An Alternative to Dark Matter and Dark Energy: Scale-Dependent Gravity in Superfluid Vacuum Theory

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Abstract: We derive an effective gravitational potential, induced by the quantum wavefunction of a physical vacuum of a self-gravitating configuration, while the vacuum itself is viewed as the superfluid described by the logarithmic quantum wave equation. We determine that gravity has a multiple-scale pattern, to such an extent that one can distinguish sub-Newtonian, Newtonian, galactic, extragalactic and cosmological terms. The last of these dominates at the largest length scale of the model, where superfluid vacuum induces an asymptotically Friedmann–Lemaître–Robertson–Walker-type spacetime, which provides an explanation for the accelerating expansion of the Universe. The model describes different types of expansion mechanisms, which could explain the discrepancy between measurements of the Hubble constant using different methods. On a galactic scale, our model explains the non-Keplerian behaviour of galactic rotation curves, and also why their profiles can vary depending on the galaxy. It also makes a number of predictions about the behaviour of gravity at larger galactic and extragalactic scales. We demonstrate how the behaviour of rotation curves varies with distance from a gravitating center, growing from an inner galactic scale towards a metagalactic scale: A squared orbital velocity's profile crosses over from Keplerian to flat, and then to non-flat. The asymptotic non-flat regime is thus expected to be seen in the outer regions of large spiral galaxies.

Keywords: quantum gravity; cosmology; superfluid vacuum; emergent spacetime; dark matter; galactic rotation curve; quantum Bose liquid; logarithmic fluid; logarithmic wave equation

1. Introduction

Astronomical observations over many length scales support the existence of a number of novel phenomena, which are usually attributed to dark matter (DM) and dark energy (DE). Dark matter was introduced to explain a range of observed phenomena at a galactic scale, such as flat rotation curves, while dark energy is expected to account for cosmological-scale dynamics, such as the accelerating expansion of the Universe. For instance, the ΛCDM model, which is currently the most popular approach used in cosmology and galaxy-scale astrophysics, makes use of both DE and cold DM concepts [1]. In spite of being a generally successful framework purporting to explain the large-scale structure of the Universe, it currently faces certain challenges [2,3].

There is also growing consensus that a convincing theory of DM- and DE-attributed phenomena cannot be a stand-alone model; but should, instead, be a part of a fundamental theory involving all known interactions. In turn, we contend that formulating this fundamental theory will be impossible without a clear understanding of the dynamical structure of the physical vacuum, which underlies all interactions that we know of. Moreover, this theory must operate at a quantum level, which necessitates us rethinking of the concept of gravity using basic notions of quantum mechanics.

One of the promising candidates for a theory of physical vacuum is superfluid vacuum theory (SVT), a post-relativistic approach to high-energy physics and gravity. Historically, it evolved from Dirac's idea of viewing the physical vacuum as a nontrivial quantum object, whose phase and derived velocity are non-observable in a quantum-mechanical sense [4]. The term 'post-relativistic', in this context, means that SVT can generally be a non-relativistic theory; which nevertheless contains relativity as a special case, or limit, with respect to some dynamical value such as momentum (akin to general relativity being a superset of the Newton's theory of gravity). Therefore, underlying three-dimensional space would not be physically observable until an observer goes beyond the above-mentioned limit, as will be discussed in more detail later in this article.

The dynamics and structure of superfluid vacuum are being studied, using various approaches which agree upon the main paradigm (physical vacuum being a background quantum liquid of a certain kind, and elementary particles being excitations thereof), but differ in their physical details, such as an underlying model of the liquid [5–7].

It is important to work with a precise definition of superfluid, to ensure that we avoid the most common misconceptions which otherwise might arise when one attempts to apply superfluid models to astrophysics and cosmology, some details can be found in Appendix A. In fact, some superfluid-like models of dark matter based on classical perfect fluids, scalar field theories or scalar-tensor gravities, turned out to be vulnerable to experimental verification [8]. Moreover, superfluids are often confused not only with perfect fluids, but also with the concomitant phenomenon of Bose-Einstein condensates (BEC), which is another kind of quantum matter occurring in low-temperature condensed matter [9]. However, even though BEC's do share certain features with superfluids, this does not imply that they are superfluidic in general.

In particular, quantum excitations in laboratory superfluids that we know of have dispersion relations of a distinctive shape called the Landau "roton" spectrum. Such a shape of the spectral curve is crucial, as it ensures the suppression of dissipative fluctuations at a quantum level [10,11], which results in inviscid flow [12,13]. If plotted as an excitation energy versus momentum, the curve starts from the origin, climbs up to a local maximum (called the maxon peak), then slightly descends to a local nontrivial minimum (called the "roton" energy gap); then grows again, this time all the way up, to the boundary of the theory's applicability range. In fact it is not the roton energy gap alone, but the energy barrier formed by the maxon peak and roton minimum in momentum space, which ensures the above-mentioned suppression of quantum fluctuations in quantum liquid and, ultimately, causes its flow to become inviscid. In other words, it is the global characteristics of the dispersion curve, not just the existence of a nontrivial local minimum and related energy gap, which is important for superfluidity to occur. Obviously, these are non-trivial properties, which cannot possibly occur in all quantum liquids and condensates. Further details and aspects are discussed in Appendix A.

This paper is organized as follows. Theory of physical vacuum based on the logarithmic superfluid model is outlined in Section 2, where we also demonstrate how four-dimensional spacetime can emerge from the three-dimensional dynamics of quantum liquid. In Section 3, we derive the gravitational potential, induced by the logarithmic superfluid vacuum in a given state, using certain simplifying assumptions. Thereafter, in Section 4, we give a brief physical interpretation of different parts of the derived gravitational potential and estimate their characteristic length scales. In Section 5, profiles of induced matter density are derived and discussed for the case of spherical symmetry. Galactic scale phenomena are discussed in Section 6, where the phenomenon of galactic rotation curves is explained without introducing any exotic matter ad hoc. In Section 7, we discuss the various mechanisms of the accelerating expansion of the Universe, as well as the cosmological singularity, "vacuum catastrophe" and cosmological coincidence problems. Conclusions are drawn in Section 8.

2. Logarithmic Superfluid Vacuum

Superfluid vacuum theory assumes that the physical vacuum is described, when disregarding quantum fluctuations, by the fluid condensate wavefunction $\Psi(\mathbf{r}, t)$, which is a three-dimensional Euclidean scalar. The state itself is described by a ray in the corresponding Hilbert space, therefore this wavefunction obeys a normalization condition

$$\langle \Psi | \Psi \rangle = \int_{\mathcal{V}} \rho \, d\mathcal{V} = \mathcal{M}, \tag{1}$$

where $\mathcal{M}$ and $\mathcal{V}$ are the total mass and volume of the fluid, respectively, and $\rho = |\Psi|^2$ is the fluid mass density. The wavefunction's dynamics is governed by an equation of a $U(1)$-symmetric Schrödinger form:

$$\left[-i\hbar \, \partial_t - \frac{\hbar^2}{2m} \boldsymbol{\nabla}^2 + V_{\text{ext}}(\mathbf{r}, t) + F(|\Psi|^2) \right] \Psi = 0, \tag{2}$$

where m is the constituent particles' mass, $V_{\text{ext}}(\mathbf{r}, t)$ is an external or trapping potential and $F(\rho)$ is a duly chosen function, which effectively takes into account many-body effects inside the fluid. This wave equation can be formally derived as a minimizing condition of an action functional with the following Lagrangian:

$$\mathcal{L} = \frac{i\hbar}{2} (\Psi \partial_t \Psi^* - \Psi^* \partial_t \Psi) + \frac{\hbar^2}{2m} |\boldsymbol{\nabla} \Psi|^2 + V_{\text{ext}}(\mathbf{r}, t) \, |\Psi|^2 + V(|\Psi|^2), \tag{3}$$

where $V(\rho)$ equals to a primitive of $F(\rho)$ up to an additive constant: $F(\rho) = V'(\rho)$; throughout the paper the prime denotes a derivative with respect to the function's argument.

In this picture, massless excitations, such as photons, are analogous to acoustic waves propagating with velocity $c_s \propto \sqrt{|p'(\rho)|}$, where fluid pressure $p = p(\rho)$ is determined via the equation of state. For the system (2), both the equation of state and speed of sound can be derived using the fluid-Schrödinger analogy, which was established for a special case in [14], and generalized for an arbitrary $F(\rho)$ in works [7,15]. In a leading-order approximation with respect to the Planck constant, we obtain

$$p = -\frac{1}{m} \int \rho F'(\rho) \, d\rho, \quad c_s^2 = \frac{1}{m} \rho |F'(\rho)|, \tag{4}$$

while higher-order corrections would induce Korteweg-type effects, thus significantly complicating the subject matter [15].

Furthermore, it is natural to require that superfluid vacuum theory must recover Einstein's theory of relativity at a certain limit. One can show that at a limit of low momenta of quantum excitations, often called a "phononic" limit by analogy with laboratory quantum liquids, Lorentz symmetry does emerge. This can be easily shown by virtue of the fluid/gravity analogy [16], which was subsequently used to formulate the BEC-spacetime correspondence [7]; it can also be demonstrated by using dispersion relations [11,17], which are generally become deformed in theories with non-exact Lorentz symmetry [18–21].

This correspondence states that Lorentz symmetry is approximate, while four-dimensional spacetime is an induced phenomenon, determined by the dynamics of quantum Bose liquid moving in Euclidean three-dimensional space. The latter is only observable by a certain kind of observer, a F(ull)-observer. Other observers, R(elativistic)-observers, perceive this superfluid as a non-removable background, which can be modeled as a four-dimensional pseudo-Riemannian manifold. What is the difference between these types of observers?

F-observers can perform measurements using objects of arbitrary momenta and "see" the fundamental superfluid wavefunction's evolution in three-dimensional Euclidean space according to Equation (2) or an analogue thereof. On the other hand, R-observers are restricted to measuring only small-momentum small-amplitude excitations of the background superfluid. This is somewhat

analogous to listening to acoustic waves (phonons) in the conventional Bose-Einstein condensates, but being unaware of higher-energy particles such as photons or neutrons.

