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On the Vacuum Structure of the $\mathcal{N}=4$ Conformal Supergravity

Ioannis Dalianis [1,2,*], Alex Kehagias [1], Ioannis Taskas [1] and George Tringas [1]

[1] Physics Division, Zografou Campus, National Technical University of Athens, 15780 Athens, Greece; kehagias@central.ntua.gr (A.K.); taskas@mail.ntua.gr (I.T.); georgiostringas@mail.ntua.gr (G.T.)
[2] Physics Department, University Campus, National and Kapodistrian University of Athens, 15784 Zografou, Greece
* Correspondence: idalianis@phys.uoa.gr

Abstract: We consider $\mathcal{N}=4$ conformal supergravity with an arbitrary holomorphic function of the complex scalar S which parametrizes the $SU(1,1)/U(1)$ coset. Assuming non-vanishings vevs for S and the scalars in a symmetric matrix E_{ij} of the $\overline{\mathbf{10}}$ of $SU(4)$ R-symmetry group, we determine the vacuum structure of the theory. We find that the possible vacua are classified by the number of zero eigenvalues of the scalar matrix and the spacetime is either Minkowski, de Sitter, or anti-de Sitter. We determine the spectrum of the scalar fluctuations and we find that it contains tachyonic states which, however, can be removed by appropriate choice of the unspecified at the supergravity level holomorphic function. Finally, we also establish that S-supersymmetry is always broken whereas Q-supersymmetry exists only on flat Minkowski spacetime.

Keywords: supergravity; conformal symmetry; weyl gravity; conformal supergravity

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1. Introduction

Conformal supergravity is the supersymmetric completion of conformal or Weyl gravity, described by the Weyl square term. It is invariant under the full superconformal group, which is the supergroup $SU(2,2|\mathcal{N})$, the real form of $SL(4|\mathcal{N})$, where $\mathcal{N}$ counts the number of supersymmetries. This number cannot be larger than four ($\mathcal{N}\leq 4$) since otherwise, among others, the theory will contain higher spin fields [1]. The bosonic part of the above supergroup is $SO(2,4)\times U(\mathcal{N})$ if $\mathcal{N}\neq 4$ or $SO(2,4)\times SU(4)$ if $\mathcal{N}=4$, whereas the fermionic generators are in the $(4,\mathcal{N})+(\bar{4},\bar{\mathcal{N}})$ for any $\mathcal{N}$. In other words, the superconformal group contains the standard generators of the conformal group (rotations—M, translations—P, conformal boosts—K, and dilations—D) as well as the usual Q and the special S supersymmetry.

Conformal supergravity employs the Weyl multiplet which is a unique off-shell multiplet and has fewer fields than the corresponding Poincaré gravity multiplet. The reason is the high degree of symmetry since the local Weyl symmetry implies that certain modes should be absent [2,3]. Four dimensional conformal supergravity actions were known for a long time for $\mathcal{N}<4$ [4–10] and the full off-shell action for the $\mathcal{N}=4$ has also been recently found [11–13]. Part of the bosonic sector of the theory has previously been obtained in [9] by utilizing the conformal anomaly of $\mathcal{N}=4$ vector multiplets while four dimensional $\mathcal{N}=4$ solutions without conformal symmetry were studied in [14,15].

Conformal supergravity can also be obtained as the massless limit $m\to 0$ of the supersymmetric completion of $m^2R+Weyl^2$ gravity [16–18] (see also [19–22]). In general, although such theories contain ghost propagating states [23–25], they are interesting as they arise in the twistor-string theory via closed strings or gauge singlet open strings [26]. It is interesting that in the $m\to 0$ limit, the spectrum is re-organized so that the symmetry is enhanced from super Poincaré to superconformal, whereas, at the same time R-symmetries get promoted to local gauge symmetries. Let us notice that there are higher curvature supergravities with physical spectrum [27–32] and a rich vacuum structure [33].

A particular feature of conformal supergravity is that, although the $\mathcal{N} < 4$ theories are unique, the $\mathcal{N} = 4$ one is not. In fact, the $\mathcal{N} = 4$ theory contains a dimensionless scalar ϕ^α, $(\alpha = 1, 2)$ that parametrize the coset $SU(1,1)/U(1)$. The $U(1)$ is realized as a local symmetry. The corresponding gauge field is composite with chiral action on the fermions and therefore, the R-symmetry group is enhanced to $SU(4) \times U(1)$.

Then, the Weyl square term, among others in the supersymmetric action, is multiplied by a holomorphic function $\mathcal{H}(\phi^\alpha)$. The existence of this ambiguity of the $\mathcal{N} = 4$ conformal supergravity was known for some time [34] and it has been explicitly worked out in [12,13]. In this work our aim is to uncover the vacuum structure of the $\mathcal{N} = 4$ conformal supergravity where scalar fields are also excited looking for maximally symmetric background solutions.

The structure of the paper is as follows: In Section 2, we describe the spectrum of the $\mathcal{N} = 4$ conformal supergravity and the corresponding action. In Section 3, we explore the vacuum structure of the $\mathcal{N} = 4$ theory, and finally, we conclude in Section 4.

2. Spectrum and Action

In order to establish notation, let us recall the spectrum of the $\mathcal{N} = 4$ confromal supergravity. Greek letters $\mu, \nu, \ldots$ denote space-time indices, $a, b, \ldots$ denote tangent space indices, and $i, j, \ldots$ are $SU(4)$ indices. The bosonic sector contains the vierbein e^a_μ, the $SU(4)$ gauge field ${V_\mu}^i{}_j$, and the gauge field b_μ which gauges the dilatations. There are also composite gauge fields describing the spin connection ${\omega_\mu}^a{}_b$, the gauge field f^a_μ associated to conformal boosts, and the composite $U(1)$ gauge field A_μ. The bosonic sector is completed with a complex anti-self-dual tensor field ${T_{ab}}^{ij}$ which is in the **6** of $SU(4)$, the complex scalars E^{ij} in the **10** and the auxiliary pseudoreal scalars ${D_{ij}}^{kl}$ in the $\mathbf{20}'$ of $SU(4)$. Finally, the bosonic sector is completed by the scalars ϕ^α $(\alpha = 1, 2)$ which parametrize the coset $SU(1,1)/U(1)$. They are invariant under dilatations and transform as a doublet under $SU(1,1)$ global transformations. The conditions and the constraints these fields satisfy are[1]

$$
\begin{aligned}
&{T_{ab}}^{ij} = -{T_{ba}}^{ij} = -{T_{ab}}^{ji}, \quad {T_{ab}}^{ij} = -\frac{1}{2}\epsilon_{ab}{}^{cd}{T_{cd}}^{ij}, \\
&D^{ij}{}_{kl} = \frac{1}{4}\epsilon^{ijmn}\epsilon_{klpq}D^{pq}{}_{mn}, \quad E^{ij} = E^{ji}, \quad E_{ij} = \left(E^{ij}\right)^*, \\
&\phi^\alpha\phi_\alpha = 1, \quad \phi_1 = (\phi^1)^*, \quad \phi_2 = -(\phi^2)^*.
\end{aligned}
\tag{1}
$$

