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Deconstructing Frame-Dragging

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Abstract: The vorticity of world-lines of observers associated with the rotation of a massive body was reported by Lense and Thirring more than a century ago. In their example, the frame-dragging effect induced by the vorticity is directly (explicitly) related to the rotation of the source. However, in many other cases, it is not so, and the origin of vorticity remains obscure and difficult to identify. Accordingly, in order to unravel this issue, and looking for the ultimate origin of vorticity associated to frame-dragging, we analyze in this manuscript very different scenarios where the frame-dragging effect is present. Specifically, we consider general vacuum stationary spacetimes, general electro-vacuum spacetimes, radiating electro-vacuum spacetimes, and Bondi–Sachs radiating spacetimes. We identify the physical quantities present in all these cases, which determine the vorticity and may legitimately be considered as responsible for the frame-dragging. Doing so, we provide a comprehensive, physical picture of frame-dragging. Some observational consequences of our results are discussed.

Keywords: frame-dragging; super–energy; gravitational radiation

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1. Introduction

The dragging of inertial frames produced by self–gravitating sources, whose existence has been recently established by observations [1], is one of the most remarkable effect predicted by the general theory of relativity (GR) (see [2,3]).

The term "frame-dragging" usually refers to the influence of a rotating massive body on a gyroscope by producing vorticity in the congruence of world-lines of observers outside the rotating object. Although the appropriateness of the term "frame-dragging" has been questioned by Rindler [4], it nevertheless has been used regularly in the literature to this day, and accordingly, we shall adopt such a term here (see also [5,6]).

The basic concept for the understanding of this effect is that of vorticity of a congruence, which describes the rotation of a gyroscope attached to the congruence, with respect to reference particles.

Two different effects may be detected by means of gyroscopes. One of these (the Fokker de Sitter effect) refers to the precession of a gyroscope following a closed orbit around a spherically symmetric mass distribution. It has been verified with a great degree of accuracy by observing the rotation of the earth–moon system around the sun [7], but this is not the frame-dragging effect we are interested in here. The other effect, the one we are concerned with in this work, is the Lense–Thirring–Schiff precession, which refers to the appearance of vorticity in the congruence of world-lines of observers in the gravitational field of a massive rotating ball. It was reported for the first time by Lense and Thirring [8], and is usually referred to as the Lense–Thirring effect (some authors suggest that it should be named the Einstein–Thirring–Lense effect instead, see [9–11]). This result led Schiff [12] to propose the use of gyroscopes to measure such an effect. Since then, this idea has been developed extensively (see [4,13–21], and references cited therein).

However, although the origin of vorticity may be easily identified in the Lense–Thirring metric, as due to the rotation of a massive object, it is not always explicitly related to the rotation of massive objects. In fact, in any vacuum stationary space time (besides the Lense–Thirring metric), we can detect a frame-dragging effect, without specifying an explicit link to the rotation of a massive body [22].

The situation is still more striking for the electro-vacuum space times. The point is that the quantity responsible for the rotational (relativistic) multipole moments in these spacetimes is affected by the mass rotations (angular momentum), as well as by the electromagnetic field—that is, it contains contributions from both (angular momentum and electromagnetic field). This explains why such a quantity does not necessarily vanish in the case when the angular momentum of the source is zero but electromagnetic fields are present.

The first known example of this kind of situation was brought out by Bonnor [23]. Thus, analyzing the gravitational field of a magnetic dipole plus an electric charge, he showed that the corresponding spacetime is stationary and a frame-dragging effect appears. As a matter of fact, all stationary electro-vacuum solutions exhibit frame-dragging [24], even though in some cases, the angular momentum of the source is zero. In this latter case, the rotational relativistic multipole moments, and thereby the vorticity, are generated by the electromagnetic field. Furthermore, as we shall see, electrodynamic radiation also produces vorticity.

Finally, it is worth recalling that vorticity is present in gravitationally radiating spacetimes. The influence of gravitational radiation on a gyroscope through the vorticity associated with the emission of gravitational radiation was put forward for the first time in [25], and has been discussed in detail since then in [26–33]. In this case too, the explicit relationship between the vorticity and the emission of gravitational radiation was established without resorting to the rotation of the source itself.

Although in many of the scenarios described above, a rotating object is not explicitly identified as the source of vorticity, the fact remains that at a purely intuitive level, one always associates the vorticity of a congruence of world-lines, under any circumstance, to the rotation of "something".

The purpose of this work is twofold: on the one hand, we shall identify the physical concept (the "something") behind all cases where frame-dragging is observed, whether or not the angular momentum of the source vanishes. On the other hand, we would like to emphasize the possible observational consequences of our results.

As we shall see below, in all possible cases, the appearing vorticity is accounted for by the existence of a flow of superenergy on the plane orthogonal to the vorticity vector, plus (in the case of electro-vacuum spacetimes) a flow of electromagnetic energy on the same plane.

Since superenergy plays a fundamental role in our approach, we shall start by providing a brief introduction of this concept in the next section.

2. Superenergy and the Super-Poynting Vector

The concept of energy is a fundamental tool in all branches of physics, allowing us to approach and solve a vast number of problems under a variety of circumstances. This explains the fact that since the early times of GR, many researchers have tried by means of very different approaches to present a convincing definition of gravitational energy, in terms of an invariant local quantity. All these attempts, as is well-known, have failed. The reason for this failure is easy to understand.

Indeed, as we know, in classical field theory, energy is a quantity defined in terms of potentials and their first derivatives. However, on the other hand, we also know that in GR it is impossible to construct a tensor expressed only through the metric tensor (the potentials) and their first derivatives (in accordance with the equivalence principle). Therefore, a local description of gravitational energy in terms of true invariants (tensors of any rank) is not possible within the context of the theory.

Thus, the following alternatives remain:

- To define energy in terms of a non–local quantity;
- To resort to pseudo–tensors; and
- To introduce a succedaneous definition of energy.

One example of the last of the above alternatives is superenergy, which may be defined either from the Bel or from the Bel–Robinson tensor [34–36] (they both coincide in vacuum), and has been shown to be very useful when it comes to explaining a number of phenomena in the context of GR.

Both the Bel and the Bel–Robinson tensors are obtained by invoking the "structural" analogy between GR and the Maxwell theory of electromagnetism. More specifically, exploiting the analogy between the Riemann tensor ($R_{\alpha\beta\gamma\delta}$) and the Maxwell tensor ($F_{\mu\nu}$), Bel introduced a four-index tensor defined in terms of the Riemann tensor in a way which is reminiscent of the definition of the energy–momentum tensor of electromagnetism in terms of the Maxwell tensor. This is the Bel tensor.

The Bel–Robinson tensor is defined as the Bel tensor, but with the Riemann tensor replaced by the Weyl tensor ($C_{\alpha\beta\gamma\delta}$) (see [37] for a comprehensive account and more recent references on this issue).

Let us now introduce the electric and magnetic parts of the Riemann and the Weyl tensors, as

$$E_{\alpha\beta} = C(R)_{\alpha\gamma\beta\delta}u^{\gamma}u^{\delta}, \tag{1}$$

$$H_{\alpha\beta} = C(R)^{*}_{\alpha\gamma\beta\delta}u^{\gamma}u^{\delta}, \tag{2}$$

where $C(R)_{\alpha\gamma\beta\delta}$ is the Weyl (Riemann) tensor, the four-vector u^{γ} in vacuum is the tangent vector to the world-lines of observers, and $C(R)^{*}_{\alpha\gamma\beta\delta}$ is the dual of the Weyl (Riemann) tensor.

A third tensor may be defined from the double dual of the Riemann tensor as

$$X_{\alpha\beta} = {}^{*}R^{*}_{\alpha\gamma\beta\delta}u^{\gamma}u^{\delta}. \tag{3}$$

The double dual of the Weyl tensor coincides with the electric part of the Weyl tensor (up to a sign).

Next, from the analogy with electromagnetism, the super-energy and super-Poynting vector are defined by

$$U(R) = \frac{1}{2}(X_{\alpha\beta}X^{\alpha\beta} + E_{\alpha\beta}E^{\alpha\beta}) + H_{\alpha\beta}H^{\alpha\beta}; \qquad U(C) = E_{\alpha\beta}E^{\alpha\beta} + H_{\alpha\beta}H^{\alpha\beta} \tag{4}$$

$$P(R)_{\alpha} = \eta_{\alpha\beta\gamma\delta}(E^{\beta}_{\epsilon}H^{\gamma\epsilon} - X^{\beta}_{\epsilon}H^{\epsilon\gamma})u^{\delta}; \qquad P(C)_{\alpha} = 2\eta_{\alpha\beta\gamma\delta}E^{\beta}_{\epsilon}H^{\gamma\epsilon}u^{\delta}, \tag{5}$$

where $R(C)$ denotes whether the quantity is defined with a Riemann (Weyl) tensor, and $\eta_{\alpha\beta\gamma\delta}$ is the Levi–Civita tensor.

In the next sections, we shall bring out the role played by the above-introduced variables in the study of the frame-dragging effect.