According to BEC-spacetime correspondence, a R-observer "sees" himself located inside four-dimensional curved spacetime with a pseudo-Riemannian metric. The latter can be written in Cartesian coordinates as [7]:

$$
g_{\mu\nu} \propto \frac{\rho}{c_s}
\begin{pmatrix}
-\left[c_s^2 - \eta^2 (\boldsymbol{\nabla} S)^2\right] & \vdots & -\eta \boldsymbol{\nabla} S \\
\cdots\cdots & \cdot & \cdots \\
-\eta \boldsymbol{\nabla} S & \vdots & \mathbf{I}
\end{pmatrix},
\tag{5}
$$

where $\eta = \hbar/m$, $S = S(\boldsymbol{r}, t) = -i \ln \left(\Psi(\boldsymbol{r}, t)/|\Psi(\boldsymbol{r}, t)|\right)$ is a phase of the condensate wavefunction written in the Madelung representation, $\Psi = \sqrt{\rho} \exp(iS)$, and $\mathbf{I}$ is a three-dimensional unit matrix. To maintain the correct metric signature in Equation (5), condition $|c_s| > \eta |\boldsymbol{\nabla} S|$ must be imposed, which indicates that c_s is the maximum attainable velocity of test particles (i.e., small-amplitude excitations of the condensate), moving along geodesics on this induced spacetime. Therefore, c_s is the velocity of those excitations of vacuum, which describe massless particles in the low-momentum limit, whereas massive test particles move along geodesics of a pseudo-Riemannian manifold with metric Equation (5). According to a R-observer, they are freely falling, independently of their properties including their rest mass.

In this approach, we interpret Einstein field equations not as differential equations for an unknown metric; but as a definition for an induced stress-energy tensor, describing some effective matter to which test particles couple. Therefore, this would be the gravitating matter observed by a R-observer. We thus obtain

$$
\widetilde{T}_{\mu\nu} \equiv \kappa^{-1} \left[R_{\mu\nu}(g) - \frac{1}{2} g_{\mu\nu} R(g) \right],
\tag{6}
$$

where $\kappa = 8\pi G/c_{(0)}^2$ is the Einstein's gravitational constant. An example of usage of this procedure will be considered in Section 7.1. While Equation (6) is in fact an assumption, it should hold not only under the validity of conventional general relativity, but also in other Lorentz-symmetric theories of gravity which are linear with respect to the Riemann tensor, because the form of Einstein equations is quite universal (up to a conformal transform). For other Lorentz-symmetric theories, whose field equations cannot be transformed into this form, definition (6) can be adjusted accordingly.

Furthermore, one can see from Equation (4), that c_s contains an unknown function $F(\rho)$. To determine its form, let us recall that one of the relativistic postulates implies that velocity c_s should not depend on density, at least in the classical limit. More specifically, at low momenta, this velocity should tend to the value $c_{(0)} \approx c$, where $c = 2.9979 \times 10^{10}$ cm s^{-1} is called the speed of light in vacuum, for historical reasons. Recalling Equation (4), this requirement can be written as a differential equation [7]:

$$
\rho |F'(\rho)| = m c_s^2 \approx \mathrm{const}(\rho),
\tag{7}
$$

where $\mathrm{const}(\rho)$ denotes a function which does not depend on density. The solution of this differential equation is a logarithmic function:

$$
F(\rho) = -b \ln (\rho/\bar{\rho}),
\tag{8}
$$

where b and $\bar{\rho}$ are generally real-valued functions of coordinates. The wave Equation (2) thus narrows down to

$$
i\hbar \partial_t \Psi = \left[-\frac{\hbar^2}{2m} \boldsymbol{\nabla}^2 + V_{\mathrm{ext}}(\boldsymbol{r}, t) - b \ln \left(|\Psi|^2/\bar{\rho}\right) \right] \Psi,
\tag{9}
$$

where b is the nonlinear coupling; $b = b(\boldsymbol{r}, t)$ in general. Correspondingly, Equation (4) yields

$$
p = -(b/m)\rho, \quad c_s = \sqrt{|b|/m},
\tag{10}
$$

thus indicating that logarithmic Bose liquid behaves like barotropic perfect fluid; but only when one neglects quantum corrections, and assumes classical averaging. This reaffirms the statement made in the previous Section about the place of perfect-fluid models when it comes to gravitational phenomena. The way gravity emerges in the superfluid vacuum picture is entirely different from those models, as will be demonstrated shortly, after we have specified our working model.

Some special cases of Equation (9), for example when $b \to b_0 = \text{const}$, were extensively studied in the past, although not for reasons related to quantum liquids [22,23]. There were also extensive mathematical studies of these equations, to mention just some very recent literature [24–37].

Interestingly, wave equations with logarithmic nonlinearity can be also introduced into fundamental physics independently of relativistic arguments [7,38,39]. This nonlinearity readily occurs in the theory of open quantum systems, quantum entropy and information [40,41]; as well as in the theory of general condensate-like materials, for which characteristic kinetic energies are significantly smaller than interparticle potentials [42].

One example of such a material is helium II, the superfluid phase of helium-4. For the latter, the logarithmic superfluid model is known to have been well verified by experimental data [10,43]. Among other things, the logarithmic superfluid model does reproduce the sought-after Landau-type spectrum of excitations, discussed in the previous Section; detailed derivations can be found in [10]. One of underlying reasons for such phenomenological success is that the ground-state wavefunction of free (trapless) logarithmic liquid is not a de Broglie plane wave, but a spatial Gaussian modulated by a de Broglie plane wave. This explains the liquid's inhomogenization followed by the formation of fluid elements or parcels; which indicates that such models do describe fluids, rather than gaseous matter [44–49].

To summarize, a large number of arguments to date, both theoretical and experimental, demonstrate the robustness of logarithmic models in the general theory of superfluidity. In the next Section we shall demonstrate the logarithmic superfluid model's capabilities when assuming superfluidity of the physical vacuum itself.

In what follows, we shall make use of a minimal inhomogeneous model for the logarithmic superfluid which was proposed in [42], based on statistical and thermodynamics arguments. In the F-observer's picture, its wave equation can be written as

$$i\hbar\partial_t \Psi = \left[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + V_{\text{ext}}(\boldsymbol{r},t) - \left(b_0 - \frac{q}{r^2}\right)\ln\left(\frac{|\Psi|^2}{\bar{\rho}}\right) \right]\Psi, \tag{11}$$

where $r = |\boldsymbol{r}| = \sqrt{\boldsymbol{r}\cdot\boldsymbol{r}}$ is a radius-vector's absolute value, and b_0 and q are real-valued constants. For definiteness, let us assume that $b_0 > 0$, because one can always change the overall signs of the nonlinear term $F(\rho)$ and the corresponding field-theoretical potential $V(\rho)$. As always, this wave equation must be supplemented with a normalization condition (1), boundary and initial conditions of a quantum-mechanical type; which ensure the fluid interpretation of Ψ [50].

One can show that nonlinear coupling $b = b(\boldsymbol{r}) = b_0 - q/r^2$ is a linear function of the quantum temperature T_Ψ, which is defined as a thermodynamic conjugate of quantum information entropy sometimes dubbed as the Everett-Hirschman information entropy. The latter can be written as $S_\Psi = -\langle\Psi|\ln\left(|\Psi|^2/\bar{\rho}\right)|\Psi\rangle = -\int_{\mathcal{V}}|\Psi|^2\ln\left(|\Psi|^2/\bar{\rho}\right)d\mathcal{V}$, where a factor $1/\bar{\rho}$ is introduced for the sake of correct dimensionality, and can be absorbed into an additive constant due to the normalization condition (1). Therefore, one can expect that the thermodynamical parameters

$$b_0 = b_0(T_\Psi), \; q = q(T_\Psi) \tag{12}$$

are constant at a fixed temperature T_Ψ. Thus, for a trapless version of the model (11) we have four parameters, but only two of them, m and $\bar{\rho}$, are *a priori* fixed, whereas the other two, b_0 and q, can vary depending on the environment.

3. Induced Gravitational Potential

Invoking model (11), while neglecting quantum fluctuations, let us assume that physical vacuum is a collective quantum state described by wavefunction $\Psi = \Psi_{\mathrm{vac}}(\boldsymbol{r}, t)$, which forms a self-gravitating configuration with a center at $\boldsymbol{r} = 0$. Therefore, for this state, the solution of Equation (11) is equivalent to the solution of the linear Schrödinger equation,

$$i\hbar \partial_t \Psi = \left[-\frac{\hbar^2}{2m} \boldsymbol{\nabla}^2 + V_{\mathrm{eff}}(\boldsymbol{r}, t) \right] \Psi, \tag{13}$$

for a particle of mass m driven by an effective potential

$$
\begin{aligned}
V_{\mathrm{eff}}(\boldsymbol{r}, t) &= V_{\mathrm{ext}}(\boldsymbol{r}, t) - b \ln\left(\frac{|\Psi_{\mathrm{vac}}(\boldsymbol{r}, t)|^2}{\bar{\rho}} \right) \\
&= V_{\mathrm{ext}}(\boldsymbol{r}, t) - \left(b_0 - \frac{q}{r^2} \right) \ln\left(\frac{|\Psi_{\mathrm{vac}}(\boldsymbol{r}, t)|^2}{\bar{\rho}} \right),
\end{aligned}
\tag{14}
$$

when written in Cartesian coordinates [42]. If working in curvilinear coordinates, the last formula must be supplemented with terms which arise after separating out the angular variables in the wave equation.

In the absence of quantum excitations and other interactions, it is natural to associate this effective quantum-mechanical potential with the only non-removable fundamental interaction that we know of: gravity. This interpretation will be further justified in Section 4. Therefore, in Cartesian coordinates one can write the induced gravitational potential as

$$\Phi(\boldsymbol{r}, t) = -\frac{1}{m} V_{\mathrm{eff}}(\boldsymbol{r}, t) = \frac{1}{m} \left(b_0 - \frac{q}{r^2} \right) \ln\left(\frac{|\Psi_{\mathrm{vac}}(\boldsymbol{r}, t)|^2}{\bar{\rho}} \right), \tag{15}$$

where we assume that the background superfluid is trapless, i.e., we set $V_{\mathrm{ext}} = 0$. It should also be remembered that in curvilinear coordinates, this formula must be modified according to the remark after Equation (14); but for now we shall disregard any anisotropy and rotation.