The fields of the $\mathcal{N} = 4$ conformal supergravity are completed by the positive chirality fermions which are the gravitini ψ^i_μ, the S-supersymmetry composite ϕ^i_μ, and the two spinor fields, Λ_i in the **4** and ${\chi^{ij}}_k$ in the **20** of $SU(4)$. The supersymmetry transformations of the $\mathcal{N} = 4$ fields can be found in [8].

The full off-shell action of the $\mathcal{N} = 4$ conformal supergravity has been constructed in [13]. The pure gravitational part contains the Weyl contribution

$$
e^{-1}\mathcal{L} = \frac{1}{2}(\mathcal{H} + \overline{\mathcal{H}})W^{\mu\nu\kappa\lambda}W_{\mu\nu\kappa\lambda} + \cdots, \tag{2}
$$

where $\mathcal{H}$ is a holomorphic function of ϕ^α. The equations of motion for vanishing fields ${T_{ab}}^{ij}$, E^{ij}, $D^{ik}{}_{kl}$ and constant ϕ^α are

$$
B_{\mu\nu} = 0, \quad (\partial_\alpha\mathcal{H} + \partial_{\bar{\alpha}}\overline{\mathcal{H}})W^{\mu\nu\kappa\lambda}W_{\mu\nu\kappa\lambda} = 0, \tag{3}
$$

where $\partial_\alpha = \partial/\partial\phi^\alpha$, $\partial_{\bar{\alpha}} = \partial/\partial\phi^{\bar{\alpha}}$ and

$$
B_{\mu\nu} = \nabla^\rho\nabla_\sigma W^\sigma{}_{\mu\rho\nu} + \frac{1}{2}R^{\rho\sigma}W_{\rho\mu\sigma\nu} \tag{4}
$$

is the Bach tensor. Clearly, conformally flat backgrounds are solutions of the equations of motion Equation (3). In particular, Minkowski, de Sitter, and Anti-de Sitter spacetimes are maximally symmetric vacuum solutions. These solutions are however, trivial in the sense that they do not involve any field other than the vierbein and are indistinguishable in the Weyl theory (they are all maximally symmetric and have vanishing Weyl tensor). Our aim here is to uncover (part of) the vacuum structure of the $\mathcal{N} = 4$ conformal supergravity where scalar fields are also excited. Since we are looking for maximally symmetric backgrounds we will only assume non-vanishing scalars E^{ij} and ϕ^α, since a non-vanishing tensor field $T_{ab}{}^{ij}$ will in general reduce the background symmetry. In this case, the relevant bosonic part of the action, in the $b_\mu = 0$ gauge, is

$$\begin{aligned}\mathcal{L} = &+ \mathcal{H}\Big[\frac{1}{2}W^{\mu\nu\rho\sigma}W_{\mu\nu\rho\sigma} + \frac{1}{4}E_{ij}D_\mu D^\mu E^{ij} + \frac{1}{8}D^{ij}{}_{kl}D^{kl}{}_{ij} - \frac{1}{16}E_{ij}E^{jk}E_{kl}E^{li} + \frac{1}{48}(E_{ij}E^{ij})^2\Big] \\ &+ \mathcal{D}\mathcal{H}\Big[+\frac{1}{16}D^{ij}{}_{kl}(E_{im}E_{jn}\epsilon^{klmn})\Big] + \frac{\mathcal{D}^2\mathcal{H}}{384}E_{ij}E_{kl}E_{mn}E_{pq}\epsilon^{ikmp}\epsilon^{jlnq} + h.c.\,,\end{aligned} \tag{5}$$

where D_μ is the (super)conformal covariant derivative. The fields $D^{ij}{}_{kl}$ are auxiliaries as they appear only algebraically in Equation (5). Integrating them out by using their equation of motion

$$D^{ij}{}_{kl} = -\frac{1}{4}\frac{\mathcal{D}\mathcal{H}}{\mathcal{H}}E_{km}E_{ln}\epsilon^{ijmn}, \tag{6}$$

we find that the lagrangian in Equation (5) is written as

$$\begin{aligned}\mathcal{L} =&\mathcal{H}\Big[\frac{1}{2}W^{\mu\nu\rho\sigma}W_{\mu\nu\rho\sigma} - \frac{1}{4}\nabla_\mu E_{ij}\nabla^\mu E^{ij} - \frac{1}{24}RE_{ij}E^{ij} - \frac{1}{16}E_{ij}E^{jk}E_{kl}E^{li} + \frac{1}{48}(E_{ij}E^{ij})^2\Big] \\ &- \frac{1}{64}\frac{(\mathcal{D}\mathcal{H})^2}{\mathcal{H}}E_{ij}E_{kl}E_{mn}E_{pq}\epsilon^{ikmp}\epsilon^{jlnq} + \frac{\mathcal{D}^2\mathcal{H}}{384}E_{ij}E_{kl}E_{mn}E_{pq}\epsilon^{ikmp}\epsilon^{jlnq} + h.c.\end{aligned} \tag{7}$$

Note that apart from the usual Weyl square term, the scalars E_{ij} are conformally coupled to the curvature in the standard way. However, in order gravity to be attractive in the infrared, vector multiplets coupled to the supergravity multiplet are needed [18,35–37].