3. Frame-Dragging in Vacuum Stationary Spacetimes

As we mentioned in the Introduction, the first case of frame-dragging analyzed in the literature was the Lense–Thirring effect. For didactical reasons, we shall start by considering first this case, and from there on, we shall consider examples of increasing complexity. Thus, afterward, we shall consider the Kerr metric, an approximation of which is the Lense–Thirring spacetime, and finally, we shall consider the general vacuum stationary spacetime case.

3.1. The Lense–Thirring Precession

The Lense–Thirring effect is based on an approximate solution to the Einstein equations, which reads [8]

$$\begin{aligned} ds^2 &= -\left(1-\frac{2m}{r}\right)dt^2 + \left(1+\frac{2m}{r}\right)\left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right) \\ &+ \frac{4J\sin^2\theta}{r}d\phi dt. \end{aligned} \tag{6}$$

The above metric corresponds to the gravitational field outside a spinning sphere of constant density, which up to the first order in m/r and J/r^2, with m and J denoting the mass and the angular momentum respectively, coincides with the Kerr metric, by identifying

$$ma = -J, \tag{7}$$

where a is the Kerr parameter [38].

The time-like vector u^α tangent to the world-lines of observers at rest in the frame of (6) is given by

$$u^\alpha = \left(\frac{1}{\sqrt{1-\frac{2m}{r}}}, 0, 0, 0\right), \tag{8}$$

and from the above expression, the vorticity vector, defined as usual by

$$\omega^\alpha = \frac{1}{2}\eta^{\alpha\eta\iota\lambda}u_\eta u_{\iota,\lambda}, \tag{9}$$

has, up to orders a/r and m/r, the following non-null components

$$\omega^r = \frac{2ma\cos\theta}{r^3}, \tag{10}$$

$$\omega^\theta = \frac{ma\sin\theta}{r^4}, \tag{11}$$

or

$$\Omega = (\omega^\alpha\omega_\alpha)^{1/2} = \frac{ma}{r^3}\sqrt{1+3\cos^2\theta}, \tag{12}$$

which at $\theta = \frac{\pi}{2}$ reads

$$\Omega = \frac{ma}{r^3}. \tag{13}$$

The above expression embodies the essence of the Lense–Thirring effect. It describes the vorticity of the world-lines of observers, produced by the rotation (J) of the source. Such vorticity, as correctly guessed by Schiff [12], could be detected by a gyroscope attached to the world-lines of our observer.

Even though in this case, the vorticity is explicitly related to the rotation of the spinning object which sources the gravitational field, the fact that this link, in many other cases, is not so explicitly established leads us to the question: what is (are) the physical mechanism(s), which explains the appearance of vorticity in the world-lines of the observer? As we shall see in the next few sections, the answer to this question may be given in terms of a flow of superenergy, plus (in the case of electro-vacuum spacetimes) a flow of electromagnetic energy.

Thus, in order to approach this conclusion, let us calculate the leading term of the super-Poynting gravitational vector at the equator. Using (5) and (6), we obtain for the only non-vanishing component (remember that in vacuum, both expressions for the super-Poynting vector coincide),

$$P^\phi \approx 9\frac{m^2}{r^2}\frac{a}{r}\frac{1}{r^5}. \tag{14}$$

It describes a flux of super-energy on the plane orthogonal to the vorticity vector. On the other hand, it follows at once from (14) that

$$P^{\phi} = 0 \Leftrightarrow a = 0 \Leftrightarrow \omega^{\alpha} = 0.$$

From the comments above, a hint about the link between superenergy and vorticity (frame-dragging) begins to appear. In order to delve deeper on this issue, let us next consider the Kerr metric.

3.2. Frame-Dragging in the Kerr Metric

The calculations performed in the previous subsection can be very easily repeated for the Kerr metric.

In Boyer–Linquist coordinates, the Kerr metric takes the form

$$\begin{aligned} ds^2 &= \left(-1+\frac{2mr}{r^2+a^2\cos^2\theta}\right)dt^2-\left(\frac{4mar\sin^2\theta}{r^2+a^2\cos^{\theta}}\right)dtd\phi+\left(\frac{r^2+a^2\cos^2\theta}{r^2-2mr+a^2}\right)dr^2+(r^2+a^2\cos^2\theta)d\theta^2 \\ &+ \left(r^2\sin^2\theta+a^2\sin^2\theta+\frac{2mra^2\sin^4\theta}{r^2+a^2\cos^2\theta}\right)d\phi^2. \end{aligned} \tag{15}$$

For this spacetime, the time-like vector u^{α} tangent to the world-lines of observers at rest in (15), reads

$$u^{\alpha} = \left(\frac{1}{\sqrt{1-\frac{2mr}{r^2+a^2\cos^2\theta}}}, 0, 0, 0\right). \tag{16}$$

The vorticity of the above-described congruence is characterized by a vorticity vector ω^{α} with components

$$\omega^r = 2mra\cos\theta(r^2-2mr+a^2)(r^2+a^2\cos^2\theta)^{-2}(r^2-2mr+a^2\cos^2\theta)^{-1}, \tag{17}$$

and

$$\omega^{\theta} = ma\sin\theta(r^2-a^2\cos^2\theta)(r^2+a^2\cos^2\theta)^{-2}(r^2-2mr+a^2\cos^2\theta)^{-1}. \tag{18}$$

The above expressions coincide with (10) and (11) up to first order in m/r and a/r.

As a final step, we obtain for the super-Poynting vector (5)

$$P^{\mu} = (P^t, 0, 0, P^{\phi}), \tag{19}$$

with

$$\begin{aligned} P^t &= -18m^3ra^2\sin^2\theta(r^2-2mr+a^2\sin^2\theta+a^2)(r^2+a^2\cos^2\theta)^{-4}(r^2-2mr+a^2\cos^2\theta)^{-2} \\ &\times \left(\frac{r^2-2mr+a^2\cos^2\theta}{r^2+a^2\cos^2\theta}\right)^{-1/2}, \end{aligned} \tag{20}$$

$$\begin{aligned} P^{\phi} &= 9m^2a(r^2-2mr-a^2\cos^2\theta+2a^2)(r^2+a^2\cos^2\theta)^{-4}(r^2-2mr+a^2\cos^2\theta)^{-1} \\ &\times \left(\frac{r^2-2mr+a^2\cos^2\theta}{r^2+a^2\cos^2\theta}\right)^{-1/2}, \end{aligned} \tag{21}$$

where the package GR-Tensor has been used (GRTensor III (https://github.com/grtensor/grtensor)).

From the above expressions, it follows, as in the precedent case, that there is an azimuthal flow of superenergy as long as $a \neq 0$, where inversely, the vanishing of such a flow implies $a = 0$. Once again, frame-dragging appears to be tightly related to a circular flow of superenergy on the plane orthogonal to the vorticity vector.

In the present case, we can delve deeper in the relationship between the source of the field and the vorticity, since a specific interior for the Kerr metric is available [39]. The remarkable fact is the presence of a non-vanishing T_t^ϕ component of the energy–momentum tensor of the source, which, defining as usual an energy–momentum flux vector as $F^\nu = -V^\mu T_{\nu\mu}$ (where V^μ denotes the four velocity of the fluid), implies that in the equatorial plane of our system (within the source), energy flows around in circles around the symmetry axis. This result, as we shall see in the next section, is reminiscent of an effect appearing in stationary Einstein–Maxwell systems. Indeed, in all stationary Einstein–Maxwell systems, there is a non-vanishing component of the Poynting vector describing a similar phenomenon [23,24] (of electromagnetic nature, in this latter case). Thus, the appearance of such a component seems to be a distinct physical property of rotating fluids, which has been overlooked in previous studies of these sources, and that is directly related to the vorticity (see Equations (8) and (18) in [39]).

3.3. Frame-Dragging in a General Stationary Vacuum Spacetime

Let us now consider the general stationary and axisymmetric vacuum case.

The line element for a general stationary and axisymmetric vacuum spacetime may be written as [40,41]

$$ds^2 = -f dt^2 + 2f\omega dt d\phi + f^{-1}e^{2\gamma}(d\rho^2 + dz^2) + (f^{-1}\rho^2 - f\omega^2)d\phi^2, \tag{22}$$

where $x^0 = t$; $x^1 = \rho$; $x^2 = z$ and $x^3 = \phi$ and metric functions depend only on ρ and z, which must satisfy the vacuum field equations:

$$\gamma_\rho = \frac{1}{4\rho f^2}\left[\rho^2\left(f_\rho^2 - f_z^2\right) - f^4\left(\omega_\rho^2 - \omega_z^2\right)\right], \tag{23}$$

$$\gamma_z = \frac{1}{2\rho f^2}\left(\rho^2 f_\rho f_z - f^4\omega_\rho\omega_z\right), \tag{24}$$

$$f_{\rho\rho} = -f_{zz} - \frac{f_\rho}{\rho} - \frac{f^3}{\rho^2}\left(\omega_\rho^2 + \omega_z^2\right) + \frac{1}{f}\left(f_\rho^2 + f_z^2\right), \tag{25}$$

$$\omega_{\rho\rho} = -\omega_{zz} + \frac{\omega_\rho}{\rho} - \frac{2}{f}(f_\rho\omega_\rho + f_z\omega_z), \tag{26}$$

where subscripts denote partial derivatives.