It should be noticed that if one regards this potential as a multiplication operator then its quantum-mechanical average would be related to the Everett-Hirschman information entropy discussed in the previous Section: $\langle \Phi \rangle \sim T_\Psi \langle \Psi | \ln\left(|\Psi|^2\right) |\Psi\rangle \sim T_\Psi S_\Psi$. This not only makes theories of entropic gravity (which are essentially based on the ideas of Bekenstein, Hawking, Jacobson and others) a subset of the logarithmic superfluid vacuum approach, but also endows them with an underlying physical meaning and origin of the entropy implied.

We can see that the induced potential maintains its form as long as the physical vacuum stays in the state $|\Psi_{\mathrm{vac}}\rangle$. If the vacuum were to transition into a different state, then it would change its wavefunction; hence the induced gravitational potential would also change. We expect that our vacuum is currently in a stable state, which is close to a ground state or at least to a metastable state, with a sufficiently large lifetime. It is thus natural to assume that the state $|\Psi_{\mathrm{vac}}\rangle$ is stationary and rotationally invariant.

As we established earlier, the wavefunction describing such a state should be the solution of a quantum wave equation containing logarithmic nonlinearity. In the case of trivial spatial topology and infinite extent, the amplitude of such a solution is known to be the product of a Gaussian function, which was mentioned in the previous Section, and a conventional quantum-mechanical part, which is a product of an exponential function, power function and a polynomial. Thus we can write the amplitude's general form as:

$$|\Psi_{\mathrm{vac}}| = \sqrt{\bar{\rho}} \left(\frac{r}{\ell} \right)^{\chi_0/2} P(r) \exp\left(-\frac{a_2}{2\ell^2} r^2 + \frac{a_1}{2\ell} r + \frac{a_0}{2} \right), \tag{16}$$

where $P(r)$ is a polynomial function, χ_0 and a's are constants, and $\bar{\ell} = (m/\bar{\rho})^{1/3}$ is a classical characteristic length scale of the logarithmic nonlinearity (alternatively, one can choose $\bar{\ell}$ being equal to the quantum characteristic length, $\hbar/\sqrt{mb_0}$, which might be more useful for $\hbar$-expansion techniques). If quantum liquid occupies an infinite spatial domain then the normalization condition (1) requires

$$a_2 > 0, \tag{17}$$

which is also confirmed by analytical and numerical studies of differential equations with logarithmic nonlinearity of various types [23–25,28,31,35,42].

Both the form of a function $P(r)$ and the values of χ_0 and a's must be determined by a solution of an eigenvalue problem for the wave equation under normalization and boundary conditions. At this stage, those conditions are not yet precisely known; even if they were, we do not yet know which quantum state our vacuum is currently in. Therefore, these constants' values remain theoretically unknown at this stage, yet can be determined empirically.

Furthermore, for the sake of simplicity, let us approximate the power-polynomial term $(r/\bar{\ell})^{\chi_0/2}P(r)$, by the single power function $(r/\bar{\ell})^{\chi/2}$, where the constant χ is the best fitting parameter. Therefore, we can approximately rewrite Equation (16) as

$$|\Psi_{\text{vac}}|^2 \approx \bar{\rho} \, \exp\left[-\frac{a_2}{\bar{\ell}^2}r^2 + \frac{a_1}{\bar{\ell}}r + \chi \ln\left(\frac{r}{\bar{\ell}}\right) + a_0\right], \tag{18}$$

which is more convenient for further analytical studies than the original expression (16). From the empirical point of view, the function (18) can be considered as a trial function, whose parameters can be fixed using experimental data following the procedure we describe below.

For the trial solution (18), the normalization condition (1) immediately imposes a constraint for one of its parameters:

$$\exp(a_0) \approx \frac{\mathcal{M}a_2^{(\chi+3)/2}}{2\pi m} \left[\, \check{Z}\left(\frac{3}{2}, \frac{1}{2}\right) + \frac{a_1}{\sqrt{a_2}} \, \check{Z}\left(2, \frac{3}{2}\right)\right]^{-1}, \tag{19}$$

where we introduced an auxiliary function $\check{Z}(a, b) = \Gamma(a + \chi/2)\, {}_1F_1(a + \chi/2, b; a_1^2/4a_2)$, where $\Gamma(a)$ and ${}_1F_1(a, b; z)$ are the gamma function and Kummer confluent hypergeometric function, respectively. If values of a's and χ are determined, e.g., empirically, then this formula can be used to estimate the ratio $\mathcal{M}/m$.

Furthermore, by substituting the trial solution (18) into the definition (15), we derive the induced gravitational potential as a sum of seven terms:

$$\begin{aligned} \Phi(r) \;=\; & \Phi_{\text{smi}}(r) + \Phi_{\text{RN}}(r) + \Phi_{\text{N}}(r) \\ & + \Phi_{\text{gal}}(r) + \Phi_{\text{mgl}}(r) + \Phi_{\text{dS}}(r) + \Phi_0, \end{aligned} \tag{20}$$

where

$$\Phi_{\text{smi}}(r) = -\frac{\chi q}{m}\frac{\ln(r/\bar{\ell})}{r^2} = -\zeta_{\chi q}c_b^2\frac{L_{\text{smi}}^2\ln(r/\bar{\ell})}{r^2}, \tag{21}$$

$$\Phi_{\text{RN}}(r) = -\frac{a_0 q}{m}\frac{1}{r^2} = -\zeta_{a_0 q}c_b^2\frac{L_{\text{RN}}^2}{r^2}, \tag{22}$$

$$\Phi_{\text{N}}(r) = -\frac{a_1 q}{m\bar{\ell}}\frac{1}{r} = -\frac{GM}{r}, \tag{23}$$

$$\Phi_{\text{gal}}(r) = \frac{\chi b_0}{m}\ln(r/\bar{\ell}) = c_b^2\chi\ln(r/\bar{\ell}), \tag{24}$$

$$\Phi_{\text{mgl}}(r) = \frac{a_1 b_0}{m\bar{\ell}}r = \zeta_{a_1}c_b^2\frac{r}{L_{\text{mgl}}}, \tag{25}$$

$$\Phi_{\text{dS}}(r) = -\frac{a_2 b_0}{m\bar{\ell}^2}r^2 = -c_b^2\frac{r^2}{L_{\text{dS}}^2}, \tag{26}$$

and

$$\Phi_0 = \frac{1}{m}\left(a_0 b_0 + \frac{a_2 q}{\bar{\ell}^2}\right) = \frac{1}{m}\left(a_0 b_0 + \frac{q}{L_{\text{dS}}^2}\right) \tag{27}$$

is the additive constant. Here, and throughout the paper, we denote the sign functions by ζ's: $\zeta_\alpha = \text{sign}(\alpha)$, and use the following notations:

$$c_b = \sqrt{\frac{b_0}{m}}, \ GM = \frac{a_1 q}{m\bar{\ell}}, \ L_{\text{smi}} = \sqrt{\frac{|\chi q|}{b_0}},$$

$$L_{\text{RN}} = \sqrt{\frac{|a_0 q|}{b_0}}, \ L_{\text{mgl}} = \frac{\bar{\ell}}{|a_1|} = \frac{|q|}{mGM}, \tag{28}$$

$$L_{\text{dS}} = \frac{\bar{\ell}}{\sqrt{a_2}}, \ L_\chi = \frac{\chi\bar{\ell}}{a_1} = \frac{\chi q}{mGM},$$

where G is the Newton's gravitational constant as per usual.

Furthermore, Lorentz symmetry emerges in the "phononic" low-momentum limit of the theory, as discussed in the previous Section. Therefore, a R-observer would perceive the gravity induced by potential (20) as curved four-dimensional spacetime, which is a local perturbation (not necessarily small) of the background flow metric, such as the one derived in Section 5.3 of [7], see Section 7.1 below. In a rotationally invariant case, the line element of this spacetime can be written in the Newtonian gauge; if $\Phi(r)/c_{(0)}^2 \ll 1$, then it can be approximately rewritten in the form

$$ds^2 \approx -c_{(0)}^2\left[1 + \frac{2\Phi(r)}{c_{(0)}^2}\right]dt^2 + \frac{dr^2}{1 + 2\Phi(r)/c_{(0)}^2} + R^2(r)d\sigma^2, \tag{29}$$

where $R(r) = r\left[1 + \mathcal{O}\left(\Phi(r)/c_{(0)}^2\right)\right] \approx r$, $d\sigma^2 = d\theta^2 + \sin^2\theta\,d\varphi^2$ is the line element of a unit two-sphere, and a leading-order approximation with respect to the Planck constant is implied, as usual. The mapping (29) is valid for regions where the induced metric maintains a signature '$-+++$', and its matrix is non-singular. In other regions, such as close vicinities of spacetime singularities or horizons, the relativistic approximation is likely to fall outside its applicability range, thus it should be replaced with the F-observer's description of reality.

The main simplifying assumptions and approximations underlying the derivation of our gravitational potential are summarized and enumerated in the Appendix B.

4. Physical Interpretation

It should be noticed that if we did not have a logarithm in the original model (11), then in Equations (14) and (15), then we would not have arrived at the polynomial functions in Equations (20)–(26), which are easily recognizable. This reaffirms our expectations that the underlying model can be successfully confirmed by experiment; but first those functions must be endowed with precise physical meaning.

In this Section, we shall assign a physical interpretation to each term of the derived gravitational potential. For the sake of brevity, we shall be omitting an additive constant Φ_0, assuming that it is small compared to $c_{(0)}^2$.

