We show Zeeman-like splitting in the energy of spinors propagating in a background gravitational field, analogous to the spinors in an electromagnetic field, otherwise termed the Gravitational Zeeman Effect. These spinors are also found to acquire a geometric phase, in a similar way as they do in the presence of magnetic fields. However, in a gravitational background, the Aharonov-Bohm type effect, in addition to Berry-like phase, arises. Based on this result, we investigate geometric phases acquired by neutrinos propagating in a strong gravitational field. We also explore entanglement of neutrino states due to gravity, which could induce neutrino-antineutrino oscillation in the first place. We show that entangled states also acquire geometric phases which are determined by the relative strength between gravitational field and neutrino masses.

1. Introduction

It is well known that if the time dependence in the Hamiltonian arises through certain parameters, namely adiabatic parameters, then the system develops a non-dynamic phase, called the Berry phase [1]. Spinors propagating in the magnetic fields are known to acquire such a Berry phase. Interestingly, a neutrino propagating through a medium also develops such a system, while the varying matter density corresponds to the adiabatic parameter. Importantly, although originally Berry phase was found in the context of adiabatic, unitary and cyclic evolutions of time-dependent quantum systems, later it was re-established for non-adiabatic, non-unitary and non-cyclic cases with its generalized definition [2–4].

Several authors have studied the geometric phases in neutrino oscillations. Although it was argued in an earlier work that the Berry phase plays no role in two-flavor neutrino oscillations in matter [5], the work was restricted to a limited region in the parameter space. However, it was shown by exploiting the spin degree of freedom that the interaction of neutrinos with the transverse magnetic field can lead to a geometric effect [6–9]. Later on, it was argued [10] that the Berry phase can only appear in the presence of non-standard (e.g., R-parity violating supersymmetry) neutrino-matter interactions for the particular case of two-flavor oscillations in matter. Essentially, all of the above papers argued that geometric phases do not arise in the two-flavor neutrino oscillation probabilities with CP conservation in vacuum or in matter, in the absence of any non-standard neutrino-matter interactions. It was, however, furthermore argued [11] that even in the absence of CP violation, neutrinos in two-flavor oscillation in vacuum in a period can acquire an overall phase consisting of a dynamical phase and a phase depended on mixing angle only. The second part of the phase, which is

of geometric origin, was called Berry phase. Note that this phase does not arise due to slowly varying parameters leading to adiabatic evolution, rather due to Schrödinger evolution of the system giving a closed loop in the Hilbert space. As the phase is a global phase at the amplitude level, it does not appear in measurable quantities like probabilities of appearance or survival of neutrinos. These cyclic geometric phases were furthermore extended by the later authors [12] to obtain non-cyclic phases for two- and three-flavor neutrinos in vacuum, which remain unobservable because of the same reason as before. Also, the geometric phases for neutrinos propagating in varying magnetic fields have been reported [13].

It is interesting to note that [14] the Berry phase has a connection to the phase discovered by Pancharatnam [15]. In fact, both of the phases can be described under the same platform [3]. Unlike the Berry phases obtained in the above work, it has however been established [16] that Pancharatnam phase can appear in detection probabilities and hence can be observed directly even in an effective two-flavor approximation. There are many other explorations over the years in various contexts of geometric phase and entanglement in neutrinos, including those with CPT violation and fluctuating matter, non-linear refraction, magnetic field, dissipative matter, etc. [17–24].

However, none of the works above considered the effects of gravity in the calculations, except one [24] which considered Newtonian self-gravitational interaction; whether the interaction of spinors and then neutrinos with gravitational field causes any effect or not. This issue particularly arises due to the fact that neutrinos interacting with background gravity may not preserve CPT [25,26], which may be shown as a natural candidate for governing the Berry phase, even in the evolution of neutrinos due to the split of dispersion energy between neutrino and antineutrino. Indeed, within the pure standard model of particle physics, the neutrino oscillations cannot be understood and hence relaxing the CPT conservation through gravitational interaction is one of the natural steps forward to beyond standard model. While the Berry phase arises in the presence of non-standard matter-neutrino interactions, neutrino spin and magnetic field interactions, it is a natural question if the coupling between spin of neutrino and in general spinor and spin connection to the background gravity generates any geometric effect.

Two-flavor neutrino oscillation in the background gravity has been discussed in various astrophysical contexts. One of the current authors explored possible Lorentz and CPT violations in the neutrino sector in the presence of background gravity and its astrophysical consequences [25–29]. Earlier, the analogy of solar neutrino oscillations with the precession of electron spin in a time-dependent magnetic field was discussed [30]. Then based on the evolution of a statistical ensemble, oscillations for neutrinos from supernovae or in the early universe in the presence of mixing and matter interactions in a thermal environment were shown to be viewed in terms of precession [31]. It was also observed [32] that spin flavor resonant transitions of neutrinos emanating from active galactic nuclei may occur in the vicinity of black holes due to gravitational effects and due to the presence of a large magnetic field. Interestingly, the matter effects therein become negligible in comparison to gravitational effects.

In the present paper, we start by recapitulating the origin of Berry phase in spinors in the presence of external magnetic fields in Section 2. Then we show the analogous effects in the presence of background gravitational fields, namely gravitational geometric phase in spinors in the same section. The subsequent plan is to apply this result in the neutrino sector. To do so, we first recapitulate the basic solutions of previous work discussing neutrino oscillations in curved spacetime [25,26] in Section 3, which are used in subsequent sections. Based on these neutrino states evolving in the gravitational background, we explore any geometric (as well as dynamic) effect/phase arising due to gravity in Section 4. Subsequently, our aim is to explore the possible entanglement of neutrino states coupled with background gravitational field and to compute the geometric phase arising in their evolution in Section 5. Finally, we discuss how the geometric phases actually vary with gravitational field in Section 6 and summarize results in Section 7.

2. Geometric Phases in the Presence of Electromagnetic and Gravitational Fields

2.1. In Electromagnetic Field

The Dirac equation, describing dynamics of spinors, in the presence of electromagnetic field and the underlying dispersion energy, Zeeman splitting and geometric/Berry phase are well-known. However, for the ease of understanding their similarities and also dissimilarities with those in the gravitational field, which is the main target here, we first recall the Dirac equation in the presence of electromagnetic field given by

$$\left[i\gamma^\mu \left(\partial_\mu - ieA_\mu\right) - m\right]\psi = 0, \tag{1}$$

where the various components of γ^μ, where $\mu = 0,1,2,3$, are Dirac matrices with their usual meaning, e is the electric charge, m is the mass of the spinor and A_μ is the electromagnetic covariant 4-vector potential. Here, we choose units $c = \hbar = 1$. For the non-trivial solution for ψ, the energies/Hamiltonians of the spin-up and spin-down particles are given by

$$(H + eA_0)^2 = (\hat{p} - e\vec{A})^2 + m^2 + e\vec{\sigma} \cdot \vec{B}, \tag{2}$$

where A_0 is the temporal component of A_μ which is basically the Coulomb potential, $\hat{p}$ is the quantum mechanical momentum operator $-i\nabla$, and $\vec{\sigma}$ is the Pauli spin matrix. In the non-relativistic limit, where m^2 is much larger than the rest of the terms in the R.H.S. of Equation (2), it reduces to

$$H = -eA_0 \pm \left[\frac{(\hat{p} - e\vec{A})^2}{2m} + m + \frac{e\vec{\sigma} \cdot \vec{B}}{2m}\right]. \tag{3}$$

Apart from the split due to the positive and negative energy solutions, clearly there is an additional split in the respective energy levels. This is basically Zeeman-splitting governed by the term with Pauli's spin matrix, in the up and down spinors for the positive and negative energy spinors induced by magnetic fields, whether we choose relativistic or non-relativistic regimes. The same governing term involved with $\vec{\sigma}$ is also responsible for the Berry phase if $\vec{B}$ is varying. For convenience, $\vec{B}$ is generally considered in the parameter space and is decomposed as $\vec{B} = |\vec{B}| \left(\hat{r} \sin\tilde{\theta}\cos\tilde{\phi} + \hat{\tilde{\theta}}\sin\tilde{\theta}\sin\tilde{\phi} + \hat{\tilde{\phi}}\cos\tilde{\theta}\right)$. Hence, the Berry phase at a fixed r is given by

$$\Phi_g = i\oint \langle\psi|\nabla|\psi\rangle.d\vec{R} = \frac{\tilde{\phi}}{2}(1 - \cos\tilde{\theta}), \tag{4}$$

where $\vec{R} \equiv (r, \tilde{\theta}, \tilde{\phi})$, the coordinate vector of the underlying Poincaré sphere. When $\vec{R}$ is constant, $\Phi_g = 0$.

Figure 1 represents the energy splitting given by Equation (3). While the primary splitting corresponds to positive and negative energy solutions, the secondary splitting corresponds to the interaction between the spin and magnetic fields.

Recapitulation of all of the above results will be useful to explore and understand the consequences of the dynamics of spinors in the gravitational field. As we will show below, although there are certain similarities between the effects of electromagnetic and gravitational fields to the spinors, there are some unique consequences in the latter.
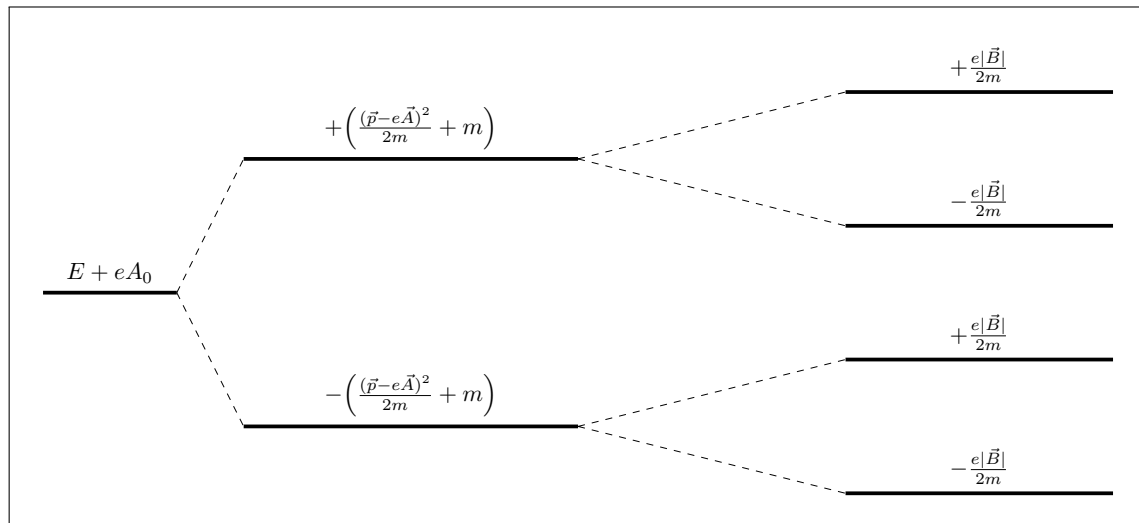
Figure 1. Zeeman-splitting in the electromagnetic case.