Next, the four-velocity vector for an observer at rest in the frame of (22) reads

$$u^\alpha = (f^{-1/2}, 0, 0, 0). \tag{27}$$

We may now proceed to calculate the super-Poynting vector for the general metric (22), without making any assumption about the matter content of the source, and one gets (using again GR–Tensor),

$$P^\mu = (P^t, 0, 0, P^\phi) \quad \text{with} \quad P^t = \omega P^\phi, \tag{28}$$

where P^ϕ is given by (again in the general case, i.e., without taking into account the field equations)

$$P^\phi = f^{3/2}e^{-4\gamma}\rho^{-5}\{A11\},$$

or using the field Equations (23)–(26) in the above expression

$$P^{\phi} = -\frac{1}{32} f^{-3/2} e^{-4\gamma} \rho^{-5} \{A12\},$$

where $A11$ and $A12$ are given in the Appendix A.

From the analysis provided in [24], and from (5), we know that

$$H_{\alpha\beta} = 0 \Leftrightarrow \omega^{\alpha} = 0 \Leftrightarrow \omega = 0, \tag{29}$$

and

$$H_{\alpha\beta} = 0 \Rightarrow P^{\mu} = 0\,. \tag{30}$$

In order to establish a link between vorticity and the super-Poynting vector of the kind already found for the Kerr (and Lense–Thirring) metric, we still need to prove that the vanishing of the super-Poynting vector implies the vanishing of the vorticity—that is, we have to prove that

$$P^{\mu} = 0 \Leftrightarrow H_{\alpha\beta} = 0 \Leftrightarrow \omega^{\alpha} = 0 \Leftrightarrow \omega = 0\,. \tag{31}$$

Such a proof has been carried out in [22], but is quite cumbersome, and therefore, we shall omit the details here.

Thus, based on (31), we conclude that for any stationary spacetime, irrespectively of its source, there is a frame-dragging effect associated to a flux of superenergy on the plane orthogonal to the vorticity vector.

We shall next analyze the electro-vacuum stationary case.

4. Frame-Dragging in Electro-Vacuum Stationary Spacetimes

electro-vacuum solutions to the Einstein equations pose a challenge concerning the frame-dragging effect. This was pointed out for the first time by Bonnor in [23] by analyzing the gravitational field produced by a magnetic dipole with an electric charge in the center. The surprising result is that, for this spacetime, the world-lines of observers at rest with respect to the electromagnetic source are endowed of vorticity (i.e., the resulting spacetime is not static, but stationary).

In order to explain the appearance of vorticity in the spacetime generated by a charged magnetic dipole, Bonnor resorts to a result pointed out by Feynmann in his Lectures on Physics [42], showing that for such a system (in the context of classical electrodynamics), there exists a non-vanishing component of the Poynting vector describing a flow of electromagnetic energy round in circles. This strange result led Feynmann to write that "it shows the theory of the Poynting vector is obviously nuts". However, some pages ahead in the same book, when discussing the "paradox" of the rotating disk with charges and a solenoid, Feynmann shows that this "circular" flow of electromagnetic energy is absolutely necessary in order to preserve the conservation of angular momentum. In other words, the theory of the Poynting vector is not only "nuts", but is necessary to reconcile the electrodynamics with the conservation law of angular momentum.

Based on the above comments, Bonnor then suggests that, in the context of GR, such a circular flow of energy affects inertial frames by producing vorticity of congruences of particles, relative to the compass of inertia. In other words, Bonnor suggests that the "something" which rotates, thereby generating the vorticity, is electromagnetic energy.

The interesting point is that this conjecture was shown to be valid for a general axially symmetric stationary electro-vacuum metric [24].

Indeed, assuming the line element (22) for the spacetime admitting an electromagnetic field, it can be shown that the variable responsible for the rotational multipole moments, which in turn determines the vorticity of the congruence of world-lines of observers, is affected by both the electromagnetic field and by the mass rotations (angular momentum) [24]. This explains why the vorticity does not necessarily vanish in the case when the angular momentum of the source is zero but electromagnetic fields are present. At any rate, it is important to stress that in such cases, the super-Poynting vector does not vanish either.

We shall next consider the presence of vorticity due to gravitational and electromagnetic radiation.

5. Vorticity and Radiation

We shall now analyze the vorticity related to the emission of gravitational and/or electromagnetic radiation. As we shall see, the emission of radiation is always accompanied by the appearance of vorticity of world-lines of observers. Furthermore, the calculations suggest that once the radiation process has stopped, there is still a remaining vorticity associated with the tail of the wave, which allows, in principle, to prove (or disprove) the violation of the Huyghens principle in a Riemannian spacetime (see [43–49] and references therein for a discussion on this issue), by means of observations.

5.1. Gravitational Radiation and Vorticity

Since the early days of GR, starting with the works of Einstein and Weyl on the linear approximation of the Einstein equations, a great deal of work has been done so far in order to provide a consistent framework for the study of gravitational radiation. In addition, important collaboration efforts have been carried out, and are now under consideration, to put in evidence gravitational waves by means of laser interferometers [50].

However, it was necessary to wait for more than half a century, until Bondi and coworkers [51] provided firm theoretical evidence of the existence of gravitational radiation without resorting to the linear approximation.

The essential "philosophy" behind the Bondi formalism consists of interpreting gravitational radiation as the physical process by means of which the source of the field "informs" about any changes in its structure. Thus, the information required to forecast the evolution of the system (besides the "initial" data) is thereby identified with radiation itself, and this information is represented by the so-called "news function". In other words, whatever happens at the source, leading to changes in the field, it can only do so by affecting the news function, and vice versa. Therefore, if the news function is zero over a time interval, there is no gravitational radiation over that interval. Inversely, non-vanishing news on an interval implies the emission of gravitational radiation during that interval. Thus, the main virtue of this approach resides in providing a clear and precise criterion for the existence of gravitational radiation.

The above-described picture is reinforced by the fact that the Bondi mass of a system is constant if, and only if there is no news.

In order to facilitate discussion, let us briefly introduce the main aspects of the Bondi approach. The starting point is the general form of the line element of an axially (and reflection) symmetric asymptotically flat spacetime, which in null (Bondi) coordinates, reads

$$\begin{aligned} ds^2 &= \left(\frac{V}{r}e^{2\beta} - U^2r^2e^{2\gamma}\right)du^2 + 2e^{2\beta}dudr \\ &+ 2Ur^2e^{2\gamma}dud\theta - r^2\left(e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta d\phi^2\right), \end{aligned} \tag{32}$$

where V, β, U, and γ are functions of u, r, and θ.

The coordinates are numbered $x^{0,1,2,3} = u, r, \theta, and\phi$, respectively. u is a time-like coordinate, such that $u = constant$ defines a null surface. In flat spacetime, this surface coincides with the null light cone open to the future. r is a null coordinate ($g_{rr} = 0$), and θ and ϕ are two angle coordinates.

Regularity conditions in the neighborhood of the polar axis ($\sin\theta = 0$) require that the functions

$$V, \beta, U/\sin\theta, \gamma/\sin^2\theta, \tag{33}$$

are regular on the polar axis.

Then, the four metric functions are assumed to be expanded in series of $1/r$, which using field equations, produces

$$\gamma = c(u,\theta)r^{-1} + \left[C(u,\theta) - \frac{1}{6}c^3\right]r^{-3} + ..., \tag{34}$$

$$U = -(c_\theta + 2c\cot\theta)r^{-2} + ..., \tag{35}$$

$$\begin{aligned} V &= r - 2M(u,\theta) \\ &- \left[N_\theta + N\cot\theta - c_\theta^2 - 4cc_\theta\cot\theta - \frac{1}{2}c^2(1+8\cot^2\theta)\right]r^{-1} + ..., \end{aligned} \tag{36}$$

$$\beta = -\frac{1}{4}c^2 r^{-2} + ..., \tag{37}$$

where letters as subscripts denote derivatives, and

$$4C_u = 2c^2c_u + 2cM + N\cot\theta - N_\theta. \tag{38}$$

The three functions c, M, and N depend on u and θ, and are further related by the supplementary conditions

$$M_u = -c_u^2 + \frac{1}{2}(c_{\theta\theta} + 3c_\theta\cot\theta - 2c)_u, \tag{39}$$

$$-3N_u = M_\theta + 3cc_{u\theta} + 4cc_u\cot\theta + c_uc_\theta. \tag{40}$$

In the static case, M equals the mass of the system, whereas N and C are closely related to the dipole and quadrupole moments, respectively.

Next, Bondi defines the mass $m(u)$ of the system as

$$m(u) = \frac{1}{2}\int_0^\pi M(u,\theta)\sin\theta d\theta, \tag{41}$$

which by virtue of (39) and (33), yields

$$m_u = -\frac{1}{2}\int_0^\pi c_u^2\sin\theta d\theta. \tag{42}$$

The two main conclusions emerging from Bondi's approach are

- If γ, M and N are known for some $u = a$(constant), and c_u (the news function) is known for all u in the interval $a \leq u \leq b$, then the system is fully determined in that interval.
- As it follows from (42), the mass of a system is constant if, and only if there are no news.

