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Ricci Linear Weyl/Maxwell Mutual Sourcing †

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Abstract: We elevate the field theoretical similarities between Maxwell and Weyl vector fields into a full local scale/gauge invariant Weyl/Maxwell mutual sourcing theory. In its preliminary form, and exclusively in four dimensions, the associated Lagrangian is dynamical scalar field free, hosts no fermion matter fields, and Holdom kinetic mixing is switched off. The mutual sourcing term is then necessarily spacetime curvature (not just metric) dependent, and inevitably Ricci linear, suggesting that a non-vanishing spacetime curvature can in principle induce an electromagnetic current. In its mature form, however, the Weyl/Maxwell mutual sourcing idea serendipitously constitutes a novel variant of the gravitational Weyl-Dirac (incorporating Brans-Dicke) theory. Counter intuitively, and again exclusively in four dimensions, the optional quartic scalar potential gets consistently replaced by a Higgs-like potential, such that the co-divergence of the Maxwell vector field resembles a conformal vacuum expectation value.

Keywords: Weyl gravity; Weyl vector field; Maxwell vector field; scale invariance; $U(1)$ gauge invariance; Ricci curvature; Holdom kinetic mixing; dilaton; Higgs potential

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Within the framework of Riemann geometry, with tensor fields serving as the fundamental objects, ordinary derivatives are consistently replaced by covariant derivatives to assure diffeomorphism invariance. Going one step further into the territory of Weyl geometry [1,2], the tensor fields are traded for so-called co-tensor fields, and the covariant derivatives are generalized into co-covariant (also known as starred $\star$) derivatives, respectively. Resembling a $U(1)$ local gauge theory, the star derivation procedure mandatorily introduces a new player into the game, the Weyl vector field $a_\mu(x)$. The prototype theory in this category is Weyl-Dirac gravity [3–5], a local scale symmetric generalization of Brans-Dicke theory [6]. A counter example is provided by C^2 conformal gravity [7,8] which, owing to the Weyl tensor $C^{\mu}_{\ \nu\lambda\sigma}$ being an in-tensor (co-tensor of weight zero), solely in 4-dimensions, does not require the presence of $a_\mu(x)$. Other theoretical directions include two scalar gravity-anti-gravity theories [9–12], and Kaluza-Klein reduced higher dimensional local scale symmetric theories [13–15]. However, while treated on equal canonical footing in the Lagrangian formalism, Maxwell vector field $A_\mu(x)$ and Weyl vector field $a_\mu(x)$ play completely different roles in theoretical physics.

From the geometric point of view, the differences between these two vector fields sharpens. While $A_\mu(x)$ constitutes an in-vector, $a_\mu(x)$ does not constitute a co-vector at all. However, in spite of their physical and geometrical differences, these two vector fields do share a similar transformation law under their corresponding local symmetries. To be specific,

$$A_\mu(x) \to A_\mu(x) - \partial_\mu \Phi(x) \, , \tag{1}$$

$$a_\mu(x) \to a_\mu(x) - \partial_\mu \varphi(x) \, , \tag{2}$$

with Φ taking values on a circle whereas φ on the real line. In turn, both their kinetic terms, namely $F_{\mu\nu}$ and $f_{\mu\nu}$ respectively, transform alike as Weyl in-scalars, and a Holdom-style kinetic mixing [16] becomes then field theoretically permissible. In this essay, however, with or without invoking the kinetic mixing term, we elevate the apparent field theoretical similarities between Maxwell and Weyl vector fields into a full local scale/gauge invariant mutual sourcing theory. In line with Einstein-Hilbert and especially with Weyl-Dirac actions, and solely in a 4-dimensional spacetime, we show that the scale symmetric Weyl/Maxwell mutual source mixing is necessarily spacetime curvature dependent (not just metric dependent), and inevitably Ricci linear. This way, a non-vanishing spacetime curvature becomes an unconventional source of the electromagnetic current.