2.2. In Gravitational Field

Dirac equation in the presence of background gravitational fields has already been shown to have many consequences (see, e.g., the work by one of the present authors [25,26,28,33,34]) and is known to have the form (see, e.g., [26,34–36])

$$\left[i\gamma^\mu \partial_\mu - m + \gamma^5 \gamma^\mu B_\mu^g\right]\psi = 0, \tag{5}$$

where B_μ^g is the gravitational covariant 4-vector potential (gravitational coupling with the spinor), given by

$$B_\mu^g = e_\mu^d B_d^g = \epsilon^{abc}_{d}\, e_{b\lambda}\left(\partial_a e_c^\lambda + \Gamma_{\gamma\mu}^\lambda e_c^\gamma e_a^\mu\right), \tag{6}$$

where e_c^λ-s and $\Gamma_{\gamma\mu}^\lambda$ are various components of vierbeins and Christoffel connection with Greek and Latin indices respectively indicating curved and local flat coordinates, ϵ^{abcd} is the 4-dimensional Levi-Civita symbol and $\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ as usual. Here we do not repeat the calculation to obtain the reduced form of the Dirac equation given by Equation (5), which is available in the existing literature, see, e.g., [28,29,37] for details. The form of Equation (5) is easy to understand in local inertial coordinates, where the Dirac $\gamma-$matrices and their relations are straight forward. However, it is not difficult to explore Dirac equation in general curvilinear coordinates (see, e.g., [38]). Nevertheless, in our first exploration of the geometric phase of spinors and neutrinos in gravitational background, for the convenience of developing the idea, we stick to the local inertial coordinates. This helps to compare results easily with geometric/Berry phase arising in the presence of magnetic field without losing any important physics. In the future, we will report geometric/Berry phase in gravitational field in global coordinates. Considering the problem in local coordinates, in brief, while expanding various terms of the Dirac Lagrangian (and equation) in curved spacetime, one obtains a Hermitian-like and anti-Hermitian-like parts, apart from the part already there in Minkowski spacetime. Hence, considering total Lagrangian consisting of that of particle and anti-particle (and corresponding equation), the anti-Hermitian part drops out and one obtains Equation (5) given above (see [37] for details). It can also be seen as the choice of an appropriate basis system [39], particularly clearer when we explore it in global non-inertial coordinates. Nevertheless, the appearance of the anti-Hermitian-like part (which need not always be anti-Hermitian, depending on the underlying spacetime) is independent of the Hermitian-like term [33] that alone could lead to the

axial-vector term given by Equation (5), which is the basic building block of the following discussion. Hence, for simplicity, here we do not consider the apparent anti-Hermitian term.

Now, like the case of electromagnetic fields, for the non-trivial solution of ψ in the spacetime not explicitly dependent on time (except the case where time dependence arises only via scale factor, like in an expanding universe, and the exploration is in a particular epoch or the interest is in the local-inertial coordinates), in the local coordinates, the energies/Hamiltonians of the spin-up and spin-down particles from Equation (5) are given by

$$(H - \vec{\sigma} \cdot \vec{B}^g)^2 = \hat{p}^2 + B_0^{g2} + m^2 + (\hat{p} \cdot \vec{B}^g) + \vec{\sigma} \cdot \left[(\hat{p} B_0^g) + 2B_0^g \hat{p} + (\nabla \times \vec{B}^g) - 2\vec{B}^g \times \nabla \right], \tag{7}$$

where B_0^g is the temporal component of B_μ^g and $\hat{p} = -i\nabla$. In the regime of weak gravity and when m^2 is much larger than the rest of the terms in the R.H.S. of Equation (7), it reduces to

$$H = \vec{\sigma} \cdot \vec{B}^g \pm \left[\frac{\hat{p}^2 + B_0^{g2} + (\hat{p} \cdot \vec{B}^g)}{2m} + m + \vec{\sigma} \cdot \left\{ \frac{(\hat{p} B_0^g) + (\nabla \times \vec{B}^g)}{2m} + \frac{B_0^g \hat{p} - \vec{B}^g \times \nabla}{m} \right\} \right]. \tag{8}$$

Equation (8) brings in a new effect due to the presence of an axial-vector term in the Dirac equation in the gravitational background as opposed to the case with electromagnetic effects involved with a vector term shown by Equation (3). The new effect induced by the axial-vector involves the spin–momentum coupling of the particle, apart from spin–gravity coupling, and other related terms, provided the background gravitational potential is non-zero. As in general gravitational potential is not constant (even if locally gravitational field is constant), the eigenfunction of Equation (8) cannot be plane-wave typed. However, if B_μ^g is slowly varying and time-independent, then the solution can be of the form

$$\psi = f(\vec{x}, \vec{p}) exp[i\vec{p} \cdot \vec{x}], \tag{9}$$

where $f(\vec{x}, \vec{p})$ is a slowly varying function. This leads the Hamiltonian to the form

$$H = \vec{\sigma} \cdot \vec{B}^g \pm \left[\frac{\vec{p}^2 - \frac{\nabla^2 f(\vec{x}, \vec{p})}{f(\vec{x}, \vec{p})} - \frac{2i(\nabla f(\vec{x}, \vec{p})) \cdot \vec{p}}{f(\vec{x}, \vec{p})} + B_0^{g2} - i(\nabla \cdot \vec{B}^g)}{2m} + m \right.$$

$$\left. + \vec{\sigma} \cdot \left\{ \frac{-i(\nabla B_0^g) + (\nabla \times \vec{B}^g)}{2m} + \frac{B_0^g \vec{p} - i\vec{B}^g \times \vec{p} - iB_0^g \frac{\nabla f(\vec{x})}{f(\vec{x}, \vec{p})} - \frac{\vec{B}^g \times \nabla f(\vec{x})}{f(\vec{x}, \vec{p})}}{m} \right\} \right]. \tag{10}$$

As $f(\vec{x}, \vec{p})$ is determined by the variation of background gravitational potential, we can suitably choose $f(\vec{x}, \vec{p})$ in terms of B_μ^g and $\vec{p}$ in such a way that the above Hamiltonian is Hermitian and terms involved with $f(\vec{x}, \vec{p})$ are removed, given by

$$H = \vec{\sigma} \cdot \vec{B}^g \pm \left[\frac{\vec{p}^2 + B_0^{g2}}{2m} + m + \vec{\sigma} \cdot \left\{ \frac{\nabla \times \vec{B}^g}{2m} + \frac{B_0^g \vec{p}}{m} \right\} \right], \tag{11}$$

where $f(\vec{x}, \vec{p})$ satisfies

$$\nabla^2 f(\vec{x}, \vec{p}) + i \left(2\vec{p} + \frac{B_0^g}{m}\vec{\sigma} \right) \cdot \nabla f(\vec{x}, \vec{p}) + \vec{\sigma} \cdot \left(\vec{B}^g \times \nabla f(\vec{x}, \vec{p}) \right)$$

$$+ i \left\{ \nabla \cdot \vec{B}^g + \vec{\sigma} \cdot \left(\frac{\nabla B_0^g}{2m} + \frac{\vec{B}^g \times \vec{p}}{m} \right) \right\} f(\vec{x}, \vec{p}) = 0. \tag{12}$$

Note that the solution for $f(\vec{x}, \vec{p})$ will turn out to be complex and its imaginary part may need to be adjusted by modifying $\vec{p}$ of the solution. Obviously, this is one of the possible solutions and is a

gauge choice, not unique. However, this will suffice for the present purpose when the aim is to show the existence of geometric phase and a possible new effect in gravitational background.

From Equation (11), there is a two-fold split in dispersion energy, governed by two terms associated with the Pauli spin matrix, between up and down spinors for the positive and negative energy spinors induced by gravitational fields, whether the field is weak or strong. The same governing terms are also responsible for the Berry phase, as is for electromagnetic fields, which in the parametric space with coordinates $\vec{R} \equiv (r, \tilde{\theta}, \tilde{\phi})$ at a fixed r is given by

$$\Phi_g = i \oint \langle \psi | \nabla | \psi \rangle \cdot d\vec{R} = \frac{\tilde{\phi}}{2} (1 - \cos \tilde{\theta}), \tag{13}$$

which was briefly introduced by one of us earlier [40].

The first term in the curly bracket of Equation (11) is the magnetic-equivalent contribution from gravitational field. Interestingly, even if B_μ^g is constant but non-zero, but $\vec{p}$ is varying—at least changing direction due to whatever reason, e.g., the presence of constant magnetic field which however does not produce any geometric phase—Φ_g survives, as seen from the last term in Equation (11). The contribution from the first term in Equation (11) adds up to Φ_g if B_μ^g varies (i.e., potential varies but field need not necessarily vary) and that from the second last term if $\nabla \times \vec{B}^g$ varies. Hence, while in electromagnetic fields the magnetic potential and hence field has to be varying, in gravitational field even the constant (but non-zero) gravitational potential (and also constant gravitational field) still would produce geometric/Berry phase.