In light of these comments, the relationship between news function and the occurrence of radiation becomes clear.

Let us now calculate the vorticity of the world-lines of observers at rest in the frame of (32). For such observers, the four-velocity vector has components

$$u_\alpha = \left(A, \frac{e^{2\beta}}{A}, \frac{Ur^2e^{2\gamma}}{A}, 0\right) \tag{43}$$

with

$$A \equiv \left(\frac{V}{r}e^{2\beta} - U^2r^2e^{2\gamma}\right)^{1/2}. \tag{44}$$

Using (9), we easily obtain

$$\omega^\alpha = (0,0,0,\omega^\phi) \tag{45}$$

with

$$\begin{aligned}\omega^{\phi} &= -\frac{e^{-2\beta}}{2r^2\sin\theta}\{2\beta_\theta e^{2\beta} - \frac{2e^{2\beta}A_\theta}{A} - \left(Ur^2e^{2\gamma}\right)_r \\ &+ \frac{2Ur^2e^{2\gamma}}{A}A_r + \frac{e^{2\beta}\left(Ur^2e^{2\gamma}\right)_u}{A^2} - \frac{Ur^2e^{2\gamma}}{A^2}2\beta_u e^{2\beta}\}\end{aligned} \tag{46}$$

and for the absolute value of ω^α, we get

$$\begin{aligned}\Omega &\equiv (-\omega_\alpha\omega^\alpha)^{1/2} = \frac{e^{-2\beta-\gamma}}{2r}\{2\beta_\theta e^{2\beta} - 2e^{2\beta}\frac{A_\theta}{A} - \left(Ur^2e^{2\gamma}\right)_r \\ &+ 2Ur^2e^{2\gamma}\frac{A_r}{A} + \frac{e^{2\beta}}{A^2}\left(Ur^2e^{2\gamma}\right)_u - 2\beta_u\frac{e^{2\beta}}{A^2}Ur^2e^{2\gamma}.\}\end{aligned} \tag{47}$$

Feeding back (34)–(37) into (47) and keeping only terms up to order $\frac{1}{r^2}$, we obtain

$$\begin{aligned}\Omega &= -\frac{1}{2r}(c_{u\theta} + 2c_u\cot\theta) \\ &+\frac{1}{r^2}[M_\theta - M(c_{u\theta} + 2c_u\cot\theta) - cc_{u\theta} + 6cc_u\cot\theta + 2c_uc_\theta].\end{aligned} \tag{48}$$

Let us now analyze the expression above. First of all, observe that, up to order $1/r$, a gyroscope in the gravitational field given by (32) will precess as long as the system radiates ($c_u \neq 0$). Indeed, if we assume

$$c_{u\theta} + 2c_u\cot\theta = 0 \tag{49}$$

then

$$c_u = \frac{F(u)}{\sin^2\theta}, \tag{50}$$

which implies, due to the regularity conditions (33)

$$F(u) = 0 \Longrightarrow c_u = 0. \tag{51}$$

In other words, the leading term in (48) will vanish if, and only if $c_u = 0$.

Let us now analyze the term of order $\frac{1}{r^2}$. It contains, besides the terms involving c_u, a term not involving news (M_θ). Let us now assume that initially (before some $u = u_0 =$ constant), the system is static, in which case

$$c_u = 0 \tag{52}$$

which implies, because of (40)

$$M_\theta = 0 \tag{53}$$

and $\Omega = 0$ (actually, in this case $\Omega = 0$ at any order), as expected for a static field. Then, let us suppose that at $u = u_0$ the system starts to radiate ($c_u \neq 0$) until $u = u_f$, when the news function vanishes again. For $u > u_f$, the system is not radiating, although (in general) $M_\theta \neq 0$, implying time-dependence of metric functions. This class of spacetimes is referred to as non-radiative motions [51].

Thus, in the interval $u \in (u_0, u_f)$, the leading term of vorticity is given by the term of the order $1/r$ in (48). For $u > u_f$, there is a vorticity term of order $\frac{1}{r^2}$ describing the effect of the tail of the wave on the vorticity. This provides an "observational" possibility to find evidence for the violation of Huygens' principle.

Following the line of arguments of the preceding sections, we shall establish a link between vorticity and a circular flow of superenergy on the plane orthogonal to the vorticity vector. For doing so, let us calculate the super-Poynting vector (P^μ), defined by (5). We obtain that the leading terms for each super-Poynting component are

$$P_r = -\frac{2}{r^2} c_{uu}^2, \tag{54}$$

$$P_\theta = -\frac{2}{r^2 \sin\theta}\{[2c_{uu}^2 c + c_{uu} c_u]\cos\theta + \left[c_{uu} c_{\theta u} + c_{uu}^2 c_\theta\right]\sin\theta\}, \tag{55}$$

$$P_\phi = P^\phi = 0. \tag{56}$$

Due to the reflection symmetry of the Bondi metric, which prevents motions in the ϕ direction, the azimuthal component (P^ϕ) should vanish. Thus, in this particular case, it is the θ component of P^μ that is the physical factor to be associated to the vorticity.

In order to further strengthen the case for the super-Poynting vector as the physical origin of the mentioned vorticity, we shall next consider the general radiative metric without axial and reflection symmetry.

The Bondi formalism was later extended to a general spacetime (without any kind of symmetries) by Sachs [52]. In this case, the line element reads (where we use the notation given in [53], which is slightly different from the original Sachs paper):

$$\begin{aligned} ds^2 = {} & (Vr^{-1}e^{2\beta} - r^2 e^{2\gamma} U^2 \cosh 2\delta - r^2 e^{-2\gamma} W^2 \cosh 2\delta - 2r^2 UW \sinh 2\delta) du^2 \\ & + 2e^{2\beta} du dr + 2r^2 (e^{2\gamma} U \cosh 2\delta + W \sinh 2\delta) du d\theta + 2r^2 (e^{-2\gamma} W \cosh 2\delta \\ & + U \sinh 2\delta) \sin\theta du d\phi - r^2 (e^{2\gamma} \cosh 2\delta d\theta^2 + e^{-2\gamma} \cosh 2\delta \sin^2\theta d\phi^2 \\ & + 2 \sinh 2\delta \sin\theta d\theta d\phi),, \end{aligned} \tag{57}$$

where β, γ, δ, U, W, V are functions of $x^0 = u$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$.

The formalism then proceeds exactly as in [51], but taking into account the additional degrees of freedom (see [52,53] for details). It is worth stressing that now there are two news functions.

Let us first calculate the vorticity for the congruence of observers at rest in (58), whose four-velocity vector is given by

$$u^\alpha = A^{-1} \delta_u^\alpha, \tag{58}$$

where now, A is given by

$$A = (Vr^{-1}e^{2\beta} - r^2 e^{2\gamma} U^2 \cosh 2\delta - r^2 e^{-2\gamma} W^2 \cosh 2\delta - 2r^2 UW \sinh 2\delta)^{1/2}. \tag{59}$$

Thus, (9) lead us to

$$\omega^\alpha = (\omega^u, \omega^r, \omega^\theta, \omega^\phi), \tag{60}$$

where

$$\begin{aligned} \omega^u = {} & -\frac{1}{2A^2 \sin\theta}\{r^2 e^{-2\beta}(WU_r - UW_r) + \\ & \left[2r^2 \sinh 2\delta \cosh 2\delta (U^2 e^{2\gamma} + W^2 e^{-2\gamma}) + 4UWr^2 \cosh^2 2\delta\right] e^{-2\beta}\gamma_r + \\ & 2r^2 e^{-2\beta}(W^2 e^{-2\gamma} - U^2 e^{2\gamma})\delta_r + e^{2\beta}[e^{-2\beta}(U \sinh 2\delta + e^{-2\gamma} W \cosh 2\delta)]_\theta - \\ & e^{2\beta}[e^{-2\beta}(W \sinh 2\delta + e^{-2\gamma} U \cosh 2\delta]_\phi\}, \end{aligned} \tag{61}$$

$$\omega^r = \frac{1}{e^{2\beta}\sin\theta}\{2r^2A^{-2}[((U^2e^{2\gamma}+W^2e^{-2\gamma})\sinh 2\delta\cosh 2\delta + UW\cosh^2 2\delta)\gamma_u + (W^2e^{-2\gamma}-U^2e^{2\gamma})\delta_u + \frac{1}{2}(WU_u - UW_u)] + A^2[A^{-2}(We^{-2\gamma}\cosh 2\delta + U\sinh 2\delta)]_\theta - A^2[A^{-2}(W\sinh 2\delta + Ue^{2\gamma}\cosh 2\delta)]_\phi\}, \tag{62}$$

$$\omega^\theta = \frac{1}{2r^2\sin\theta}\{A^2e^{-2\beta}[r^2A^{-2}(U\sinh 2\delta + We^{-2\gamma}\cosh 2\delta)]_r - e^{2\beta}A^{-2}[e^{-2\beta}r^2(U\sinh 2\delta + e^{-2\gamma}W\cosh 2\delta)]_u + e^{2\beta}A^{-2}(e^{-2\beta}A^2)_\phi\}, \tag{63}$$

and

$$\omega^\phi = \frac{1}{2r^2\sin\theta}\{A^2e^{-2\beta}[r^2A^{-2}(W\sinh 2\delta + Ue^{2\gamma}\cosh 2\delta)]_r - e^{2\beta}A^{-2}[r^2e^{-2\beta}(W\sinh 2\delta + Ue^{2\gamma}\cosh 2\delta)]_u + A^{-2}e^{2\beta}(A^2e^{-2\beta})_\theta\}. \tag{64}$$

Using the same scheme employed for the Bondi metric, we get, for the leading term of the absolute value of ω^μ,

$$\Omega = -\frac{1}{2r}[(c_{\theta u} + 2c_u\cot\theta + d_{\phi u}\csc\theta)^2 + (d_{\theta u} + 2d_u\cot\theta - c_{\phi u}\csc\theta)^2]^{1/2}, \tag{65}$$

which, as expected, coincides with (48) in the Bondi (axially and reflection symmetric) case ($d = c_\phi = 0$). It is worth stressing the fact that now we have two news functions (c_u, d_u).