4.1. Potential Φ_N and Gravitational Mass Generation

We begin with term (23), which has the most obvious meaning. In a non-relativistic picture, it represents Newton's model of gravity. According to the BEC-spacetime correspondence manifested through the mapping (29), a R-observer can observe an effect of the potential Φ_N by measuring probe particles moving along geodesics in the Schwarzschild spacetime:

$$ds_{(N)}^2 \approx -c_{(0)}^2 \left(1 - \frac{r_H}{r}\right) dt^2 + \frac{dr^2}{1 - r_H/r} + r^2 d\sigma^2, \tag{30}$$

where $r_H = 2GM/c_{(0)}^2$ is the Schwarzschild radius. Therefore, in absence of asymptotically non-vanishing terms, M can be interpreted as the gravitational mass of the configuration.

This mass can be expressed in terms of superfluid parameters as

$$r_H = \frac{2a_1 q}{mc_{(0)}^2 \bar{\ell}}, \quad \text{sign}\,(a_1 q) = \begin{cases} 1 & \text{gravity}, \\ -1 & \text{anti-gravity}, \end{cases} \tag{31}$$

thus assigning physical meaning to a combination of parameters $a_1 q/m\bar{\ell}$. In particular, one can see that a sign of the product $a_1 q$ determines whether the Φ_N interaction is attractive (gravity) or repulsive (anti-gravity). For most systems that we know of, anti-gravitational effects have not yet been observed, therefore one can assume that $M > 0$ or

$$a_1 q > 0 \tag{32}$$

from now on.

Nevertheless, it should be emphasized that the anti-gravity case is not *a priori* forbidden in superfluid vacuum theory. Indeed, the spacetime singularity occurs at $r = 0$ in a relativistic picture only, which poses certain issues for a R-observer, especially in the case of anti-gravity when a singularity is not covered by an event horizon (the existence of naked singularities is often doubted, on grounds of the cosmic censorship hypothesis). However, a F-observer would see no singular behavior in either case, because the wavefunction Ψ_{vac} remains regular and normalizable at each point of space and at any given time—as it should be in a quantum-mechanical theory. This reaffirms the fact that spacetime singularities are an artifact of incomplete information accessible to observers operating with relativistic particles [7].

Thus, the mapping from Equations (23)–(30) can be used to reformulate black hole phenomena in the language of continuum mechanics and the theory of superfluidity; which can resolve certain long-standing problems occurring in the relativistic theory of gravity. For instance, neglecting asymptotically non-flat terms for simplicity, one can view Equations (23), (30) and (31) as the gravitational mass generation mechanism: such mass is not a fundamental notion, but a composite quantum phenomenon induced by the background superfluid's dynamics (through the elementary inertial mass m and critical density $\bar{\rho}$), its quantum temperature (through q), and an exponential part of the condensate's wavefunction (through a_1). Such a mechanism can be thus considered as the

quantum-mechanical version of the Mach principle [7]. For example, if either a_1 or q vanish, then the system would not possess any gravitational mass, but it still would be gravitating in a non-Newtonian way, if other potentials from Equation (20) are non-zero.

4.2. Potential Φ_{RN} and Abelian Charges

Equation (22) represents another potential which can be easily recognized. According to the mapping (29), potential Φ_{RN} is observed by a R-observer as Reissner–Nordström spacetime, when taken together with the Φ_N potential:

$$ds^2_{(N+RN)} \approx -c^2_{(0)} \left(1 - \frac{r_H}{r} + \frac{r^2_Q}{r^2} \right) dt^2 + \frac{dr^2}{1 - r_H/r + r^2_Q/r^2} + r^2 d\sigma^2, \tag{33}$$

where r_Q is a characteristic length scale:

$$r_Q = \sqrt{2} L_{RN} \frac{c_b}{c_{(0)}} = \sqrt{\frac{2|a_0 q|}{m c^2_{(0)}}}, \tag{34}$$

provided $a_0 q < 0$.

The Reissner–Nordström metric is known to be associated with the gravitational field caused by a charge related to an abelian group. An example would be an electric charge Q, which is related to the abelian group $U(1)$ of electromagnetism. This charge can be thus revealed through the formula

$$Q^2 = \frac{r^2_Q c^4_{(0)}}{k_e G} \approx \frac{2|a_0 q| c^2}{k_e G m}, \tag{35}$$

where k_e is the Coulomb constant. In other words, potential Φ_{RN} describes, together with Φ_N, the gravitational field created by an object of a charge Q and gravitational mass M.

Thus, from a F-observer's viewpoint, an electrical charge is not an elementary notion, but a composite quantum phenomenon, induced by the background superfluid's dynamics (through the elementary inertial mass m), its quantum temperature (through q), and overall constant coefficient of the condensate's wavefunction (through a_0).

It should be noted also that Φ_{RN} is a short-range potential, therefore, it becomes substantial only at those microscopical length scales, of an order $r_{QH} = r^2_Q/r_H = \bar{\ell} |a_0/a_1|$ or below. Since $r_Q < r_H$ for most objects we know of, we have $0 \leqslant r_{QH} < r_H$. Thus, those scales would be causally inaccessible to a R-observer, but a F-observer would have no problem accessing them, per usual.

4.3. Potential Φ_{smi} and Strong Gravity

As the distance from a gravitating center decreases, it is term (21) which eventually predominates. According to the mapping (29), this potential Φ_{smi}, when taken together with the Φ_N and Φ_{RN} potentials, induces spacetime with the line element:

$$
\begin{aligned}
ds^2_{(N+RN+smi)} \approx & -c^2_{(0)} \left[1 - \frac{r_H}{r} + \frac{r^2_Q}{r^2} - \zeta_{\chi q} \frac{r^2_W}{r^2} \ln\left(\frac{r}{\bar{\ell}}\right) \right] dt^2 \\
& + \frac{dr^2}{1 - r_H/r + r^2_Q/r^2 - \zeta_{\chi q} r^2_W \ln(r/\bar{\ell})/r^2} + r^2 d\sigma^2,
\end{aligned} \tag{36}
$$

where r_W is a characteristic length scale:

$$r_W = \sqrt{2} L_{smi} \frac{c_b}{c_{(0)}} = \sqrt{\frac{2|\chi q|}{m c^2_{(0)}}}. \tag{37}$$

The potential Φ_{smi} has a distinctive property: unlike other sub-Newtonian potentials in Equation (20), it can switch between repulsive and attractive regimes, depending on whether the distance is larger or smaller than $\bar{\ell}$.

The magnitudes of Φ_{smi} and Φ_{RN} become comparable at two values of r, shown by the formula

$$r_{WQ}^{(\pm)} = \bar{\ell}\exp\left(\pm r_Q^2/r_W^2\right) = \bar{\ell}\exp\left(\pm|a_0/\chi|\right), \tag{38}$$

which indicates that $|\Phi_{\mathrm{smi}}|$ overtakes $|\Phi_{\mathrm{RN}}|$ either at $r < r_{WQ}^{(-)}$ or at $r > r_{WQ}^{(+)}$. If $|a_0/\chi|$ is large then $r_{WQ}^{(+)}$ is exponentially large and $r_{WQ}^{(-)}$ is exponentially small.

Magnitudes of Φ_{smi} and Φ_{N} become comparable at a certain value of r, which is shown by the formula

$$r_{WM} \;=\; -\ell_W\,\mathcal{W}\left(-\bar{\ell}/\ell_W\right) \leqslant \ell_W, \tag{39}$$

where $\ell_W \equiv r_W^2/|r_H| = |L_\chi|$, and by $\mathcal{W}(a)$ we denoted the Lambert function. The function $|\Phi_{\mathrm{smi}}(r)|$ becomes larger than $|\Phi_{\mathrm{N}}(r)|$ at $r < r_{WM} \leqslant |L_\chi|$.

Furthermore, considering Φ_{smi} as a perturbation of Φ_{N}, one can deduce that the effective gravitational coupling,

$$G_{\mathrm{eff}} \approx G\left[1 + \zeta_{\chi q}L_\chi\frac{\ln\left(r/\bar{\ell}\right)}{r}\right], \tag{40}$$

becomes larger as r gets smaller. Here an approximation symbol reminds us that we are working with wavefunction (18), and assume a leading-order approximation with respect to the Planck constant.

One can see that gravity naturally becomes stronger at shorter scales, without introducing any additional effects or matter, which suggests the way towards resolving the hierarchy problem. Indeed, the latter states the large discrepancy between magnitudes of forces of Standard Model interactions and classical gravity, but in this case, the gravitational force grows stronger than inverse square as r decreases, as $\ln r/r^3$. Moreover, it is possible to have even stronger small-length behaviour in our approach: if one goes beyond a minimal model (11) and generalizes nonlinear coupling to a series: $b(r) = b_0 - q/r^2 + q_1/r^3 + ...$, then this will induce terms $\ln r/r^4$, $\ln r/r^5$, *et cetera*.

On the other hand, as r grows, the function G_{eff} converges to G, especially if $|L_\chi|$ or $1/\bar{\ell}$ are sufficiently small; which makes G_{eff} approximately constant for a large range of values of r.

4.4. Potential Φ_{dS} and de Sitter Spacetime

Let us turn our attention to terms which do not tend to zero at $r \to \infty$. The physical meaning of one of these terms, given by Equation (26), becomes clear upon using the mapping (29):

$$ds_{(\mathrm{dS})}^2 \approx -c_{(0)}^2\left(1 - \frac{r^2}{R_{\mathrm{dS}}^2}\right)dt^2 + \frac{dr^2}{1 - r^2/R_{\mathrm{dS}}^2} + r^2d\sigma^2, \tag{41}$$

where

$$R_{\mathrm{dS}} = \frac{1}{\sqrt{2}}\frac{c_{(0)}}{c_b}L_{\mathrm{dS}} = \bar{\ell}\sqrt{\frac{mc_{(0)}^2}{2a_2b_0}} \tag{42}$$

is a radius of de Sitter horizon.