Figure 2 represents the energy splitting given by Equation (11), which was already introduced briefly by one of us [34,40] (note, however, a typo in the notation of the first splitting in those works). Here, the splittings are different to those in electromagnetic case. The total splittings are involved with the interaction between the spin, background gravitational potential and gravitational field. Hence, the gravitational "Zeeman-effect" appears to be different to the conventional electromagnetic Zeeman-effect. While the electromagnetic Zeeman-effect and geometric phase depend on the underlying magnetic field only, their gravitational counter-parts in general involve both gravitational potential and field. Therefore, gravitational effects reveal an Aharonov-Bohm type effect in addition to Berry-like phase. Nevertheless, the total energy of the system of particles remains conserved in both electromagnetic and gravitational cases (which indeed should be in the time-independent spacetime). Also, in the local inertial frame, at a given epoch if the process is considered in expanding universe, gravitational potential B_μ^g appears to be constant, acting as a background effect.
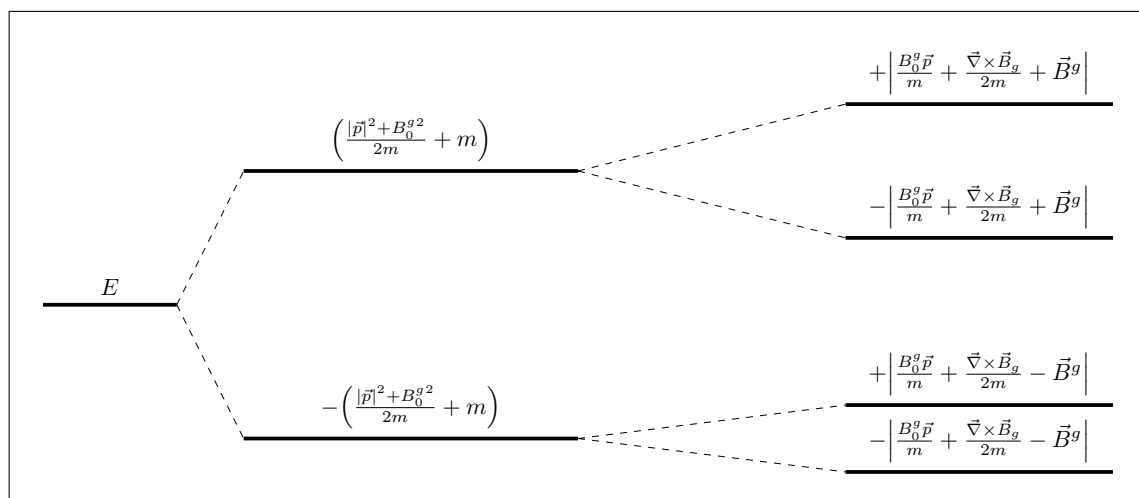

Figure 2. Gravitational "Zeeman-splitting".

Note that B_μ^g can be computed for various spacetime metrics, as given by previous work [25–29,33]. In order to have a non-zero B_μ^g, spherical symmetry has to be broken and hence in the Schwarzschild geometry (and hence for the spacetime around a non-rotating black hole), it vanishes. On the other hand, in the Kerr geometry (and hence for the spacetime around a rotating black hole), it survives independent of the choice of coordinates: in the Boyer-Lindquist as well as the Kerr-Schild [25,27]. Also, it survives in other natural spacetimes breaking spherical symmetry, e.g., in early universe under gravity wave perturbation, the Bianchi II, VIII and IX anisotropic universe, in the Fermi-normal coordinates up to second order correction [25,26,29,33]. In the Kerr-Schild coordinates, the temporal part of gravitational potential reads as [25,26]

$$B_0^g = -\frac{4a\sqrt{Mz}}{\bar{\rho}^2\sqrt{2r^3}}, \tag{14}$$

where $\bar{\rho}^2 = 2r^2 + a^2 - x^2 - y^2 - z^2$, r is the radial coordinate of the system and M and a are respectively the mass and the angular momentum per unit mass of the black hole. Naturally, B_0^g survives (and is varying with space coordinates) for any non-spinning black hole leading to gravitational Zeeman effect and Berry phase independent of spatial part $\vec{B}^g$. Similarly, non-zero $\vec{B}^g$ leads to gravitational Zeeman effect and Berry phase (when $\vec{B}^g$ has to be varying as well), independent of B_0^g. In the Bianchi II spacetime with, e.g., even equal scale-factors in all directions, B_0^g survives as [25,29]

$$B_0^g = \frac{4 + 3y^2 - 2y}{8 + 2y^2}, \tag{15}$$

leading to both gravitational Zeeman splitting and Berry phase even though $\vec{B}^g = 0$.

Also see similar exploration in varied contexts [41–44]. Hence, the axial vector term in Equation (5) and the related terms involving with $\vec{\sigma}$ in Equation (8) contribute as long as the spacetime naturally has some handedness, independent of the choice of coordinates.

3. Neutrino States in the Presence of Gravitational Field

As discussed in the Introduction, over the years there have been several explorations of neutrinos in the presence of gravitational field. For the present purpose, we particularly use some of earlier results [26] obtained by one of the present authors and further modify as required.

3.1. Neutrino–Antineutrino Mixing

Recalling the work by Sinha and Mukhopadhyay [26] describing the mixing of neutrinos (ψ) and antineutrinos (ψ^c) in the presence of gravitational coupling, based on the formalism discussed above, let us write down the mass eigenstates ν_1 and ν_2 for a particular flavor at $t = 0$

$$|\nu_1(0)\rangle = \cos\theta\,|\psi^c(0)\rangle + e^{i\phi}\,\sin\theta\,|\psi(0)\rangle \tag{16a}$$
$$|\nu_2(0)\rangle = -\sin\theta\,|\psi^c(0)\rangle + e^{i\phi}\,\cos\theta\,|\psi(0)\rangle, \tag{16b}$$

when

$$\tan\theta = \frac{m}{B_0 + \sqrt{B_0^2 + m^2}}, \quad \phi = arg(-m), \tag{17}$$

where B_0 is the gravitational scalar coupling potential and m the Majorana mass of the neutrino. Henceforth, by B_μ we will mean B_μ^g itself, defined in the previous section, in order keep the same

notation as of previous papers. The large B_0 corresponds to $\theta \to 0$, hence no mixing and thence no oscillation. However, at an arbitrary time t the mass eigenstates are

$$|\nu_1(t)\rangle = \cos\theta \, e^{-iE_{\psi^c}t} \, |\psi^c(0)\rangle + e^{i\phi} \, \sin\theta \, e^{-iE_\psi t} \, |\psi(0)\rangle \qquad (18a)$$

$$|\nu_2(t)\rangle = -\sin\theta \, e^{-iE_{\psi^c}t} \, |\psi^c(0)\rangle + e^{i\phi} \, \cos\theta \, e^{-iE_\psi t} \, |\psi(0)\rangle, \qquad (18b)$$

where E_ψ and E_{ψ^c} are dispersion energies of neutrino and anti-neutrino respectively, in local coordinates neglecting their variation, given by (see also, e.g., [34])

$$E_\psi = \sqrt{(\vec{p} - \vec{B})^2 + m^2} + B_0,$$

$$E_{\psi^c} = \sqrt{(\vec{p} + \vec{B})^2 + m^2} - B_0, \qquad (19)$$

where $\vec{B}$ is the gravitational vector coupling potential and $\vec{p}$ the momentum of the neutrinos. In the absence of a gravitational field, neutrinos and antineutrinos mix in the same angle, hence there is no neutrino–antineutrino oscillation. This is indeed in accordance with the experimental finding [45].

The underlying oscillation length can also be recalled, for ultra-relativistic and non-relativistic (or weakly gravitating) neutrinos, as

$$\lambda = \frac{\pi}{B_0 - |\vec{B}|} \quad \text{and} \quad \frac{\pi}{B_0} \quad \text{respectively.} \qquad (20)$$

This depends only on the strength of the gravitational field. If we consider neutrinos to be coming out off the inner accretion disk, of few factors times Schwarzschild radii, around a spinning black hole of mass $M = 10M_\odot$, where $M_\odot$ is the mass of Sun, then $B_0 - |\vec{B}| = \tilde{B} = 10^{-19}$ GeV [25], which clearly argues neutrinos not to be influenced by the gravitational field of black hole as $m >> 10^{-10}$ eV. However, for $\tilde{B} << m$ the energy difference also turns out to be $2B_0$ (which is $2B_0 - 2|\vec{B}|$ for ultra-relativistic neutrinos), which leads to $\lambda \sim 10$ km. If the disk is around a supermassive black hole of $M = 10^8 M_\odot$, i.e., in an Active Galactic Nucleus (AGN), then λ may increase to 10^8 km. Therefore, an oscillation between mass eigenstates may complete from a few factors to hundreds of Schwarzschild radii in the disk, depending upon the size of the inner edge where neutrinos come out and angular momentum of the black hole. This may produce copious antineutrinos over neutrinos and may cause overabundance of neutrons and positrons, which may further have consequences in core-collapse supernovae. However, neutrinos around a primordial black hole of mass $M_p \sim 10^{17}$ gm [1] in the site of temperature 1 MeV are ultra-relativistic and could lead to an oscillation length as small as $\lambda \sim 10^{-16}$ km. Note that (ν_1, ν_2) is just the transformed spinor of original (ψ^c, ψ).

3.2. Mixing of Mass Eigenstates

Following previous work [26], neutrino and anti-neutrino states can also be in principle written as a linear combination of suitable mass states at $t = 0$ as

$$|\psi^c(0)\rangle = \cos\theta \, |\nu_1(0)\rangle - e^{i\phi} \, \sin\theta \, |\nu_2(0)\rangle \qquad (21a)$$

$$|\psi(0)\rangle = \sin\theta \, |\nu_1(0)\rangle + e^{i\phi} \, \cos\theta \, |\nu_2(0)\rangle \qquad (21b)$$

and at an arbitrary time t as

$$|\psi^c(t)\rangle = \cos\theta \, e^{-iE_1 t} \, |\nu_1(0)\rangle - e^{i\phi} \, \sin\theta \, e^{-iE_2 t} \, |\nu_2(0)\rangle \qquad (22a)$$

$$|\psi(t)\rangle = \sin\theta \, e^{-iE_1 t} \, |\nu_1(0)\rangle + e^{i\phi} \, \cos\theta \, e^{-iE_2 t} \, |\nu_2(0)\rangle, \qquad (22b)$$

---

[1]   Note that the corresponding temperature $T_p \sim 10^{-20} M_\odot / M_p$ GeV [46].

where in ultra-relativistic limits the energies of the mass eigenstates are

$$E_{(1,2)} \approx |\vec{p}| + |\vec{B}| + \frac{m^2_{(1,2)}}{2|\vec{p}|},$$ (23)

when the corresponding masses

$$m_1 = m_n - \sqrt{B_0^2 + m^2}, \quad m_2 = m_n + \sqrt{B_0^2 + m^2},$$ (24)

in the presence of lepton number conserving mass (m_n) and violating mass (m). Here, $|\vec{p}| \sim E$, the mean energy of the neutrinos, and θ and ϕ are the same as in Equation (17). The oscillation length is

$$\lambda_n = \frac{\pi E}{m_n \sqrt{B_0^2 + m^2} + |\vec{B}|E},$$ (25)

which indicates that only for $B_0 \gtrsim m$ (when $|\vec{B}|$ is very small, as could be in a certain spacetime, e.g., Bianchi II universe with equal scale factors, shown above), the gravitational field could affect the oscillation, which is again in accordance with experiments done in the laboratory [45]. However, this condition could easily be satisfied in few factors to few tens of Schwarzschild radii away from a primordial black hole of mass $\lesssim 10^{24}$–10^{20} gm.

