

# Parton Distribution Functions and Tensorgluons

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**Abstract:** We derive the regularised evolution equations for the parton distribution functions that include tensorgluons.

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## 1. Introduction

The Bjorken scaling [1] is broken by a logarithmically dependent function [2,3] of the transverse momentum  $Q^2$  and is due to the interaction of quarks and gluons inside the hadrons [2–16]. In this article, we shall consider a possibility [9] that inside hadrons there are additional partons—tensorgluons, which can carry a part of the proton momentum [17–24]. The extension of the Yang–Mills theory was formulated in terms of the gauge invariant Lagrangian [18–21]. For tensorgluons of rank-2, it has the following form [18–20]:

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4}G_{\mu\nu}^a G_{\mu\nu}^a - \frac{1}{4}G_{\mu\nu,\lambda}^a G_{\mu\nu,\lambda}^a - \frac{1}{4}G_{\mu\nu}^a G_{\mu\nu,\lambda\lambda}^a \\ & + \frac{1}{4}G_{\mu\nu,\lambda}^a G_{\mu\lambda,\nu}^a + \frac{1}{4}G_{\mu\nu,\nu}^a G_{\mu\lambda,\lambda}^a + \frac{1}{2}G_{\mu\nu}^a G_{\mu\lambda,\nu\lambda}^a + \dots \end{aligned} \quad (1)$$

where the field strength tensors have the form:

$$\begin{aligned} G_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c, \\ G_{\mu\nu,\lambda}^a &= \partial_\mu A_{\nu\lambda}^a - \partial_\nu A_{\mu\lambda}^a + g f^{abc} (A_\mu^b A_{\nu\lambda}^c + A_{\mu\lambda}^b A_\nu^c), \\ G_{\mu\nu,\lambda\rho}^a &= \partial_\mu A_{\nu\lambda\rho}^a - \partial_\nu A_{\mu\lambda\rho}^a + g f^{abc} (A_\mu^b A_{\nu\lambda\rho}^c + A_{\mu\lambda}^b A_{\nu\rho}^c + A_{\mu\rho}^b A_{\nu\lambda}^c + A_{\mu\lambda\rho}^b A_\nu^c), \\ & \dots \end{aligned} \quad (2)$$

The first term in (1) corresponds to the standard Yang–Mills Lagrangian. The expression for the full Lagrangian can be found in [18–20]. For illustration purposes, we shall present the next term of the Lagrangian that describes the interaction of the rank-3 tensorgluons

$$\begin{aligned} \mathcal{L} = & -\frac{1}{4}G_{\mu\nu,\lambda\rho}^a G_{\mu\nu,\lambda\rho}^a - \frac{1}{8}G_{\mu\nu,\lambda\lambda}^a G_{\mu\nu,\rho\rho}^a - \frac{1}{2}G_{\mu\nu,\lambda}^a G_{\mu\nu,\lambda\rho\rho}^a - \frac{1}{8}G_{\mu\nu}^a G_{\mu\nu,\lambda\lambda\rho\rho}^a + \\ & + \frac{1}{3}G_{\mu\nu,\lambda\rho}^a G_{\mu\lambda,\nu\rho}^a + \frac{1}{3}G_{\mu\nu,\nu\lambda}^a G_{\mu\rho,\rho\lambda}^a + \frac{1}{3}G_{\mu\nu,\nu\lambda}^a G_{\mu\lambda,\rho\rho}^a + \\ & + \frac{1}{3}G_{\mu\nu,\lambda}^a G_{\mu\lambda,\nu\rho\rho}^a + \frac{2}{3}G_{\mu\nu,\lambda}^a G_{\mu\rho,\nu\lambda\rho}^a + \frac{1}{3}G_{\mu\nu,\nu}^a G_{\mu\lambda,\lambda\rho\rho}^a + \frac{1}{3}G_{\mu\nu}^a G_{\mu\lambda,\nu\lambda\rho\rho}^a. \end{aligned} \quad (3)$$

