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Communication

Solving Linear Tensor Equations

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Abstract: We develop a systematic way to solve linear equations involving tensors of arbitrary rank. We start off with the case of a rank 3 tensor, which appears in many applications, and after finding the condition for a unique solution we derive this solution. Subsequently, we generalize our result to tensors of arbitrary rank. Finally, we consider a generalized version of the former case of rank 3 tensors and extend the result when the tensor traces are also included.

Keywords: tensor equations; tensor algebra; linear systems of tensors

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1. Introduction

In many applications a tensorial equation of the form

$$a_1 N_{\alpha\mu\nu} + a_2 N_{\nu\alpha\mu} + a_3 N_{\mu\nu\alpha} + a_4 N_{\alpha\nu\mu} + a_5 N_{\nu\mu\alpha} + a_6 N_{\mu\alpha\nu} = B_{\alpha\mu\nu} \tag{1}$$

appears, where $B_{\alpha\mu\nu}$ is some given (i.e., known) tensor field, $a_i's$, $i = 1, 2, \ldots, 6$ are some given scalar fields and $N_{\alpha\mu\nu}$ is the unknown tensor field one wishes to solve for. For instance, this tensorial equation is encountered when one varies the quadratic Metric–Affine Gravity action [1–3] with respect to the affine connection[1]. There, $B_{\alpha\mu\nu}$ represents the (known) hypermomentum source and $N_{\alpha\mu\nu}$ is the distortion tensor [8] in which spacetime torsion and non-metricity are encoded. Then, having solved for $N_{\alpha\mu\nu}$ entirely in terms of the sources, that is, combinations of $B_{\alpha\mu\nu}$, one can easily obtain the forms of torsion and non-metricity, namely the non-Riemannian [9] parts of the geometry. In order to solve this equation, one could go about and split $N_{\alpha\mu\nu}$ into its irreducible decomposition and then take contractions, symmetrizations and so forth, in order to find the various pieces in terms of $B_{\alpha\mu\nu}$ and its contractions. Even though this may work in some cases, it will be a difficult task in general. Moreover, this procedure will fall short quickly if one wishes to generalize the above considerations and ask for the general solution (N in terms of B) of the rank-n tensorial equation:

$$\underbrace{a_1 N_{\mu_1\mu_2\mu_3\ldots\mu_n} + a_2 N_{\mu_n\mu_1\mu_2\ldots\mu_{n-1}} + \cdots + a_n N_{\mu_2,\mu_3,\ldots\mu_n\mu_1} + \ldots}_{n!-terms} = B_{\mu_1\mu_2\mu_3\ldots\mu_n}. \tag{2}$$

Evidently, one easily realizes that it would be impossible to solve the latter by resorting to some decomposition scheme for N[2]. It is then natural to ask, is there a systematic and practical way to solve equations of the form (1) or more generally (2). It is the purpose of this letter to answer this question. As we show, under a fairly general non-degeneracy condition, it is always possible to find the unique solution of (1), or more generally of (2), by following a certain procedure that we develop below along with some extensions/generalizations.

2. The Theorems

In what follows, we present 3 Theorems. In the first one, the systematic way to solve Equation (1) for N is proved. We then extend this result to tensors N of arbitrary rank (i.e., not necessary 3) and solve equations of the form (2). Finally we derive the solution of a generalized version of (1), where the traces of N are also included. We have the following.

Theorem 1. *Consider the tensor equation:*

$$a_1 N_{\alpha\mu\nu} + a_2 N_{\nu\alpha\mu} + a_3 N_{\mu\nu\alpha} + a_4 N_{\alpha\nu\mu} + a_5 N_{\nu\mu\alpha} + a_6 N_{\mu\alpha\nu} = B_{\alpha\mu\nu}, \tag{3}$$

where a_i, $i = 1, 2, \ldots, 6$ are scalars, $B_{\alpha\mu\nu}$ is a given (known) tensor and $N_{\alpha\mu\nu}$ are the components of the unknown tensor[3] N. Define the matrix:

$$A := \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_2 & a_3 & a_1 & a_5 & a_6 & a_4 \\ a_3 & a_1 & a_2 & a_6 & a_4 & a_5 \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 \\ a_5 & a_4 & a_6 & a_2 & a_1 & a_3 \\ a_4 & a_6 & a_5 & a_1 & a_3 & a_2 \end{pmatrix} \tag{4}$$

If the system is non-degenerate, that is, if[4]

$$det(A) \neq 0 \tag{5}$$

holds true, then the general and unique solution of (3) reads:

$$N_{\alpha\mu\nu} = \tilde{a}_{11} B_{\alpha\mu\nu} + \tilde{a}_{12} B_{\nu\alpha\mu} + \tilde{a}_{13} B_{\mu\nu\alpha} + \tilde{a}_{14} B_{\alpha\nu\mu} + \tilde{a}_{15} B_{\nu\mu\alpha} + \tilde{a}_{16} B_{\mu\alpha\nu}, \tag{6}$$

where the $\tilde{a}_{1i}$'s are the first row elements of the inverse matrix A^{-1}.