3.3. Flavor Mixing

As each flavor acquires two mass states through particle–antiparticle mixing, a two-generation electron–muon neutrino system effectively will have four flavor states, governing the Lagrangian density [26]

$$
\begin{aligned}
(-g)^{-1/2}\mathcal{L}_m \;=\; & -\frac{1}{2}\left(v_{e1}^\dagger m_{e1} v_{e1} \;+\; v_{e2}^\dagger m_{e2} v_{e2} \;+\; v_{\mu1}^\dagger m_{\mu1} v_{\mu1} \;+\; v_{\mu2}^\dagger m_{\mu2} v_{\mu2} \right. \\
& \left. -\; v_{\mu1}^\dagger m_{\mu e} v_{e1} \;-\; v_{e1}^\dagger m_{\mu e} v_{\mu1} \;+\; v_{\mu2}^\dagger m_{\mu e} v_{e2} \;+\; v_{e2}^\dagger m_{\mu e} v_{\mu2} \right),
\end{aligned}
$$ (26)

in the presence of a mixing mass $m_{\mu e}$. From Equation (24)

$$m_{(e,\mu)1} \;=\; m_n - \sqrt{B_0^2 + m_{e,\mu}^2}, \quad m_{(e,\mu)2} \;=\; m_n + \sqrt{B_0^2 + m_{e,\mu}^2},$$ (27)

which no longer are the definite masses of electron and muon neutrinos due to the presence of $m_{\mu e}$. In terms of mass states $|\chi_i\rangle$, when $i = 1 - 4$, the sets of two Majorana neutrino flavor eigenstates are described as

$$|v_{e1}\rangle \;=\; \cos\theta_1 |\chi_1\rangle \;-\; e^{i\phi_1} \sin\theta_1 |\chi_2\rangle$$ (28a)
$$|v_{\mu1}\rangle \;=\; \sin\theta_1 |\chi_1\rangle \;+\; e^{i\phi_1} \cos\theta_1 |\chi_2\rangle$$ (28b)

and

$$|v_{e2}\rangle \;=\; \cos\theta_2 |\chi_3\rangle \;-\; e^{i\phi_2} \sin\theta_2 |\chi_4\rangle$$ (29a)
$$|v_{\mu2}\rangle \;=\; \sin\theta_2 |\chi_3\rangle \;+\; e^{i\phi_2} \cos\theta_2 |\chi_4\rangle,$$ (29b)

where the corresponding mixing parameters are given by

$$\tan\theta_{1,2} \;=\; \frac{\mp 2m_{\mu e}}{(m_{e(1,2)} - m_{\mu(1,2)}) + \sqrt{(m_{e(1,2)} - m_{\mu(1,2)})^2 + 4m_{\mu e}^2}}, \quad \phi_{1,2} = arg(\mp m_{\mu e}).$$ (30)

At an arbitrary time t, $|\chi_{1,3}(0)\rangle \to |\chi_{1,3}(t)\rangle = e^{-iE_{e(1,2)}t}|\chi_{1,3}(0)\rangle$ and $|\chi_{2,4}(0)\rangle \to |\chi_{2,4}(t)\rangle = e^{-iE_{\mu(1,2)}t}|\chi_{2,4}(0)\rangle$, when $E_{e(1,2)} = \sqrt{|\vec{p}|^2 + m_{e(1,2)}^2}$ and $E_{\mu(1,2)} = \sqrt{|\vec{p}|^2 + m_{\mu(1,2)}^2}$.

The oscillation lengths can be recalled as [26]

$$\lambda_{fg1} = \frac{4\pi E}{|M_1^2 - M_2^2|}, \quad \lambda_{fg2} = \frac{4\pi E}{|M_3^2 - M_4^2|}, \tag{31}$$

where

$$\Delta M^2 = |M_{1,3}^2 - M_{2,4}^2|$$
$$= \left(\sqrt{B_0^2 + m_{\mu(1,2)}^2} + \sqrt{B_0^2 + m_{e(1,2)}^2} \right)$$
$$\times \sqrt{\left\{ \left(\sqrt{B_0^2 + m_{\mu(1,2)}^2} - \sqrt{B_0^2 + m_{e(1,2)}^2} \right)^2 + 4m_{e\mu}^2 \right\}}, \tag{32}$$

which indicates that only for $B_0 \gtrsim m_{e(1,2)}, m_{\mu(1,2)}$, the gravitational field could affect the oscillation. For $B_0 >> m$, mixing is maximum and the oscillation length turns out to be $\lambda_{fg1,2} \sim \pi E / B_0 m_{e\mu}$. For other quantitative details see [26].

In the more realistic three-flavor case, the above discussions remain valid, but with the emergence of three sets of mass eigenstates. All the underlying masses will also be modified by the gravitational coupling term, similar to those given by Equations (24) and (27). Accordingly, mixing angle will become complicated and also the oscillation probability, but the influence of gravitational field will remain there, depending on, e.g., the black hole mass.

4. Dynamic and Geometric Phases

Let us consider the wavefunction $\Psi(t)$ of a system evolving over a time interval $t \in [0, \tau]$, where $\Psi(0)$ is its initial state and $\Psi(\tau)$ being the final. The total phase accumulated over the entire evolution is given by $\Phi_t = arg(\langle \Psi(0)|\Psi(\tau)\rangle)$ and the corresponding dynamic phase is given by $\Phi_d = -\int_0^\tau \langle \Psi(t)|i\partial_t|\Psi(t)\rangle dt$. The difference between the two phases is defined as the geometric phase [4], given by

$$\Phi_g = arg(\langle \Psi(0)|\Psi(\tau)\rangle) + \int_0^\tau \langle \Psi(t)|i\partial_t|\Psi(t)\rangle dt. \tag{33}$$

In a situation where the system could oscillate back-and-forth between $\Psi(t)$ and $\tilde{\Psi}(t)$ (which is antiparticle of $\Psi(t)$), e.g., the case of neutrino oscillation, we define new total and dynamic phases respectively given by $\Phi_{to} = arg(\langle \tilde{\Psi}(0)|\Psi(\tau)\rangle)$ and $\Phi_{do} = -\int_0^\tau \langle \tilde{\Psi}(t)|i\partial_t|\Psi(t)\rangle dt$. We term them as respective oscillation phases, where the geometric oscillation phase is given by

$$\Phi_{go} = arg(\langle \tilde{\Psi}(0)|\Psi(\tau)\rangle) + \int_0^\tau \langle \tilde{\Psi}(t)|i\partial_t|\Psi(t)\rangle dt. \tag{34}$$

Below we use these definitions to evaluate various phases in the neutrino sector. More precisely, we evaluate Φ_t, Φ_d, Φ_g and Φ_{to}, Φ_{do}, Φ_{go} for various neutrino states recalled in the previous section.

The phases, as we show below, depend on B_μ, which furthermore is determined by the nature of underlying spacetime and the corresponding parameter values. For the explicit computations of B_μ, see previous papers, e.g., [25–29,33]. Nevertheless, for the present purpose we do not consider the contribution due to the spatial variation of neutrino states at $t = 0$, which is obvious from Section 2. Our interest rather is the contribution to the geometric phases due to mixing and oscillation of states, which arise due to the effect of spacetime curvature on to the time evolution of neutrino states.

4.1. Neutrino-Antineutrino Mixing

The total phase for $|\nu_1\rangle$ is

$$\Phi_{t1} = arg\left(\langle\nu_1(0)|\nu_1(\tau)\rangle\right) = \tan^{-1}\left(-\frac{\cos^2\theta\sin E_\psi\tau + \sin^2\theta\sin E_{\psi^c}\tau}{\cos^2\theta\cos E_\psi\tau + \sin^2\theta\cos E_{\psi^c}\tau}\right) \tag{35}$$

and for $|\nu_2\rangle$

$$\Phi_{t2} = arg\left(\langle\nu_2(0)|\nu_2(\tau)\rangle\right) = \tan^{-1}\left(-\frac{\sin^2\theta\sin E_\psi\tau + \cos^2\theta\sin E_{\psi^c}\tau}{\sin^2\theta\cos E_\psi\tau + \cos^2\theta\cos E_{\psi^c}\tau}\right). \tag{36}$$

The other phase is given by for $|\nu_1\rangle$ as

$$-\Phi_{d1} = \int_0^\tau \langle\nu_1(t)|i\partial_t|\nu_1(t)\rangle dt = -\phi\,\sin^2\theta + \int_0^\tau\left(E_\psi\cos^2\theta + E_{\psi^c}\sin^2\theta\right)dt \tag{37}$$

and for $|\nu_2\rangle$ as

$$-\Phi_{d2} = \int_0^\tau \langle\nu_2(t)|i\partial_t|\nu_2(t)\rangle dt = -\phi\,\cos^2\theta + \int_0^\tau\left(E_\psi\sin^2\theta + E_{\psi^c}\cos^2\theta\right)dt, \tag{38}$$

when θ is independent of time. Even if θ is not constant, the part outside the integral in either of the Equations (37) and (38) always contributes to Φ_{g1} and Φ_{g2} respectively, revealing a τ-independent phase, as long as ϕ in the neutrino states is not constant. For $\theta = 0$ which corresponds to $B_0 >> m$, the total geometric phases for ν_1 and ν_2 turn out to be $n\pi$ and $n\pi - \phi$ respectively with $n = 0, 1, 2, 3 \cdots$. For $\theta = \pi/4$ which corresponds to $B_0 << m$, they are $n\pi - \phi/2$. However, generally speaking neutrino mass does not vary with time and hence ϕ remains fixed throughout the propagation. Thus, all the terms associated with ϕ actually vanish and any geometric contribution to the phase arises from other terms in, e.g., $\Phi_{t1} - \Phi_{d1}$.

For the oscillation between mass eigenstates, total phase

$$\Phi_{to1} = arg\left(\langle\nu_2(0)|\nu_1(\tau)\rangle\right) = \frac{\pi}{2} - \frac{(E_\psi + E_{\psi^c})\tau}{2} \tag{39}$$

and the other phase is given by

$$-\Phi_{do1} = \int_0^\tau \langle\nu_2(t)|i\partial_t|\nu_1(t)\rangle dt = \phi\,\cos\theta\sin\theta + \int_0^\tau\left(E_\psi - E_{\psi^c}\right)\cos\theta\sin\theta\,dt. \tag{40}$$

As before, the term outside the integral in Φ_{do1} survives only if ϕ varies with time, which generally may not be the case as the neutrino mass is fixed.