In the same way, for the super-Poynting vector, we obtain in this case,

$$P_\mu = (0, P_r, P_\theta, P_\phi). \tag{66}$$

Since the explicit expressions are too long (see [28] for details), we shall just present the leading terms for each super-Poynting component, and they read

$$P_r = -\frac{2}{r^2}(d_{uu}^2 + c_{uu}^2), \tag{67}$$

$$P_\theta = -\frac{2}{r^2\sin\theta}\{[2(d_{uu}^2+c_{uu}^2)c + c_{uu}c_u + d_{uu}d_u]\cos\theta + \left[c_{uu}c_{\theta u} + d_{uu}d_{\theta u} + (c_{uu}^2 + d_{uu}^2)c_\theta\right]\sin\theta + c_{uu}d_{\phi u} - d_{uu}c_{\phi u} + (d_{uu}^2 + c_{uu}^2)d_\phi\}, \tag{68}$$

$$P_\phi = \frac{2}{r^2}\{2[c_{uu}^2d_u - d_{uu}c_u - (d_{uu}^2 + c_{uu}^2)d]\cos\theta + \left[c_{uu}d_{\theta u} - d_{uu}c_{\theta u} - (c_{uu}^2 + d_{uu}^2)d_\theta\right]\sin\theta + (d_{uu}^2 + c_{uu}^2)c_\phi - (c_{uu}c_{\phi u} + d_{uu}d_{\phi u})\}, \tag{69}$$

from which it follows,

$$P^\phi = -\frac{2}{r^4\sin^2\theta}\{\sin\theta[d_{u\theta}c_{uu} - d_{uu}c_{u\theta}] + 2\cos\theta[c_{uu}d_u - d_{uu}c_u] - [c_{uu}c_{u\phi} + d_{uu}d_{u\phi}]\}. \tag{70}$$

This component of course vanishes in the Bondi case.

From the expressions above, we see that the main conclusion established for the Bondi metric is also valid in the most general case—namely, there is always a non-vanishing component of P^{μ} on the plane orthogonal to a unit vector along which there is a non-vanishing component of vorticity, and inversely, P^{μ} vanishes on a plane orthogonal to a unit vector along which the component of vorticity vector vanishes. The link between the super-Poynting vector and vorticity is thereby firmly established.

So far, we have shown the appearance of vorticity in stationary vacuum spacetimes, stationary electro-vacuum spacetimes, and in radiative vacuum spacetimes (Bondi–Sachs), and have succeeded in exhibiting the link between this vorticity and a circular flow of electromagnetic and/or super-energy on the plane orthogonal to the vorticity vector. It remains for us to analyze the possible role of electromagnetic radiation in the appearance of vorticity. The next section is devoted to this issue.

5.2. Electromagnetic Radiation and Vorticity

The relationship between electromagnetic radiation and vorticity has been unambiguously established in [54]. The corresponding calculations are quite cumbersome, and we shall not reproduce them here. Instead, we shall highlight the most important results emerging from such calculations.

The formalism used to study the general electro-vacuum case (including electromagnetic radiation) was developed by van der Burg in [55]. It represents a generalization of the Bondi–Sachs formalism for the Einstein–Maxwell system.

Thus, the starting point is the Einstein–Maxwell system of equations, which reads

$$R_{\mu\gamma} + T_{\mu\gamma} = 0, \tag{71}$$

$$F_{[\mu\nu,\delta]} = 0, \tag{72}$$

$$F^{\mu\nu}_{;\nu} = 0, \tag{73}$$

where $R_{\mu\gamma}$ is the Ricci tensor, and the energy momentum tensor $T_{\mu\gamma}$ of the electromagnetic field is given, as usual, by

$$T_{\mu\nu} = \frac{1}{4} g_{\mu\nu} F_{\gamma\delta} F^{\gamma\delta} - g^{\gamma\delta} F_{\mu\gamma} F_{\nu\delta}. \tag{74}$$

Then, following the script indicated in [51], that is, expanding the physical and metric variables in power series of $1/r$ and using the Einstein–Maxwell equations, one arrives at the conclusion that if a specific set of functions is prescribed on a given initial hypersurface $u = constant$, the evolution of the system is fully determined, provided the four functions, c_u, d_u, X, Y are given for all u. These four functions are the news functions of the system. The first two (c_u, d_u) are the gravitational news functions already mentioned before for the purely gravitational case, whereas X and Y are the two news functions corresponding to the electromagnetic field, and these appear in the series expansion of F_{12}, F_{13}. Thus, whatever happens at the source leading to changes in the field, it can only do so by affecting the four news functions, and vice versa.

Following the same line of arguments, an equation for the decreasing of the mass function due to the radiation (gravitational and electromagnetic) similar to (42) can be obtained, and it reads

$$m_u = -\int_0^{2\pi} \int_0^{\pi} (c^*_u \bar{c}^*_u + \frac{1}{2} X^* \bar{X}^*) \sin\theta d\theta d\phi, \tag{75}$$

where

$$c^* = c + id, \qquad X^* = X + iY, \tag{76}$$

and the bar denotes a complex conjugate.

Having arrived at this point, we can now proceed to calculate the vorticity, the super-Poynting vector, and the electromagnetic Poynting vector. The resulting expressions are available in [54], and since they are extremely long here, we shall focus on the main conclusions emerging from them.

First, the vorticity vector (9) is calculated for the four-vector u^α given by (58). The important point to stress here is that the absolute value of ω^μ can be written generically as

$$\Omega = \Omega_{\mathcal{G}} r^{-1} + \cdots + \Omega_{\mathcal{GEM}} r^{-3} + \cdots , \tag{77}$$

where subscripts $\mathcal{G}$, $\mathcal{GEM}$, and $\mathcal{EM}$ stand for gravitational, gravito-electromagnetic, and electromagnetic. The "gravitational" subscript refers to those terms exclusively containing functions appearing in the purely gravitational case and their derivatives. "Electromagnetic" terms are those exclusively containing functions appearing in $F_{\mu\nu}$ and their derivatives, whereas the "gravito-electromagnetic" subscript refers to those terms containing functions of either kind and/or a combination of both.

Finally, we calculate the electromagnetic Poynting vector defined by

$$S^\alpha = T^{\alpha\beta} u_\beta, \tag{78}$$

and the super-Poynting vector defined by (5). Since we are not operating in vacuum, $P(C)_\alpha$ and $P(R)_\alpha$ are different, and we shall use $P(C)_\alpha$ for the discussion.

The resulting expressions are deployed in [54]. Let us summarize the main information contained in such expressions.

First, we notice that the leading terms for each super-Poynting (contravariant) component are

$$\begin{aligned} P^u &= P^u_{\mathcal{G}} r^{-4} + \cdots , \\ P^r &= P^r_{\mathcal{G}} r^{-4} + \cdots , \\ P^\theta &= P^\theta_{\mathcal{G}} r^{-4} + \cdots + P^\theta_{\mathcal{GEM}} r^{-6} + \cdots , \\ P^\phi &= P^\phi_{\mathcal{G}} r^{-4} + \cdots + P^\phi_{\mathcal{GEM}} r^{-6} + \cdots , , \end{aligned} \tag{79}$$

whereas for the electromagnetic Poynting vector, we can write

$$S^u = S^u_{\mathcal{EM}} r^{-4} + S^u_{\mathcal{GEM}} r^{-5} \cdots , \tag{80}$$

$$S^r = S^r_{\mathcal{EM}} r^{-2} + S^r_{\mathcal{GEM}} r^{-3} + \cdots , \tag{81}$$

$$S^\theta = S^\theta_{\mathcal{EM}} r^{-4} + S^\theta_{\mathcal{GEM}} r^{-5} + \cdots , and \tag{82}$$

$$S^\phi = S^\phi_{\mathcal{EM}} r^{-4} + S^\phi_{\mathcal{GEM}} r^{-5} + \cdots . \tag{83}$$

Next, there are explicit contributions from the electromagnetic news functions in $\Omega_{\mathcal{GEM}}$, as well as in $P^\phi_{\mathcal{GEM}}$ and $P^\theta_{\mathcal{GEM}}$. More so, the vanishing of these contributions in $P^\phi_{\mathcal{GEM}}$ and $P^\theta_{\mathcal{GEM}}$ implies the vanishing of the corresponding contribution in $\Omega_{\mathcal{GEM}}$, and vice versa.