We are interested in maximally symmetric vacuum solutions (Minkowski, de Sitter, or anti-de Sitter) so that

$$R_{\mu\nu\rho\sigma} = \Lambda\big(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\big)\,, \quad E_{ij} = \text{const.}\,, \quad \phi^\alpha = \text{const.}\,, \tag{8}$$

where Λ is the cosmological constant which, in our conventions, it is positive, negative, or zero for de Sitter, anti-de Sitter, or Minkowski spacetime, respectively. In this case, the equations of motion which follow from the Lagrangian of Equation (7) are

$$AB_{\mu\nu} + C\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \frac{1}{2}Vg_{\mu\nu} = 0\,, \tag{9}$$

$$R\partial_I C - \partial_I V = 0\,, \tag{10}$$

where the index I enumerates collectively all scalar fields and $B_{\mu\nu}$ is the Bach tensor defined in Equation (4). We have also introduced the functions

$$A(\phi^\alpha, \phi_\alpha) = \mathcal{H} + \overline{\mathcal{H}}\,, \quad C(\phi^\alpha, \phi_\alpha, E_I) = -\frac{1}{24}A\,E_{ij}E^{ij}\,,$$

$$G(\phi^\alpha, \phi_\alpha) = \Big(\frac{(\mathcal{D}\mathcal{H})^2}{\mathcal{H}} - \frac{\mathcal{D}^2\mathcal{H}}{6}\Big)\,, \tag{11}$$

which, for shorthand, we will refer to them simply as A, C, and G. The scalar potential V turns out to be

$$\begin{aligned} V =& \mathcal{H}\Big(\frac{1}{16}E_{ij}E^{jk}E_{kl}E^{li} - \frac{1}{48}(E_{ij}E^{ij})^2\Big) \\ &+ \frac{1}{64}\Big(\frac{(\mathcal{D}\mathcal{H})^2}{\mathcal{H}} - \frac{\mathcal{D}^2\mathcal{H}}{6}\Big)E_{ij}E_{kl}E_{mn}E_{pq}\epsilon^{ikmp}\epsilon^{jlnq} + h.c. \end{aligned} \tag{12}$$

Since the Bach tensor vanishes for conformally flat geometries as the ones we are after (Minkowski, de Sitter, or anti-de Sitter), Equations (9) and (10) are simply

$$\Lambda = \frac{V}{2C}\,, \qquad 2V\partial_I C - C\partial_I V = 0\,. \tag{13}$$

The first equation of Equation (13) specifies the cosmological constant Λ and the second one can be written as

$$\frac{C^2}{V}\partial_I \frac{V}{C^2} = \frac{1}{V_{eff}}\partial_I V_{eff} = 0\,. \tag{14}$$

Therefore, the vacua of the theory are specified by the extrema of the "effective" potential

$$V_{eff} = \frac{V}{C^2}\,. \tag{15}$$

Our aim in the following is to determine these extrema.

3. Vacuum Solutions

In order to find the vacuum structure we should minimize the effective scalar potential in Equation (15), which is explicitly written as

$$\begin{aligned} V_{eff} = \Big(\frac{24}{(\mathcal{H}+\bar{\mathcal{H}})E_{ij}E^{ij}}\Big)^2 \Big\{&\mathcal{H}\Big(\frac{1}{16}E_{ij}E^{jk}E_{kl}E^{li} - \frac{1}{48}(E_{ij}E^{ij})^2\Big) \\ &+ \frac{1}{64}\Big(\frac{(\mathcal{D}\mathcal{H})^2}{\mathcal{H}} - \frac{\mathcal{D}^2\mathcal{H}}{6}\Big)E_{ij}E_{kl}E_{mn}E_{pq}\epsilon^{ikmp}\epsilon^{jlnq}\Big\} + h.c. \end{aligned} \tag{16}$$

Clearly, we cannot proceed without specifying the exact form of the holomorphic function $\mathcal{H}(\phi^\alpha)$. Therefore, we need to choose the explicit form of $\mathcal{H}(\phi^\alpha)$ as, at this point, it is totally arbitrary, although it is expected to be specified in a more fundamental theory. However, before, we need to elaborate on its properties and structure which will be discussed in the next sections.

3.1. Structure of $\mathcal{H}$

The derivatives $\mathcal{D}$ on the scalar manifold $SU(1,1)/U(1)$ which appear in the lagrangian Equation (5) are defined as [12]

$$\mathcal{D} = -\phi^a\epsilon_{ab}\frac{\partial}{\partial\phi_b}\,, \qquad \overline{\mathcal{D}} = +\phi_a\epsilon^{ab}\frac{\partial}{\partial\phi^b}\,, \qquad \mathcal{D}^0 = \phi^\alpha\frac{\partial}{\partial\phi_a} - \phi_\alpha\frac{\partial}{\partial\phi_\alpha}\,. \tag{17}$$

Clearly with the definitions above, the derivative $\mathcal{D}\mathcal{H}$ of the holomorphic function $\mathcal{H}(\phi^\alpha)$ is also holomorphic ($\bar{\mathcal{D}}\mathcal{D}\mathcal{H} = 0$). Now, due to the constraint $\phi^\alpha\phi_\alpha = 1$, functions of the form $\mathcal{H}(\phi_2/\phi_1)$ are holomorphic. Therefore it is convenient to define the fields S and ψ as

$$S = \frac{\phi_2}{\phi_1}\,, \quad \overline{S} = -\frac{\phi^2}{\phi^1}\,, \quad e^{2i\psi} = \frac{\phi^1}{\phi_1}\,, \tag{18}$$

In this parametrization, the field S parametrize the Poincare disk $0 \leq |S| < 1$ whereas the phase ψ describes the $U(1)$. The derivatives Equation (17) are expressed now as

$$\begin{aligned}\mathcal{D} &= -e^{+2i\psi}\Big((1-S\overline{S})\partial_s + \frac{i}{2}\overline{S}\partial_\psi\Big)\,,\\ \overline{\mathcal{D}} &= (\mathcal{D})^*\,, \quad \mathcal{D}^0 = -i\partial_\psi\,. \end{aligned} \tag{19}$$

Then, the operator $\mathcal{D}^2$ turns out to be

$$\begin{aligned}\mathcal{D}^2 = \overline{S}e^{2i\psi}\mathcal{D} + e^{4i\psi}\Big(&-\overline{S}(1-S\overline{S})\partial_S + \frac{i}{2}\overline{S}(1-S\overline{S})\{\partial_S,\partial_\psi\}\\ &-\frac{1}{4}\overline{S}^2\partial_\psi^2 + (1-S\overline{S})^2\partial_S^2\Big)\,. \end{aligned} \tag{20}$$

One can see that for any choice of $\mathcal{H}(S)$ the first order derivative is also holomorphic $\overline{\mathcal{D}}\mathcal{D}\mathcal{H} = 0$. With the expressions in Equations (19) and (20), we have for the quantities $\mathcal{D}\mathcal{H}$ and $\mathcal{D}^2\mathcal{H}$ that enter in the lagrangian

$$\mathcal{D}\mathcal{H} = -e^{+2i\psi}(1-S\overline{S})\partial_S\mathcal{H}\,, \tag{21}$$

$$\mathcal{D}^2\mathcal{H} = -e^{+4i\psi}(1-S\overline{S})\Big(2\overline{S}\partial_S\mathcal{H} - (1-S\overline{S})\partial_S^2\mathcal{H}\Big)\,, \tag{22}$$