From previous work [25,28], B_0 can be computed for the spacetime around black holes as

$$B_0 \sim 10^{-18}\frac{M_\odot}{M}\text{GeV}. \tag{41}$$

Therefore, for a black hole in an X-ray binary with $M = 10M_\odot$, $B_0 << m$, when the Majorana mass of a neutrino $m \sim 10^{-2}$eV. In this case, there is apparently no effect of gravity on the geometric and dynamic phases and Φ_{do1} turns out to be τ-independent and arises due to the Majorana nature of the neutrino because $E_\psi = E_\psi^c$. This is purely the consequence of the mixing of neutrino and antineutrino, which occurs due to the presence of Majorana mass. The same is true for black holes at the center of AGNs.

For primordial black holes with $M \leq 10^{24}$ gm, on the other hand, $B_0 \geq 1$ eV so that $B_0 >> m$. Therefore, $\theta \to 0$ and hence the part outside the integral of $\Phi_{d1} \to 0$ and and that of $\Phi_{d2} \to -\phi$.

Moreover, $\Phi_{do1} \to 0$. In this case, gravitational field removes any possibility of mixing and then oscillation, which however affects geometric and dynamic phases.

When the mass of a primordial black hole increases to $M = 10^{26}$ gm, $B_0 \sim m$, which alters the mixing angle compared to that in the absence of gravitational effect, and hence affects the phases. An important point to note is that the larger the mass of black hole, the larger its radius, and hence the smaller the density in the surrounding disk. Therefore, in order to affect geometric and dynamic phases due to gravitational effect, the gravitational mass should not be more than $\sim 10^{-6} M_\odot$.

4.2. Mixing of Mass Eigenstates

In this case, for $|\psi^c\rangle$ the total phase

$$\Phi_{t1} = arg\left(\langle \psi^c(0)|\psi^c(\tau)\rangle\right) = \tan^{-1}\left(-\frac{\cos^2\theta \sin E_2\tau + \sin^2\theta \sin E_1\tau}{\cos^2\theta \cos E_2\tau + \sin^2\theta \cos E_1\tau}\right) \tag{42}$$

and the dynamical phase containing a term which does not explicitly depend on τ due to non-zero neutrino phase ϕ, given by

$$-\Phi_{d1} = \int_0^\tau \langle \psi^c(t)|i\partial_t|\psi^c(t)\rangle dt = -\phi \, \sin^2\theta + \int_0^\tau \left(E_2\cos^2\theta + E_1\sin^2\theta\right) dt. \tag{43}$$

Similarly, for $|\psi\rangle$

$$\Phi_{t2} = arg\left(\langle \psi(0)|\psi(\tau)\rangle\right) = \tan^{-1}\left(-\frac{\sin^2\theta \sin E_2\tau + \cos^2\theta \sin E_1\tau}{\sin^2\theta \cos E_2\tau + \cos^2\theta \cos E_1\tau}\right), \tag{44}$$

$$-\Phi_{d2} = \int_0^\tau \langle \psi(t)|i\partial_t|\psi(t)\rangle dt = -\phi \, \cos^2\theta + \int_0^\tau \left(E_2\sin^2\theta + E_1\cos^2\theta\right) dt. \tag{45}$$

For neutrino-antineutrino oscillation, the total phase

$$\Phi_{to1} = arg\left(\langle \psi^c(0)|\psi(\tau)\rangle\right) = \frac{\pi}{2} - \frac{(E_1 + E_2)\tau}{2} \tag{46}$$

and the other phase

$$-\Phi_{do1} = \int_0^\tau \langle \psi^c(t)|i\partial_t|\psi(t)\rangle dt = \phi \, \cos\theta \sin\theta + \int_0^\tau (E_1 - E_2)\cos\theta \sin\theta \, dt. \tag{47}$$

Here θ is assumed to be independent of time. If, in general, θ is not a constant, then other terms will contribute to $\Phi_{d1,2}$ and Φ_{do1}. For $B_0 >> m$, $\Phi_{do1} \to 0$, while for $m >> B_0$, all parts of the phases survive. Oscillation is also possible for $m >> B_0$, as long as $m_n \neq 0$. Simultaneously, oscillation and modified geometric and dynamic phases due to gravity are revealed, only when $B_0 \sim m, m_n$, which is possible in the site of, e.g., primordial black holes. Note that θ and ϕ are the same as that for the cases of neutrino-antineutrino mixing for the various parameters of spacetime, e.g., the mass of the black hole. More so, as mentioned before, any term in the phase associated with ϕ does not survive if ϕ is not a time-varying function, which generally is the case for neutrinos whose mass is assumed to be fixed.

4.3. Flavor Mixing

In this case, the total phase for $|\nu_{e1,2}\rangle$

$$\Phi_{t1} = arg\left(\langle \nu_{e(1,2)}(0)|\nu_{e(1,2)}(\tau)\rangle\right) = \tan^{-1}\left(-\frac{\cos^2\theta_{1,2}\sin E_{e(1,2)}\tau + \sin^2\theta_{1,2}\sin E_{\mu(1,2)}\tau}{\cos^2\theta_{1,2}\cos E_{e(1,2)}\tau + \sin^2\theta_{1,2}\cos E_{\mu(1,2)}\tau}\right) \tag{48}$$

and the other phase containing a τ-independent part, given by

$$-\Phi_{d1} = \int_0^\tau \langle \nu_{e(1,2)}(t)|i\partial_t|\nu_{e(1,2)}(t)\rangle dt = -\phi_{1,2}\,\sin^2\theta_{1,2} + \int_0^\tau \left(E_{e(1,2)}\cos^2\theta_{1,2} + E_{\mu(1,2)}\sin^2\theta_{1,2} \right) dt. \quad (49)$$

For $|\nu_{\mu(1,2)}\rangle$, they are

$$\Phi_{t2} = arg\left(\langle \nu_{\mu(1,2)}(0)|\nu_{\mu(1,2)}(\tau)\rangle \right) = \tan^{-1}\left(-\frac{\sin^2\theta_{1,2}\sin E_{e(1,2)}\tau + \cos^2\theta_{1,2}\sin E_{\mu(1,2)}\tau}{\sin^2\theta_{1,2}\cos E_{e(1,2)}\tau + \cos^2\theta_{1,2}\cos E_{\mu(1,2)}\tau} \right), \quad (50)$$

$$-\Phi_{d2} = \int_0^\tau \langle \nu_{\mu(1,2)}(t)|i\partial_t|\nu_{\mu(1,2)}(t)\rangle dt = -\phi_{1,2}\,\cos^2\theta_{1,2} + \int_0^\tau \left(E_{e(1,2)}\sin^2\theta_{1,2} + E_{\mu(1,2)}\cos^2\theta_{1,2} \right) dt, \quad (51)$$

when $\theta_{1,2}$ are independent of time. As before, for $\theta_{1,2}$ not being constant, other complicated dynamical terms will contribute to $\Phi_{g1,2}$, and the phases associated with ϕ would not contribute eventually as the neutrino mass does not vary with time in general.

For oscillation, total phase

$$\Phi_{to1} = arg\left(\langle \nu_{\mu(1,2)}(0)|\nu_{e(1,2)}(\tau)\rangle \right) = \frac{\pi}{2} - \frac{(E_{e(1,2)} + E_{\mu(1,2)})\tau}{2} \quad (52)$$

and other phase

$$-\Phi_{do1} = \int_0^\tau \langle \nu_{\mu(1,2)}(t)|i\partial_t|\nu_{e(1,2)}(t)\rangle dt = \phi_{1,2}\cos\theta_{1,2}\sin\theta_{1,2} + \int_0^\tau (E_{e(1,2)} - E_{\mu(1,2)})\cos\theta_{1,2}\sin\theta_{1,2}\,dt, \quad (53)$$

where $E_{e(1,2)} - E_{\mu(1,2)} = \Delta M^2/2|\vec{p}|$ and $E_{e(1,2)} + E_{\mu(1,2)} = 2|\vec{p}| + (m^2_{e(1,2)} + m^2_{\mu(1,2)})/2|\vec{p}|$ in the weak gravity limit. When $B_0 >> m_n, m_{e(1,2)}, m_{\mu(1,2)}$, interestingly $\theta_{1,2}$ given in Equation (30) become constant and equal to $\mp\pi/4$ (when the lepton number conserving mass m_n is the same in both the electron and muon sectors). This brings the part outside the integrals in $\Phi_{d1,2}$ and Φ_{do1} as a constant which furthermore turns out to be the same as the geometric phase $\Phi_{g1,2}$. They are independent of whether the spacetime is stationary (e.g., around a rotating black hole) or time-dependent (e.g., of early universe). Note that in the absence of gravity ($B_0 << m_n, m_{e1,2}, m_{\mu1,2}$), $\theta_{1,2}$ depend on specific values of neutrino masses. However, if $m_{e1,2} = m_{\mu1,2}$, then again $\theta_{1,2} = \mp\pi/4$ (when the lepton number conserving mass m_n is the same in both the electron and muon sectors). Also, the dynamical parts of flavor oscillation phase survive whether B_0 dominating neutrino masses or otherway round, however their variation depends on the value of B_0.

5. Entanglement of Neutrino States and Corresponding Geometric Phases

We begin by showing that neutrino (ψ) and antineutrino (ψ^c) combined system, as given by Equation (21) (also Equation (16)), forms entanglement. As $\psi^c = -i\sigma_2\psi^*$, and if ψ^c is purely spin-down with only one component non-zero then ψ is purely spin-up,

$$\begin{pmatrix} \psi^c \\ \psi \end{pmatrix} = c \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} - c^* \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (54)$$

where c is the non-zero component of ψ and we choose Weyl representation for convenience. As it stands, the first and second terms cannot be decomposed into the direct-product of two independent states, hence they entangle. Similarly, the combined mass eigenstates in the presence of gravitational field and Majorana mass, given by Equation (16), can be shown to exhibit entangled states.

Now in the presence of flavor mixing, as given by Equations (28) and (29), the states ν_{e1} and $\nu_{\mu1}$ are orthogonal to each other and the states ν_{e2} and $\nu_{\mu2}$ do so. Also without mixing term, ν_{e1} and ν_{e2} form two orthogonal mass eigenstates for neutrino–antineutrino mixing in the electron sector and $\nu_{\mu1}$ and $\nu_{\mu2}$ in the muon sector (when we consider only two flavors for simplicity).