A spinor helicity technique [25–42] was used to calculate tensorgluon scattering amplitudes [22] and to extract the splitting amplitudes of gluons and tensorgluons [23]. The tensorgluon splitting amplitudes are

singular at the boundary values similar to the case of the standard splitting amplitudes in QCD, though the singularities are of higher order compared to the standard case and require a special regularisation technique to be developed before they can be placed into the equations which are describing the evolution of the generalised parton distribution functions (10). Here, we shall further develop a technique proposed earlier in [9] which will allow for regularising the tensorgluon splitting amplitudes.

The present paper is organised as follows. In Section 2, the basic formulae for scattering amplitude and splitting functions are recalled, definitions and notations are specified, and the details of the regularisation scheme are presented. In Section 3, we derive the regularised evolution equations for the parton distribution functions that take into account the creation of tensorgluons. Section 4 contains concluding remarks and summarises the physical consequences of the tensorgluons' hypothesis.

## 2. Splitting Functions

It was proposed in [9] that a possible emission of tensorgluons inside a hadron will produce a tensorgluon "cloud" inside a hadron in addition to the quark and gluon "clouds". Our goal here is to specify the regularisation of the generalised evolution equations introduced in [9] such that it will be consistent with the regularisation scheme used to regularised the DGLAP equations [4–8,10–12]. The splitting probabilities for tensorgluons have the form [9]:

$$\begin{aligned}
 P_{TG}(z) &= C_2(G) \left[ \frac{z^{2s+1}}{(1-z)_+^{2s-1}} + \frac{(1-z)^{2s+1}}{z^{2s-1}} \right], \\
 P_{GT}(z) &= C_2(G) \left[ \frac{1}{z(1-z)_+^{2s-1}} + \frac{(1-z)^{2s+1}}{z} \right], \\
 P_{TT}(z) &= C_2(G) \left[ \frac{z^{2s+1}}{(1-z)_+} + \frac{1}{(1-z)_+ z^{2s-1}} \right].
 \end{aligned}
 \tag{4}$$

The invariant operator  $C_2$  for the representation  $R$  is defined by the equation  $t^a t^a = C_2(R) 1$  and  $tr(t^a t^b) = T(R) \delta^{ab}$ . These functions satisfy the relations

$$P_{TG}(z) = P_{TG}(1-z), \quad P_{GT}(z) = P_{TT}(1-z), \quad z < 1.
 \tag{5}$$

One should define the regularisation procedure for the singular factors  $(1-z)^{-2s+1}$  and  $z^{-2s+1}$  reinterpreting them as the distributions  $(1-z)_+^{-2s+1}$  and  $z_+^{-2s+1}$ . The regularisation has been defined in the following way [9]:

$$\begin{aligned}
 \int_0^1 dz \frac{f(z)}{(1-z)_+^{2s-1}} &= \int_0^1 dz \frac{f(z) - \sum_{k=0}^{2s-2} \frac{(-1)^k}{k!} f^{(k)}(1) (1-z)^k}{(1-z)^{2s-1}}, \\
 \int_0^1 dz \frac{f(z)}{z_+^{2s-1}} &= \int_0^1 dz \frac{f(z) - \sum_{k=0}^{2s-2} \frac{1}{k!} f^{(k)}(0) z^k}{z^{2s-1}}, \\
 \int_0^1 dz \frac{f(z)}{z_+(1-z)_+} &= \int_0^1 dz \frac{f(z) - (1-z)f(0) - zf(1)}{z(1-z)},
 \end{aligned}
 \tag{6}$$

where  $f(z)$  is an arbitrary test function that is sufficiently regular at the points  $z = 0$  and  $z = 1$  and, as one can be convinced, the defined subtraction guarantees the convergence of the integrals. Using the same arguments as in the standard case [4], we will add the delta function terms into the definition