3.2. *Vacua of the $\mathcal{N}=4$ Conformal Supergravity*

The scalars E_{ij} are in the $\overline{\mathbf{10}}$ of $SU(4)$ and therefore can be represented as a complex symmetric 4×4 matrix. One of the simplest configuration is the one where E_{ij} is diagonal and takes therefore the form

$$E_{ij} = \begin{pmatrix} E_1 & 0 & 0 & 0\\ 0 & E_2 & 0 & 0\\ 0 & 0 & E_3 & 0\\ 0 & 0 & 0 & E_4\end{pmatrix} = E_i\delta_{ij} \quad (\text{no summation})\,. \tag{23}$$

In order to proceed now, we will distinguish two different cases for the holomorphic function $\mathcal{H}$: (1) $\mathcal{D}\mathcal{H} = \mathcal{D}\mathcal{H}_0 = 0$, and (2) $\mathcal{D}\mathcal{H} \neq const$. We will examine these cases separately below.

3.2.1. Constant Holomorphic Function

The first case corresponds to a constant homolophic function $\mathcal{H}$. We can take here $\mathcal{H}(S) \equiv \mathcal{H}_0 = \text{const.}$, so that

$$A = \mathcal{H}_0 + \overline{\mathcal{H}}_0 = \frac{1}{\alpha^2}\,, \qquad C = -\frac{1}{24\alpha^2}E_{ij}E^{ij}\,, \qquad G = 0\,. \tag{24}$$

Then, the effective potential in Equation (16) and the cosmological constant in Equation (13) turn out to be

$$V_{eff} = -12\alpha^2\frac{\big(\sum_i |E_i|^2\big)^2 - 3\sum_i |E_i|^4}{\big(\sum_i |E_i|^2\big)^2}\,, \tag{25}$$

and

$$\Lambda = +\frac{1}{4}\frac{\big(\sum_i |E_i|^2\big)^2 - 3\sum_i |E_i|^4}{\sum_i |E_i|^2} \tag{26}$$

for $i = 1\ldots 4$, respectively. Note that, unlike the effective potential, the cosmological constant does not depend on the overall factor A as it should for a constant holomorphic $\mathcal{H}$.

The effective potential, V_{eff} is minimized at several points with different gauge symmetry breaking patterns and different values of the cosmological constant reported below.

The extrema of the effective potential are a solution of Equation (14) and they can be classified according to the number of zero eigenvalues of the scalar-field matrix E_{ij}. These extrema fall in the following classes:

(a) The first class consists of configurations of only one non-zero eigenvalue of E_{ij} (let say $|E_1| = |E_2| = |E_3| = 0$ and $|E_4| = |E| \neq 0$). The associated cosmological constant is negative

$$\Lambda = -\frac{|E|^2}{2}\,, \tag{27}$$

giving rise to an anti-de Sitter background, an unbroken $SU(3)$ and positive effective potential $\langle V_{eff}\rangle = +24\alpha^2$.

(b) A second class of solutions is when two eigenvalues of the matrix are zero (let say $|E_1| = |E_2| = 0$) while the others have equal modulus ($|E_3| = |E_4| = |E|$). This breaks $SU(4) \to SU(2)$ and the cosmological constant turns out to be again negative

$$\Lambda = -\frac{|E|^2}{4}\,, \tag{28}$$

corresponding to an anti-de Sitter background. The effective potential is positive in this case $\langle V_{eff}\rangle = +6\alpha^2$.

(c) A third class of solutions is obtained when there is a single zero eigenvalue (let say $|E_1| = 0$) and the modulus of the other eigenvalues are equal ($|E_2| = |E_3| = |E_4|$). In this case, it can easily be verified that the cosmological constant in Equation (26) vanishes

$$\Lambda = 0\,, \tag{29}$$

corresponding to a Minkowski vacuum and an unbroken $U(1)$ symmetry.

(d) A final class of extrema of the effective potential contains scalars E_{ij} with non-zero eigenvalues but with equal modulus ($|E_1| = |E_2| = |E_3| = |E_4| = |E|$). The cosmological constant is positive in this case and turns out to be

$$\Lambda = +\frac{|E|^2}{4}\,, \tag{30}$$

corresponding to a de Sitter background whereas the $SU(4)$ symmetry is in generally broken. However, when the fields E_i $(i = 1, 2, 3, 4)$ are equal and not only their modulus, the $SU(4)$ is broken down to $O(4)$. The effective energy turns out to be $\langle V_{eff}\rangle = -3\alpha^2$.

The number of zero eigenvalues, the vacuum energy and the symmetry breaking for each case are collected in the Table 1:

Table 1. The vacua of the theory in the case of a diagonal scalar matrix E_{ij} are collectively presented in this table. As we will see later, most of the vacua are non-supersymmetric.

Vacuum Case	Zero Eigenvalues	VEV of Effective Potential	Background	Unbroken Subgroup of SU(4)
a	3	$\langle V_{eff}\rangle > 0$	AdS	SU(3)
b	2	$\langle V_{eff}\rangle > 0$	AdS	SU(2)
c	1	$\langle V_{eff}\rangle = 0$	Minkowski	U(1)
d	0	$\langle V_{eff}\rangle < 0$	dS	completely broken

3.2.2. Non-Constant Holomorphic Function

Let us now examine the effective potential in Equation (16) when the holomorphic function $\mathcal{H}$ is non-constant. In this case, the effective potential for a diagonal scalar-field matrix E_{ij} of the form (23) turns out to be

$$V_{eff} = -\frac{12}{A(S,\bar{S})}\frac{\left(\sum_i |E_i|^2\right)^2 - 3\sum_i |E_i|^4}{\left(\sum_i |E_i|^2\right)^2} + \frac{216}{A(S,\bar{S})^2}\left\{\frac{\prod_i E_i \times G}{\left(\sum_i |E_i|^2\right)^2} + h.c.\right\} . \tag{31}$$

Comparing the above potential to the one of the previous section (constant $\mathcal{H}$), we see that the first term is the same but the function $A \equiv A(S,\bar{S})$ is not constant anymore. The extra contribution in the brackets appears due to the presence of the derivatives of the holomorphic function $\mathcal{H}$. It is important to notice that since the second term in the effective potential in Equation (31) is a product of the eigenvalues of the scalar matrix E_{ij}, it vanishes when the determinant of E_{ij} is zero, i.e., when at least one of its eigenvalues vanishes.