This metric represents de Sitter spacetime (written in static coordinates), which belongs to a class of Friedmann–Lemaître–Robertson–Walker (FLRW) spacetimes. Indeed, by applying a coordinate transformation $\tau - t = \tau_{\mathrm{dS}}\ln\left(1 - r^2/R_{\mathrm{dS}}^2\right)$, $\varrho = \alpha_0^{-1}r\exp(-t/2\tau_{\mathrm{dS}})\left(1 - r^2/R_{\mathrm{dS}}^2\right)^{-1/2}$, one can rewrite Equation (41) in isotropic coordinates

$$ds_{(\mathrm{dS})}^2 \to -c_{(0)}^2 d\tau^2 + \alpha_0^2\exp\left(\frac{\tau}{\tau_{\mathrm{dS}}}\right)\left(d\varrho^2 + \varrho^2d\sigma^2\right), \tag{43}$$

where $\tau_{\mathrm{dS}} = R_{\mathrm{dS}}/2c_{(0)}$ and α_0 is an integration constant.

The physical implications of the term Φ_{dS} will be further examined in Section 7.

4.5. Potentials Φ_{mgl} and Φ_{gal} and Gravity on Astronomical Scales

The remaining asymptotically non-vanishing potentials are given by Equations (24) and (25). With respect to the dependence upon radial distance from the gravitating center, they occupy an intermediate place between de Sitter term (26) and Newtonian potential (23). It is thus natural to expect that these terms are responsible for large scale dynamics—from galaxies (a kiloparsec scale) to metagalactic objects, such as voids and superclusters (a megaparsec scale).

According to the mapping (29), the terms Φ_{gal} and Φ_{mgl} modify de Sitter metric (41): causing the resulting spacetime to be asymptotically de Sitter only. The line element, which corresponds to the terms Φ_{dS}, Φ_{mgl} and Φ_{gal} taken together, can be written in static coordinates as

$$
\begin{aligned}
ds^2_{(\mathrm{gal+mgl+dS})} \;\approx\; & -c^2_{(0)}\left[1 + \beta^2_\chi \ln\!\left(\tfrac{r}{\bar{\ell}}\right) + \tfrac{r}{R_{\mathfrak{M}}} - \tfrac{r^2}{R^2_{\mathrm{dS}}}\right] dt^2 \\
& + \frac{dr^2}{1 + \beta^2_\chi \ln\left(r/\bar{\ell}\right) + r/R_{\mathfrak{M}} - r^2/R^2_{\mathrm{dS}}} + r^2 d\sigma^2,
\end{aligned}
\tag{44}
$$

where $\beta_\chi = \sqrt{2\chi\, b_0 / mc^2_{(0)}} = \sqrt{2\chi}\, c_b/c_{(0)}$, and

$$
R_{\mathfrak{M}} = \frac{mc^2_{(0)}\bar{\ell}}{2a_1 b_0} = \frac{\zeta_{a_1}}{2}\frac{c^2_{(0)}}{c^2_b} L_{\mathrm{mgl}}
\tag{45}
$$

is a characteristic length scale constant, which can be positive or negative depending on the sign of a_1. Note that in Equation (44), we included the Φ_{dS}-induced (de Sitter) term to remind us that at large r we still have a spacetime of a FLRW type.

Interestingly, linear terms in metrics occur also in an alternative theory of gravity, Weyl gravity [51,52]. This coincidence can be explained by conformal symmetry, which often emerges in logarithmic models at the relativistic limit, see for example Section 7.1. However, Weyl gravity does not produce a logarithmic term in metric, while in our theory it is induced by Φ_{gal}. This term is responsible for the flat rotation curves phenomenon in galaxies, which will be discussed, along with other astronomical implications of the terms Φ_{mgl} and Φ_{gal}, in further detail in Section 6.

It is also useful to know that the metric induced by the term (25) alone,

$$
ds^2_{(\mathrm{mgl})} \;\approx\; -c^2_{(0)}\left(1 + \frac{r}{R_{\mathfrak{M}}}\right) dt^2 + \frac{dr^2}{1 + r/R_{\mathfrak{M}}} + r^2 d\sigma^2,
\tag{46}
$$

transforms to the conformally FLRW-type metric

$$
ds^2_{(\mathrm{mgl})} \;\rightarrow\; \frac{(1 + \varrho/4R_{\mathfrak{M}})^2}{a^2_{\mathfrak{M}}(\tau)\,(1 - \varrho/4R_{\mathfrak{M}})^2}\left[-c^2_{(0)} d\tau^2 + \frac{a^2_{\mathfrak{M}}(\tau)}{(1 - \varrho^2/16R^2_{\mathfrak{M}})^2}\left(d\varrho^2 + \varrho^2 d\sigma^2\right)\right],
\tag{47}
$$

upon applying the coordinate transformation $r = \varrho/(1 - \varrho/4R_{\mathfrak{M}})^2$, $t = \int d\tau/a_{\mathfrak{M}}(\tau)$. For R-observers, this represents a surrounding homogeneous and isotropic spacetime with scale factor $a_{\mathfrak{M}}(\tau)$ and negative-definite spatial scalar curvature, $-1/(2R_{\mathfrak{M}})^2$.

Finally, one could mention that potentials of type (24) were studied, albeit in the context of a linear Schrödinger equation, in [53,54].

5. Density of Effective Gravitating Matter

In this Section, let us derive spherically-symmetric density profiles of effective gravitating matter ρ_Φ, which formally corresponds to the superfluid-vacuum induced potential (20). Contrary to the directly observable value of orbital velocity derived in Section 6, such density depends more substantially on the choice of an observer. It can be defined either via the Lorentz-covariant definition (6), or via the Poisson equation, which is a non-relativistic version of Equation (6). Correspondingly, one would obtain different results, which will be discussed below.

In accordance with the last paragraph of Appendix B, this Section's computations cannot take into account any secondary induced matter, such as the equilibrium configurations of mass-energy emerging as a result of interaction between scalar and tensor modes of superfluid vacuum's excitations. One can show that such equilibria do exist, manifesting themselves in a form of general relativistic nonsingular horizon-free stellar-like objects or particle-like Q-balls, therefore, in reality such objects would definitely contribute to the density associated with dark matter. In other words, here we are restricting ourselves to background values of density.

In this Section only, we shall be temporarily assuming that $M = M(r)$, otherwise the corresponding contribution to the density profile would be identically zero, and thus non-indicative.

5.1. Galilean Symmetry

In this case, one defines an effective matter density by virtue of the Poisson equation. Taking the whole potential (20) and assuming spherical symmetry, we obtain

$$
\begin{aligned}
\rho_\Phi(r) &\equiv \frac{1}{4\pi G}\left[\Phi''(r) + \frac{2}{r}\Phi'(r)\right] \\
&= \rho_{\text{smi}}(r) + \rho_{\text{RN}}(r) + \rho_{\text{N}}(r) \\
&\quad + \rho_{\text{gal}}(r) + \rho_{\text{mgl}}(r) + \rho_{\text{dS}}(r),
\end{aligned}
\tag{48}
$$

where

$$
\rho_{\text{smi}}(r) = \frac{3\chi\, q}{4\pi G m r^4}\left[1 - \frac{2}{3}\ln\left(\frac{r}{\bar{\ell}}\right)\right], \tag{49}
$$

$$
\rho_{\text{RN}}(r) = -\frac{a_0 q}{2\pi G m r^4} = \frac{k_e Q^2}{4\pi r^4}, \tag{50}
$$

$$
\rho_{\text{N}}(r) = -\frac{1}{4\pi r}M''(r), \tag{51}
$$

$$
\rho_{\text{gal}}(r) = \frac{\chi\, b_0}{4\pi G m r^2} = \frac{v_{\text{gal}}^2}{4\pi G r^2}, \tag{52}
$$

$$
\rho_{\text{mgl}}(r) = \frac{a_1 b_0}{2\pi G m \bar{\ell} r} = \frac{\chi\, b_0}{2\pi G m L_\chi r}, \tag{53}
$$

$$
\rho_{\text{dS}}(r) = -\frac{3 a_2 b_0}{2\pi G m \bar{\ell}^2} = -\frac{3 b_0}{2\pi G m L_{\text{dS}}^2} = \text{const}, \tag{54}
$$

whereby the sum of the last three densities can be regarded as a density corresponding to the astronomical-scale "dark matter" and "dark energy", which will be further justified in Sections 6 and 7.

Furthermore, these formulae were derived on the assumption that the gravitational coupling constant G is the same for all length scales; which is valid when any influence from the term (21) can be disregarded. However, this term might cause an additional effect, discussed in Section 4.3: it makes the gravitational coupling constant vary with distance. If this does happen, then in Equations (48)–(54) one should replace G with a running constant given by Equation (40).

5.2. Lorentz Symmetry

In this case, one defines effective matter density by virtue of Einstein field equations with the induced stress-energy tensor defined by Equation (6). Using it together with Equation (20) and metric (29), we obtain

$$
\begin{aligned}
\tilde{\rho}_\Phi(r) &\equiv -\frac{1}{4\pi G r}\left[\Phi'(r) + \frac{1}{r}\Phi(r)\right] \\
&= \tilde{\rho}_{\mathrm{smi}}(r) + \tilde{\rho}_{\mathrm{RN}}(r) + \tilde{\rho}_{\mathrm{N}}(r) \\
&\quad + \tilde{\rho}_{\mathrm{gal}}(r) + \tilde{\rho}_{\mathrm{mgl}}(r) + \tilde{\rho}_{\mathrm{dS}}(r) + \tilde{\rho}_{\Phi_0}(r),
\end{aligned}
\tag{55}
$$

where

$$
\tilde{\rho}_{\mathrm{smi}}(r) = \frac{\chi q}{4\pi G m r^4}\left[1 - \ln\left(\frac{r}{\ell}\right)\right],
\tag{56}
$$

$$
\tilde{\rho}_{\mathrm{RN}}(r) = -\frac{a_0 q}{4\pi G m r^4} = \frac{k_e Q^2}{8\pi r^4},
\tag{57}
$$

$$
\tilde{\rho}_{\mathrm{N}}(r) = \frac{1}{4\pi r^2}M'(r),
\tag{58}
$$

$$
\tilde{\rho}_{\mathrm{gal}}(r) = -\frac{\chi b_0}{4\pi G m r^2}\left[1 + \ln\left(\frac{r}{\ell}\right)\right],
\tag{59}
$$

$$
\tilde{\rho}_{\mathrm{mgl}}(r) = -\frac{a_1 b_0}{2\pi G m \bar{\ell} r} = -\frac{\chi b_0}{2\pi G m L_\chi r},
\tag{60}
$$

$$
\tilde{\rho}_{\mathrm{dS}}(r) = \frac{3 a_2 b_0}{4\pi G m \bar{\ell}^2} = \frac{3 b_0}{4\pi G m L_{\mathrm{dS}}^2} = \mathrm{const},
\tag{61}
$$

$$
\tilde{\rho}_{\Phi_0}(r) = -\frac{\Phi_0}{4\pi G r^2} = -\frac{1}{4\pi G m r^2}\left(a_0 b_0 + \frac{q}{L_{\mathrm{dS}}^2}\right),
\tag{62}
$$

whereby the sum of last four densities can be regarded corresponding to the astronomical-scale "dark matter" and "dark energy", which will be further justified in Sections 6 and 7. Density (62) is a somewhat surprising contribution, because it corresponds to the constant term Φ_0 in Equation (20), which is not supposed to affect trajectories; at least, in classical mechanics. Unless its presence can be confirmed by observations, it must be regarded as a gauge term, or as an artifact of approximations underlying Equation (29).