of the diagonal kernels so that they will completely determine the behaviour of  $P_{qq}(z)$ ,  $P_{GG}(z)$  and  $P_{TT}(z)$  functions with the coefficients that can be determined by using the momentum sum rule [9]:

$$\begin{aligned}
 P_{qq}(z) &= C_2(R) \left[ \frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(z-1) \right], \\
 P_{GG}(z) &= 2C_2(G) \left[ \frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) \right] + \frac{\sum_s (12s^2 - 1)C_2(G) - 4n_f T(R)}{6} \delta(z-1), \\
 P_{TT}(z) &= C_2(G) \left[ \frac{z^{2s+1}}{(1-z)_+} + \frac{1}{(1-z)_+ z^{2s-1}} + \sum_{j=1}^{2s+1} \frac{1}{j} \delta(z-1) \right].
 \end{aligned} \tag{7}$$

For completeness, we shall present also quark and gluon splitting functions [4]:

$$P_{Gq}(z) = C_2(R) \frac{1 + (1-z)^2}{z}, \tag{8}$$

$$P_{qG}(z) = T(R) [z^2 + (1-z)^2], \tag{9}$$

where  $C_2(G) = N, C_2(R) = \frac{N^2-1}{2N}, T(R) = \frac{1}{2}$  for the SU(N) groups.

### 3. Regularisation of Generalised DGLAP Equations

The deep inelastic structure functions can be expressed in terms of parton distribution densities [4–8,10–12]. If  $q^i(x, Q^2)$  is the density of quarks of type  $i$  (summed over colors) inside a nucleon target with fraction  $x$  of the proton longitudinal momentum in the infinite momentum frame, then the unpolarised structure functions can be represented in the following form:

$$2F_1(x, Q^2) = F_2(x, Q^2)/x = \sum_i e_i^2 [q^i(x, Q^2) + \bar{q}^i(x, Q^2)].$$

The  $Q^2$  dependence of the parton densities is described by the integro-differential equations for quark  $q^i(x, t)$  and gluon densities  $G(x, t)$ , where  $t = \ln(Q^2/Q_0^2)$  [4–8,10–12]. If there is an additional emission of tensorgluons in the proton, then one should introduce the corresponding density  $T(x, t)$  of tensorgluons and the integro-differential equations that describe the  $Q^2$  dependence of parton densities in this general case has the following form [9]:

$$\begin{aligned}
 \frac{dq^i(x, t)}{dt} &= \frac{\alpha(t)}{2\pi} \int_x^1 \frac{dy}{y} \left[ \sum_{j=1}^{2n_f} q^j(y, t) P_{q^i q^j} \left( \frac{x}{y} \right) + G(y, t) P_{q^i G} \left( \frac{x}{y} \right) \right], \\
 \frac{dG(x, t)}{dt} &= \frac{\alpha(t)}{2\pi} \int_x^1 \frac{dy}{y} \left[ \sum_{j=1}^{2n_f} q^j(y, t) P_{Gq^j} \left( \frac{x}{y} \right) + G(y, t) P_{GG} \left( \frac{x}{y} \right) + T(y, t) P_{GT} \left( \frac{x}{y} \right) \right], \\
 \frac{dT(x, t)}{dt} &= \frac{\alpha(t)}{2\pi} \int_x^1 \frac{dy}{y} \left[ G(y, t) P_{TG} \left( \frac{x}{y} \right) + T(y, t) P_{TT} \left( \frac{x}{y} \right) \right].
 \end{aligned} \tag{10}$$