Proof. Starting from (3), we perform the 5 independent possible permutations on the indices, and also including (3), we end up with the system:

$$\begin{aligned} a_1 N_{\alpha\mu\nu} + a_2 N_{\nu\alpha\mu} + a_3 N_{\mu\nu\alpha} + a_4 N_{\alpha\nu\mu} + a_5 N_{\nu\mu\alpha} + a_6 N_{\mu\alpha\nu} &= B_{\alpha\mu\nu} \\ a_1 N_{\nu\alpha\mu} + a_2 N_{\mu\nu\alpha} + a_3 N_{\alpha\mu\nu} + a_4 N_{\mu\alpha\nu} + a_5 N_{\alpha\nu\mu} + a_6 N_{\nu\mu\alpha} &= B_{\nu\alpha\mu} \\ a_1 N_{\mu\nu\alpha} + a_2 N_{\alpha\mu\nu} + a_3 N_{\nu\alpha\mu} + a_4 N_{\nu\mu\alpha} + a_5 N_{\mu\alpha\nu} + a_6 N_{\alpha\nu\mu} &= B_{\mu\nu\alpha} \\ a_1 N_{\alpha\nu\mu} + a_2 N_{\mu\alpha\nu} + a_3 N_{\nu\mu\alpha} + a_4 N_{\alpha\mu\nu} + a_5 N_{\mu\nu\alpha} + a_6 N_{\nu\alpha\mu} &= B_{\alpha\nu\mu} \\ a_1 N_{\nu\mu\alpha} + a_2 N_{\alpha\nu\mu} + a_3 N_{\mu\alpha\nu} + a_4 N_{\nu\alpha\mu} + a_5 N_{\alpha\mu\nu} + a_6 N_{\mu\nu\alpha} &= B_{\nu\mu\alpha} \\ a_1 N_{\mu\alpha\nu} + a_2 N_{\nu\mu\alpha} + a_3 N_{\alpha\nu\mu} + a_4 N_{\mu\nu\alpha} + a_5 N_{\nu\alpha\mu} + a_6 N_{\alpha\mu\nu} &= B_{\mu\alpha\nu}. \end{aligned} \tag{7}$$

Then, defining the matrix:

$$A := \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_2 & a_3 & a_1 & a_5 & a_6 & a_4 \\ a_3 & a_1 & a_2 & a_6 & a_4 & a_5 \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 \\ a_5 & a_4 & a_6 & a_2 & a_1 & a_3 \\ a_4 & a_6 & a_5 & a_1 & a_3 & a_2, \end{pmatrix} \tag{8}$$

along with the columns

$$\mathcal{N} := (N_{\alpha\mu\nu}, N_{\nu\alpha\mu}, N_{\mu\nu\alpha}, N_{\alpha\nu\mu}, N_{\nu\mu\alpha}, N_{\mu\alpha\nu})^T \tag{9}$$

and

$$\mathcal{B} := (B_{\alpha\mu\nu}, B_{\nu\alpha\mu}, B_{\mu\nu\alpha}, B_{\alpha\nu\mu}, B_{\nu\mu\alpha}, B_{\mu\alpha\nu})^T \tag{10}$$

we may express the above system in matrix form as:

$$A\mathcal{N} = \mathcal{B}. \tag{11}$$

In the above, $\mathcal{N}$ is the column consisting of the unknown elements we wish to find. Since, by hypothesis, we have a non-degenerate system, it follows that $det(A) \neq 0$ and as a result the inverse A^{-1} exists. We then formally multiply the above equation by A^{-1} from the left to get:

$$\mathcal{N} = A^{-1}\mathcal{B}. \tag{12}$$

The above is a column equation and of course each element on the left column must be equal to each element on the right. Equating the first element we arrive at the stated result:

$$N_{\alpha\mu\nu} = \tilde{a}_{11}B_{\alpha\mu\nu} + \tilde{a}_{12}B_{\nu\alpha\mu} + \tilde{a}_{13}B_{\mu\nu\alpha} + \tilde{a}_{14}B_{\alpha\nu\mu} + \tilde{a}_{15}B_{\nu\mu\alpha} + \tilde{a}_{16}B_{\mu\alpha\nu}, \tag{13}$$

where the $\tilde{a}_{1i}\,'s$ are the elements of the first row of the inverse matrix A^{-1} which, of course, depend on $a_1, a_2, \ldots, a_6$. Note that the equations we get for the rest of the column elements will be related to the above one with cyclic permutations and will therefore give nothing new. Concluding, (13) is the general solution of (3). Some comments are now in order.