Furthermore, a running gravitational coupling constant can not be implemented in the relativistic case as simply as in Section 5.1. To preserve Lorentz invariance, one has to associate this coupling with a four-dimensional scalar; which automatically upgrades general relativity to a scalar-tensor gravity with a non-minimally coupled scalar field. This theory cannot be written by hand, but it must be derived in a way which is similar to that used in Section 7.1.

Comparing results of Sections 5.1 and 5.2, one can conclude that the definition of effective matter density is somewhat ambiguous: in particular, it drastically depends on symmetry assumptions. Therefore, further experimental studies should help to empirically establish which symmetry is more appropriate to use when dealing with "dark" phenomena.

6. Galactic Rotation Curves

In this Section, we demonstrate how induced gravitational potential can explain various phenomena, which are usually attributed to dark matter. Let us focus on the terms (24) and (25), which were partially discussed in Section 4.5. Because they become significant at a galactic scale and above (i.e., a kiloparsec to megaparsec scale), it is natural to conform them to astronomical observations; such as those of rotation curves in galaxies.

In a spherically symmetric case, velocity curves of stars orbiting with non-relativistic velocities on a plane in a central gravitational potential $\Phi(r)$ can be estimated using a simple formula

$v^2 = R a_c = R \Phi'(R)$, where v is the orbital velocity, a_c is the centripetal acceleration, and R is the orbit's radius. The cylindrically symmetric case can be considered by analogy, by assuming various disk models [55,56].

Considering the terms (24) and (25) in conjunction with the Newtonian term (23), we thus obtain

$$
\begin{aligned}
v(R) &= \left\{ R\frac{d}{dR}\left[\Phi_{\mathrm{N}}(R) + \Phi_{\mathrm{gal}}(R) + \Phi_{\mathrm{mgl}}(R)\right]\right\}^{1/2} \\
&= \sqrt{v_{\mathrm{N}}^2 + v_{\mathrm{gal}}^2 + \Phi_{\mathrm{mgl}}(R)},
\end{aligned}
\tag{63}
$$

where

$$
v_{\mathrm{N}}^2 = \frac{GM}{R} = \frac{a_1 q}{m\bar{\ell}}\frac{1}{R} = -\Phi_{\mathrm{N}}(R),
\tag{64}
$$

$$
v_{\mathrm{gal}}^2 = \chi c_b^2 = \frac{\chi b_0}{m} = \mathrm{const},
\tag{65}
$$

while the contribution from the term (26) is disregarded for now, due the assumed smallness of the "local" cosmological constant $1/R_{\mathrm{dS}}^2$; and the contribution from the term (21) is disregarded due the assumed smallness of the corresponding characteristic length, according to discussion in Section 4.3.

In the case of a galaxy, the contribution from the Newtonian term Φ_{N} rapidly decreases as R grows. Correspondingly, the main contribution would then come from the second term in a row, Φ_{gal}, and then from the third term, Φ_{mgl}. From Equation (65) one can see that the contribution from Φ_{gal} is constant, which explains the average flatness of galactic rotation curves.

Notice that the value of velocity v_{gal} depends on one of the wavefunction parameters χ and one of quantum temperature parameters b_0. Both are not *a priori* fixed parameters of the model, cf. Equations (12) and (18), but vary depending on the environment and conditions: background superfluid gets affected by the gravitational potential it induces, because this potential acts upon the surrounding conventional matter, thus creating density inhomogeneity. Therefore, χ and b_0 should generally be different for each galaxy; and hence should v_{gal} be.

Similar to the case of v_{gal}, the parameter a_1 hence a value $\Phi_{\mathrm{mgl}}(R)$ at a fixed R will also be dependent on the gravitating object they refer to. This potential should usually be negligible on the inner scale length of a galaxy, but as R grows towards the extragalactic length scale, $R \gtrsim 10\,\mathrm{kpc}$, rotation curves should start to deviate from flat:

$$
v(R) \approx v_{\mathrm{gal}}\sqrt{1 + \frac{R}{L_\chi}},
\tag{66}
$$

which can be used for estimating the combination of superfluid vacuum parameters $\chi\bar{\ell}/a_1$ empirically. In cases where the contribution from other terms of the induced potential cannot be neglected, Equation (63) must be generalized to include those too.

Possible galactic-scale regions, where this non-flat asymptotics should become visible, depending on a value L_χ, are the outer regions of large spiral galaxies, such as M31 or M33 [57–61], where $\Phi_{\mathrm{mgl}}(r)$ can not only overtake $\Phi_{\mathrm{N}}(r)$ but also become comparable with $\Phi_{\mathrm{gal}}(r)$.

7. Accelerating Expansion of the Universe

This phenomenon is usually explained by introducing exotic forms of relativistic matter, such as dark energy; usually modeled by various long-range scalar fields, which are assumed not to affect the numerous particle physics experiments on the Earth. The superfluid vacuum approach offers a simple framework, which can explain the Universe's expansion as an observer-dependent effect, without involving any matter other than the background superfluid itself.

7.1. Conformally Flat Spacetime and Dilaton Field

Following work [7], let us consider the most simple possible special case: laminar flow of a logarithmic background condensate in a state $|\Psi_{(0)}\rangle$, described by Equation (9) at $b = \text{const}$, with a constant velocity $\boldsymbol{u}^{(0)}$, if viewed as, from the F-observer's perspective, an embedding into underlying Euclidean space. On the other hand, what does a R-observer see?

Due to a well-known separability property of the logarithmic Schrödinger equation [23,25,28,31,42], the phase of its simplest ground-state solutions is a linear function of a radius-vector:

$$i \ln \left(\Psi_{(0)}(\boldsymbol{r}, t) / |\Psi_{(0)}(\boldsymbol{r}, t)| \right) \propto \boldsymbol{u}^{(0)} \cdot \boldsymbol{r} + f(t), \tag{67}$$

where $\boldsymbol{u}^{(0)}$ is a constant 3-vector, and $f(t)$ is an arbitrary function of time. In this case, the fluid-Schrödinger analogy confirms that the background condensate does flow with a constant velocity $\boldsymbol{u} = -i\eta \boldsymbol{\nabla} \ln \left(\Psi_{(0)} / |\Psi_{(0)}| \right) \propto \boldsymbol{u}^{(0)}$. Recalling Equation (5), it means that the background geometry induced by such solutions is conformally flat:

$$ds^2_{(0)} \propto \frac{1}{\rho} |\Psi_{(0)}(\boldsymbol{r}, t)|^2 \left[-c_s^2 dt^2 + \left(d\boldsymbol{r} - \boldsymbol{u}^{(0)} dt \right)^2 \right], \tag{68}$$

where c_s is given by Equation (10), in a leading-order approximation with respect to the Planck constant.

Spacetime of a type (68) lies within a large class of manifolds with the vanishing Weyl tensor—a type **O** in the Petrov classification. This is the class that all FLRW spacetimes belong to, including those which describe the Universe with accelerating expansion—simply written in conformally-flat coordinates instead of comoving ones.

Using definition (6), we obtain an induced stress-energy tensor for our system:

$$\kappa \widetilde{T}_{\mu\nu} = \tilde{D} \left[\nabla_\mu \nabla_\nu \phi - \nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu} \left(\nabla_\lambda \nabla^\lambda \phi + \tfrac{1}{2}(\tilde{D} - 1) \nabla_\lambda \phi \nabla^\lambda \phi \right) \right], \tag{69}$$

where $\tilde{D} = D - 2 = 2$, ∇ is a covariant derivative with respect to metric g, and by ϕ we denote the induced scalar field:

$$\phi = \ln \left(|\Psi_{(0)}(\boldsymbol{r}, t)|^2 / \bar{\rho} \right), \tag{70}$$

up to an additive constant.

This stress-energy tensor strongly resembles the one occurring in the theory of gravity with a scalar field. One can verify that it can be indeed derived from the following scalar-tensor gravity action functional

$$\tilde{\mathcal{S}}[g, \phi] \propto \int d^D x \sqrt{-g} \, e^{\tilde{D}\phi} \left[R + \tilde{D}(\tilde{D} + 1)(\nabla \phi)^2 \right], \tag{71}$$

where the notation "ϕ" reminds us that the field ϕ is fixed by the solution of the original quantum wave equation, cf. Equation (70), while the variation of action must be taken with respect to the metric only. In other words, both metric g and dilaton ϕ are induced by the superfluid vacuum being in a state described by $\Psi_{(0)}(\boldsymbol{r}, t)$.

Thus, we have found yet another example of the differences between the F-observer's and R-observer's pictures of reality. While the former sees a background quantum fluid flowing with a constant velocity in three-dimensional Euclidean space, the latter observes itself as being inside four-dimensional spacetime governed by a Lorentz-covariant scalar-tensor gravity.

An action functional (71) therefore explains why covariant models involving scalars provide a robust description of the large-scale evolution of the Universe, agreeing with current observational data; yet no quanta of relativistic dilaton have thus far been detected.