Interestingly, it is clear from Equations (18) and (19) that gravitational field converts ν_{e1} (and $\nu_{\mu1}$) to ν_{e2} (and $\nu_{\mu2}$) by oscillation, leading to both of them being present at an arbitrary time. Hence, Equations (28) and (29) show that gravitational effect brings out two independent sets of flavor neutrinos, $\{\nu_{e1}, \nu_{\mu1}\}$ and $\{\nu_{e2}, \nu_{\mu2}\}$, satisfying respective orthogonality conditions between electron and muon neutrinos in the respective Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ independently. Hence, the neutrino states in $\mathcal{H}_1$ should entangle with those in $\mathcal{H}_2$ which are non-interacting. Therefore, following the conventional approach (e.g., [47]) we can construct the entangled states at $t = 0$

$$|\psi_1(0)\rangle = \cos\alpha\, |\nu_{e1}(0)\rangle |\nu_{e2}(0)\rangle + e^{i\beta}\, \sin\alpha\, |\nu_{\mu1}(0)\rangle |\nu_{\mu2}(0)\rangle, \tag{55a}$$

$$|\psi_2(0)\rangle = -\sin\alpha\, |\nu_{e1}(0)\rangle |\nu_{e2}(0)\rangle + e^{i\beta}\, \cos\alpha\, |\nu_{\mu1}(0)\rangle |\nu_{\mu2}(0)\rangle, \tag{55b}$$

$$|\psi_3(0)\rangle = \cos\alpha\, |\nu_{e1}(0)\rangle |\nu_{\mu2}(0)\rangle + e^{i\beta}\, \sin\alpha\, |\nu_{\mu1}(0)\rangle |\nu_{e2}(0)\rangle, \tag{55c}$$

$$|\psi_4(0)\rangle = -\sin\alpha\, |\nu_{e1}(0)\rangle |\nu_{\mu2}(0)\rangle + e^{i\beta}\, \cos\alpha\, |\nu_{\mu1}(0)\rangle |\nu_{e2}(0)\rangle, \tag{55d}$$

when $|\nu_{e1}\rangle$ and $|\nu_{e2}\rangle$ (and $|\nu_{\mu1}\rangle$ and $|\nu_{\mu2}\rangle$) in Equation (55a) are two points on the Poincaré sphere and so on for others equations. The angle α determines the degree of entanglement. As is the case in the Poincaré sphere of a single spin$-1/2$ particle, the above equation suggests that α and β parameterize a two-sphere called Schmidt sphere.

Note that various quantum correlations of the system of basic one-flavor and two-flavor neutrino and antineutrino states in the presence of gravitational field and hence gravitational Zeeman splitting, as described in Section 3, were already explored by one of the present authors [34]. Flavor entropy has been used therein to probe the entanglement in the system, which gets suppressed with the increase of gravitational field.

At an arbitrary time t, the entangled states, defined in Equation (55a–d), go to $|\psi_{1,2,3,4}(t)\rangle$ which have the same form as in Equation (55a–d), except $|\nu_{e1,2}(0)\rangle$ and $|\nu_{\mu1,2}(0)\rangle$ replaced by $|\nu_{e1,2}(t)\rangle$ and $|\nu_{\mu1,2}(t)\rangle$ respectively, as given by Equations (28) and (29) in terms of $|\chi_{1,3}(t)\rangle$ and $|\chi_{2,4}(t)\rangle$ generically.

Therefore, based on the definitions given in the beginning of Section 4, the total, dynamic and geometric parts of the phases in the evolution of entangled states $\psi_{1,2,3,4}$ can be obtained from

$$\Phi_{t1,2,3,4} = arg\left(\langle\psi_{1,2,3,4}(0)|\psi_{1,2,3,4}(\tau)\rangle\right), \quad -\Phi_{d1,2,3,4} = \int_0^\tau \langle\psi_{1,2,3,4}(t)|i\partial_t|\psi_{1,2,3,4}(t)\rangle dt. \tag{56}$$

Explicitly, for $\alpha = \pi/4$ and $\beta = 0$, we obtain

$$\Phi_{t1,2_{\pi/4}} = \tan^{-1}$$
$$\left[-\frac{\cos^2\theta_{12}^{\mp}\{\sin(E_{e1}+E_{e2})\tau + \sin(E_{\mu1}+E_{\mu2})\tau\} + \sin^2\theta_{12}^{\mp}\{\sin(E_{\mu1}+E_{e2})\tau + \sin(E_{\mu2}+E_{e1})\tau\}}{\cos^2\theta_{12}^{\mp}\{\cos(E_{e1}+E_{e2})\tau + \cos(E_{\mu1}+E_{\mu2})\tau\} + \sin^2\theta_{12}^{\mp}\{\cos(E_{\mu1}+E_{e2})\tau + \cos(E_{\mu2}+E_{e1})\tau\}}\right], \tag{57}$$

$$\Phi_{t3,4_{\pi/4}} = \tan^{-1}$$
$$\left[-\frac{\sin^2\theta_{12}^{\pm}\{\sin(E_{e1}+E_{e2})\tau + \sin(E_{\mu1}+E_{\mu2})\tau\} + \cos^2\theta_{12}^{\pm}\{\sin(E_{\mu1}+E_{e2})\tau + \sin(E_{\mu2}+E_{e1})\tau\}}{\sin^2\theta_{12}^{\pm}\{\cos(E_{e1}+E_{e2})\tau + \cos(E_{\mu1}+E_{\mu2})\tau\} + \cos^2\theta_{12}^{\pm}\{\cos(E_{\mu1}+E_{e2})\tau + \cos(E_{\mu2}+E_{e1})\tau\}}\right], \tag{58}$$

when $\cos^2\theta_{12}^{\mp} = \cos^2(\theta_1 \mp \theta_2)$ and $\sin^2\theta_{12}^{\mp} = \sin^2(\theta_1 \mp \theta_2)$. For $\theta_1 = \theta_2$ and $\theta_1 = -\theta_2$, respectively for $\Phi_{t1,4_{\pi/4}}$ and $\Phi_{t2,3_{\pi/4}}$, the phases furthermore reduce to

$$\Phi_{t1,2,3,4_{\pi/4}} = -\left(E_{e1}+E_{e2}+E_{\mu1}+E_{\mu2}\right)\frac{\tau}{2} \quad \text{or} \quad \pi - \left(E_{e1}+E_{e2}+E_{\mu1}+E_{\mu2}\right)\frac{\tau}{2}. \tag{59}$$

For constant and arbitrary $\alpha, \theta_1, \theta_2$

$$-\Phi_{d1} =$$
$$\cos^2\alpha\left[-\phi_1\sin^2\theta_1 - \phi_2\sin^2\theta_2 + \int_0^\tau \left(E_{\mu1}\cos^2\theta_1 + E_{e1}\sin^2\theta_1 + E_{\mu2}\cos^2\theta_2 + E_{e2}\sin^2\theta_2\right)dt\right] +$$
$$\sin^2\alpha\left[-\phi_1\cos^2\theta_1 - \phi_2\cos^2\theta_2 + \int_0^\tau \left(E_{\mu1}\sin^2\theta_1 + E_{e1}\cos^2\theta_1 + E_{\mu2}\sin^2\theta_2 + E_{e2}\cos^2\theta_2\right)dt\right]$$
$$-\beta\sin^2\alpha \tag{60}$$

and $-\Phi_{d2}$ is the same except $\cos^2 \alpha$ and $\sin^2 \alpha$ are interchanged. Similarly,

$$
\begin{aligned}
-\Phi_{d3} = &\cos^2 \alpha \left[-\phi_1 \sin^2 \theta_1 - \phi_2 \cos^2 \theta_2 + \int_0^\tau \left(E_{\mu 1} \sin^2 \theta_1 + E_{e1} \cos^2 \theta_1 + E_{\mu 2} \cos^2 \theta_2 + E_{e2} \sin^2 \theta_2 \right) dt \right] + \\
&\sin^2 \alpha \left[-\phi_1 \cos^2 \theta_1 - \phi_2 \sin^2 \theta_2 + \int_0^\tau \left(E_{\mu 1} \cos^2 \theta_1 + E_{e1} \sin^2 \theta_1 + E_{\mu 2} \sin^2 \theta_2 + E_{e2} \cos^2 \theta_2 \right) dt \right] \\
&-\beta \sin^2 \alpha
\end{aligned}
\tag{61}
$$

and $-\Phi_{d4}$ is the same except $\cos^2 \alpha$ and $\sin^2 \alpha$ are interchanged.

For $\alpha = \pi/4$, $\Phi_{d1,2,3,4}$ reduce as

$$
-\Phi_{d1,2,3,4_{\pi/4}} = \frac{1}{2} \left[-\phi_1 - \phi_2 + \int_0^\tau \left(E_{e1} + E_{e2} + E_{\mu 1} + E_{\mu 2} \right) dt - \beta \right].
\tag{62}
$$

For oscillation between entangled states, the total phases $\Phi_{to1} = arg(\langle \psi_2(0) | \psi_1(\tau) \rangle)$ and $\Phi_{to2} = arg(\langle \psi_4(0) | \psi_3(\tau) \rangle)$, for $\alpha = \pi/4$ and $\beta = 0$, are

$$
\Phi_{to1,2_{\pi/4}} = \frac{\pi}{2} - \left(E_{e1} + E_{e2} + E_{\mu 1} + E_{\mu 2} \right) \frac{\tau}{2}.
\tag{63}
$$

The other phases, containing a τ-independent part during oscillation, $-\Phi_{do1} = \int_0^\tau \langle \psi_2(t) | i\partial_t | \psi_1(t) \rangle dt$ and $-\Phi_{do2} = \int_0^\tau \langle \psi_4(t) | i\partial_t | \psi_3(t) \rangle dt$, are given by

$$
\begin{aligned}
-\Phi_{do1,2} = &-\cos\alpha \sin\alpha \left[\cos 2\theta_1 \left\{ \phi_1 + \int_0^\tau \left(E_{e1} - E_{\mu 1} \right) dt \right\} \pm \cos 2\theta_2 \left\{ \phi_2 + \int_0^\tau \left(E_{e2} - E_{\mu 2} \right) dt \right\} + \beta \right].
\end{aligned}
\tag{64}
$$

For varying mixing parameters, the terms appearing outside the integral in all Φ_d-s will also always contribute to the respective phases. Note interestingly that

$$
\Phi_{d1} + \Phi_{d2} = \Phi_{d3} + \Phi_{d4} = - \left(E_{e1} + E_{\mu 1} + E_{e2} + E_{\mu 2} \right) \tau + \left(\beta + \phi_1 + \phi_2 \right).
\tag{65}
$$