We will determine now the cosmological constant and compare it to Equation (26) of the previous section. The effective potential in Equation (31) has schematically the form

$$V_{eff} = \frac{1}{A(S,\bar{S})}V^H + \frac{1}{A(S,\bar{S})^2}V^{DH} , \tag{32}$$

where V^H and V^{DH} can be read off from Equation (31) to be

$$V^H = -12\frac{\left(\sum_i |E_i|^2\right)^2 - 3\sum_i |E_i|^4}{\left(\sum_i |E_i|^2\right)^2} , \tag{33}$$

$$V^{DH} = 216\left\{\frac{\prod_i E_i \times G}{\left(\sum_i |E_i|^2\right)^2} + h.c.\right\} . \tag{34}$$

Then, from Equations (11) and (13) we see that the cosmological constant can always be written as

$$\begin{aligned}\Lambda &= -\frac{V_{eff}}{48}A(S,\bar{S})\sum_i |E_i|^2 \\ &= -\frac{1}{48}\left(V^H + \frac{1}{A(S,\bar{S})}V^{DH}\right)\sum_i |E_i|^2 .\end{aligned} \tag{35}$$

When the derivatives of $\mathcal{H}$ vanish as was the case in the previous section, the cosmological constant depends only on the vevs of the E_{ij} fields as in the case with the constant holomorphic function $\mathcal{H}$. The derivative terms just add an extra contribution that depends on the vev of the S field which parametrizes the manifold $SU(1,1)/U(1)$.

3.2.3. Vacua for Non-Constant (General) Holomorphic Function

For non-constant holomorphic function and proceeding as before, the non-trivial critical points of the effective potential in Equation (31) turns out to be (with $i,j,k,l = 1,2,3,4$ and $i \neq j \neq k \neq l$)

$$(a) \quad |E_i| = |E_j| = |E_j| = 0 , \quad |E_k| = E \neq 0 \tag{36}$$

$$(b) \quad |E_i| = |E_j| \neq 0 , \quad |E_k| = |E_l| = 0 . \tag{37}$$

$$\begin{aligned}&(c) \quad E_i|E_i|^2 = \frac{A}{3G}E_j^* E_k^* E_l^* , \quad \text{or} \\ &E_i|E_i|^2 = \frac{27G^*|G|^2}{A^3}E_j^* E_k^* E_l^* \quad \text{and} \quad E_j|E_j|^2 = \frac{A}{3G}E_i^* E_k^* E_l^* ,\end{aligned} \tag{38}$$

$$(d) \quad E_i|E_i|^2 = \left(\frac{G^*}{G}\right)^{1/2} E_j^* E_k^* E_l^*, \tag{39}$$

where, the functions A, G are evaluated at the S-critical points. The latter can be determined whenever the explicit form of $\mathcal{H}(S)$ is known. At the above points of Equations (36)–(39), the vacua can be classified according to the number of zero eigenvalues of E_{ij} as follows:

(a) With three zero and one non-zero eigenvalue E, corresponding to Equation (36), the cosmological constant turns out to be

$$\Lambda = -\frac{1}{2}|E|^2\,, \tag{40}$$

and the effective potential is

$$V_{eff} = \frac{24}{A(S,\overline{S})}\,. \tag{41}$$

(b) For two zero (let say $E_1 = E_2 = 0$) and two non-zero eigenvalues of equal modulus ($|E_3| = |E_4| = |E|$), corresponding to Equation (37), the cosmological constant is

$$\Lambda = -\frac{1}{4}|E|^2\,, \tag{42}$$

with

$$V_{eff} = \frac{6}{A(S,\overline{S})}\,. \tag{43}$$

As dictated by Equation (35) the cosmological constants found in these last two classes of solutions are equal to those of Equations (27) and (28) respectively.

(c) For non-zero eigenvalues which satisfy the relations in Equation (38), the effective potential and the cosmological constant have the form

$$V_{eff} = \frac{72}{A(S,\overline{S})}\frac{|G|^2}{A(S,\overline{S})^2 + 3|G|^2}\,, \qquad \Lambda = -\frac{3}{2}\frac{|G|^2}{A(S,\overline{S}) + 3|G|^2}\sum_i |E_i|^2\,. \tag{44}$$

In this case the effective potential is always positive and the cosmological constant is always negative. Since the derivatives of $\mathcal{H}$ also shape the vacuum structure, it is interesting to study them explicitly using the expression in Equation (11) and the derivative operator in Equation (21)

$$G = e^{4i\psi}(1-|S|^2)\Big(\frac{(1-|S|^2)\mathcal{H}'^2}{\mathcal{H}} + \frac{2\overline{S}\mathcal{H}' - (1-|S|^2)\mathcal{H}''}{6}\Big)\,, \tag{45}$$

where the prime denotes partial derivative with respect to the S field. The first part of this function is always positive definite since the target space of S is the Poincare disk, while the sign of the second part depends on the choice of the holomorphic function. One can require the function G to vanish or only the second part to vanish which leads to a second-order differential equation for $\mathcal{H}(S)$. Solving both these cases the solution is a non-holomorphic function thus neither the second part nor the whole G can vanish with a proper selection of $\mathcal{H}(S)$.

(d) Finally, if eigenvalues that satisfy Equation (39), the effective potential and cosmological constant turn out to be

$$V_{eff} = -\frac{3}{A(S,\overline{S})} \mp \frac{27|G|}{A(S,\overline{S})^2}\,, \qquad \Lambda = \frac{1}{16}\Big(1 \pm \frac{9|G|}{A(S,\overline{S})}\Big)\sum_i |E_i|^2\,. \tag{46}$$

Whether the cosmological constant is zero, positive or negative depends on the values of G and A. In the first case, corresponding to the plus sign in Equation (46), the cosmological constant is always positive. In the second case, corresponding to the minus sign in Equation (46), there are three possibilities according to the value of $\lambda = A - 9|G|$: anti-de Sitter for $\lambda < 1$, de Sitter for $\lambda > 1$ and Minkowski for $\lambda = 0$.

We sum up our results in the Table 2:

Table 2. In this table, we collectively present the vacua of the theory, indicating the number of eigenvalues and their relation to the cosmological constant and the symmetry breaking pattern. The Minkowski vacuum exists only if we fine tune $\lambda = 0$.