Let our starting point be the familiar 4-dimensional action involving a linear electromagnetic coupling term

$$\mathcal{I}_{EM} = \int \left(\mathcal{L}_G - \frac{1}{4}F^2 - J^\mu A_\mu\right)\sqrt{-g}\, d^4x \,, \tag{3}$$

with J^μ serving as the external electromagnetic source current, and $\mathcal{L}_G$ denoting the yet unspecified gravitational part of the Lagrangian. To keep gauge invariance manifest already at the Lagrangian level, one may invoke a Lagrange multiplier η, and simply replace A_μ by $A_\mu - \eta_{;\mu}$, such that $\eta \to \eta + \Phi$. It is only at the stage when gauge fixing becomes permissible, e.g., at the level of the equations of motion, that one may set $\eta = 0$. As dictated by the self consistency of the associated Maxwell equations $F^{\mu\nu}_{;\nu} = J^\mu$, and directly by the variation with respect to η, the theory maintains gauge invariance only provided J^μ is locally conserved $J^\mu_{;\mu} = 0$. The action Equation (3) is furthermore local scale invariant if $\mathcal{L}_G$ is such, and if J_μ happens to be a co-covariant vector of power

$$[J_\mu] = -2 \iff [J^\mu] = -4 \,, \tag{4}$$

where in our Weyl-Dirac notations,

$$[g_{\mu\nu}] = 2 \,,\; [g^{\mu\nu}] = -2 \implies [\sqrt{-g}] = 4 \,, \tag{5}$$

$$[A_\mu] = 0 \,,\; [A^\mu] = -2 \implies [F_{\mu\nu}] = 0 \,. \tag{6}$$

The last formula deserves some attention. Consider a co-covariant vector V_μ of power $[V_\mu] = n$, and recall that its covariant derivative $V_{\mu;\nu}$ does not form a co-tensor. Alternatively, one invokes a co-covariant starred derivative, and show that the corresponding co-tensor role is then taken by

$$V_{\mu\star\nu} = V_{\mu;\nu} - (n-1)a_\nu V_\mu + a_\mu V_\nu - g_{\mu\nu}a^\lambda V_\lambda \,. \tag{7}$$

In particular, notice the antisymmetric combination

$$V_{\mu\star\nu} - V_{\nu\star\mu} = V_{\mu;\nu} - V_{\nu;\mu} + n(a_\mu V_\nu - a_\nu V_\mu) \,, \tag{8}$$

telling us that antisymmetric $F_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu} = A_{\mu\star\nu} - A_{\nu\star\mu}$ is in fact an in-tensor simply because A_μ is an in-vector (meaning $n = 0$) to start with. By the same token, if U^μ is a co-contravariant vector of power $[U^\mu] = n$, its star derivative is given by

$$U^\mu_{\star\nu} = U^\mu_{;\nu} - (n+1)a_\nu U^\mu + a^\mu U_\nu - g^\mu{}_\nu a^\lambda U_\lambda \,. \tag{9}$$

In particular, its co-divergence is given by

$$U^\mu_{\star\mu} = U^\mu_{;\mu} - (n+4)a_\mu U^\mu \,. \tag{10}$$

It is only for the special case of $n = -4$, that we face the advantage of $J^\mu_{\star\mu} = J^\mu_{;\mu}$.

The fact that A_μ and a_μ share similar transformation laws under their corresponding local symmetries, and exhibit kinetic terms of the one and the same structure, may prematurely suggest, in analogy with Equation (3), an action á la

$$\int \left[\mathcal{L}_G - \frac{1}{4}f^2 - j^\mu a_\mu\right]\sqrt{-g}\, d^4x\,. \tag{11}$$

The trouble is that, while $f_{\mu\nu} = a_{\mu;\nu} - a_{\nu;\mu}$ turns out to be a legitimate in-tensor, the Weyl vector a_μ itself does not transform like a co-vector at all. Unlike the Weyl vector which transforms a la Equation (2), a power n co-vector gets scaled by a factor $e^{n\varphi(x)}$. This is to say that the action Equation (11) is not invariant under arbitrary local scale transformations.

In search of a tenable coupling term to replace the problematic $j^\mu a_\mu$, we first recall that co-covariant starred derivatives are generically linear in a_μ. For example, let S be a co-scalar of power n, then

$$S_{\star\mu} = S_{;\mu} - n a_\mu S\,, \tag{12}$$

with the bonus of having $[S_{\star\mu}] = n$ as well. Thus, a coupling term of the form

$$\mathcal{L}_{int} = j^\mu S_{\star\mu} \propto j^\mu a_\mu + ... \tag{13}$$

can certainly do, but only provided (i) $n \neq 0$ on self consistency grounds, and (ii) The source current j^μ must constitutes a co-vector of the exact power

$$[j^\mu] = -(n+4) \quad \Rightarrow \quad [j_\mu] = -(n+2)\,. \tag{14}$$

Now, aiming towards Weyl/Maxwell mutual sourcing, one would like to identify j_μ with A_μ. This is our goal, but for this to be the case, recalling that $[A_\mu] = 0$, we must first find a suitable candidate for S, such that

$$[A_\mu] = 0 \implies [S] = -2\,. \tag{15}$$

What are the options?