Comment 1. Note that if $B_{\alpha\mu\nu} = 0$ and the matrix A is non-singular we have that $N_{\alpha\mu\nu} = 0$ as a unique solution. It should be emphasized that the demand that $det(A) \neq 0$ is all essential in order for the full $N_{\alpha\mu\nu}$ tensor field to be vanishing. If the last requirement is not fulfilled, the full N tensor may as well not be identically vanishing since in this case, not the full N but certain (anti)-symmetrizations of it appear in (3). In such an occasion, only certain parts of $N_{\alpha\mu\nu}$ will be vanishing.

Comment 2. If the components $N_{\alpha\mu\nu}$ are symmetric or antisymmetric in any pair of indices then the system is greatly simplified and the 6×6 matrix A is reduced to a 3×3 matrix instead.[5] □

Theorem 2. *In a $d-$dimensional space, consider the tensor equation (with $n \leq d$):*

$$\underbrace{a_1 N_{\mu_1\mu_2\mu_3\ldots\mu_n} + a_2 N_{\mu_n\mu_1\mu_2\ldots\mu_{n-1}} + \cdots + a_n N_{\mu_2,\mu_3,\ldots\mu_n\mu_1} + \ldots}_{n!-terms} = B_{\mu_1\mu_2\mu_3\ldots\mu_n}, \tag{14}$$

where a_i, $i = 1, 2, \ldots, n$ are scalars, $B_{\mu_1\mu_2\ldots\mu_n}$ are the components of a given (known) tensor and $N_{\mu_1\mu_2\mu_3\ldots\mu_n}$ are the components of the unknown tensor N of rank n. Define the square $n! \times n!$ matrix

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n!} \\ a_2 & a_n & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+1} & a_{n!} & \cdots & a_2. \end{pmatrix} \tag{15}$$

Given that the system is non-degenerate, that is, $\det A \neq 0$, *then the general and unique solution of (14) is given by:*

$$N_{\mu_1\mu_2\mu_3\ldots\mu_n} = \tilde{a}_{11}B_{\mu_1\mu_2\mu_3\ldots\mu_n} + \cdots + \tilde{a}_{1n}B_{\mu_2\mu_1\mu_3\ldots\mu_n}, \tag{16}$$

where the $\tilde{a}'_{1i}s$ are the first row elements of the inverse matrix A^{-1}.

Proof. In an identical manner to the proof of Theorem 1, we now start from (14) and perform the $(n! - 1)$ possible independent permutations to end up with the system of $n!$ equations[6]

$$a_1 N_{\mu_1\mu_2\mu_3\ldots\mu_n} + a_2 N_{\mu_n\mu_1\mu_2\ldots\mu_{n-1}} + \cdots + a_n N_{\mu_2,\mu_3,\ldots\mu_n\mu_1} + \ldots. = B_{\mu_1\mu_2\mu_3\ldots\mu_n}$$
$$a_1 N_{\mu_2\mu_3\mu_4\ldots\mu_1} + a_2 N_{\mu_1\mu_2\mu_3\ldots\mu_n} + \cdots + a_n N_{\mu_3\mu_4\ldots\mu_1\mu_2} + \ldots. = B_{\mu_2\mu_3\mu_4\ldots\mu_1}$$
$$\ldots \tag{17}$$
$$\ldots$$
$$\ldots$$
$$\ldots$$
$$(all - possible - permutations) \tag{18}$$

We then define the square $n! \times n!$ matrix:

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n!} \\ a_2 & a_n & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+1} & a_{n!} & \cdots & a_2 \end{pmatrix} \tag{19}$$

and the columns

$$\mathcal{N} := (N_{\mu_1\mu_2\mu_3\ldots\mu_n}, N_{\mu_n\mu_1\mu_2\ldots\mu_{n-1}}, \ldots.)^T \tag{20}$$

$$\mathcal{B} := (B_{\mu_1\mu_2\mu_3\ldots\mu_n}, B_{\mu_n\mu_1\mu_2\ldots\mu_{n-1}}, \ldots.)^T, \tag{21}$$

we may express the above system in the matrix form:

$$A\mathcal{N} = \mathcal{B}. \tag{22}$$

As in Theorem 1, we then formally multiply the above equation by A^{-1} from the left to get:

$$\mathcal{N} = A^{-1}\mathcal{B} \tag{23}$$

and by equating the first row element of the left and right hand sides of the above we arrive at the stated result:

$$N_{\mu_1\mu_2\mu_3\ldots\mu_n} = \tilde{a}_{11} B_{\mu_1\mu_2\mu_3\ldots\mu_n} + \tilde{a}_{12} B_{\mu_n\mu_1\ldots\mu_{n-1}} \cdots + \tilde{a}_{1n} B_{\mu_2\mu_1\mu_3\ldots\mu_n}, \tag{24}$$

where $\tilde{a}_{11}, \tilde{a}_{11}, \ldots, \tilde{a}_{1n}$ are the elements of the first row of the inverse matrix A^{-1}. □

Remark 1. *Again, if the tensor N has some symmetry property over some pair(s) of its indices, the dimension of the matrix A will be lowered accordingly.*

Now, going back to the case of a rank 3 tensor, one may ask how does the situation change when the traces of $N_{\alpha\mu\nu}$ also appear in (3). Defining the three traces,

$$N^{(1)}_\mu := N_{\alpha\beta\mu} g^{\alpha\beta} \;\; , \;\; N^{(2)}_\mu := N_{\alpha\mu\beta} g^{\alpha\beta} \;\; , \;\; N^{(3)}_\mu := N_{\mu\alpha\beta} g^{\alpha\beta}, \tag{25}$$

the generalized version of (3), still linear in N, including the above traces reads

$$a_1 N_{\alpha\mu\nu} + a_2 N_{\nu\alpha\mu} + a_3 N_{\mu\nu\alpha} + a_4 N_{\alpha\nu\mu} + a_5 N_{\nu\mu\alpha} + a_6 N_{\mu\alpha\nu} + \sum_{i=1}^{3} \left(a_{7i} N^{(i)}_\mu g_{\alpha\nu} + a_{8i} N^{(i)}_\nu g_{\alpha\mu} + a_{9i} N^{(i)}_\alpha g_{\mu\nu} \right) = B_{\alpha\mu\nu}. \tag{26}$$

As we show below the appearance of the these extra terms does not introduce any serious technical difficulty and one can always solve for $N_{\alpha\mu\nu}$ in terms of a modified version of $B_{\alpha\mu\nu}$ that includes its traces. We have the following result.

Theorem 3. *Consider the 15 parameter linear tensor Equation (26), where $N_{\alpha\mu\nu}$ are the components of the unknown tensor field. Define the matrices*[7]

$$\Gamma := \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix} \tag{27}$$

and

$$A := \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_3 & a_1 & a_2 & a_5 & a_6 & a_4 \\ a_2 & a_3 & a_1 & a_6 & a_4 & a_5 \\ a_4 & a_6 & a_5 & a_1 & a_3 & a_2 \\ a_5 & a_4 & a_6 & a_2 & a_1 & a_3 \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1. \end{pmatrix} \tag{28}$$

Then, given that both of the above matrices are non-singular, the unique solution to (26) reads:

$$N_{\alpha\mu\nu} = \tilde{a}_{11} B_{\alpha\mu\nu} + \tilde{a}_{12} \hat{B}_{\nu\alpha\mu} + \tilde{a}_{13} \hat{B}_{\mu\nu\alpha} + \tilde{a}_{14} \hat{B}_{\alpha\nu\mu} + \tilde{a}_{15} \hat{B}_{\nu\mu\alpha} + \tilde{a}_{16} \hat{B}_{\mu\alpha\nu}, \tag{29}$$

where

$$\hat{B}_{\alpha\mu\nu} = B_{\alpha\mu\nu} - \sum_{i=1}^{3}\sum_{j=1}^{3} \left(a_{7i}\tilde{\gamma}_{ij} B^{(j)}_{\mu} g_{\alpha\nu} + a_{8i}\tilde{\gamma}_{ij} B^{(j)}_{\nu} g_{\alpha\mu} + a_{9i}\tilde{\gamma}_{ij} B^{(j)}_{\alpha} g_{\mu\nu} \right). \tag{30}$$