This correspondence also reveals the limitations of the relativistic description itself: if the superfluid vacuum goes into a different quantum state, then one gets a different expression for the induced metric, scalar, stress-energy tensor and covariant action. In fact, for more complicated

superfluid flows, even the condition (67), leading to a conformal flatness, can become relaxed to an asymptotic one. Therefore, depending on the physical configuration (determined by external potential, if any, and boundary conditions), nonlinear coupling behaviour and the quantum state the vacuum is in, small fluctuations and probe particles would obey different covariant actions. Consequently, a R-observer would have to tweak its field-theoretical models by hand; with the unified picture being observable only at the level of a F-observer.

7.2. Cosmological Constant and Local Expansion Mechanism

In Section 7.1 we considered an example of the global superfluid flow which would be seen by a relativistic observer as the accelerating expansion of the observable Universe. What about locally induced gravity (20), can it cause similar effects?

Let us consider once again the induced potentials of Sections 4.4 and 4.5. The de Sitter term from Section 4.4 predominates if we consider the physical vacuum alone, without any generated matter therein. This is perhaps only valid for the early Universe, such as the one which existed during the inflationary epoch. In the current epoch, our spacetime can only be de Sitter asymptotically or even approximately; therefore the potential Φ_{dS} must be negligible, unless one considers very large length scales. For example, if a length scale R_{dS}, which generally depends on a massive object defining the frame of reference of the function (16), is comparable to a size of the observable Universe ($\sim$10 Gpc), then R_{dS} can be related to the cosmological constant Λ as

$$R_{\mathrm{dS}}^{(\cos)} = \sqrt{3/\Lambda}, \ \Lambda \sim 10^{-56} \ \mathrm{cm}^{-2}, \tag{72}$$

therefore, the term (26) becomes substantial only at a gigaparsec scale. This relation yields an empirical constraint for a combination of characteristic parameters of superfluid vacuum, average quantum temperature and Ψ_{vac}:

$$\frac{a_2^{(\cos)} b_0^{(\cos)}}{m \bar{\ell}^2} \approx \frac{1}{6} \Lambda c^2 \sim 10^{-36} \ \mathrm{s}^{-2}, \tag{73}$$

see also the discussion around Equation (12). All this essentially means that Λ is not a fundamental constant of Nature, but a combination of various parameters of superfluid vacuum, including quantum temperature and Gaussian width of the condensate function Ψ_{vac}. Therefore, its smallness, sometimes referred as "vacuum catastrophe" [1], can easily be explained by the fact that: either average quantum temperature across the Universe, or wavefunction's width, or both, are sufficiently small compared to mc^2 and $\bar{\ell}$, respectively; thus resulting in the overall smallness of the ratio on the left-hand side of Equation (73).

The line element, which results from taking Φ_{dS}, Φ_{mgl}, Φ_{gal}, Φ_0 and Φ_N together, can be written in static coordinates as

$$
\begin{aligned}
ds^2_{(\cos+\mathrm{N})} \approx & -c^2_{(0)} \left[1 + \delta_0 - \frac{r_H}{r} + \beta_\chi^2 \ln\left(\frac{r}{\bar{\ell}}\right) + \frac{r}{R_{\mathfrak{M}}} - \frac{r^2}{R_{\mathrm{dS}}^2} \right] dt^2 \\
& + \frac{dr^2}{1 + \delta_0 - r_H/r + \beta_\chi^2 \ln\left(r/\bar{\ell}\right) + r/R_{\mathfrak{M}} - r^2/R_{\mathrm{dS}}^2} + r^2 d\sigma^2,
\end{aligned}
\tag{74}
$$

where $\delta_0 = 2\Phi_0/c^2_{(0)}$. In this metric, the Schwarzschild term ensures that spacetime singularity at $r = 0$ is "dressed" by the black hole horizon, while at large r we still have spacetime of a FLRW type. While from the viewpoint of a F-observer, no spacetime singularity would pose a problem, because quantum wavefunction remains regular and normalizable at any non-negative value of r. This reaffirms our earlier statement that cosmological singularity is an artifact of the low-momentum approximation of superfluid vacuum [7,62].

According to Equation (47), the linear potential term (25), when taken alone, induces the universal acceleration $c^2_{(0)}/2R_{\mathfrak{M}}$, occurring due to the spatial curvature, when seen by a R-observer in its own local static coordinate system. This contributes to the Hubble expansion induced by the quadratic

term (26). Thus, in the relativistic picture, the non-small fluctuation of superfluid vacuum produces an effect at the center of a gravitating configuration; and therefore contributes to the explicit rotational motions of the stars inside this configuration, which can be seen as a consequence of curved spacetime.

7.3. Expansion Mechanisms and Cosmological Coincidences

　　Comparing Sections 7.1 and 7.2, one can see that they describe different expansion mechanisms. The mechanism of Section 7.1 occurs due to the global laminar flow of background superfluid, assumed to be laminar, which is "seen" by a R-observer as a FLRW-type spacetime; the resolution of various cosmological problems related to this mechanism was discussed in Section 5 of Ref. [7]. On the other hand, in Section 7.2, expansion is explained as a cumulative effect from terms in metric (74), which do not vanish at spatial infinity, induced by a "local" wavefunction associated with a gravitating configuration or body. This wavefunction can be regarded as a fluctuation (not necessarily small) of the global wavefunction from Section 7.1.

　　The interplay between these mechanisms depends on the length scales of the quadratic and linear terms, R_{dS} and $R_{\mathfrak{M}}$. Unless a cumulative expansion effect from asymptotically non-vanishing terms taken together is, by some extraordinary coincidence, exactly equal to the expansion due to the global flow mechanism, its rate must be different from that of the global flow-induced expansion.

　　The occurrence of an additional expansion mechanism, at the scale of a supercluster, such as our Virgo or Laniakea, could explain the remarkable discrepancy between measurements of the Hubble constant using different methods from those based on the whole Universe expansion, such as cosmic microwave background radiation (CMB). Among non-CMB methods one could mention Cepheid calibration, time-delay cosmography, and geometric distance measurements to megamaser-hosting galaxies [63–65]. From a theoretical point of view, a scenario with different expansion rates seems slightly more plausible, because it does not require an explanation why accelerations from different mechanisms should be exactly equal to each other (this coincidence should not be confused with the conventional cosmological coincidence which we will discuss next).

　　Furthermore, our approach offers a simple explanation of the cosmological coincidence problem itself [1], in both the simplified and quantitative versions thereof.

　　The simplified formulation of the cosmological coincidence states that if dark matter and dark energy were different kinds of matter, then during the Universe's evolution they should have evolved independently of each other; therefore their distributions would be uncorrelated by now—which does not seem to be the case. Superfluid vacuum theory trivially resolves this paradox: because "dark matter" and "dark energy" are actually induced phenomena and manifestations of the same object, superfluid vacuum, they cannot be independent from each other.

　　The quantitative formulation of the coincidence problem is an explanation requirement for why the ratio of DM- and DE-associated densities is of order one, despite the reasons given in the simplified formulation. In our approach, we can regard all terms of the induced potential, which do not vanish at spatial infinity, as being associated with "dark" effects; but inside this group we cannot unambiguously separate DM-attributed effects from DE ones. For example, potential (25) is intermediate between de Sitter and logarithmic, thus it affects both galactic rotation curves and Hubble expansion, cf. Sections 6 and 7.2.

　　If, for simplicity, we consider only the local expansion mechanism of Section 7.2 and omit the contribution from Φ_0, then the cosmological coincidence can be reformulated as the following condition:

$$\frac{\widetilde{\Omega}_{gal} + \widetilde{\Omega}_{mgl}}{\widetilde{\Omega}_{mgl} + \widetilde{\Omega}_{dS}} = \mathcal{O}(1), \tag{75}$$

where $\widetilde{\Omega}$'s are average values corresponding to densities from Section 5.2. The numerator of this ratio represents average density of effective "dark matter", while the denominator represents effective "dark energy" density; or at least the predominating proportions thereof.

In general, this condition simply imposes yet another constraint for the parameters of the theory. It is trivially satisfied if the involved parts of the gravitational potential, hence the associated densities are of the same order of magnitude, if averaged on a large scale. Moreover, even if $\widetilde{\Omega}_{gal}$ or $\widetilde{\Omega}_{dS}$ are much smaller than the remaining involved densities, but the value $\widetilde{\Omega}_{mgl}$ is substantial, then the relation (75) still holds, due to the presence of $\widetilde{\Omega}_{mgl}$ in both parts of the ratio.

8. Conclusions

Working within the framework of the post-relativistic theory of physical vacuum, based on the logarithmic superfluid model, we derived induced gravitational potential, corresponding to a generic quantum wavefunction of the vacuum. This mechanism is radically different from the one used in models of relativistic classical fluids and fields, which are based on modifying the stress-energy tensor in Einstein field equations.

The form of such a wavefunction is motivated by ground-state solutions of quantum wave equations of a logarithmic type. Such equations find fruitful applications in the theory of strongly-interacting quantum fluids, and have been successfully applied to laboratory superfluids [10,43,49]. We note that, in principle, one is not precluded from adding other types of nonlinearity, such as polynomial ones, into the condensate wave equations, but the role of logarithmic nonlinearity is crucial.

Thus, we used a logarithmic superfluid model with variable nonlinear coupling, because it accounts for an effect of the environment in a more realistic way than the logarithmic model with a constant coupling. As a result, for the trapless version of our model, we have four parameters, but only two of them are *a priori* fixed, whereas the other two can vary, depending on the quantum thermodynamic properties of the environment under consideration. Additionally, a number of parameters come from the wavefunction solution itself. Those are not independent parameters of the theory, but functions thereof. Because we do not yet know the exact form of the superfluid wavefunction, see remarks at the end of Section 3, we leave those parameters to be empirically estimated, or bound, at the stage of current knowledge.