Vacuum Case	Number of Zero Eigenvalues	VEVs of E_{ij}	Cosmological Constant Λ	Symmetry Breaking of SU(4)
a	3	$\langle V_{eff} \rangle > 0$	AdS	SU(3)
b	2	$\langle V_{eff} \rangle > 0$	AdS	SU(2)
c	0	$\langle V_{eff} \rangle > 0$	AdS	completely broken
d	0	$\langle V_{eff} \rangle < 0$ $\langle V_{eff} \rangle = 0$ $\langle V_{eff} \rangle > 0$	dS Minkowski AdS	completely broken

3.2.4. Explicit Examples for Non-Constant Holomorphic Function

So far we have kept the discussion general and have classified the possible vacua according to the number of eigenvalues that vanish in the vacuum. In the case where the holomorphic function is constant the effective potential and the cosmological constant are independent of $\mathcal{H}(S)$ (since in this case $\mathcal{H}$ is an overall coupling constant) and the vacua are defined in Section 3.2.1. On the other hand, when the holomorphic function is non-constant, the value of the cosmological constant depends on the choice of $\mathcal{H}(S)$ which shapes the vacuum structure in a different way.

As we have discussed above, the function $\mathcal{H}(S)$ is arbitrary and is expected to be specified in a more fundamental theory. However, in order to be more explicit and for illustrative purposes, we will explore here some explicit examples with different forms of the function $\mathcal{H}(S)$. For this, we have to distinguish the possible vacua into two groups, group I, which contains the cases (a) and (b) and group II which contains the cases (c) and (d). The reason is that the effective potential in group I is determined entirely in terms of the function A, whereas the effective potential for group II is determined from both functions A and G. We start from the vacua I in Equations (40) and (42) where the cosmological constant does not depending on the S field. If we choose the holomorphic function to be linear

$$\mathcal{H}(S) = S\,, \tag{47}$$

it is obvious that the effective potential

$$V_{eff}(S,\overline{S}) = \frac{3}{Re\,S}\,, \tag{48}$$

has a runaway behavior and no critical points. Critical point of the potential exist only when the function $\mathcal{H}(S)$ has critical points itself. As a particular example for

$$\mathcal{H}(S) = \pm S^2 + S\,, \tag{49}$$

the effective potential has its extrema at $S_0 = \mp\frac{1}{2}$ for the solution in Equation (42)

$$\langle V_{eff} \rangle = \frac{6}{\mp S^2 + S + h.c.}\Big|_{S_0=\overline{S}_0} = \mp 12, \qquad \Lambda = -\frac{|E_3|^2}{2}\,. \tag{50}$$

Note that the critical points of the effective potential should lie inside the Poincare disk $0 \leq |S| < 1$.

Next, we examine the vacua II where all the eigenvalues of E_{ij} are non-zero. For convenience we assume that $E_{ij} = E\delta_{ij}$ and by minimizing the effective potential in Equation (31) we find

$$E^* = \pm E\left(\frac{G}{G^*}\right)^{1/4} \quad \text{with} \quad \Lambda = \frac{1}{4}\left(1 + \frac{9|G|}{A(S,\bar{S})}\right)|E|^2\,, \tag{51}$$

$$E^* = \pm iE\left(\frac{G}{G^*}\right)^{1/4} \quad \text{with} \quad \Lambda = \frac{1}{4}\left(1 - \frac{9|G|}{A(S,\bar{S})}\right)|E|^2\,. \tag{52}$$

In the previous subsection we saw that the derivatives of $\mathcal{H}$, which are contained in the functions G and G^* in Equation (11), give an extra contribution to the effective potential and the cosmological constant. By setting them to zero we arrive at the maximally symmetric solutions independent of the S fields. The critical points agree with the general solutions in Equation (39). The solutions in Equations (51) and (52) belong to the class d) (i.e., Equation (46)) where all the maximally symmetric cases are possible and thus the holomorphic function $\mathcal{H}(S)$ has to be specified in order to find the vacuum. To see how this works and in order to proceed further, we choose a power-law form for the holomorphic function $\mathcal{H}(S) = S^n$ as an example. Note that in the case we are discussing, since both functions A and G appear in the effective potential, it is not necessary $\mathcal{H}$ to have a critical point. Then, the cosmological constant in Equation (51) turns out to be

$$\Lambda = \frac{1}{4}\left(1 + \frac{3}{2}\frac{\sqrt{n^2 S^{-2+n}\bar{S}^{-2+n}(-1+|S|^2)^2(-1-5n+(-1+5n)|S|^2)^2}}{S^n+\bar{S}^n}\right)|E|^2\,. \tag{53}$$

The value of S is determined by minimizing the effective potential. Indeed, the critical points of the effective potential can be found for integer values of n at $S = S_0 = \bar{S}_0$. We find for example that for $n = -1$, there are two critical points $S_0 \sim -0.87$ with $\Lambda \sim 0.2|E|^2$ and $S_0 \sim 0.93$ with $\Lambda \sim 0.28|E|^2$, for $n = -2$, $|S_0| \sim 0.97$ with $\Lambda \sim 0.28|E|^2$, in general, S_0 lies within the Poincaré disk for $n \leq 0$. At these points, the cosmological constant in Equation (53) is always positive ($\Lambda > 0$) corresponding to a de Sitter background and increases for larger values of n. For $n > 0$, S_0 is outside the Poincaré disk and should not be considered. Similarly, the cosmological constant in Equation (52) and for our specific choice of function $\mathcal{H}$ leads to de Sitter backgrounds (for small negative values of n) and both de Sitter and anti-de Sitter solutions (for large negative values of n).