Proof. We begin by tracing Equation (26) three times independently with $g^{\alpha\mu}$, $g^{\alpha\nu}$ and $g^{\mu\nu}$ to get:

$$\sum_{i=1}^{3} \gamma_{1i} N^{(i)}_{\mu} = B^{(1)}_{\mu} \ , \ \sum_{i=1}^{3} \gamma_{2i} N^{(i)}_{\mu} = B^{(2)}_{\mu} \ , \ \sum_{i=1}^{3} \gamma_{3i} N^{(i)}_{\mu} = B^{(3)}_{\mu}, \tag{31}$$

where we have renamed all free indices to μ and the $\gamma'_{ij}s$ are some shorthand notations for certain linear combinations of the $a'_i s$, whose relations are given in the Appendix A. Now, defining the matrix Γ with coefficients the $\gamma'_{ij}s$ along with the columns $\eta = (N^{(1)}_{\mu}, N^{(2)}_{\mu}, N^{(3)}_{\mu})^T$ and $b = (B^{(1)}_{\mu}, B^{(2)}_{\mu}, B^{(3)}_{\mu})^T$ we may express the above system in matrix form as:

$$\Gamma\eta = b. \tag{32}$$

By hypothesis, the matrix Γ is non-singular (i.e., $det(\Gamma) \neq 0$) and therefore the inverse Γ^{-1} exists and we may formally solve for η as

$$\eta = \Gamma^{-1} b, \tag{33}$$

which in component notation translates to

$$N^{(i)}_{\mu} = \sum_{j=1}^{3} \tilde{\gamma}_{ij} B^{(i)}_{\mu} \tag{34}$$

with the $\tilde{\gamma}'_{ij}s$ being the elements of Γ^{-1}. Then, substituting these last relations back in (26), we fully eliminate the N traces in favour of the traces of B, ending up with

$$a_1 N_{\alpha\mu\nu} + a_2 N_{\nu\alpha\mu} + a_3 N_{\mu\nu\alpha} + a_4 N_{\alpha\nu\mu} + a_5 N_{\nu\mu\alpha} + a_6 N_{\mu\alpha\nu} = \hat{B}_{\alpha\mu\nu}, \tag{35}$$

where

$$\hat{B}_{\alpha\mu\nu} = B_{\alpha\mu\nu} - \sum_{i=1}^{3}\sum_{j=1}^{3} \left(a_{7i}\tilde{\gamma}_{ij} B^{(j)}_{\mu} g_{\alpha\nu} + a_{8i}\tilde{\gamma}_{ij} B^{(j)}_{\nu} g_{\alpha\mu} + a_{9i}\tilde{\gamma}_{ij} B^{(j)}_{\alpha} g_{\mu\nu} \right). \tag{36}$$

We are pretty much done now since we can apply the result of Theorem 1 to the modified tensor field $\hat{B}_{\alpha\mu\nu}$ in place of $B_{\alpha\mu\nu}$, completing therefore the proof

$$N_{\alpha\mu\nu} = \tilde{a}_{11}\hat{B}_{\alpha\mu\nu} + \tilde{a}_{12}\hat{B}_{\nu\alpha\mu} + \tilde{a}_{13}\hat{B}_{\mu\nu\alpha} + \tilde{a}_{14}\hat{B}_{\alpha\nu\mu} + \tilde{a}_{15}\hat{B}_{\nu\mu\alpha} + \tilde{a}_{16}\hat{B}_{\mu\alpha\nu}, \tag{37}$$

where the modified components $\hat{B}_{\alpha\mu\nu}$ are given by (36). □

3. Conclusions

We have formulated an analytical method that allows one to solve tensorial equations of the form (1), for the unknown tensor components $N_{\alpha\mu\nu}$ of the rank-3 tensor N. In particular, we proved that under a fairly general non-degeneracy condition (i.e., $\det(A) \neq 0$) one can always solve equations of the form ((1) by simply finding the inverse of the matrix A, which is built from the coefficients a_i appearing in the same equation. Subsequently, we generalized our result for arbitrary rank tensors and similarly obtained the solution of (14). Finally, we extended the result we obtained for the first case (Equation (1)) to the 15 parameter linear tensor Equation (26), including also the traces of $N_{\alpha\mu\nu}$ and obtained the unique solution for this case as well.

As we already mentioned in the introduction, these results find a natural application in geometric extensions of General Relativity that take into account the non-Riemannian structure of spacetime (torsion and non-metricity). In this context of Metric-Affine Theories of Gravitation, equations of the form (1), or more generally (26), relate the distortion tensor to the hypermomentum (source). Therefore, the technique we developed here will be proven to be essential for finding how the matter sources produce spacetime torsion and non-metricity. This last point is under consideration now.