From the above, it is clear that electromagnetic radiation, as described by electromagnetic news functions, does produce vorticity. Furthermore, we have identified the presence of electromagnetic news, both in the Poynting and the super-Poynting components orthogonal to the vorticity vector. Doing so, we have proved that a Bonnor-like mechanism for generating vorticity is at work in this case too, but with the important difference that now vorticity is generated by the contributions of both the Poynting and the super-Poynting vectors, on the planes orthogonal to the vorticity vector.

6. Discussion

We started this manuscript with two goals in mind. On the one hand, we wanted to identify the fundamental physical phenomenon which, being present in all scenarios exhibiting frame-dragging, could be considered as that responsible for the frame-dragging effect. In other words, we wanted to identify the factor that mediates between the source of the gravitational field and the appearance of vorticity, in any scenario.

On the other hand, we wanted to explore the observational consequences that could be derived from our analysis.

To meet our first goal, we have "deconstructed" the frame-dragging effect in several different scenarios, in order to identify a basic physical phenomenon inherent to all those situations. Doing so, it has been clearly established that in vacuum, the appearance of vorticity is always related to the existence of circular flow of super-energy in the plane orthogonal to the vorticity vector. This is true for all stationary vacuum spacetimes, as well as for general Bondi–Sachs radiative spacetimes.

In the case of electro-vacuum spacetimes, we have circular flows of super-energy, as well as circular flows of electromagnetic energy in the plane orthogonal to the vorticity vector. This is true in stationary electro-vacuum spacetimes, as well as in spacetimes admitting both gravitational and electromagnetic radiation. Particularly remarkable is the fact that electromagnetic radiation does produce vorticity.

All this having been said, a natural question arises concerning our second goal, namely, what observational consequences could be derived from the analysis presented so far?

First of all, it should be clear that the established fact that the emission of gravitational radiation always entails the appearance of vorticity in the congruence of the world-lines of observers, provides a mechanism for detecting gravitational radiation. Thus, any experimental device intended to measure rotations could be a potential detector of gravitational radiation as well. We are well-aware of the fact that extremely high sensitivities have to be reached for these detectors to be operational. Thus, from the estimates displayed in [25], we see that for a large class of possible events leading to the emission of gravitational radiation, the expected values of Ω range from $\Omega \approx 10^{-15} s^{-1}$ to $\Omega \approx 10^{-19} s^{-1}$. Although these estimates are 20 years old and deserve to be updated, we believe that probably the sensitivity of the actual technology is still below the range of expected values of vorticity. Nevertheless, the intense activity deployed in recent years in this field, invoking ring lasers, atom interferometers, atom lasers, anomalous spin-precession, trapped atoms, and quantum interference (see References [56–68] and references therein), besides the incredible sensitivities obtained so far in gyroscope technology and exhibited in the Gravity Probe B experiment [1], make us optimistic in that these kinds of detectors may be operating in the foreseeable future.

In the same order of ideas, the established link between vorticity and electromagnetic radiation has potential observational consequences which should not be overlooked. Indeed, intense electromagnetic outbursts are expected from hyperenergetic phenomena, such as collapsing hypermassive neutron stars and Gamma Ray Bursts (see [69] and references therein). Therefore, although the contributions of the $\mathcal{GEM}$ terms in (77) are of order $1/r^3$, in contrast with the $\mathcal{G}$ terms which are of order $1/r$, the coefficient of the former terms usually exceeds the latter by many orders of magnitude, which opens up the possibility to detect them more easily.

Finally, the association of the sources of electromagnetic fields (charges and currents) with vorticity suggests the possibility to extract information about the former, by measuring the latter. Thus, in [23], using the data corresponding to the earth, Bonnor estimates that the vorticity would be of the order of $\Omega \approx 4 \times 10^{-33} s^{-1}$. Although this figure is so small that we do not expect to be able to measure it in the near future, the strength of electromagnetic sources in very compact objects could produce vorticity many orders of magnitude larger.

To summarize, if we adopt the usual meaning of the verb "to explain" (a phenomenon), as referred to the action of expressing such a phenomenon in terms of fundamental concepts, then we can say that we have succeeded in explaining the frame-dragging effect as due to circular flows of super-energy and electromagnetic energy (whenever present) in planes orthogonal to vorticity vector. This result, in turn, creates huge opportunities to obtain information from self-gravitating systems by measuring the vorticity of the congruence of world-lines of observers.

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Appendix A

$$
\begin{aligned}
A11 \;=\;& \Big[-2\rho\omega_\rho(\omega_\rho^2+\omega_z^2)\gamma_\rho - 2\rho\omega_z(\omega_\rho^2+\omega_z^2)\gamma_z - \omega_z^2\omega_{\rho\rho}\rho \\
&+\omega_z^2\omega_{zz}\rho + 4\omega_z\omega_\rho\omega_{\rho z}\rho - \omega_\rho^2\omega_{zz}\rho - \omega_\rho^3 + \omega_\rho^2\omega_{\rho\rho}\rho\Big] f^4 \\
&+3\rho(\omega_\rho^2+\omega_z^2)(\omega_z f_z + \omega_\rho f_\rho) f^3 - 2\rho(-2\rho\gamma_z\omega_{\rho z} + 2\gamma_z^2\omega_\rho\rho \\
&+2\rho\gamma_\rho^2\omega_\rho + \gamma_z\omega_z + \gamma_\rho\omega_\rho + \rho\gamma_\rho\omega_{zz} - \rho\gamma_\rho\omega_{\rho\rho}) f^2 \\
&+\Big[4\rho^3(f_z\omega_z + f_\rho\omega_\rho)\gamma_\rho^2 + 2\rho^2(\rho f_{zz}\omega_\rho - 2\rho f_{\rho z}\omega_z - 2f_z\omega_z \\
&+4f_\rho\omega_\rho - 2\rho f_z\omega_{\rho z} - \rho f_{\rho\rho}\omega_\rho - \rho f_\rho\omega_{\rho\rho} + \rho f_\rho\omega_{zz})\gamma_\rho \\
&+4\rho^3(f_z\omega_z + f_\rho\omega_\rho)\gamma_z^2 + 2\rho^2(4f_\rho\omega_z + \rho f_z\omega_{\rho\rho} - 2\rho f_{\rho z}\omega_\rho \\
&+\rho f_{\rho\rho}\omega_z - 2\rho f_\rho\omega_{\rho z} - \rho f_{zz}\omega_z - \rho f_z\omega_{zz} + 2\omega_\rho f_z)\gamma_z \\
&+4\rho^3 f_{\rho z}\omega_{\rho z} - \rho^3 f_{zz}\omega_{\rho\rho} - \rho^3 f_{\rho\rho}\omega_{zz} + \rho^2 f_{zz}\omega_\rho - 2\rho^2 f_{\rho z}\omega_z \\
&-\rho^2 f_{\rho\rho}\omega_\rho + \rho^3 f_{zz}\omega_{zz} + \rho^3 f_{\rho\rho}\omega_{\rho\rho}\Big] f - 6\rho^3(f_\rho^2+f_z^2)\omega_\rho\gamma_\rho \\
&-6\rho^3(f_\rho^2+f_z^2)\omega_z\gamma_z + 3\rho^3(f_{\rho\rho}f_\rho\omega_\rho + f_{zz}f_z\omega_z + 2f_{\rho z}f_z\omega_\rho \\
&-f_{\rho\rho}f_z\omega_z + 2f_{\rho z}f_\rho\omega_z - f_{zz}f_\rho\omega_\rho)
\end{aligned}
\qquad \text{(A1)}
$$