In (10), we ignore contribution of the high-spin fermions  $\bar{q}^i$  of spin  $s + 1/2$ , which are the partners of the standard quarks [18–21], supposing that they are even heavier than the top quark. In this article, we shall limit ourselves by considering only emissions that always involve the standard gluons and spin-2 tensorgluons ignoring infinite “stairs” of transitions between tensorgluons of higher spin. In (10), the  $\alpha(t)$  is the running coupling ( $\alpha = g^2/4\pi$ ) and has the following form [17]:

$$\frac{\alpha}{\alpha(t)} = 1 + b \alpha t, \tag{11}$$

where

$$b = \frac{\sum_s (-1)^{2s} (12s^2 - 1) C_2(G) - 4n_f T(R)}{12\pi}, \quad s = 0, 1/2, 1, 3/2, 2, \dots \tag{12}$$

is the one-loop Callan–Symanzik coefficient [17]. In particular, the presence of the spin-two tensorgluons in the proton will give

$$b = \frac{58C_2(G) - 4n_f T(R)}{12\pi}. \tag{13}$$

The tensorgluon density  $T(x, t)$  changes when a gluon splits into two tensorgluons or when a tensorgluon radiates a gluon. This process is described by the third equation in (10).

The tensorgluon kernels (4) are singular at the boundary values similar to the case of the standard kernels (8), though the singularities are of higher order compared to the standard case. The ‘+’ prescription in

$$P(z) = \begin{pmatrix} P_{qq}(z) & 2n_f P_{qG}(z) & 0 \\ P_{Gq}(z) & P_{GG}(z) & P_{GT}(z) \\ 0 & P_{TG}(z) & P_{TT}(z) \end{pmatrix} \tag{14}$$

is defined as

$$[g(x)]_+ = g(x) - \delta(1-x) \int_0^1 g(z) dz, \tag{15}$$

and so

$$\int_x^1 f(z) [g(z)]_+ dz = \int_x^1 [f(z) - f(1)] g(z) dz - f(1) \int_x^1 g(z) dz. \tag{16}$$

Considering the splitting probabilities for spin two tensorgluons, we have to define ‘+++’ prescription as

$$[g(x)]_{+++} = g(x) - \delta(1-x) \int_0^1 g(z) dz - \delta'(1-x) \int_0^1 g(z)(1-z) dz - \frac{1}{2} \delta''(1-x) \int_0^1 g(z)(1-z)^2 dz, \tag{17}$$

and so

$$\int_x^1 f(z) [g(z)]_{+++} dz = \int_x^1 [f(z) - f(1) + f'(1)(1-z) - \frac{1}{2} f''(1)(1-z)^2] g(z) dz - \int_0^x [f(1) - f'(1)(1-z) + \frac{1}{2} f''(1)(1-z)^2] g(z) dz. \tag{18}$$

For the helicity-2 tensorgluons, the  $s = 2$  in (4), we will have

$$\begin{aligned} P_{TG}(z) &= C_2(G) \left[ \frac{z^5}{(1-z)^3} + \frac{(1-z)^5}{z^3} \right], \\ P_{GT}(z) &= C_2(G) \left[ \frac{1}{z(1-z)^3} + \frac{(1-z)^5}{z} \right], \\ P_{TT}(z) &= C_2(G) \left[ \frac{z^5}{(1-z)} + \frac{1}{(1-z)z^3} + \sum_{j=1}^5 \frac{1}{j} \delta(1-z) \right] \end{aligned} \tag{19}$$

and the regularisation of these kernels can be performed using the regularisation prescription (18). The regular splitting functions will take the following form:

$$\int_x^1 P_{TG}(z)f\left(\frac{x}{z}\right)dz = \int_x^1 \left[ \frac{z^5}{(1-z)^3} + \frac{(1-z)^5}{z^3} \right] f\left(\frac{x}{z}\right)dz = I_1 + I_2 + I_3,$$

$$I_1 = \int_x^1 \frac{dz}{(1-z)^3} \left( z^5 f\left(\frac{x}{z}\right) - f(x) + (5f(x) - xf'(x))(1-z) - (10f(x) - 4xf'(x) + \frac{1}{2}x^2f''(x))(1-z)^2 \right),$$

$$I_2 = - \int_0^x \frac{dz}{(1-z)^3} \left( f(x) - (5f(x) - xf'(x))(1-z) + (10f(x) - 4xf'(x) + \frac{1}{2}x^2f''(x))(1-z)^2 \right),$$

$$I_3 = \int_x^1 \frac{(1-z)^5}{z^3} f\left(\frac{x}{z}\right)dz.$$
(20)