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Appendix A. The $\gamma's$

The relations between the elements of Γ and the parameters a_i read

$$\gamma_{11} = a_1 + a_3 + a_{71} + na_{81} + a_{91} \;,\; \gamma_{12} = a_2 + a_4 + a_{72} + na_{82} + a_{92} \;,\; \gamma_{13} = a_5 + a_6 + a_{73} + na_{83} + a_{93}$$
$$\gamma_{21} = a_2 + a_5 + na_{71} + a_{81} + a_{91} \;,\; \gamma_{22} = a_1 + a_6 + na_{72} + a_{82} + a_{92} \;,\; \gamma_{23} = a_3 + a_4 + na_{73} + a_{83} + a_{93}$$
$$\gamma_{31} = a_5 + a_6 + a_{71} + a_{81} + na_{91} \;,\; \gamma_{32} = a_3 + a_4 + a_{72} + a_{82} + na_{92} \;,\; \gamma_{31} = a_1 + a_2 + a_{73} + a_{83} + na_{93}$$

Note that here we used n instead of d to specify the space dimension.

Appendix B. The Determinant of A

For the matrix A as given by (4), after some factorizations, its determinant is found to be (the use of Wolfram Mathematica [13] makes things easier here)

$$det(A) = \sigma_1\sigma_2\sigma_3^2 \tag{A1}$$

with

$$\sigma_1 = \sum_{i=1}^{6} a_i \tag{A2}$$

$$\sigma_2 = \sum_{i=1}^{3}(a_i - a_{i+3}) \tag{A3}$$

$$\sigma_3 = \sum_{i=1}^{3}(a_i^2 - a_{i+3}^2) - \sum_{i=1}^{3}\sum_{j=1,\, j>i}^{3}(a_i a_j - a_{i+3}a_{j+3}) \tag{A4}$$

Note that the determinant is of 6th order on the $a_i's$ as expected. Now, in order for the matrix to be non-singular all four sums above must be non-zero at the same time. If one (or more) of those sums is zero, this implies that in (3) a certain (anti) symmetrization occurs in $N_{\alpha\mu\nu}$ which in turn means that the latter equation can only give certain parts of $N_{\alpha\mu\nu}$ and not the full tensor. We see therefore that non-degeneracy is essential in order to obtain the full tensor N. Below we give a trivial example where such a degeneracy occurs.

Appendix C. Examples

Example 1.

Consider the equation

$$N_{\alpha\mu\nu} - N_{\mu\alpha\nu} = B_{\alpha\mu\nu} \tag{A5}$$

Here we have $a_1 = -a_3 = 1$ and $a_2 = a_4 = a_5 = a_6 = 0$ and as a result $\sigma_1 = 0$ meaning that the matrix A is singular and as a result Equation (3) is not enough to specify all the components $N_{\alpha\mu\nu}$. Of course in this example the incapability of (3) to fully specify all the components N of was obvious since we could write the above as $2N_{[\alpha\mu]\nu} = B_{\alpha\mu\nu}$ meaning that only the antisymmetric part in the first indices of N can be obtained. In more complicated cases, however, it would be quite difficult to spot certain symmetrizations that might occur, especially for the generalized version (14). In these cases the determinant criterion would be of great use in determining whether the given tensor equation can give the components of the full N tensor or not.

Example 2.

Let us now apply the result of our first Theorem in a trivial example one encounters in introductory courses of tensor calculus. There, the metric compatibility condition implies

$$\Gamma_{\nu\mu\alpha} + \Gamma_{\mu\nu\alpha} = \partial_\alpha g_{\mu\nu} \quad , \quad where \ \ \Gamma_{\nu\mu\alpha} := g_{\lambda\nu}\Gamma^{\lambda}{}_{\mu\alpha} \tag{A6}$$

which along with the torsionlessness of the connection $\Gamma^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\nu\mu}$ give us the usual Levi-Civita form of the connection. Recall that the trick to solve for $\Gamma^{\lambda}{}_{\mu\nu}$ there was to consider two subsequent cyclic permutations of the (A6) and subtract them from the latter. Let us reproduce this result here by applying the Theorem 1. In this case, as we have already mentioned the fact that $\Gamma_{\nu\mu\alpha}$ is symmetric in its last two indices, reduces A to a 3×3 matrix and we might as well set $a_4 = a_5 = a_6 = 0$. Now from (A6) we read off the coefficients $a_1 = 0$, $a_2 = a_3 = 1$ and as a result

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \tag{A7}$$

From which we see that $\det(A) = -2 \neq 0$ and we straightforwardly calculate the first row elements of the inverse matrix to be $\tilde{a}_{11} = -1/2$, $\tilde{a}_{12} = \tilde{a}_{13} = 1/2$. Then substituting these into (13) and with the identifications[8] $N_{\alpha\mu\nu} = \Gamma_{\alpha\mu\nu}$ and $B_{\alpha\mu\nu} = \partial_\alpha g_{\mu\nu}$ we arrive at the well known result

$$\Gamma_{\alpha\mu\nu} = \frac{1}{2}\Big(-\partial_\alpha g_{\mu\nu} + \partial_\nu g_{\mu\alpha} + \partial_\mu g_{\nu\alpha}\Big) \tag{A8}$$

for the Levi-Civita connection. Of course in this case the same result can be obtained trivially by the classical method we mentioned above. Our intention with these examples here is to illustrate how our general method works. Next we consider a more advanced application.