$$
\begin{aligned}
A12 \;=\;& \omega_\rho(-7\omega_z^4 - 6\omega_\rho^2\omega_z^2 + \omega_\rho^4) f^9 \\
&+\Big[-\rho\omega_\rho f_\rho(\omega_z^4 + \omega_\rho^4 + 2\omega_\rho^2\omega_z^2) - \rho f_z\omega_z(\omega_\rho^4 + 2\omega_z^2\omega_\rho^2 + \omega_z^4)\Big] f^8 \\
&+\Big[-4\rho\omega_{zz}(\omega_\rho^2 + 3\omega_z^2) + 4\omega_\rho(-2\rho\omega_z\omega_{\rho z} + \omega_z^2)\Big] f^7 \\
&+\Big[4\rho\omega_z(-8\omega_z^2 f_z + \rho\omega_z^2 f_{\rho z} - 3\rho\omega_\rho\omega_z f_{zz} - 3\rho\omega_\rho^2 f_{\rho z} - \rho f_\rho\omega_z\omega_{zz} \\
&-5\omega_\rho^2 f_z - 2\rho f_\rho\omega_\rho\omega_{\rho z} - 2\omega_z\omega_\rho f_\rho - 2\rho f_z\omega_\rho\omega_{zz} + \rho f_z\omega_z\omega_{\rho z}) \\
&+4\rho\omega_\rho(-\rho\omega_\rho f_z\omega_{\rho z} + \rho\omega_\rho f_\rho\omega_{zz} + \omega_\rho^2 f_\rho + \rho\omega_\rho^2 f_{zz})\Big] f^6 \\
&+\Big[-6\rho^2\omega_\rho^3(f_z^2 + f_\rho^2) - 2\rho^2 f_z\omega_\rho\omega_z(2\omega_\rho f_\rho + 5\omega_z f_z) \\
&+2\rho^2 f_\rho\omega_z^2(2\omega_z f_z - \omega_\rho f_\rho)\Big] f^5 + \Big[8\rho^2(f_{\rho z}\omega_z - f_\rho\omega_{zz}) \\
&-16\rho^3(f_{\rho z}\omega_{\rho z} + f_{zz}\omega_{zz}) + 10\rho^3 f_z\omega_z f_\rho\omega_\rho(f_z\omega_z + f_\rho\omega_\rho) \\
&-2\rho^3 f_z(f_\rho f_z\omega_\rho^3 + f_\rho^2\omega_z^3) + 2\rho^3 f_\rho^3(\omega_\rho^3 - \omega_\rho\omega_z^2) \\
&+2\rho^3 f_z^3(\omega_z^3 - \omega_z\omega_\rho^2)\Big] f^4 + \Big[-24\rho^3 f_{\rho z}(\omega_\rho f_z + \omega_z f_\rho) \\
&+4\rho^2 f_\rho(f_\rho\omega_\rho - 4f_z\omega_z) + 4\rho^3(3f_\rho^2\omega_{zz} + 2f_{zz}f_\rho\omega_\rho - 2\omega_{\rho z}f_\rho f_z \\
&+f_z^2\omega_{zz} - 10f_{zz}\omega_z f_z)\Big] f^3 + \Big[4\rho^4 f_{\rho z}(f_\rho^2\omega_z - f_z^2\omega_z + 2\omega_\rho f_\rho f_z) \\
&+4\rho^4 f_{zz}(f_z^2\omega_\rho - f_\rho^2\omega_\rho + 2\omega_z f_\rho f_z) + 4\rho^4\omega_{\rho z}(-f_z^3 + 3f_z f_\rho^2) \\
&+4\rho^4\omega_{zz}(-f_\rho^3 + 3f_\rho f_z^2) + 4\rho^3(4f_\rho^2 f_z\omega_z - 3f_z^2 f_\rho\omega_\rho \\
&-2f_\rho^3\omega_\rho + 3f_z^3\omega_z)\Big] f^2 + \Big[\rho^4\omega_\rho(14f_z^2 f_\rho^2 - 7f_z^4 + 5f_\rho^4) \\
&+4\rho^4\omega_z(f_\rho^3 f_z + 5f_z^3 f_\rho)\Big] f - \rho^5\omega_z(2f_z^3 f_\rho^2 + f_z^5 + f_z f_\rho^4) \\
&-\rho^5\omega_\rho(2f_\rho^3 f_z^2 + f_\rho^5 + f_\rho f_z^4)
\end{aligned}
\qquad \text{(A2)}
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References

1. Everitt, C.W.F.; DeBra, D.B.; Parkinson, B.W.; Turneaure, J.P.; Conklin, J.W.; Heifetz, M.I.; Keiser, G.M.; Silbergleit, A.S.; Holmes, T.; Kolodziejczak, J.; et al. Gravity Probe B: Final Results of a Space Experiment to Test General Relativity. *Phys. Rev. Lett.* **2011**, *106*, 221101. [CrossRef]
2. Debono, I.; Smoot, G.F. General Relativity and Cosmology: Unsolved Questions and Future Directions. *Universe* **2016**, *2*, 23. [CrossRef]
3. Vishwakarma, R.G. Einstein and Beyond: A Critical Perspective on General Relativity. *Universe* **2016**, *2*, 11. [CrossRef]
4. Rindler, W. The case against space dragging. *Phys. Lett. A* **1997**, *233*, 25–29. [CrossRef]
5. Rindler, W. The Lense-Thirring effect exposed as anti-Machian. *Phys. Lett. A* **1994**, *187*, 236–238. [CrossRef]
6. Bondi, H.; Samuel, J. The Lense-Thirring effect and Mach's principle. *Phys. Lett. A* **1997**, *228*, 121–126. [CrossRef]
7. Shapiro, I.I.; Reasenberg, R.D.; Chandler, J.F.; Babcock, R.W. Measurement of the de Sitter precession of the Moon: A relativistic three-body effect. *Phys. Rev. Lett.* **1988**, *61*, 2643. [CrossRef]
8. Lense, J.; Thirring, H. Uber den einfluss der eigenrotation der zentralkorper auf die bewuegung der planeteren und monde nach der Einsteinschen gravitationstheorie. *Phys. Z.* **1918**, *19*, 156–163.
9. Pfister, H. On the history of the so-called Lense-Thirring effect. *Gen. Relativ. Gravit.* **2007**, *39*, 1735–1748. [CrossRef]
10. Pfister, H. Editorial note to: Hans Thirring, On the formal analogy between the basic electromagnetic equations and Einstein's gravity equations in first approximation. *Gen. Relativ. Gravit.* **2012**, *44*, 3217–3224. [CrossRef]
11. Pfister, H. The History of the So-Called Lense–Thirring Effect, and of Related Effects. In *General Relativity and John Archibald Wheeler*; Ciufolini, I., Matzner, R., Eds.; Springer: Dordrecht, The Netherlands, 2010; Volume 367.
12. Schiff, L.I. Possible New Experimental Test of General Relativity Theory. *Phys. Rev. Lett* **1960**, *4*, 215–217. [CrossRef]
13. Rindler, W.; Perlick, V. Rotating coordinates as tools for calculating circular geodesics and gyroscopic precession. *Gen. Rel. Grav.* **1990**, *22*, 1067–1081. [CrossRef]
14. Pavlis, E.C.; Iorio, L. The impact of tidal errors on the determination of the Lense–Thirring effect from satellite laser ranging. *Int. J. Mod. Phys. D.* **2002**, *11*, 599–618. [CrossRef]
15. Iorio, L. On the Lense-Thirring test with the Mars Global Surveyor in the gravitational field of Mars. *Central Eur. J. Phys.* **2010**, *8*, 509–513. [CrossRef]
16. Iorio, L.; Lichtenegger, H.; Ruggiero, M.; Corda, C. Phenomenology of the Lense–Thirring effect in the Solar System. *Astrophys. Space Sci.* **2011**, *331*, 351–395. [CrossRef]
17. Iorio, L. LETSGO: A spacecraft-based mission to accurately measure the solar angular momentum with frame-dragging. *Acta Astronaut.* **2013**, *86*, 149–157. [CrossRef]
18. Strohaber, J. frame-dragging with optical vortices. *Gen. Relativ. Grav.* **2013**, *45*, 2457–2465. [CrossRef]
19. Iorio, L. On a recent preliminary study for the measurement of the Lense–Thirring effect with the Galileo satellites. *Int. J. Mod. Phy. D* **2014**, *23*, 1450028. [CrossRef]
20. Iorio, L. Measuring general relativistic dragging effects in the Earth's gravitational field with ELXIS: A proposal. *Class. Quantum Grav.* **2019**, *36*, 035002. [CrossRef]
21. Iorio, L. Is there still something left that Gravity Probe B can measure? *Universe* **2020**, *6*, 85. [CrossRef]
22. Herrera, L.; Di Prisco, A.; Carot, J. Frame-dragging and super–energy. *Phys. Rev. D* **2007**, *76*, 044012-5. [CrossRef]
23. Bonnor, W.B. Dragging of inertial frames by a charged magnetic dipole. *Phys. Lett. A* **1991**, *158*, 23–26. [CrossRef]
24. Herrera, L.; González, G.A.; Pachón, L.A.; Rueda, J.A. frame-dragging, vorticity and electromagnetic fields in axially symmetric stationary spacetimes. *Class. Quantum Grav.* **2006**, *23*, 2395–2408. [CrossRef]
25. Herrera, L.; Hernández-Pastora, J.L. On the influence of gravitational radiation on a gyroscope. *Class. Quantum Grav.* **2000**, *17*, 3617–3625. [CrossRef]
26. Sorge, F.; Bini, D.; de Felice, F. Gravitational waves, gyroscopes and frame-dragging. *Class. Quantum Grav.* **2001**, *18*, 2945–2958. [CrossRef]
27. Herrera, L.; Santos, N.O.; Carot, J. Gravitational radiation, vorticity and the electric and magnetic part of Weyl tensor. *J. Math. Phys.* **2006**, *47*, 052502. [CrossRef]
28. Herrera, L.; Barreto, W.; Carot, J.; Di Prisco, A. Why does gravitational radiation produces vorticity? *Class. Quantum Grav.* **2007**, *24*, 2645–2651. [CrossRef]
29. Herrera, L. Radiation and vorticity: The missing link. *Gen. Relativ. Gravit.* **2014**, *46*, 1654–1657. [CrossRef]
30. Bicak, J.; Katz, J.; Lynden–Bell, D. Gravitational waves and dragging effects. *Class. Quantum Grav.* **2008**, *25*, 165017. [CrossRef]
31. Bini, A.D.; Geralico, A.; Plastino, W. Cylindrical gravitational waves: C–energy, super-energy and associated dynamical effects. *Class. Quantum Grav.* **2019**, *36*, 095012. [CrossRef]
32. Herrera, L. Gravitational radiation , vorticity and super–energy. A conspicuous threesome. *Universe* **2019**, *5*, 164. [CrossRef]
33. Ruggiero, M.L.; Ortolan, A. Gravito–magnetic resonance in the field of a gravitational wave. *Phys. Rev. D* **2020**, *102*, 101501(R). [CrossRef]
34. Bel, L. Sur la radiation gravitationelle. *Comp. Rend. Acad. Sci.* **1958**, *247*, 1094–1096.
35. Bel, L. Radiation states and the problem of energy in general relativity. *Gen. Relativ. Gravit.* **2000**, *32*, 2047–2078. [CrossRef]
36. Bel, L. Introduction d'un tenseur du quatrieme order. *Comp. Rend. Acad. Sci. Paris* **1959**, *248*, 1297–1300.