At this stage, fundamental scalar fields are yet to be introduced. In fact, the option of not introducing fundamental scalar fields into the theory is exclusively viable in four spacetime dimensions. So, in the absence of scalar fields, the answer to the above question must come from the geometry of the underlying 4-dim curved spacetime. The simplest curvature scalar to think of is no doubt the Ricci scalar R. However, unfortunately, R cannot enter the game as is, but must be traded for its $\tilde{R}$ scale symmetric co-scalar variant. In 4-dimensions, it is given by

$$\tilde{R} = R + 6a^\mu_{;\mu} - 6a^\mu a_\mu\,. \tag{16}$$

Note that we prefer the notation $\tilde{R}$, instead of the original $R^\star$ or ${}^\star R$, leaving the star symbol solely for denoting co-derivation. The crucial observation now is that $[\tilde{R}] = -2$, and the same is true for its co-derivative

$$\tilde{R}_{\star\mu} = \tilde{R}_{;\mu} + 2a_\mu \tilde{R}\,. \tag{17}$$

In turn, the master requirement Equation (15) can now be satisfied by naturally choosing $S = \tilde{R}$. It is straightforward to verify that other powers of $\tilde{R}$, as well as higher order curvature co-scalars, such as $\tilde{R}^{\mu\nu}\tilde{R}_{\mu\nu}$ and $\tilde{R}^{\mu\nu\lambda\sigma}\tilde{R}_{\mu\nu\lambda\sigma}$, will not do.

We can now close the circle. Rather than assigning external non-dynamical source currents J^μ and j^μ, we let the Maxwell vector field A_μ and the Weyl vector field a_μ source each other. The result is the simplest dynamical scalar free local gauge/scale invariant Weyl/Maxwell mixing theory described by the action

$$I = -\int \left[\frac{1}{4}F^2 + \frac{1}{4}f^2 + \frac{1}{2}eA^\mu \tilde{R}_{\star\mu}\right]\sqrt{-g}\, d^4x\,, \tag{18}$$

where e is a universal dimensionless constant. Recall that, in analogy with the note following Equation (3), A_μ is to be replaced by $A_\mu - \eta_{\star\mu} = A_\mu - \eta_{;\mu}$, with $[\eta] = 0$, whenever is needed (like here) to make gauge invariance manifest already at the Lagrangian level. It is crucial to notice that $A^\mu \tilde{R}$ happens to be a co-contravariant vector of the special power $[A^\mu \tilde{R}] = -4$. Hence, by recalling Equation (10) twice, we find

$$A^\mu \tilde{R}_{\star\mu} = -A^\mu_{\star\mu} \tilde{R} + (A^\mu \tilde{R})_{\star\mu} = -(A^\mu_{;\mu} - 2A^\mu a_\mu)\tilde{R} + (A^\mu \tilde{R})_{;\mu} \, . \tag{19}$$

Up to a total divergence, and by no coincidence, also up to a total co-divergence, Equation (18) can be now re-written in the attractive $\tilde{R}$-linear form

$$I_0 = -\int \left[\frac{1}{4}F^2 + \frac{1}{4}f^2 - \frac{1}{2}eA^\mu_{\star\mu}\tilde{R} \right] \sqrt{-g}\, d^4x \tag{20}$$

The Weyl/Maxwell mutual sourcing can take a more conventional form by introducing yet a non-dynamical real scalar field ϕ (accompanied by a suitable Lagrange multiplier λ, such that $[\lambda] = -2$) into the theory

$$I_1 = -\int \left[\frac{1}{4}F^2 + \frac{1}{4}f^2 - \phi^2 \tilde{R} + \lambda \left(\phi^2 - \frac{1}{2}eA^\mu_{\star\mu} \right) \right] \sqrt{-g}\, d^4x \, . \tag{21}$$

However, for the scalar field to become dynamical, a supplementary in-scalar kinetic term is mandatory, and following Dirac, the Brans-Dicke coefficient ω of such a term can be fully arbitrary, not necessarily critical. This leads us to

$$I_2 = -\int \left[\frac{1}{4}F^2 + \frac{1}{4}f^2 - \phi^2 \tilde{R} + \omega g^{\mu\nu}\phi_{\star\mu}\phi_{\star\nu} + \lambda \left(\phi^2 - \frac{1}{2}eA^\mu_{\star\mu} \right) \right] \sqrt{-g}\, d^4x \, , \tag{22}$$

which, up to the λ-term, establishes contact with the Weyl-Dirac theory.