Like the cases in previous section, as neutrino mass is not expected to vary with time, the phases associated with ϕ do not contribute generally.
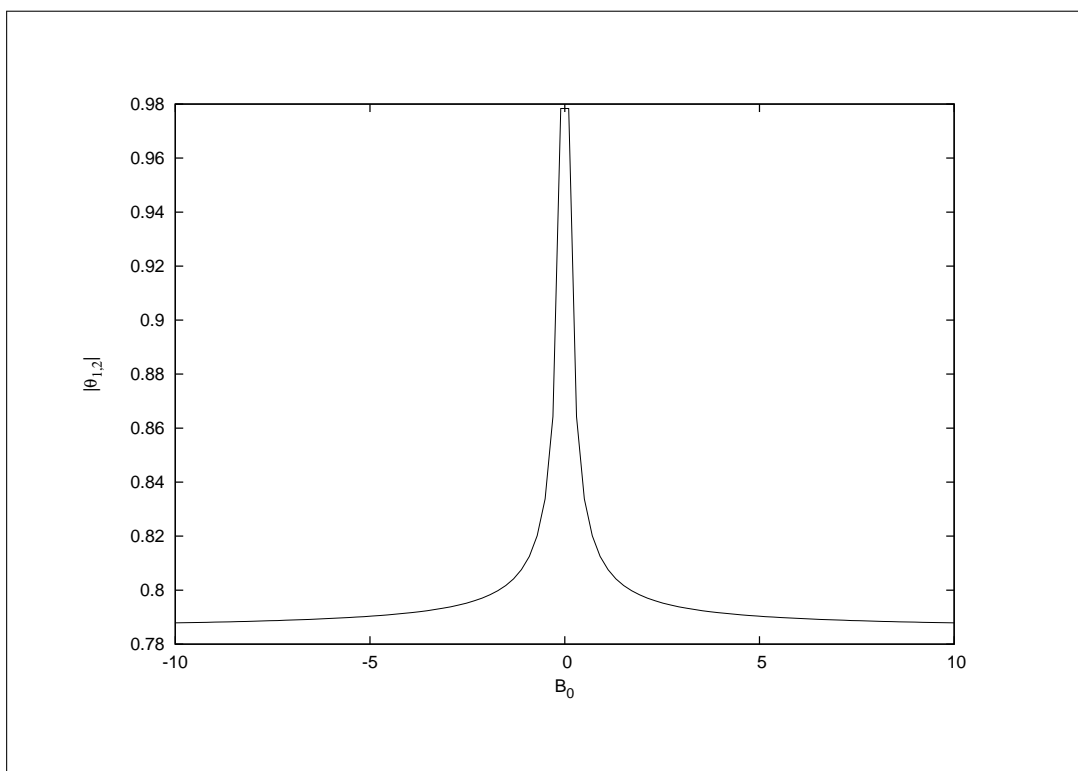
6. Variation of Mixing Angles with Gravitational Field

The phases independent of τ are associated with mixing angles and phases of neutrinos, e.g., $\theta_1, \theta_2, \alpha$ (also θ) and ϕ_1, ϕ_2, β (also ϕ). Therefore, depending on θ-s, which are determined by gravitational field and the physical nature of spacetime geometry, the τ-independent parts of phases vary. In the absence of gravitational field and in the presence of lepton number violating interaction and hence Majorana mass, neutrino and antineutrino mix with $\theta = \pi/4$. Figure 3 shows that how the mixing angle of the basic neutrino-antineutrino states changes with gravitational coupling, which furthermore controls the geometric Berry-like phases associated with $\Phi_{t1,2} - \Phi_{d1,2}$ given in Sections 4.1 and 4.2. While a stronger gravity effect kills oscillation, it still leads to a non-zero geometric phase.

In the presence of very strong gravitational effect ($B_0 >> m_e, m_\mu$) (see, e.g., [25,26]), $\theta_{1,2} \to \pi/4$ and for entangled states $\Phi_{do1,2} \to \beta \cos\alpha \sin\alpha$. Similarly, the τ-independent part of $\Phi_{d1,2,3,4}$ of entangled states survives even at a very strong gravitational field for an arbitrary α. Figure 4 shows that θ_1 and θ_2 decrease with the increase of $|B_0|$, which furthermore controls geometric Berry-like phases associated with entangled states given in Section 5. Figure 5 shows how the corresponding τ-independent part of Φ_{d1} varies with the change of the strength of gravitational coupling. This confirms that while $\theta_{1,2}$ decrease with increasing B_0, mixing and also various phases still survive.
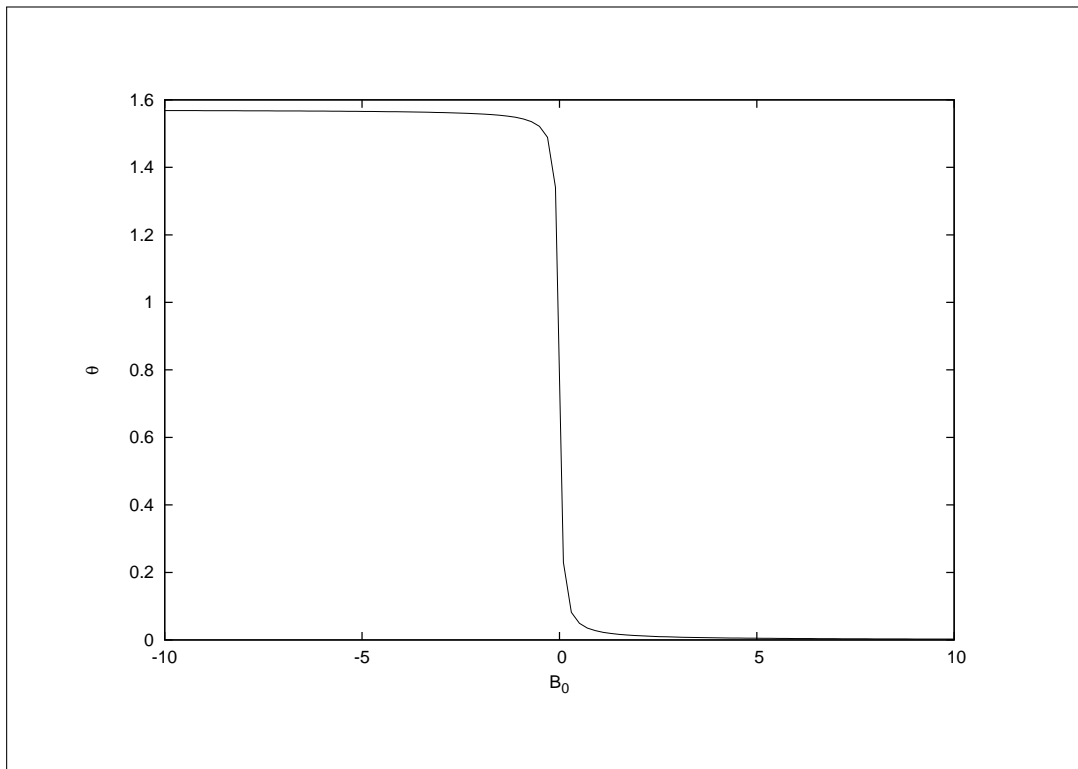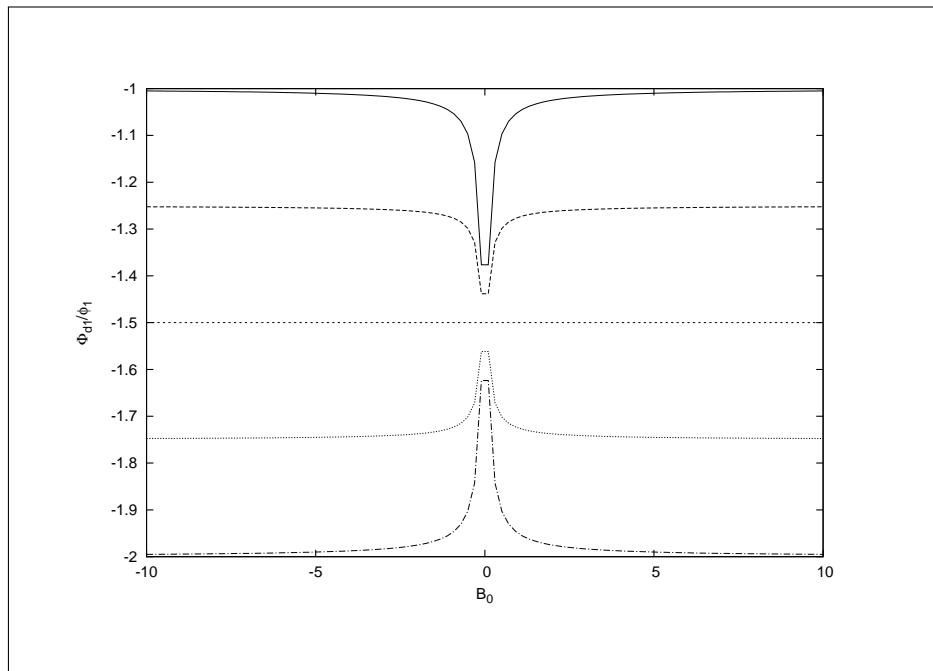
Figure 3. Variation of mixing angle in radian of basic neutrino–antineutrino mixing as a function of gravitational coupling in units of eV with $m = 0.05$ eV.

Figure 4. Variation of flavor mixing angles in radian as functions of gravitational coupling in units of eV with $m_e = 0.01$ eV, $m_\mu = 0.1$ eV and $m_{\mu e} = 0.05$ eV.

Figure 5. Variation of τ-independent part of Φ_{d1} for entangled states as a function of gravitational coupling in units of eV with $m_e = 0.01$ eV, $m_\mu = 0.1$ eV and $m_{\mu e} = 0.05$ eV, and $\phi_1 = \phi_2 = \beta$. From top to bottom, lines are for $\alpha = 0, \pi/6, \pi/4, \pi/3$ and $\pi/2$.

7. Summary

Spinors interacting with background gravity of arbitrary strength in an arbitrary spacetime are known to be split in energies between up and down spinors and furthermore into the states of positive and negative energies. The only requirement is that the spacetime should not be spherically symmetric. It has been shown that such spinors acquire a geometric phase due to background gravitational field in the same way as they do in a magnetic field. The necessary condition for such a situation is either the spacetime curvature coupling to the spinor (gravitational 4-vector potential) is not constant or the momentum of the spinor is not constant, along with a non-zero (even constant) temporal part of curvature coupling.