It turns out that gravitational interaction has a multiple-scale structure in our theory: induced potential is dominated by different terms at each length scale; such that one can distinguish sub-Newtonian, Newtonian (inverse-law), galactic (logarithmic-law), metagalactic (linear-law), and cosmological (square-law) parts. A relativistic observer, who operates with low-momentum small-amplitude fluctuations of superfluid vacuum, observes this induced potential by measuring the trajectories of probe particles moving along geodesics in induced four-dimensional pseudo-Riemannian spacetime. The metric of the latter is determined by virtue of the BEC-spacetime correspondence and fluid-Schrödinger analogy, applied jointly.

The sub-Newtonian part of the induced gravitational potential is defined as one which grows faster than the inverse law, as distance tends towards zero. It can be naturally divided into the following two parts. One part has an inverse square law behaviour, and thus can be associated with the gravitational field caused by a $U(1)$ gauge charge, such as an electric charge. On a relativistic level, it is described by Reissner–Nordström spacetime. The other part has 'inverse square times logarithm' law behaviour, which might become substantial at both ultra-short and macroscopic distances, depending on the values of the corresponding parameters. If it "survives" at macroscopic distances, then it upgrades Newton's gravitational constant to a function of length, such that gravity has both strong and weak regimes.

With the potential or spacetime metric in hand, one can, in principle, assign effective fictitious matter density to our potential, which corresponds to "dark matter" and "dark energy". This can be done in two ways: either by Einstein field equations in a relativistic case, or the Poisson equation in a non-relativistic one. It should be noted that the resulting density in each of the cases can be modified, depending on whether the gravitational constant is considered to be running or not. This will require more verification from future experimental and theoretical studies.

Furthermore, on a galactic scale and above, the potential is dominated by non-Newtonian terms, which do not vanish at spatial infinity. This explains the non-Keplerian behaviour of rotation curves in galaxies, which is often attributed to dark matter. Our model, not only explains the average flatness of galactic rotation curves, but also makes a number of new predictions. One of them is the approximately linear law behavior of gravitational potential on a metagalactic scale, which is an intermediate scale between galactic distances and the size of the observable universe. This should partially affect galactic rotation curves too: as the distance from the gravitating center grows further towards the metagalactic length scale, a squared velocity's profile asymptotically changes from being flat towards linear, cf. Equation (66).

On the other hand, at the largest length scale, the induced potential displays square law behaviour. If the quadratic term is negative-definite, then the corresponding metric describes (asymptotically) de Sitter space, merely written in static coordinates. Taken together with the contribution from the linear potential term, this explains the accelerating expansion of the corresponding spacetime region, which is usually associated with dark energy.

Such expansion could supplement the "global" one, caused by laminar flow of background logarithmic superfluid absent any other matter, which induces a FLRW-type spacetime. The occurrence of more than one type of expansion mechanism, could be responsible for the discrepancy between measurements of the Hubble constant using different methods.

The relevant problems, such as smallness of cosmological constants and cosmological coincidence, were also discussed.

To conclude, we used the BEC-spacetime correspondence and fluid-Schrödinger analogy to argue that the description of reality and fundamental symmetry crucially depend on the choice of an observer. We demonstrated that both dark matter and dark energy are related phenomena, and different manifestations of the same object, superfluid vacuum, which acts by inducing both gravitational potential and spacetime.

Conflicts of Interest: The author declares no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

BEC	Bose-Einstein condensate
CDM	Cold dark matter
CMB	Cosmic microwave background
DE	Dark energy
DM	Dark matter
dS	de Sitter
FLRW	Friedmann–Lemaître–Robertson–Walker
F-observer	Full observer
RN	Reissner–Nordström
R-observer	Relativistic observer
SVT	Superfluid vacuum theory

Appendix A. Superfluid or Not

When it comes to astrophysics and cosmology, including dark matter and dark energy applications, the notion of superfluid is probably the most misunderstood and misused term of all. In addition to having a catchy, but not exactly informative, prefix, the first thing which causes misunderstanding of the term is inviscid flow. This, the most striking feature of superfluids, is often regarded as the

definition thereof. As a result, superfluids are often confused with perfect fluids, which have no viscosity by definition, and can be easily implemented into Einstein field equations by virtue of a stress-energy tensor borrowed from classical fluid mechanics.

However, perfect fluids, being classical hydrodynamical objects by construction, can not properly reflect the essentially quantum nature of the superfluidic matter that we know of. Therefore, perfect or non-quantum inviscid fluids can be used as a crude approximation, at best, of superfluids.

Additionally, superfluids are often also confused with the concomitant phenomenon of Bose-Einstein condensation (BEC), which is another kind of quantum matter occurring in low-temperature condensed matter, such as cold cesium atoms in a trap. However, in the superfluid helium phase, for example, the BEC's content comprises only about ten per cent. Therefore, this condensate alone cannot fully account for dissipation-free flow and the other distinctive features of superfluids. Even though Bose-Einstein condensates do share certain features with superfluids, this does not imply that they are superfluidic in general.

In particular, quantum excitations in laboratory superfluids are known to have dispersion relations of a distinctive shape called the Landau "roton" spectrum. This shape of the spectral curve is crucial, as it ensures the suppression of dissipative fluctuations at a quantum level [10]. If plotted as an excitation energy versus momentum, the curve starts from the origin, climbs up to a local maximum (called the *maxon* peak), then slightly descends to a local nontrivial minimum (called the "roton" energy gap); and then grows again, this time all the way up, to the boundary of the theory's applicability range, cf. a solid curve in the Figure 1a from [11]. In fact it is not the roton energy gap alone, but the energy barrier formed by the maxon peak and roton minimum in momentum space, which ensures the above-mentioned suppression of quantum fluctuations in quantum liquid and, ultimately, causes its flow to become inviscid. In other words, it is the global characteristics of the curve, not just the existence of a nontrivial local minimum and related energy gap, which are required for superfluidity to occur. Obviously, Landau's shape is a non-trivial property which cannot possibly occur in all quantum liquids.

The final reason for the misuse of the term 'superfluid' is the extensive, but not always careful, utilization of relativistic scalar field models in astrophysics and cosmology. Historically, the four-dimensional scalar field came about as a bold extrapolation of a non-relativistic wavefunction into the realm of Lorentz-symmetric theories. However, when it comes to quantum liquids and condensates, there is a fundamental difference between relativistic scalar field and condensate wavefunction, which makes a correspondence between them far from isomorphic.

The fluid condensate wavefunction obeys both a normalization condition (1) and a wave Equation (2), therefore, it is a three-dimensional Euclidean scalar related to a ray in the associated Hilbert space. One can see that Equations (1)–(3) are essentially non-relativistic and three-dimensional. While one can still make the last two relativistic, by replacing derivative parts with the Lorentz-covariant analogues thereof, Equation (1) strongly violates the Lorentz invariance. This condition requires the foliation of a spacetime manifold into $(3+1)$-dimensional spacelike hypersurfaces, and makes mass-energy a three-dimensional scalar, not a time component of a four-dimensional vector.

As a result, relativistic scalar fields models offer a useful, but approximate description of superfluidic phenomena (which is valid for small wavefunction amplitudes and low momenta of excitations, running ahead), while the rigorous extrapolation of BEC and superfluid notions into the relativistic domain requires special treatment. Therefore, it is not surprising that some "superfluid" models of dark matter; which are based on classical perfect fluid models, scalar field theories or scalar-tensor theories of gravity, turn out to be vulnerable to experimental verification [8].

Appendix B. Assumptions and Approximations

Let us summarize and enumerate the main simplifying assumptions and approximations underlying the derivation of our gravitational potential. We used them to keep our calculations as analytical and non-perturbative as possible, which is crucial for an essentially nonlinear theory, such as ours.

First, it should be emphasized that superfluid vacuum theory, even when narrowed down to its logarithmic version, is a framework which potentially contains a set of models. Our chosen underlying quantum superfluid model, defined by Equation (11), is a minimal one. It can easily be expanded by adding polynomial terms $|\Psi|^k$ to the logarithmic nonlinearity in wave equations, to make it describe the phenomena in a more precise way. The reason for this is that the logarithmic nonlinearity is a leading-order approximation for the condensate-like matter, as discussed in [42]; but non-logarithmic terms can also come into play: one example is to be found in [43]. The nonlinear coupling function $b = b(\mathbf{r}, t)$ can also be made more detailed, to account for the thermodynamic environment in a more realistic way.

Second, for a chosen model, the signs of coupling parameters can be changed, which is often equivalent to changing the topological sector a model belongs to. Because we do not precisely know the topological structure of the physical vacuum we live in, it essentially doubles the number of candidate models to be tested empirically. For example, changing the sign of the field-theoretical potential; or, alternatively, the overall sign of the nonlinear coupling b; switches between the topologically trivial, which is considered in this paper, and the topologically non-trivial sectors. In a logarithmic liquid model with constant nonlinear coupling, switching to the topologically non-trivial sector of the theory means that wavefunction changes from a droplet-like non-topological soliton type to a bubble-like topological soliton type, as discussed in [47,49]. Consequently, the forms of a trial wavefunction and induced potential in Section 3 would also change.

Third, we have avoided the problem of ambiguity in choosing trapping potential and boundary conditions by making the former identically zero, and the latter of a conventional quantum-mechanical type on an infinite spatial domain of trivial topology $\mathbb{R}^3$. These are simplifying assumptions which are yet to be proven to work, otherwise they must be replaced with something more sophisticated.

Fourth, even if we have chosen our model correctly, there still remains the ambiguity of how to determine the state of the vacuum—is it in a ground state, an excited but metastable state, a pure or mixed state, a superposition of states, or even in a quantum transition between states? Any new empirical information about this could drastically change the assumptions underlying the computations in Section 3.

Fifth, the technical approximation which led us from Equations (16)–(18) might oversimplify the picture. It is suitable for the purposes of this study, but should be modified in more precise considerations.

Finally, throughout the paper, we consider the physical vacuum alone: assuming that its small excitations, which would be observed by a R-observer as relativistic matter with deformed dispersion relations [11,17–21], do not back-react, for example, via interaction between scalar and tensor modes. This is obviously an over-simplified picture of reality, which is sufficient for our current purpose, but unlikely to be valid in general.

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