With such an observation in hand, the latest action needs not be the final word, as the Weyl-Dirac theory is known to further allow for a quartic scalar potential. Consequently, we cannot resist replacing the quadratic ϕ^2-constraint by a quartic ϕ^4 potential, and by consistently doing so, trading the auxiliary co-scalar λ for a dimensionless constant coefficient Λ. The resulting theory reads

$$I_3 = -\int \left[\frac{1}{4}F^2 + \frac{1}{4}f^2 - \phi^2 \tilde{R} + \omega g^{\mu\nu}\phi_{\star\mu}\phi_{\star\nu} + \Lambda \left(\phi^2 - \frac{1}{2}eA^\mu_{\star\mu} \right)^2 \right] \sqrt{-g}\, d^4x \tag{23}$$

with $e = 0$ signaling the exact Weyl-Dirac limit. In fact, and perhaps counter intuitively,

$$v^2 \equiv \frac{1}{2}eA^\mu_{\star\mu} = e(\frac{1}{2}A^\mu_{;\mu} - a_\mu A^\mu) \tag{24}$$

highly resembles (and can be referred to as) a conformal vacuum expectation value. The former constraint $\phi^2 = \frac{1}{2}eA^\mu_{\star\mu}$ is now realized as the minimum (for $\Lambda > 0$) of a tenable Higgs potential. We note here again that, in all evolving action versions $I_{0,1,2,3}$, in order to make gauge invariance manifest already at the Lagrangian level, one consistently replaces A_μ by $A_\mu - \eta_{\star\mu} = A_\mu - \eta_{;\mu}$. While the presence of the η is mandatory as long as the $U(1)$ coupling is non-minimal, it can eventually be integrated out by gauge fixing. The situation may look somewhat reminiscent of the Stueckelberg action for a massive vector field, but recall that the present theory is a priori free of any mass scale. As far as the physical interpretation of η is concerned, it should be clarified that it cannot be regarded a new independent dynamical scalar field. The reason being that it is just the one and only combination $A_\mu - \eta_{;\mu}$ which actually enters the Lagrangian.

We modestly aimed towards Weyl/Maxwell mutual sourcing, and have automatically been driven into its unified Weyl/Dirac/Maxwell/Higgs embedding. Gravity just cannot stay out of the game.

There may be however a price to pay. While the situation is apparently somewhat similar to the Weyl-Dirac theory, the differential equations of motion are beyond second order (a counter example is provided by the non-trivial local scale invariant extension of the 4-dim Gauss-Bonnet theory). If this is the case, then the Ricci linear coupling may introduce ghosts and render the minimal theory sick.

It has not escaped our attention that, while sticking to 4-dimensions, one is always free to add curvature quadratics terms, for example $\mathcal{L}_G = \tilde{R}^2$ or $\mathcal{L}_G = C^2$ without violating local scale invariance. Another pretentious attempt would be to add Equation (20) to the standard Einstein-Hilbert $\mathcal{L}_G = R$, which obviously does not respect Weyl scale symmetry. This would mean revising Einstein-Maxwell into Einstein-Weyl/Maxwell theory, and modifying even the Reissner-Nordstrom solution accordingly. Such generalizations are however beyond the scope (and even beyond the rationale) of the present paper. On pedagogical and simplicity grounds, however, we hereby set $\mathcal{L}_G = 0$ and first study the action Equation (20) on its own merits.

At any rate, here are some distinctive features of the simplest Weyl/Maxwell mutual sourcing theory I_0 prescribed by the action Equation (20):

- The highlight is, roughly speaking, the construction of the Maxwell conserved current J_μ from spacetime curvature (involving a_μ dependence). The variation with respect to A_μ is straight forward, giving rise to the conformal conservation law

$$\left(F^{\mu\nu} - \frac{1}{2}eg^{\mu\nu}\tilde{R}\right)_{\star\nu} = 0\,, \tag{25}$$

where one can make use of the identity $F^{\mu\nu}_{\star\nu} = F^{\mu\nu}_{;\nu}$. Self consistency (and $g^{\mu\nu}_{\star\nu} = 0$) then dictates the complementary co-scalar constraint

$$g^{\mu\nu}\tilde{R}_{\star\mu\star\nu} = 0\,. \tag{26}$$

Here again, owing to $[\tilde{R}_{\star\mu}] = -2$, one can take advantage of $g^{\mu\nu}\tilde{R}_{\star\mu\star\nu} = g^{\mu\nu}\tilde{R}_{\star\mu;\nu}$, and recall Equation (16) to further probe the structure of the Maxwell current