Neutrinos, as a class of spinors in nature, are shown to acquire geometric as well as dynamic phases during their propagation under background gravitational field. To have a non-trivial phase induced by the gravitational field of compact objects (e.g., black hole), the mass of the object producing gravitational fields must not be more than a millionth of a solar mass, i.e., primordial in nature. There are many missions to constrain the evidence for primordial black holes, e.g., observing specific small interference patterns within gamma-ray bursts by the *Fermi* Gamma-ray Space Telescope could be the indirect evidence for primordial black holes.

In the flavor sector, when the background gravity is much stronger than the lepton number violating (Majorana) masses, the mixing parameters $\theta_{1,2}$ become a constant equal to $\pi/4$, hence maximum mixing, which could lead to ϕ-dependent geometric phases if ϕ varies. However, for a weak background gravity, $\theta_{1,2}$ are found to depend on the specific values of the neutrino masses. The combined neutrino–antineutrino states form an entangled system whose phases have also been calculated.

Author Contributions: Conceptualization, B.M. and S.K.G.; methodology, B.M. and S.K.G.; software, B.M.; validation, B.M.; formal analysis, B.M. and S.K.G.; investigation, B.M.; resources, B.M. and S.K.G.; data curation, B.M.; writing—original draft preparation, B.M.; writing—review and editing, B.M.; visualization, B.M.; supervision, B.M.; project administration, B.M.; funding acquisition, B.M. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Berry, M.V. Quantal phase factors accompanying adiabatic changes. *Proc. R. Soc. A* **1984**, *392*, 45.
2. Aharonov, Y.; Anandan, J. Phase change during a cyclic quantum evolution. *Phys. Rev. Lett.* **1987**, *58*, 1593. [CrossRef] [PubMed]
3. Samuel, J.; Bhandari, R. General setting for Berry's phase. *Phys. Rev. Lett.* **1988**, *60*, 2339. [CrossRef] [PubMed]
4. Mukunda, N.; Simon, R. Quantum kinematic approach to the geometric phase. I. General formalism. *Ann. Phys. (N. Y.)* **1993**, *228*, 205. [CrossRef]
5. Nakagawa, N. Geometrical phase factors and higher-order adiabatic approximations. *Ann. Phys. (N. Y.)* **1987**, *179*, 145. [CrossRef]
6. Vidal, J.; Wudka, J. Non-dynamical contributions to left-right transitions in the solar neutrino problem. *Phys. Lett. B* **1990**, *249*, 473. [CrossRef]
7. Aneziris, C.; Schechter, J. Three Majorana neutrinos in a twisting magnetic field. *Phys. Rev. D* **1992**, *45*, 1053. [CrossRef]
8. Smirnov, A.Y. The geometrical phase in neutrino spin precession and the solar neutrino problem. *Phys. Lett. B* **1991**, *260*, 161. [CrossRef]
9. Guzzo, M.M.; Bellandi, J. On the question of neutrino spin precession in a magnetic field. *Phys. Lett. B* **1992**, *294*, 243.
10. He, X.-G.; Li, X.-Q.; McKellar, B.H.J.; Zhang, Y. Berry phase in neutrino oscillations. *Phys. Rev. D* **2005**, *72*, 053012. [CrossRef]
11. Blasone, M.; Henning, P.A.; Vitiello, G. Berry phase for oscillating neutrinos. *Phys. Lett. B* **1999**, *466*, 262. [CrossRef]
12. Wang, X.-B.; Kwek, L.C.; Liu, Y.; Oh, C.H. Noncyclic phase for neutrino oscillation. *Phys. Rev. D* **2001**, *63*, 053003. [CrossRef]
13. Joshi, S.; Jain, S. Geometric phase for neutrino propagation in magnetic field. *Phys. Lett. B* **2016**, *754*, 135. [CrossRef]
14. Ramaseshan, S.; Nityananda, R. The interference of polarized light as an early example of Berry's phase. *Curr. Sci.* **1986**, *55*, 1225.
15. Pancharatnam, S. Generalized theory of interference, and its applications. *Proc. Indian Acad. Sci. A* **1956**, *44*, 247. [CrossRef]
16. Mehta, P. Topological phase in two flavor neutrino oscillations. *Phys. Rev. D* **2009**, *79*, 096013. [CrossRef]
17. Dajka, J.; Syska, J.; Łuczka, J. Geometric phase of neutrino propagating through dissipative matter. *Phys. Rev. D* **2011**, *83*, 097302. [CrossRef]
18. Syska, J.; Dajka, J.; Łuczka, J. Interference phenomenon and geometric phase for Dirac neutrino in π^+ decay. *Phys. Rev. D* **2013**, *87*, 117302. [CrossRef]
19. Joshi, S.; Jain, S.R. Noncyclic geometric phases and helicity transitions for neutrino oscillations in a magnetic field. *Phys. Rev. D* **2017**, *96*, 096004. [CrossRef]
20. Johns, L.; Fuller, G.M. Geometric phases in neutrino oscillations with nonlinear refraction. *Phys. Rev. D* **2017**, *95*, 043003. [CrossRef]
21. Wang, Z.; Pan, H. Exploration of CPT violation via time-dependent geometric quantities embedded in neutrino oscillation through fluctuating matter. *Nuc. Phys. B* **2017**, *915*, 414. [CrossRef]
22. Dixit, K.; Alok, A.K.; Banerjee, S.; Kumar, D. Geometric phase and neutrino mass hierarchy problem. *J. Phys. G* **2018**, *45*, 085002. [CrossRef]
23. Capolupo, A.; Giampaolo, S.M.; Hiesmayr, B.C.; Vitiello, G. Geometric phase of neutrinos: Differences between Dirac and Majorana neutrinos. *Phys. Lett. B* **2018**, *780*, 216. [CrossRef]

24. Simonov, K.; Capolupo, A.; Giampaolo, S.M. Gravity, entanglement and CPT-symmetry violation in particle mixing. *Eur. Phys. J. C* **2019**, *79*, 902. [CrossRef]

25. Mukhopadhyay, B. Gravity-induced neutrino-antineutrino oscillation: CPT and lepton number non-conservation under gravity. *Class. Quantum Gravity* **2007**, *24*, 1433. [CrossRef]

26. Sinha, M.; Mukhopadhyay, B. CPT and lepton number violation in the neutrino sector: Modified mass matrix and oscillation due to gravity. *Phys. Rev. D* **2008**, *77*, 025003. [CrossRef]

27. Singh, P.; Mukhopadhyay, B. Gravitationally induced neutrino asymmetry. *Mod. Phys. Lett. A* **2003**, *18*, 779. [CrossRef]

28. Mukhopadhyay, B. Neutrino asymmetry around black holes: Neutrinos interact with gravity. *Mod. Phys. Lett. A* **2005**, *20*, 2145. [CrossRef]

29. Debnath, U.; Mukhopadhyay, B.; Dadhich, N. Spacetime Curvature Coupling of Spinors in Early Universe:. Neutrino Asymmetry and a Possible Source of Baryogenesis. *Mod. Phys. Lett. A* **2006**, *21*, 399. [CrossRef]

30. Kim, C.W.; Sze, W.K.; Nussinov, S. Neutrino oscillations and the Landau-Zener formula. *Phys. Rev. D* **1987**, *35*, 4014. [CrossRef]

31. Stodolsky, L. Treatment of neutrino oscillations in a thermal environment. *Phys. Rev. D* **1987**, *36*, 2273. [CrossRef] [PubMed]

32. Píriz, D.; Roy, M.; Wudka, J. Neutrino oscillations in strong gravitational fields. *Phys. Rev. D* **1996**, *54*, 1587. [CrossRef] [PubMed]

33. Mohanty, S.; Mukhopadhyay, B.; Prasanna, A.R. Experimental tests of curvature couplings of fermions in general relativity. *Phys. Rev. D* **2002**, *65*, 122001. [CrossRef]

34. Dixit, K.; Naikoo, J.; Mukhopadhyay, B.; Banerjee, S. Quantum correlations in neutrino oscillations in curved spacetime. *Phys. Rev. D* **2019**, *100*, 055021. [CrossRef]

35. Birrell, N.D.; Davies, P. *Quantum Fields in Curved Space*; Cambridge University Press: Cambridge, UK, 1982.

36. Kaku, M. *Quantum Field Theory*; Oxford University Press: Oxford, UK, 1993.

37. Schwinger, J. *Particles, Sources, and Fields III*; Addison-Wesley: Redwood City, CA, USA, 1989.

38. Obukhov, Y.N. Spin, Gravity, and Inertia. *Phys. Rev. Lett.* **2001**, *86*, 192. [CrossRef]

39. Huang, X.; Parker, L. Hermiticity of the Dirac Hamiltonian in curved spacetime. *Phys. Rev. D* **2009**, *79*, 024020. [CrossRef]

40. Mukhopadhyay, B. Exploring the Universe: From Near Space to Extra-Galactic. In *Astrophysics and Space Science Proceedings*; Mukhopadhyay, B., Sasmal, S., Eds.; Springer: Berlin/Heidelberg, Germany, 2018; Volume 53, p. 3.

41. Colladay, D.; Kostelecký, V.A. CPT violation and the standard model. *Phys. Rev. D* **1997**, *55*, 6760. [CrossRef]

42. Ellis, J.; Mavromatos, N.E. Role of space-time foam in breaking supersymmetry via the Barbero-Immirzi parameter. *Phys. Rev. D* **2011**, *84*, 085016. [CrossRef]

43. Mavromatos, N.E.; Sarkar, S. CPT-violating leptogenesis induced by gravitational defects. *Eur. Phys. J. C* **2013**, *73*, 2359. [CrossRef]

44. Mosquera Cuesta, H.J. Neutrino astrophysics in slowly rotating spacetimes permeated by nonlinear electrodynamics fields. *Astrophys. J.* **2017**, *835*, 215. [CrossRef]

45. Diaz, J.S.; Katori, T.; Spitz, J.; Conrad, J.M. Search for neutrino-antineutrino oscillations with a reactor experiment. *Phys. Lett. B* **2013**, *727*, 412. [CrossRef]

46. Shapiro, S.L.; Teukolsky, S.A. *Black Holes, White Dwarfs, and Neutron Stars: The Physics of Compact Objects*; John Wiley & Sons: New York, NY, USA, 1983.

47. Sjöqvist, E. Geometric phase for entangled spin pairs. *Phys. Rev. A* **2000**, *62*, 022109. [CrossRef]