For the gluon–tensor splitting function, we will get:

$$\int_x^1 P_{GT}(z)f\left(\frac{x}{z}\right)dz = \int_x^1 \left[ \frac{3-3z+z^2}{(1-z)^3} + \frac{1+(1-z)^5}{z} \right] f\left(\frac{x}{z}\right)dz = J_1 + J_2 + J_3,$$

$$J_1 = \int_x^1 \frac{1}{(1-z)^3} \left( (3-3z+z^2)f\left(\frac{x}{z}\right) - f(x) - (f(x) + xf'(x))(1-z) - (f(x) + 2xf'(x) + \frac{1}{2}x^2f''(x))(1-z)^2 \right) dz$$

$$J_2 = - \int_0^x \frac{dz}{(1-z)^3} [f(x) + (f(x) + xf'(x))(1-z) + (f(x) + 2xf'(x) + \frac{1}{2}x^2f''(x))(1-z)^2]$$

$$J_3 = \int_x^1 \frac{1+(1-z)^5}{z} f\left(\frac{x}{z}\right)dz.$$
(21)

Using the regularisation (16) for the tensor-tensor splitting function, we will get the following expression:

$$\int_x^1 P_{TT}(z)f\left(\frac{x}{z}\right)dz = \int_x^1 \left[ \frac{1+z^5}{(1-z)} + \frac{1+z+z^2}{z^3} + \sum_{j=1}^5 \frac{1}{j} \delta(1-z) \right] f\left(\frac{x}{z}\right)dz$$

$$= \int_x^1 \frac{dz}{(1-z)} \left[ (1+z^5)f\left(\frac{x}{z}\right) - 2f(x) \right] + f(x) \left[ 2\ln(1-x) + \frac{137}{60} \right]$$

$$+ \int_x^1 \frac{1+z+z^2}{z^3} f\left(\frac{x}{z}\right)dz.$$
(22)

All new splitting functions which have been added to the standard evolution equation are now well defined and can be calculated using the above Equations (20)–(22).

#### 4. Discussion

The gluon density  $G(x, t)$  inside the hadrons is one of the least constrained functions since it does not couple directly to the photon in deep-inelastic scattering measurements. The process of gluon splitting leads to the emission of tensorgluons and therefore a part of the proton momentum that is carried by the neutral constituents can be shared between gluons and tensorgluons and the density of neutral partons is the sum of two density functions:  $G(x, t) + T(x, t)$ . Because tensorgluons have a larger spin, they can influence the spin structure of the nucleon. The details can be found in [9,43]. To disentangle the contributions of gluons and tensorgluons to the partons densities of a nucleon and to decide which piece of the neutral partons is generated by gluons and which one by tensorgluons, one should measure the helicities of the neutral components.

In supersymmetric extensions of the Standard Model [44,45], the gluons and quarks have natural partners—gluinos of spin  $s = 1/2$  and squarks of spin  $s = 0$ . If the gluinos appear as elementary constituents of the hadrons, then the theory predicts the existence of new hadronic states, the R-hadrons [46,47]. The experimental data provide the evidence that most probably they have to be very heavy [48,49].

The existence of tensorgluon partons inside the proton does not predict a new hadronic state, a proton remains a proton. The tensorgluons will alternate the parton distribution functions of a proton. The question is to which extent the tensorgluons will change the parton distribution functions. The regularisation of the splitting amplitudes developed above (20)–(22) will allow for solving the generalised DGLAP evolution Equation (10) for the parton distribution functions that takes into account the processes of emission of tensorgluons by gluons. The integration can now be performed using the algorithms developed in [50–53] and to find out the ratio of densities between gluons and tensorgluons.

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