Example 3.

Probably the most useful application of our Theorem to physical systems is the analysis of the connection field equations in Metric-Affine Gravity. There our method will be proven to be all essential in solving for torsion and non-metricity in terms of their sources. To briefly further expand on this last point let us consider a subclass of the 11 parameter quadratic MAG Theory consisting of the usual Einstein-Hilbert term and the three quadratic torsion terms[9]. The latter reads

$$S[g,\Gamma,\Phi]=\frac{1}{2\kappa}\int d^n x\sqrt{-g}\Big[R+b_1 S_{\alpha\mu\nu}S^{\alpha\mu\nu}+b_2 S_{\alpha\mu\nu}S^{\mu\nu\alpha}+b_3 S_\mu S^\mu\Big]+S_M[g,\Gamma,\Phi] \tag{A9}$$

In the above $S_{\mu\nu}{}^{\lambda}:=\Gamma^{\lambda}{}_{[\mu\nu]}$ is the torsion tensor, b_i are dimensionless parameters and Φ collectively denotes the matter fields. Our definitions for the various geometrical and physical objects will be those of [1]. For convenience let us also mention that our definition for non-metricity is $Q_{\alpha\mu\nu}:=-\nabla_\alpha g_{\mu\nu}$ and the affine connection can be split according to

$$\Gamma^{\lambda}{}_{\mu\nu}=\widetilde{\Gamma}^{\lambda}{}_{\mu\nu}+\frac{1}{2}g^{\alpha\lambda}(Q_{\mu\nu\alpha}+Q_{\nu\alpha\mu}-Q_{\alpha\mu\nu})-g^{\alpha\lambda}(S_{\alpha\mu\nu}+S_{\alpha\nu\mu}-S_{\mu\nu\alpha})=\widetilde{\Gamma}^{\lambda}{}_{\mu\nu}+N^{\lambda}{}_{\mu\nu} \tag{A10}$$

where $\widetilde{\Gamma}^{\lambda}{}_{\mu\nu}$ is the usual Levi-Civita connection and $N^{\lambda}{}_{\mu\nu}$ is the so-called distortion tensor representing how much the general connection deviates from the Levi-Civita. In terms of the latter, torsion and non-metricity are readily computed by

$$S_{\mu\nu\alpha}=N_{\alpha[\mu\nu]}\;\;,\quad Q_{\nu\alpha\mu}=2N_{(\alpha\mu)\nu} \tag{A11}$$

Out of torsion and non-metricity we can construct the following vectors

$$S_\mu:=S_{\mu\lambda}{}^{\lambda}\;\;,\quad Q_\alpha:=Q_{\alpha\mu\nu}g^{\mu\nu}\;,\;\; q_\nu=Q_{\alpha\mu\nu}g^{\alpha\mu} \tag{A12}$$

and from distortion

$$N^{(1)}_\mu:=N_{\alpha\beta\mu}g^{\alpha\beta}\;\;,\;\;N^{(2)}_\mu:=N_{\alpha\mu\beta}g^{\alpha\beta}\;\;,\;\;N^{(3)}_\mu:=N_{\mu\alpha\beta}g^{\alpha\beta} \tag{A13}$$

Let us now come back to our example. After varying action (A9) with respect to the independent affine connection we get

$$\begin{aligned}\Big(\frac{Q_\lambda}{2}+2S_\lambda\Big)g^{\mu\nu}-Q_\lambda{}^{\mu\nu}-2S_\lambda{}^{\mu\nu}+\Big(q^\mu-\frac{Q^\mu}{2}-2S^\mu\Big)\delta^\nu_\lambda+4a_1Q^{\nu\mu}{}_\lambda+2a_2(Q^{\mu\nu}{}_\lambda+Q_\lambda{}^{\mu\nu})\\ +2b_1S^{\mu\nu}{}_\lambda+2b_2S_\lambda{}^{[\mu\nu]}+2b_3S^{[\mu}\delta^{\nu]}_\lambda=\kappa\Delta_\lambda{}^{\mu\nu}\end{aligned} \tag{A14}$$

where

$$\Delta_\lambda{}^{\mu\nu}:=-\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta\Gamma^\lambda{}_{\mu\nu}} \tag{A15}$$