$$J_\mu = e(a_\mu\tilde{R} + \frac{1}{2}\tilde{R}_{;\mu})\,. \tag{27}$$

An important question is then which conformal metrics might admit a non-vanishing R.H.S. of Equation (27), or even better: Which geometries will not do so? Apart from some special cases, e.g., conformal Schwarzschild and Schwarzschild-deSitter metrics [7], the general answer is still unknown. We emphasize that the conservation of the co-vector J^μ needs not be considered an external constraint, but rather be a legitimate consistency condition which does not break local scale invariance. This only requires though, as noted earlier, the replacement of A_μ by $A_\mu - \eta_{;\mu}$

- By the same token, the variation with respect to a_μ leads to the field equation

$$f^{\mu\nu}_{\star\nu} = j^\mu\,. \tag{28}$$

It takes some algebra though to establish the analogy with the Maxwell current, and verify that the Weyl current is indeed proportional to A_μ, and is given explicitly by

$$j_\mu = e(A_\mu\tilde{R} + 3A^\nu_{\star\nu\star\mu})\,. \tag{29}$$

- The co-divergence of the Maxwell vector field resembles a dilaton, with the formal definition being the coefficient of $\tilde{R}$ in the Lagrangian Equation (20), namely

$$\phi^2 = \frac{1}{2} e A^{\mu}_{\star\mu} = e(\frac{1}{2} A^{\mu}_{;\mu} - a_{\mu} A^{\mu}) \,. \tag{30}$$

  The fact that the roots of such a dilaton-like configuration are electromagnetic in origin is a natural consequence of the Weyl/Maxwell mutual sourcing. It is only in the intermediate stages, as expressed by the successive actions $I_{1,2}$, that ϕ becomes an independent scalar dilaton field on its own merits. Later on, as demonstrated by actions I_3, in analogy with the Higgs mechanism, Equation (30) represents the vacuum of the theory.
- Finally, imitating Holdom's $U(1) \otimes U'(1)$ kinetic mixing, one may switch on the analogous scale/gauge symmetric Weyl/Maxwell kinetic mixing [17]

$$\mathcal{L}_{\epsilon} = \frac{1}{2} \epsilon g^{\mu\lambda} g^{\nu\sigma} F_{\mu\nu} f_{\lambda\sigma} \,, \tag{31}$$

  parametrized by some dimensionless coefficient ϵ. No dramatic effects are expected as long as minimally coupled charged scalar fields or fermion fields are not introduced. Once they do enter the theory, reflecting the opposite transformation laws of $A_{\mu} \to -A_{\mu}$ versus $a_{\mu} \to +a_{\mu}$, the discrete CP symmetry gets explicitly violated.

To summarize, the general idea of Weyl/Maxwell mutual sourcing has been formulated on two field theoretical levels. They are: (1) A preliminary theory, free of fundamental scalars and fermion fields, and (2) A full Weyl-Dirac variant theory incorporating a genuine real dilaton scalar field. The main message is that the Weyl/Maxwell mutual sourcing term is necessarily spacetime curvature (not just metric) dependent and inevitably Ricci linear, thereby suggesting that a non-vanishing spacetime curvature can in principle induce an electric current. A central (and quite a novel) role is played in the theory by the co-divergence of the Maxwell vector field $A^{\mu}_{\star\mu}$. In the basic version, prescribed by the action I_0 (see Equation (20)), serving as the coefficient of the Ricci curvature term, it effectively resembles a dilaton field ϕ^2 whose roots are thus counter intuitively electromagnetic in origin. The idea elegantly and most naturally fits into the Weyl-Dirac (incorporating Brans-Dicke) theory. Originally, the latter exclusively allows for the quartic potential term $\Lambda\phi^4$, but in the Weyl/Maxwell mutual sourcing extension, prescribed by the action I_3, it is consistently traded for the Higgs-like potential $\Lambda(\phi^{\dagger}\phi - \frac{1}{2} e A^{\mu}_{\star\mu})^2$ without upsetting the local scale invariance. In other words, $A^{\mu}_{\star\mu}$ serves as (to be referred to) a conformal vacuum expectation value. Bearing in mind that a spontaneous local scale symmetry breaking mechanism is still very much at large, we can only hope that the theory discussed may hopefully contribute (currently under extensive investigation) in this field theoretical direction.

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