37. García–Parrado Gómez Lobo, A. Dynamical laws of superenergy in general relativity. *Class. Quantum Grav.* **2008**, *25*, 015006. [CrossRef]
38. Kerr, R.P. Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics. *Phys. Rev. Lett.* **1963**, *11*, 237. [CrossRef]
39. Hernández-Pastora, J.L.; Herrera, L. Interior solution for the Kerr metric. *Phys. Rev. D* **2017**, *95*, 024003. [CrossRef]
40. Papapetrou, A. Eine rotationssymmetrische losung in der allgmeinen relativitatstheorie. *Ann. Phys.* **1953**, *12*, 309–315. [CrossRef]
41. Stephani, H.; Kramer, D.; MacCallum, M.; Honselaers, C.; Herlt, E. *Exact Solutions to Einstein's Field Equations*, 2nd ed.; Cambridge University Press: Cambridge, UK, 2003; p. 309.
42. Feynman, R.P.; Leighton, R.B.; Sand, M. *Lectures on Physics II*; Addison–Wesley: Reading, PA, USA, 1964; pp. 27–28.
43. DeWitt, B.S.; Brehme, R. Radiation damping in a gravitational field. *Ann. Phys.* **1960**, *9*, 220–259. [CrossRef]
44. Kundt, W.; Newman, E.T. Hyperbolic Differential Equations in Two Dimensions. *J. Math. Phys.* **1968**, *9*, 2193–2210 . [CrossRef]
45. Bonnor, W.B. Approximate methods and gravitational radiation. In *Proceedings of the Meeting on General Relativity*; Barbera, G., Ed.; Comitato Nazionale per le Manifeestazioni Celebrative: Florence, Italy, 1965; pp. 119–142.
46. Couch, W.; Torrence, R.; Janis, A.; Newman, E.T. Tail of a Gravitational Wave. *J. Math. Phys.* **1968**, *9*, 484. [CrossRef]
47. Bonnor, W.B. Gravitational wave tails. In *Ondes et Radiations Gravitationelles*; Editions du Centre National de la Recherche Scientifique: Paris, France, 1974; pp. 73–81.
48. Blanchet, L.; Shafer, G. Gravitational wave tails and binary star systems. *Class. Quantum. Grav.* **1993**, *10*, 2699–2722. [CrossRef]
49. Marchand, T.; Blanchet, L.; Faye, G. Gravitational-wave tail effects to quartic non-linear order. *Class. Quantum. Grav.* **2016**, *33*, 244003. [CrossRef]
50. Abbot, B.P.; Abbot, R.; Abbot, T.D.; Abernathy, M.R.; Acernese, F.; Ackley, K.; Adams, C.; Adams, T.; Addesso, P.; Adhikari, R.X.; et al. (LIGO Scientific Collaboration and Virgo Collaboration). Observation of Gravitational Waves from a Binary Black Hole Merger. *Phys. Rev. Lett.* **2016**, *116*, 061102. [CrossRef]
51. Bondi, H.; van der Burg, M.G.J.; Metzner, A.W.K. Gravitational waves in general relativity VII. Waves from axi–symmetric isolated systems. *Proc. R. Soc. A* **1962**, *269*, 21–52.
52. Sachs, R. Gravitational waves in general relativity VIII. Waves in asymptotically flat space–time. *Proc. R. Soc. A* **1962**, *270*, 103–125.
53. Van der Burg, M.G.J. Gravitational Waves in General Relativity. IX. Conserved Quantities. *Proc. R. Soc.* **1966**, *A294*, 112–122.
54. Herrera, L.; Barreto, W. Electromagnetic radiation produces frame-dragging. *Phys. Rev. D* **2012**, *86*, 064014. [CrossRef]
55. Van der Burg, M.G.J. Gravitational Waves in General Relativity X. Asymptotic Expansions for the Einstein-Maxwell Field. *Proc. R. Soc. A* **1969**, *310*, 221–230.
56. Scully, M.O.; Zubairy, M.S.; Haugan, M.P. Proposed optical test of metric gravitation theories. *Phys. Rev. A* **1981**, *24*, 2009. [CrossRef]
57. Lenef, A.; Hammond, T.D.; Smith, E.T.; Chapman, M.S.; Rubenstein, R.A.; Pritchard, D.E. Rotation Sensing with an Atom Interferometer. *Phys. Rev. Lett.* **1997**, *78*, 760. [CrossRef]
58. Gustavson, T.L.; Bouyer, P.; Kasevich, M.A. Precision Rotation Measurements with an Atom Interferometer Gyroscope. *Phys. Rev. Lett.* **1997**, *78*, 2046. [CrossRef]
59. Stedman, G.E.; Schreiber, K.U.; Bilger, H.R. On the detectability of the Lense–Thirring field from rotating laboratory masses using ring laser gyroscope interferometers. *Class. Quantum Grav.* **2003**, *20*, 2527–2540. [CrossRef]
60. Stedman, G.E.; Hurst, R.B.; Schreiber, K.U. On the potential of large ring lasers. *Opt. Commun.* **2007**, *279*, 124–129. [CrossRef]
61. Di Virgilio, A.; Schreiber, K.U.; Gebauer, A.; Wells, J.-P.R.; Tartaglia, A.; Belfi, J.; Beverini, N.; Ortolan, A. A laser gyroscope system to detect the gravitomagnetic effect on earth. *Int. J. Mod. Phys. D* **2010**, *19*, 2331–2343. [CrossRef]
62. Bosi, F.; Cella, G.; Di Virgilio, A.; Ortolan, A.; Porzic, A.; Solimenof, S.; Cerdonio, M.; Zendri, J.P.; Allegrini, M.; Bel, J.; et al. Measuring gravito-magnetic efects by multiring–laser gyroscope. *Phys. Rev. D* **2011**, *84*, 122002. [CrossRef]
63. Robins, N.P.; Altin, P.A.; Debs, J.E.; Close, J.D. Atom lasers: Production, properties and prospects for precision inertial measurement. *Phys. Rep.* **2013**, *529*, 265–296. [CrossRef]
64. Tartaglia, A. Experimental determination of gravitomagnetic effects by means of ring lasers. *J. Phys. Conf. Ser.* **2013**, *453*, 012019. [CrossRef]
65. Di Virgilio, A.; Allegrini, M.; Beghi, A.; Belfi, J.; Beverini, N.; Bosi, F.; Bouhadef, B.; Calamai, M.; Carelli, G.; Cuccato, D.; et al. A ring laser array for fundamental physics. *Comp. Rend. Phys.* **2014**, *15*, 866–874. [CrossRef]
66. Campbell, W.C.; Hamilton, P. Rotation sensing with trapped ions. *J. Phys. B* **2017**, *50*, 064002. [CrossRef]
67. Tartaglia, A.; di Virgilio, A.; Belfi, J.; Beverini, N.; Ruggiero, M.L. Testing general relativity by means of ringlasers. *Eur. Phys. J. Plus* **2017**, *132*, 73. [CrossRef]
68. di Virgilio, A.; Beverini, N.; Carelli, G.; Ciampini, D.; Fuso, F.; Maccioni, E. Analysis of ring lasers gyroscopes including laser dynamics. *Eur. Phys. J. C* **2019**, *79*, 573. [CrossRef]
69. Lehner, L.; Palenzuela, C.; Liebling, S.L.; Thompson, C.; Hanna, C. Intense electromagnetic outbursts from collapsing hypermassive neutron stars. *Phys. Rev. D* **2012**, *86*, 104035. [CrossRef]