is the hypermomentum source that encodes the microstructure of matter. Ideally one wishes to solve the above equation for torsion and non-metricity in terms of the sources (hypermomentum). Even though the situation seems a little complicated with a small trick we will see how our results here may be applied to provide the solution effortlessly. The subtle point is to observe that on using Equation (A12) and their contractions we may express the connection field equations solely in terms of the distortion. A straightforward calculation reveals

$$\begin{aligned}b_1N_{\alpha\mu\nu}+\Big(-1+\frac{b_2}{2}\Big)N_{\nu\alpha\mu}+\Big(-1+\frac{b_2}{2}\Big)N_{\mu\nu\alpha}\\ -b_1N_{\alpha\nu\mu}-\frac{b_2}{2}N_{\nu\mu\alpha}-\frac{b_2}{2}N_{\mu\alpha\nu}-\frac{b_3}{2}g_{\nu\alpha}N^{(1)}_\mu+\frac{b_3}{2}g_{\nu\alpha}N^{(2)}_\mu+g_{\nu\alpha}N^{(3)}_\mu\\ +\frac{b_3}{2}g_{\mu\alpha}N^{(1)}_\nu-\frac{b_3}{2}g_{\mu\alpha}N^{(2)}_\mu+g_{\mu\nu}N^{(2)}_\alpha=\kappa\Delta_{\alpha\mu\nu}\end{aligned} \tag{A16}$$

We see now that the above equation falls exactly into the category of Theorem 3. Then, with the obvious identifications among the parameters of the theory and those of Equation (26) by applying Theorem 3 and setting $B_{\alpha\mu\nu} = \kappa\Delta_{\alpha\mu\nu}$, we find the exact solution for the distortion

$$N_{\alpha\mu\nu} = \kappa\tilde{a}_{11}\hat{\Delta}_{\alpha\mu\nu} + \kappa\tilde{a}_{12}\hat{\Delta}_{\nu\alpha\mu} + \kappa\tilde{a}_{13}\hat{\Delta}_{\mu\nu\alpha} + \kappa\tilde{a}_{14}\hat{\Delta}_{\alpha\nu\mu} + \kappa\tilde{a}_{15}\hat{\Delta}_{\nu\mu\alpha} + \kappa\tilde{a}_{16}\hat{\Delta}_{\mu\alpha\nu} \tag{A17}$$

where

$$\hat{\Delta}_{\alpha\mu\nu} = \Delta_{\alpha\mu\nu} - \sum_{i=1}^{3}\sum_{j=1}^{3}\left(a_{7i}\tilde{\gamma}_{ij}\Delta_{\mu}^{(j)}g_{\alpha\nu} + a_{8i}\tilde{\gamma}_{ij}\Delta_{\nu}^{(j)}g_{\alpha\mu} + a_{9i}\tilde{\gamma}_{ij}\Delta_{\alpha}^{(j)}g_{\mu\nu}\right) \tag{A18}$$

Then, on substituting the latter into (A10) and (A12) we find the exact solutions of the affine-connection, torsion and non-metricity in therms of the hypermomentum sources. We see therefore how powerful and useful the Theorems we provided here are in analysing and simplifying the study of Metric-Affine Theories of Gravity.

Notes

1 Third order tensors appear also in mechanics, see for instance [4]. Some other recent advances in Mathematical Physics include [5–7].

2 For decompositions of rank-3 tensors see [10] and for the geometric picture, one may consult [11]. In addition, a nice review on tensor calculus can be found in [12].

3 Of course, the result holds true even when $N_{\alpha\mu\nu}$ are the components of a tensor density instead, or even of a connection given that $B_{\alpha\mu\nu}$ are also of the same kind.

4 A necessary condition for this to happen is that $\sum_{i=1}^{6} a_i \neq 0$. However, this condition alone is not sufficient, since the latter quantity can be non-vanishing but it may be so that the full determinant still vanishes. See Appendices A–C for more details on this feature.

5 This is easily realized as follows. Without loss of generality let us suppose that N is symmetric in its first two indices, i.e., $N_{\alpha\mu\nu} = N_{\mu\alpha\nu}$. Then, with this relation and circle permutations of it, it is trivial to see that only three combinations of N appear in (3) and, as a result, the system reduces to a 3×3. Of course, the same goes also when N is antisymmetric in any pair of its indices.

6 The first one is Equation (14) itself.

7 The elements γ_{ij} are linear combinations of the parameters a_i and their exact relations are given in the Appendices A–C.

8 Recall that our result holds true not only for tensor but also for tensor densities and connection coefficients as well.

9 Here for the sake of illustration (of applications) we restrict ourselves to the case where only three quadratic pieces are included. The full 11 parameter Theory will be studied elsewhere.

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