3.3. Stability

In order to determine whether the vacua found in the previous sections are stable, we have to calculate the masses of the fluctuations around these vacua. We consider the simplest case where $\mathcal{H} = const$ and the only scalar fields considered are E_{ij} and their conjugates since this is the case where the masses can be calculated analytically. In the more general case, numerical calculations are necessary. A small perturbation δE_{ij} around the vacuum satisfies the equation

$$\nabla_\mu \nabla^\mu \delta E_{ij} - \left(\frac{2}{3}\Lambda + 4\frac{\partial^2 V}{\partial E^{ij}\partial E_{kl}}\delta E_{kl} + 4\frac{\partial^2 V}{\partial E^{ij}\partial E^{kl}}\delta E^{kl}\right) = 0\,, \tag{54}$$

where the second derivative of the potential is calculated on the vacuum and we have used that $R = 4\Lambda$. The 20 real degrees of freedom of δE_{ij} corresponding to the $\mathbf{10} + \overline{\mathbf{10}}$ fields can be arranged so that Equation (54) can be written as $\nabla^2\delta E - \mathcal{M}^2\delta E = 0$. The square of the 20×20 mass matrix $\mathcal{M}^2$ is of the form

$$\mathcal{M}^2 = \begin{pmatrix} M^2_{ij\overline{kl}} & M^2_{ijkl} \\ M^2_{\overline{ijkl}} & M^2_{\overline{ij}kl} \end{pmatrix}\,. \tag{55}$$

Stability requires the eigenvalues of the matrix $\mathcal{M}^2$ to be non-zero for Minkowski and de Sitter backgrounds. However, in the case of AdS vacua Equations (27) and (28) the eigenvalues m^2 should satisfy the BF-bound [38,39]

$$m_{ij}^2 \geq -\frac{3}{4}|\Lambda| \,. \tag{56}$$

For the AdS vacuum obtained from an E_{ij} with one non-zero eigenvalue of Equation (27), we find that the following eigenvalues of the mass matrix

$$m_{11}^2 = \frac{1}{6}(-|E|^2 + 4\Lambda) = -\frac{1}{2}|E|^2 \,, \quad m = 6 \tag{57}$$

$$m_{22}^2 = \frac{1}{2}m_{11}^2 = -\frac{1}{4}|E|^2 \,, \quad m = 6 \tag{58}$$

$$m_{33}^2 = \frac{1}{6}(2|E|^2 + 4\Lambda) = 0 \,, \quad m = 6 \tag{59}$$

$$m_{44}^2 = \frac{1}{12}(2|E|^2 + 4\Lambda) = 0 \,, \quad m = 1 \tag{60}$$

$$m_{55}^2 = \frac{1}{12}(6|E|^2 + 4\Lambda) = \frac{1}{3}|E|^2 \,, \quad m = 1 \tag{61}$$

where m is the multiplicity of the eigenvalues. There are six negative eigenvalues in Equation (57) which do not satisfy the BF bound and therefore the corresponding AdS background is unstable. The rest of the vacua have more complicated structures since the matrix E_{ij} has more than one non-zero eigenvalues and extra non-diagonal terms appear in the mass matrix $\mathcal{M}^2$. For the second AdS vacuum in Equation (28), the eigenvalues are

$$m_{11}^2 = \frac{1}{6}(|E|^2 + 4\Lambda) = 0 \,, \quad m = 9 \tag{62}$$

$$m_{22}^2 = \frac{1}{2}m_{11}^2 = 0 \,, \quad m = 2 \tag{63}$$

$$m_{33}^2 = \frac{1}{6}(-2|E|^2 + 4\Lambda) = -\frac{1}{2}|E|^2 \,, \quad m = 2 \tag{64}$$

$$m_{44}^2 = \frac{1}{2}m_{33}^2 = -\frac{1}{4}|E|^2 \,, \quad m = 4 \tag{65}$$

$$m_{55}^2 = \frac{1}{6}(7|E|^2 + 4\Lambda) = |E|^2 \,, \quad m = 1 \tag{66}$$

$$m_{66}^2 = \frac{1}{24}(6|E|^2 + 8\Lambda) = \frac{1}{6}|E|^2 \,, \quad m = 1 \tag{67}$$

$$m_{77}^2 = \frac{1}{24}(14|E|^2 + 8\Lambda) = \frac{1}{2}|E|^2 \,, \quad m = 1 \tag{68}$$

where again two of the negative eigenvalues violate the BF bound leading to an unstable AdS. Similarly, for the Minkowski vacuum of Equation (29), the eigenvalues of the mass matrix are

$$m_{11}^2 = 0 \,, \quad m = 13 \tag{69}$$

$$m_{22}^2 = -\frac{1}{4}|E|^2 \,, \quad m = 2 \tag{70}$$

$$m_{33}^2 = \frac{1}{2}|E|^2 \,, \quad m = 2 \tag{71}$$

$$m_{44}^2 = |E|^2 \,. \quad m = 3 \tag{72}$$

and therefore the Minkowski vacuum is unstable due to tachyonic modes. Lastly, the de Sitter vacuum of Equation (30) is also unstable since the mass spectrum is

$$m_{11}^2 = \frac{1}{6}(-|E|^2 + 4\Lambda) = 0\,, \qquad m = 6 \tag{73}$$

$$m_{22}^2 = \frac{1}{2}m_{11}^2 = 0\,, \qquad m = 4 \tag{74}$$

$$m_{33}^2 = \frac{1}{6}(5|E|^2 + 4\Lambda) = |E|^2\,, \qquad m = 6 \tag{75}$$

$$m_{44}^2 = \frac{1}{2}m_{33}^2 = \frac{1}{2}|E|^2\,, \qquad m = 3 \tag{76}$$

$$m_{55}^2 = \frac{1}{12}(-3|E|^2 + 4\Lambda) = -\frac{1}{6}|E|^2\,, \quad m = 1 \tag{77}$$

and contains one tachyonic mode.

The tachyonic modes in the spectrum can be lifted by considering non-constant holomorphic function $\mathcal{H}$. Indeed, when $\mathcal{H}$ has not trivial derivatives the equation for the scalar fluctuations takes the form

$$\nabla_\mu \nabla^\mu \delta E_{ij} - \left\{ \frac{2}{3}\Lambda + \frac{4}{\mathcal{H} + \overline{\mathcal{H}}} \left(\frac{\partial^2 V}{\partial E^{ij} \partial E_{kl}} \delta E_{kl} + \frac{\partial^2 V}{\partial E^{ij} \partial E^{kl}} \delta E^{kl} \right) \right\} = 0\,, \tag{78}$$

where the potential V now is given by Equation (16). Clearly, a suitable non-trivial holomorphic function $\mathcal{H}$ can shift the masses of the perturbations such that the tachyonic states of the spectrum are removed. Some simple examples we worked out indicate that when the contribution of the derivatives of $\mathcal{H}$ to the potential is appropriately positive, the masses are shifted accordingly. However, since the exact form of $\mathcal{H}$ is not known, we can not say something more concrete at this point. We should note that it is also expected that when matter fields are coupled to the theory, the above instabilities will be further removed.

3.4. Non-Diagonal E_{ij}

The previous results exclude all vacua with a diagonal form of the scalars E_{ij}. More general forms of E_{ij} may also be considered but at the cost of increasing complexity. A relatively simple case that can be solved analytically is for a non-diagonal E_{ij} of the form

$$E_{ij} \;:\; \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & E_{33} & E_{34} \\ 0 & 0 & E_{43} & E_{44} \end{pmatrix}. \tag{79}$$

Analogously to the diagonal matrix with two non-zero eigenvalues, the gauge symmetry breaks in $SU(4) \to SU(2)$ and the effective potential takes the form

$$V_{eff} = -\frac{12}{A(S,\overline{S})} + \frac{36}{A(S,\overline{S})} \frac{\sum_{ij} |E_{ij}|^4 + 2\Big(\sum_i |E_{ii}|^2 |E_{34}|^2 + h.c.) + (E_{44}^* E_{33}^* E_{34}^2 + h.c.) \Big)}{\Big(\sum_{ij} |E_{ij}|^2 \Big)^2} \tag{80}$$

Then, the critical points of the potential turns out to be

$$(1) \quad E_{33} E_{44} = E_{34}^2 \tag{81}$$

where

$$V_{eff}(S,\overline{S}) = \frac{24}{A(S,\overline{S})}\,, \quad \Lambda = -\frac{1}{2} \sum_{ij} |E_{ij}|^2 \tag{82}$$

and

$$(2) \quad E_{44}E_{34}^{*} = -E_{33}^{*}E_{34}, \quad |E_{44}|^2 = |E_{33}|^2 \tag{83}$$

where

$$V_{eff}(S,\overline{S}) = \frac{6}{A(S,\overline{S})}\,, \quad \Lambda = -\frac{1}{8}\sum_{ij}|E_{ij}|^2 \tag{84}$$

The minimization of the effective potential V_{eff} leads to two different vacua of the same form as the 2×2 diagonal case with a negative cosmological constant (corresponding to an anti-de Sitter vacuum). Note that the cosmological constant here differs to the diagonal case because of the non-diagonal elements contribution as we have noticed above. In the contrary, comparing the effective potential energy in Equations (82) and (84) to the case of the 2×2 diagonal matrix given in Equations (41) and (43), we see that they are equal.

3.5. Partial Supersymmetry Breaking

Let us now examine whether supersymmetry is preserved by the vacua we found above. The Q and S- supersymmetry transformations are generated by the opposite chirality spinors ϵ^i and η^i, respectively. The fermion shifts under Q and S- supersymmetry, when only the ϕ^a and E_{ij} fields are turned on, are the following[2]

$$\delta_Q\Lambda_i = 2\epsilon^{ab}\phi_a \not{D}\phi_b\epsilon_i + E_{ij}\epsilon^j\,, \tag{85}$$

$$\delta_Q{\chi^{ij}}_k = -\frac{1}{2}\epsilon^{ijlm}\not{D}E_{kl}\epsilon_m + {D^{ij}}_{kl}\epsilon^l\,, \tag{86}$$

$$\delta_Q{\psi_\mu}^i = 2(\partial_\mu\epsilon^i + \frac{1}{2}b_\mu\epsilon^i - \frac{1}{2}\omega_\mu\cdot\sigma\epsilon^i - {V_\mu}^i{}_j\epsilon^j)\,, \tag{87}$$

and

$$\delta_S\Lambda_i = 0\,, \tag{88}$$

$$\delta_S{\chi^{ij}}_k = -\frac{1}{2}\epsilon^{ijlm}E_{kl}\eta_m\,, \tag{89}$$

$$\delta_S{\psi_\mu}^i = -\gamma_\mu\eta^i\,, \tag{90}$$

respectively. Clearly, S-supersymmetry is always broken since the gravitino shifts are non-zero for not trivial η^i. Similarly, the conditions for unbroken Q-supersymmetry are

$$E_{ij}\epsilon^j = 0\,, \tag{91}$$

$$\frac{1}{4}\frac{\mathcal{DH}}{\mathcal{H}}E_{kt}E_{lf}\epsilon^{ijtf}\epsilon^l = 0\,, \tag{92}$$

$$\left(\partial_\mu - \frac{1}{2}\omega_\mu\cdot\sigma\right)\epsilon^i = 0. \tag{93}$$

Then, the supersymmetric background are necessarily Minkowski and the scalars matrix $(E)_{ij} = E_{ij}$ should satisfy

$$\det E_{ij} = 0. \tag{94}$$

In particular, the number of unbroken supersymmetries is the number of zero eigenvalues. We should also mention that another possibility is the fermionic shifts under Q-supersymmetry to be canceled by an S-fermionic shift. However, one can show that there are no no-trivial supersymmetry parameters ϵ^i and η^i in the same direction (same index i) that would allow for anti-de Sitter supersymmetric backgrounds. Therefore, the only supersymmetric backgrounds in Weyl superconformal supergravity are Minkowski spacetimes.

4. Conclusions

We have studied possible vacua of maximal $\mathcal{N} = 4$ conformal supergravity which is the supersymmetric completion of conformal or Weyl gravity. It is invariant under the full superconformal group $SU(2,2|4)$, the real form of $SL(4|4)$. Although such theories are considered to need UV completion, they may emerge as a low-energy theory of string theory [17]. In particular, it has been claimed that it is not originating from closed strings, but it is an effective open string theory, localized on D3-branes. We should notice that the superconformal symmetry we discuss here is a classical symmetry. The latter is broken by quantum effects since, although the theory is power-counting renormalizable, it has non-vanishing one-loop beta-functions [34]. Thus it suffer from conformal anomaly so that (super)conformal symmetry is broken. However, since conformal symmetry is a gauge symmetry here, it poses a threat and leads to inconsistencies [40,41].

We have studied the vacuum of this theory by turning on the scalars E_{ij} in the $\overline{\mathbf{10}}$ of $SU(4)$ and the scalars ϕ^a which parametrize $SU(1,1)/U(1)$ coset. The scalars E_{ij} have Weyl weight $w = +1$ and therefore, their non-zero vev breaks both conformal and $SU(4)$ symmetry. We have found that the theory admits de Sitter, anti-de Sitter, and Minkowski vacua determined by the vev of the scalars E_{ij} and ϕ^a. In addition, S-supersymmetry is always broken, whereas Q-supersymmetry is preserved only on Minkowski backgrounds. The vacua we have found are unstable as the fluctuations around them are tachyonic. This pathology indicates that a UV completion is necessary which will remove the instability and project out the ghost massive graviton state inherited in Weyl gravity.

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Notes

1 We use the notation

$$\epsilon_{ab} = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}, \quad \epsilon^{ab} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \eta^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2 The full fermionic transformations can be found in [8].

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